RBE 2002: Unified Robotics II
Analog Filtering

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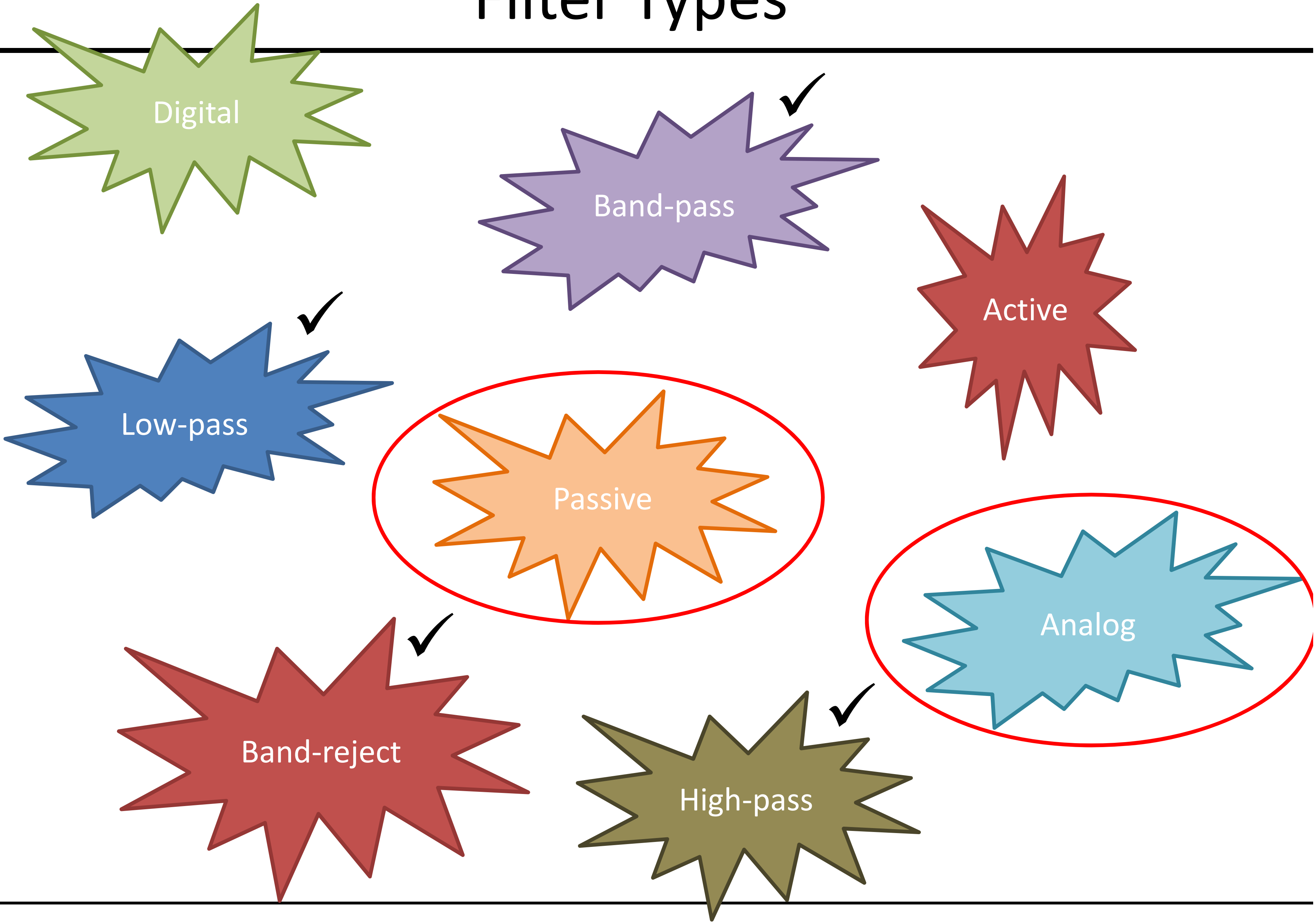
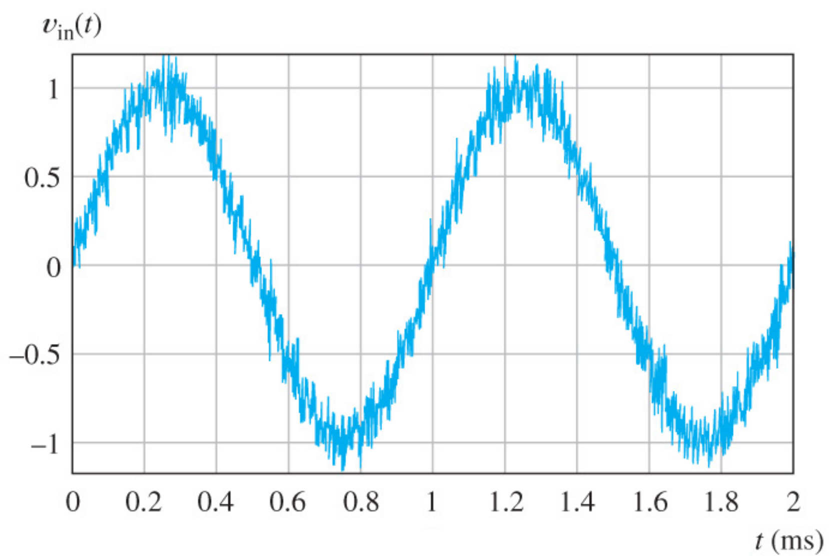
Filter Types



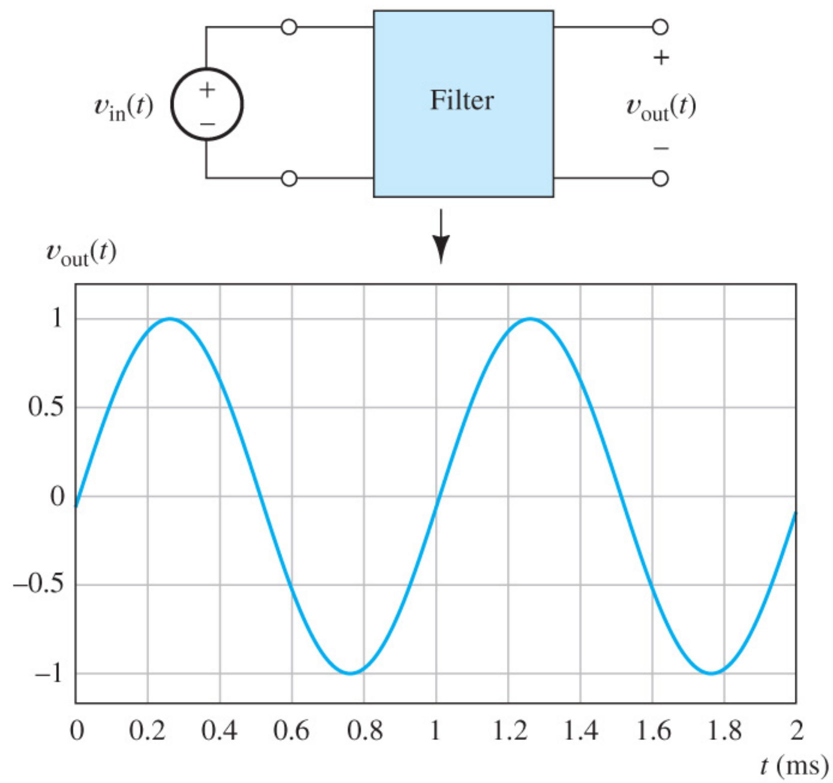
Real World Application

- ✓ Suppose you have a sensor on your robot that is generating a sinusoidal signal
- ✓ Also suppose there is some high frequency noise that is being picked up by the sensor & wiring
- ✓ How might you deal with this?



Real World Application

- ✓ Send the noisy signal through a low pass filter
- ✓ If the filter is set up properly, the high frequencies are removed and you are left with a clean signal

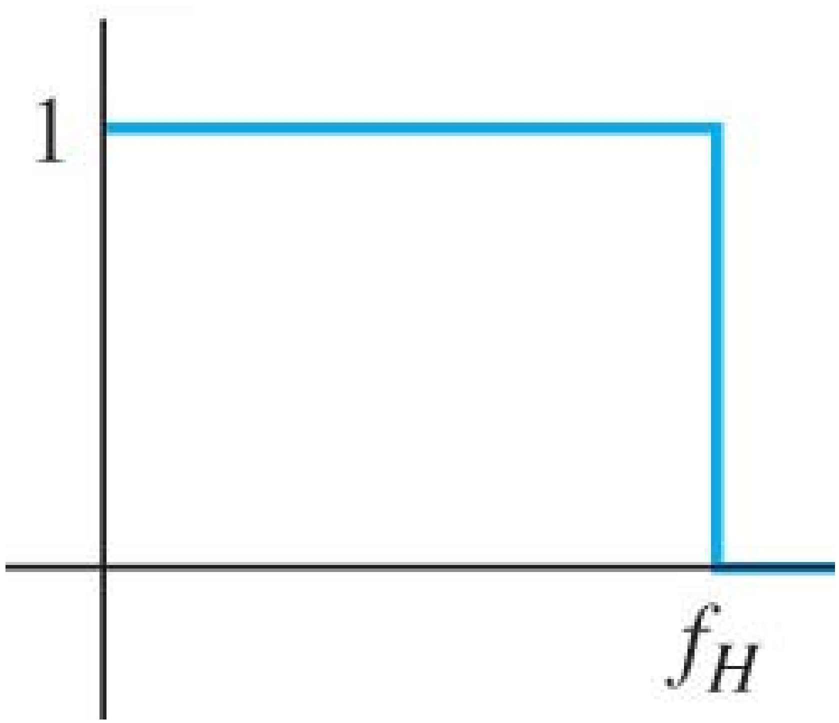


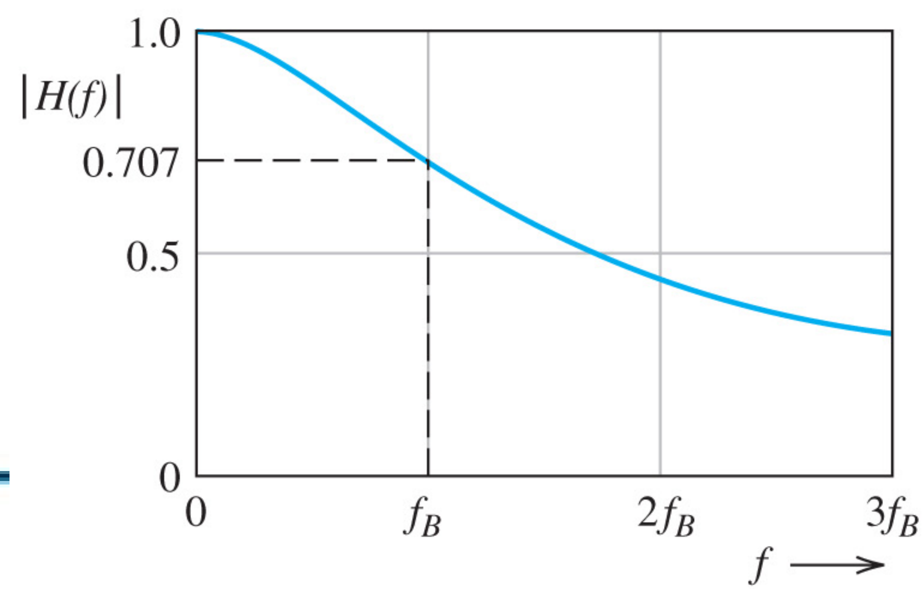
Ideal Filters

- ✓ An ideal filter passes frequencies in the desired range with no change at all, and completely rejects frequencies outside of that range
- ✓ What kinds of filters might there be?
  - Low pass
  - High pass
  - Band pass
  - Band reject

Real Filters

- ✓ It is difficult to create real filters that behave in the same way as ideal filters
- ✓ That said, real filters can be (and are) very effective
- ✓ We will concentrate today on passive, analog filters





Frequency Characteristics

- ✓ We will construct our passive, analog filters out of the following components:
  - Resistors
  - Capacitors
  - Inductors
- ✓ First we will need to examine the frequency characteristics of these devices
- ✓ We also need to define the term *impedance*

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

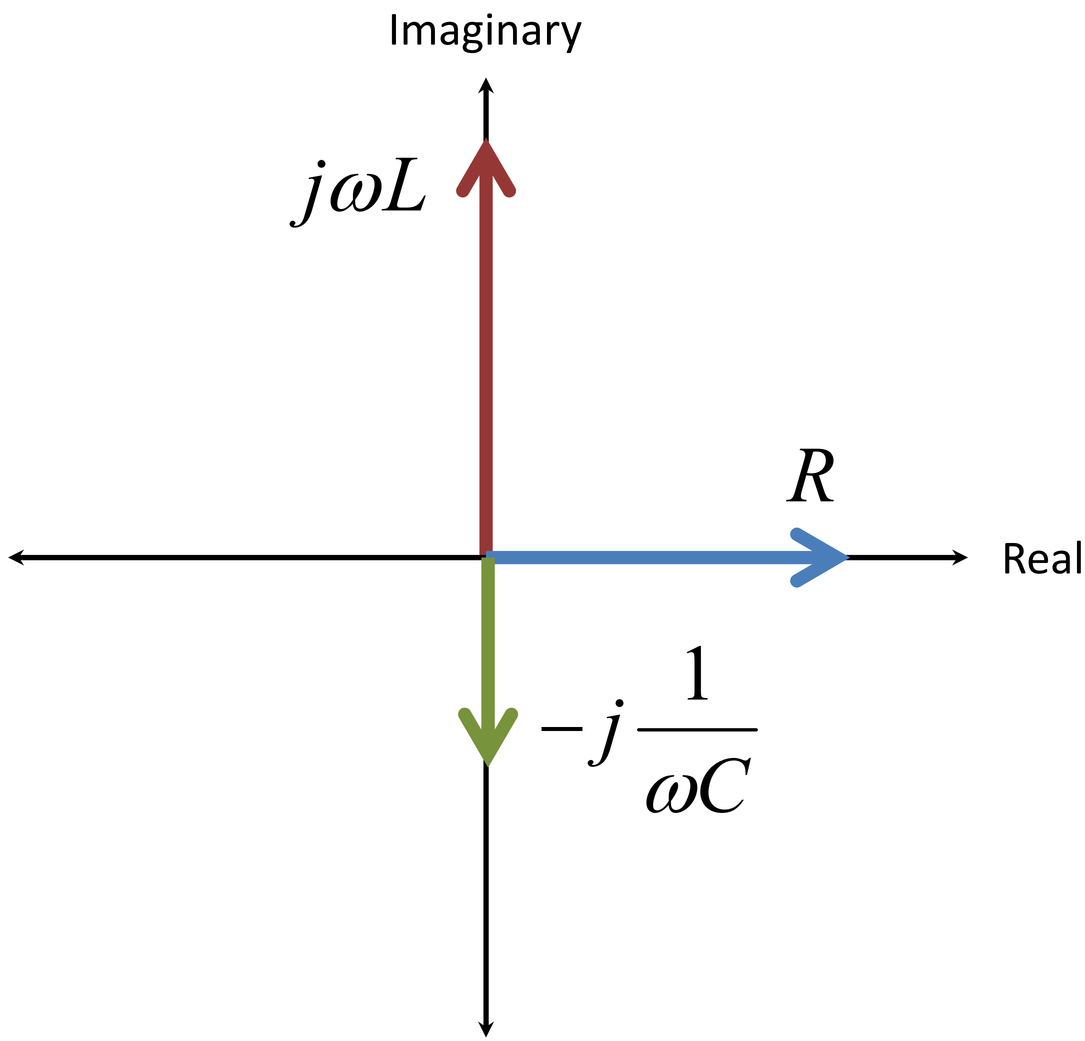
Impedance Characteristics

- ✓ Resistors $$\frac{\mathbf{V}}{\mathbf{I}} = R$$

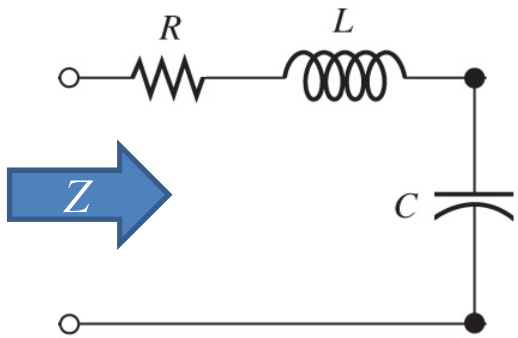
- ✓ Capacitors $$\frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} = \frac{1}{j2\pi fC} = -j\frac{1}{2\pi fC}$$

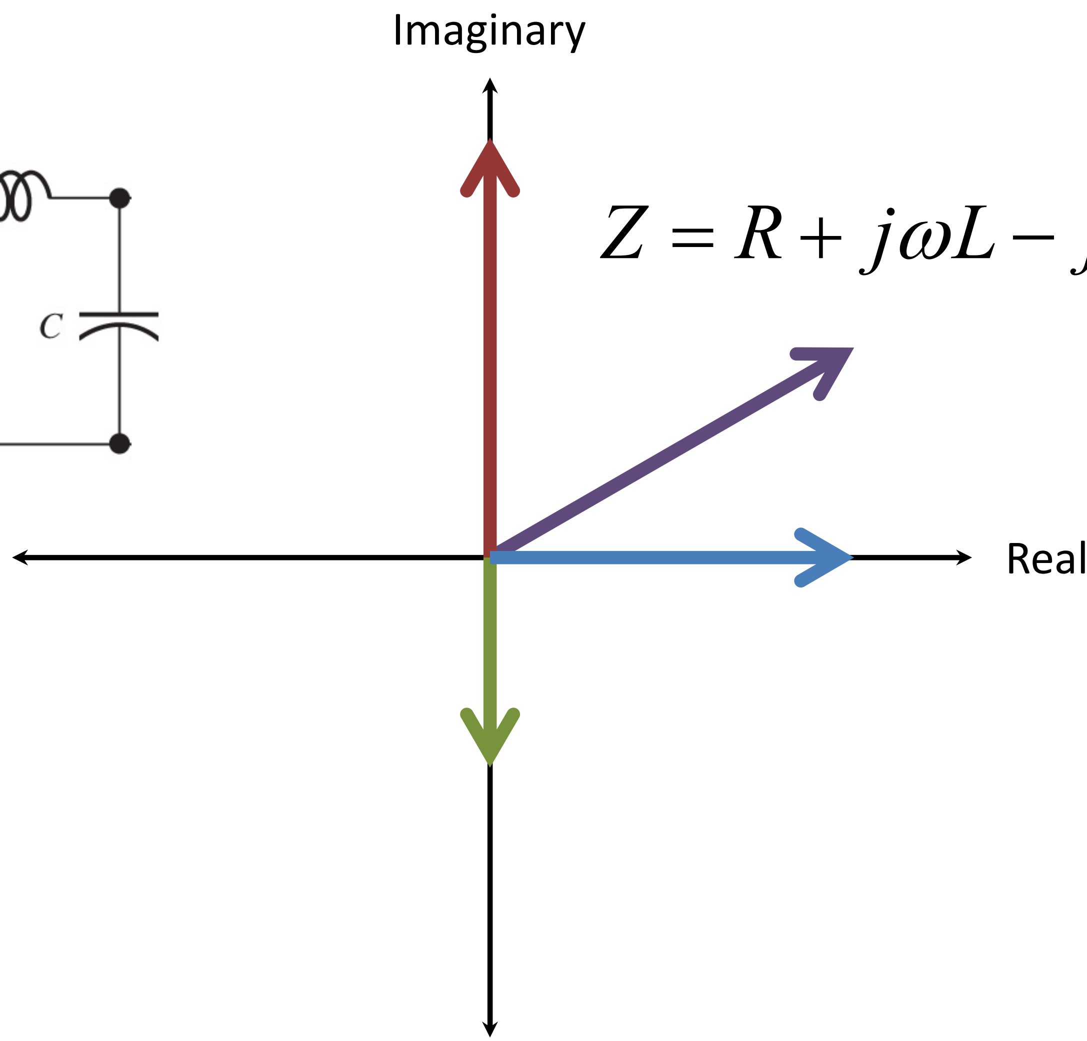
- ✓ Inductors $$\frac{\mathbf{V}}{\mathbf{I}} = j\omega L = j2\pi fL$$

Impedance Characteristics



Impedance Characteristics





$$Z = R + j\omega L - j\frac{1}{\omega C}$$

Frequency Characteristics

✓ Capacitors $$\frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$$

- At low frequencies: $$\lim_{f \to 0} \frac{1}{j2\pi fC} \to \infty$$

- At high frequencies: $$\lim_{f \to \infty} \frac{1}{j2\pi fC} \to 0$$

Frequency Characteristics

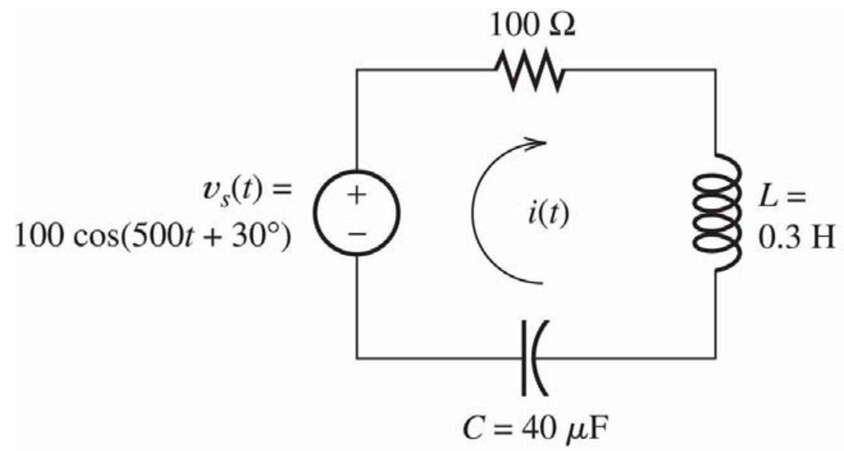
✓ Inductors $$\frac{\mathbf{V}}{\mathbf{I}} = j\omega L = j2\pi fL$$

- At low frequencies: $$\lim_{f \to 0} j2\pi fl \to 0$$

- At high frequencies: $$\lim_{f \to \infty} j2\pi fl \to \infty$$

Introduction to Phasors

- ✓ Imagine a circuit with a steady-state sinusoidal excitation



- ✓ In the general case, we can write this as:

$$v(t) = A\cos(\omega t + \phi)$$

Introduction to Phasors

- ✓ Actually, let's suppose that things are in reality more complicated – that the excitation signal is as follows:

$$v(t) = 10\cos(\omega t) + 5\sin(\omega t + 60^\circ) + 5\cos(\omega t + 90^\circ)$$

- ✓ You could use trig identities and with a fair amount of work get to an answer
- ✓ But there is an easier way to approach the problem

Introduction to Phasors

- ✓ For a sinusoidal voltage of the form:

$$v_1(t) = A_1 \cos(\omega t + \phi_1)$$

we define the phasor as:

$$\mathbf{V}_1 = V_1 \cos \angle \phi_1$$

- ✓ A phasor for a sinusoid is therefore a complex number having a magnitude equal to the peak value and having a phase angle the same as the sinusoid

Introduction to Phasors

- ✓ Suppose we have a sinusoidal voltage of the form:

$$v_2(t) = A_2 \sin(\omega t + \phi_2)$$

- ✓ In this case we first convert it back to a cosine function using the trig identity:

$$\sin(z) = \cos(z - 90^\circ)$$

- ✓ This gives us:

$$v_2(t) = A_2 \cos(\omega t + \phi_2 - 90^\circ)$$

$$\mathbf{V}_2 = V_2 \cos\angle(\phi_2 - 90^\circ)$$

Introduction to Phasors

- ✓ We are now in a position to deal with our original complicated case:

$$v(t) = 10\cos(\omega t) + 5\sin\left(\omega t + 60^\circ\right) + 5\cos\left(\omega t + 90^\circ\right)$$

- ✓ We begin by converting everything back to cosine functions:

$$v(t) = 10\cos(\omega t) + 5\cos\left(\omega t - 30^\circ\right) + 5\cos\left(\omega t + 90^\circ\right)$$

Introduction to Phasors

- ✓ Recall Euler's identity:

$$\cos(\theta) = \mathrm{Re}\left(e^{j\theta}\right) = \mathrm{Re}\left[\cos(\theta) + j\sin(\theta)\right]$$

- ✓ We can write our original equation:

$$v(t) = 10\cos(\omega t) + 5\cos\left(\omega t - 30^\circ\right) + 5\cos\left(\omega t + 90^\circ\right)$$

as:

$$v(t) = 10\,\mathrm{Re}\left[e^{j\omega t}\right] + 5\,\mathrm{Re}\left[e^{j\left(\omega t - 30^\circ\right)}\right] + 5\,\mathrm{Re}\left[e^{j\left(\omega t + 90^\circ\right)}\right]$$

Introduction to Phasors

- ✓ We can bring the scalar inside, resulting in:

$$v(t) = \mathrm{Re}\left[10e^{j\omega t}\right] + \mathrm{Re}\left[5e^{j\left(\omega t - 30^\circ\right)}\right] + \mathrm{Re}\left[5e^{j\left(\omega t + 90^\circ\right)}\right]$$

  and then group the terms as:

$$v(t) = \mathrm{Re}\left[10e^{j\omega t} + 5e^{j\left(\omega t - 30^\circ\right)} + 5e^{j\left(\omega t + 90^\circ\right)}\right]$$

- ✓ We now can factor out the common term $e^{j\omega t}$ to get:

$$v(t) = \mathrm{Re}\left[\left(10 + 5e^{-j30^\circ} + 5e^{j90^\circ}\right)e^{j\omega t}\right]$$

Introduction to Phasors

✓ The complex numbers can now be put in polar form:

$$v(t) = \mathrm{Re}\left[\left(10\angle\left(0^\circ\right) + 5\angle\left(-30^\circ\right) + 5\angle\left(90^\circ\right)\right)e^{j\omega t}\right]$$

✓ We now combine the complex numbers:

$$\begin{aligned} 10\angle\left(0^\circ\right) + 5\angle\left(-30^\circ\right) + 5\angle\left(90^\circ\right) &= 10 + 4.33 - j2.5 + j5 \\ &= 14.33 + j2.5 \\ &= 14.54\angle 9.90^\circ \\ &= 14.54e^{j9.90^\circ} \end{aligned}$$

Introduction to Phasors

✓ The result is:

$$v(t) = \text{Re}\left[\left(14.54e^{j9.90^\circ}\right)e^{j\omega t}\right]$$

$$= \text{Re}\left[14.54e^{j\left(\omega t + 9.90^\circ\right)}\right]$$

$$= 14.54\cos\left(\omega t + 9.90^\circ\right)$$

Whew!!!

That actually wasn't
all that easy…

Introduction to Phasors

- ✓ Let's try this once again, but this time we'll take a shortcut…
- ✓ Here is our original signal:

$$v(t) = 10\cos(\omega t) + 5\sin\left(\omega t + 60^\circ\right) + 5\cos\left(\omega t + 90^\circ\right)$$

- ✓ The shortcut is to simply write them as phasors using this notation:

$$\mathbf{V}_x = V_x \cos \angle \phi_x$$

Introduction to Phasors

✓ The gives us:

$$v(t) = 10\cos(\omega t) + 5\sin(\omega t + 60^\circ) + 5\cos(\omega t + 90^\circ)$$

$$\mathbf{V}_1 = V_1 \cos\angle\phi_1 = 10\angle(0^\circ)$$

$$\mathbf{V}_2 = V_2 \cos\angle\phi_2 = 5\angle(-30^\circ)$$

$$\mathbf{V}_3 = V_3 \cos\angle\phi_3 = 5\angle(90^\circ)$$

Introduction to Phasors

- ✓ We can now add the phasors as before:

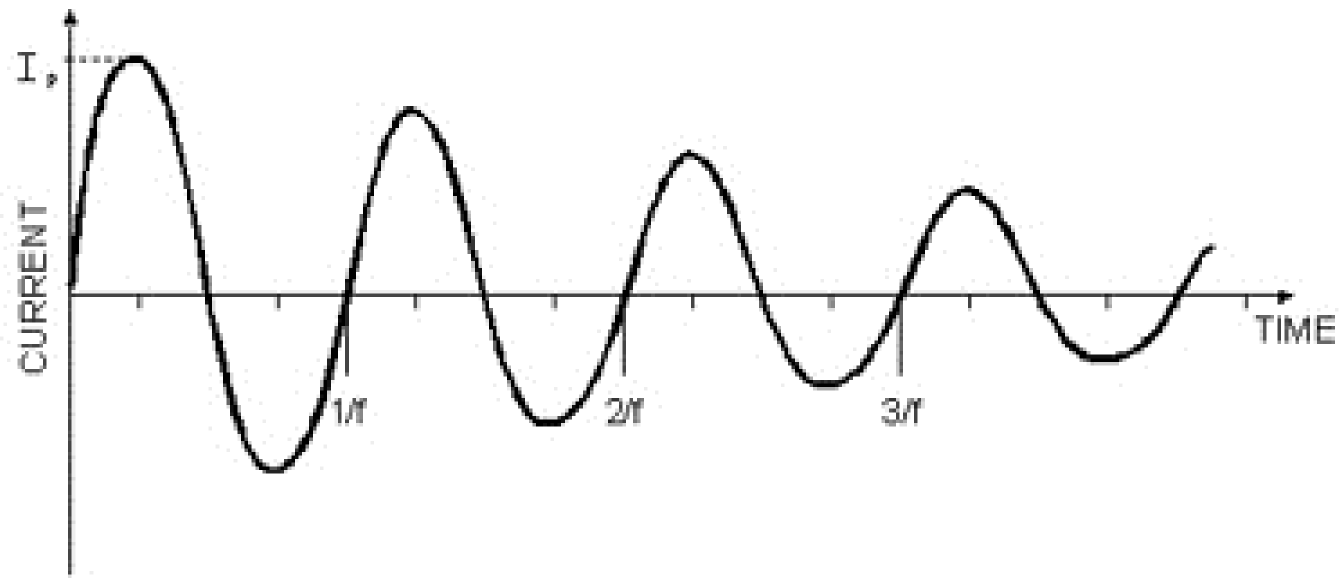
$$\begin{aligned}\mathbf{V} &= \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \\ &= 10\angle\left(0^\circ\right) + 5\angle\left(-30^\circ\right) + 5\angle\left(90^\circ\right) \\ &= 10 + 4.33 - j2.5 + j5 \\ &= 14.33 + j2.5 \\ &= 14.54\angle\left(9.90^\circ\right)\end{aligned}$$

- ✓ Converting back to the time domain, we get:

$$v\left(t\right) = 14.54\cos\left(\omega t + 9.90^\circ\right)$$

Introduction to Phasors

- ✓ We said this was true for the case of steady-state sinusoidal excitation
- ✓ Let's now extend this to another interesting class of waveforms, the damped sinusoid



Introduction to Phasors

- ✓ The most general form of a damped sinusoid is:

$$v(t) = Ae^{\sigma t}\cos(\omega t + \phi)$$

  where the constant σ is real and (usually) negative or zero

- ✓ If σ is zero, this corresponds to an undamped sinusoid
- ✓ We can easily extend our existing definitions of phasor voltages and currents to account for damped sinusoids if we make just a slight change…

Introduction to Phasors

- ✓ We define a new variable s which we will call…
- ✓ …the complex frequency
- ✓ We define it as:

$$s = \sigma + j\omega$$

- ✓ Let's look at the special case where $\sigma = 0$

$$s = j\omega$$

Introduction to Phasors

- ✓ We can start with a time-domain representation of a circuit's behavior:

$$v(t) = A\cos(\omega t + \phi)$$

- ✓ This in turn can be represented using phasor notation:

$$\mathbf{V}(j\omega) = Ae^{j\phi} = A\angle\phi$$

- ✓ Then with another simple change of variable we end up with:

$$\mathbf{V}(s) = Ae^{j\phi} = A\angle\phi$$

- ✓ From this we see that phasor analysis is equivalent to using the Laplace Transform

Laplace Transform

- ✓ Resistors $$Z_R(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = R$$

- ✓ Capacitors $$Z_C(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = \frac{1}{j\omega C} = \frac{1}{sC}$$

- ✓ Inductors $$Z_L(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = j\omega L = sL$$

Example: Laplace Transform

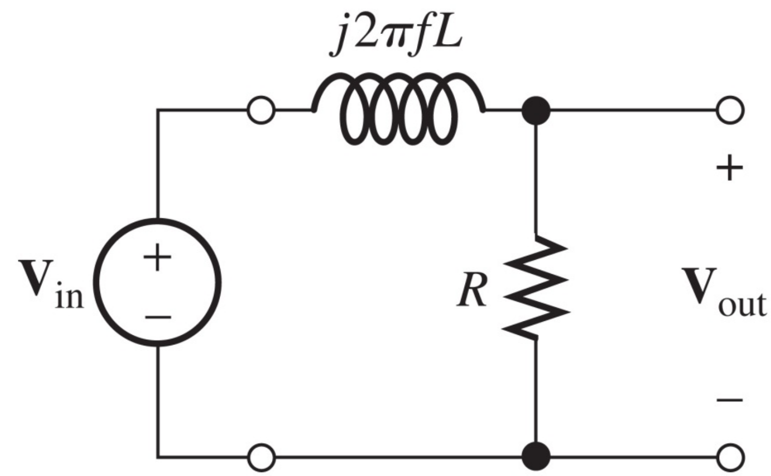
- ✓ Lets determine the frequency response of a series RL circuit using the Laplace Transform
- ✓ Assume:

$$v_s(t) = 10e^{-2t}\cos(5t)\,\mathrm{V}$$

where

$$R = 4\Omega;\ L = 2H$$

- ✓ Therefore: $s = -2 + 5j$



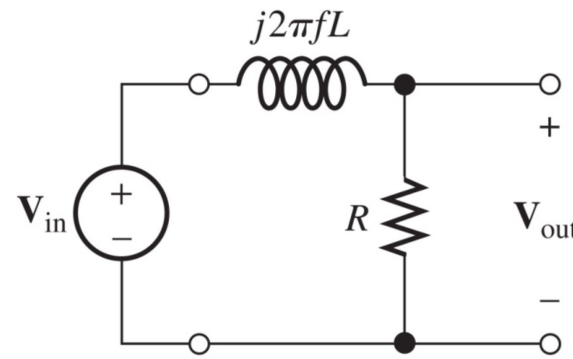
Example: Laplace Transform

✓ The input phasor is given by:

$$\mathbf{V}(s) = 10\angle 0\mathrm{V}$$

✓ The impedance seen by the voltage source is:

$$Z(s) = R + sL = 4 + 2s$$

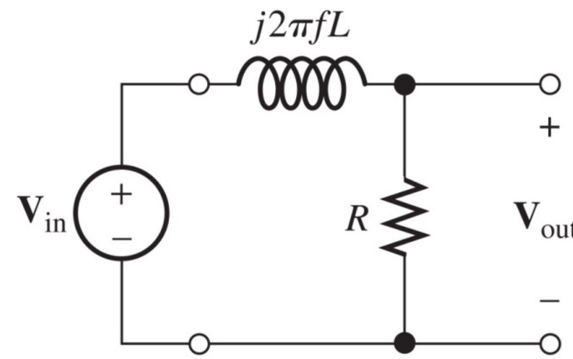


Example: Laplace Transform

✓ The series current is therefore:

$$\mathbf{I}(s) = \frac{\mathbf{V}(s)}{Z(s)} = \frac{10}{4+2s} = \frac{10}{4+2(-2+j5)}$$

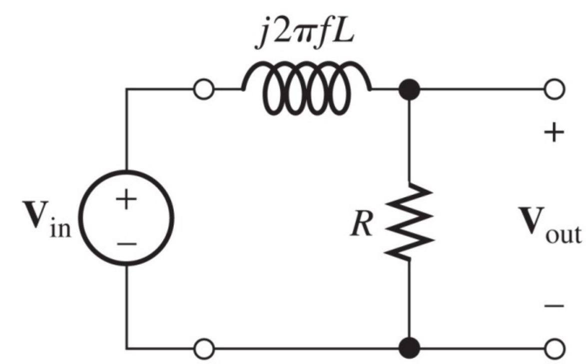
$$= \frac{10}{j10} = j1 = 1\angle\left(-\frac{\pi}{2}\right)$$



Example: Laplace Transform

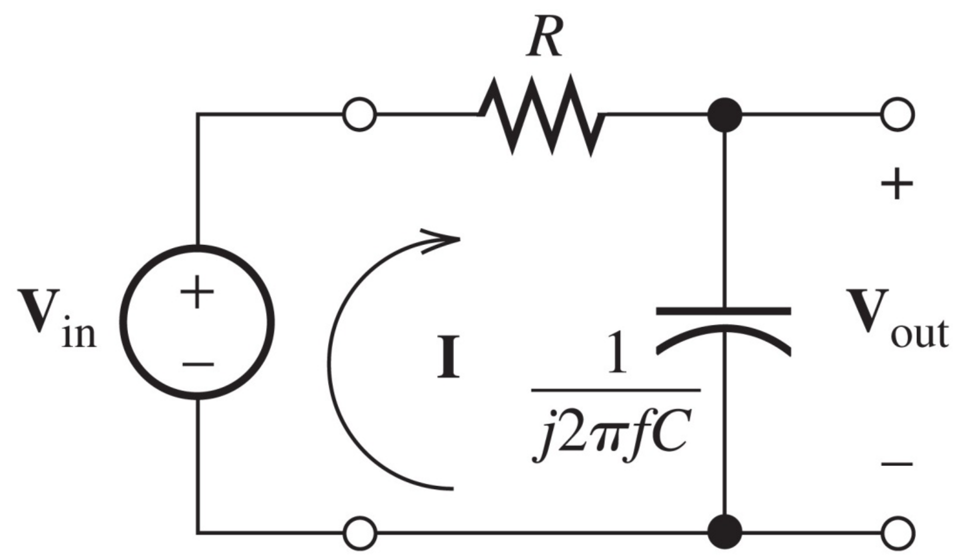
- ✓ Finally, the time-domain expression for the current is given by:

$$i_s(t) = e^{-2t}\cos\left(5t - \frac{\pi}{2}\right)\text{A}$$

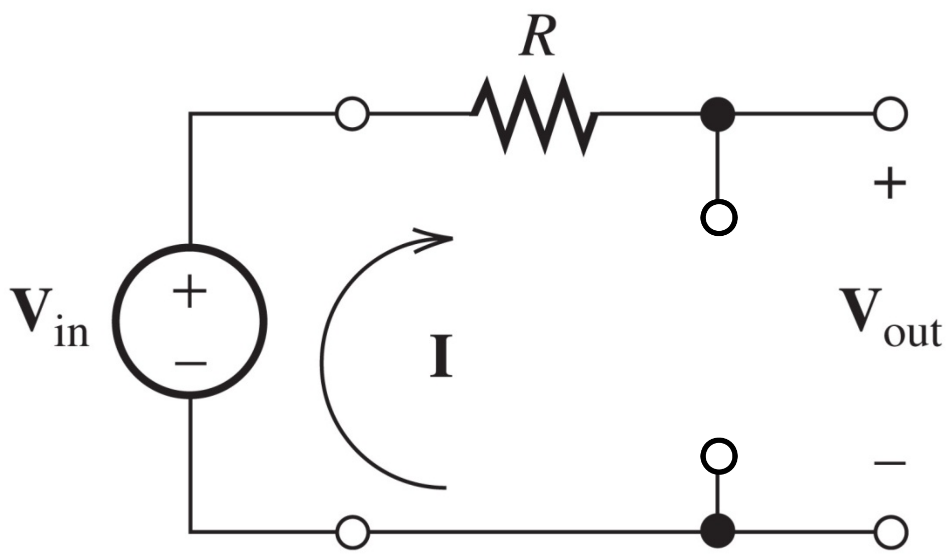


Low-Pass Filters

- ✓ Now that we have the background behind us, lets design a low-pass filter
- ✓ As the name implies, we want to pass the low frequencies
- ✓ The circuit shown at the right will do this

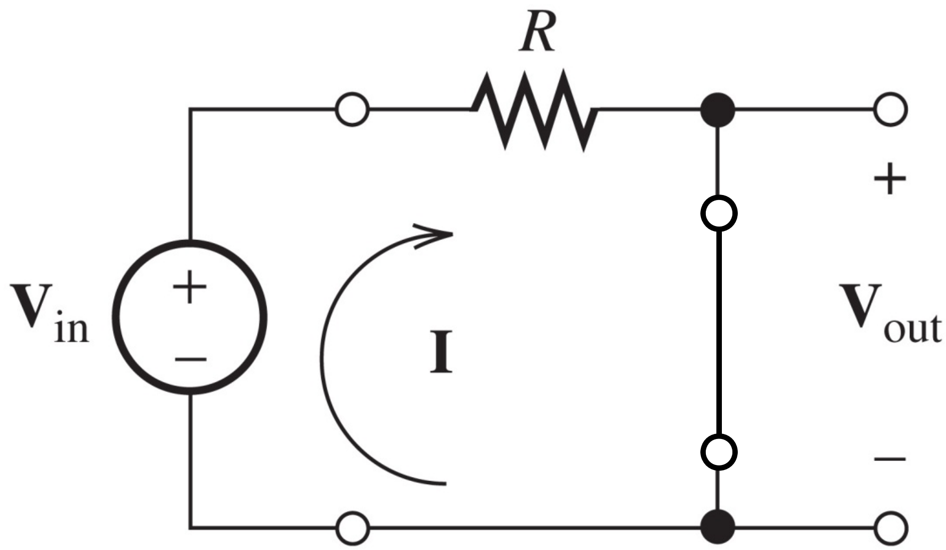


First Order RC Low-Pass Filter



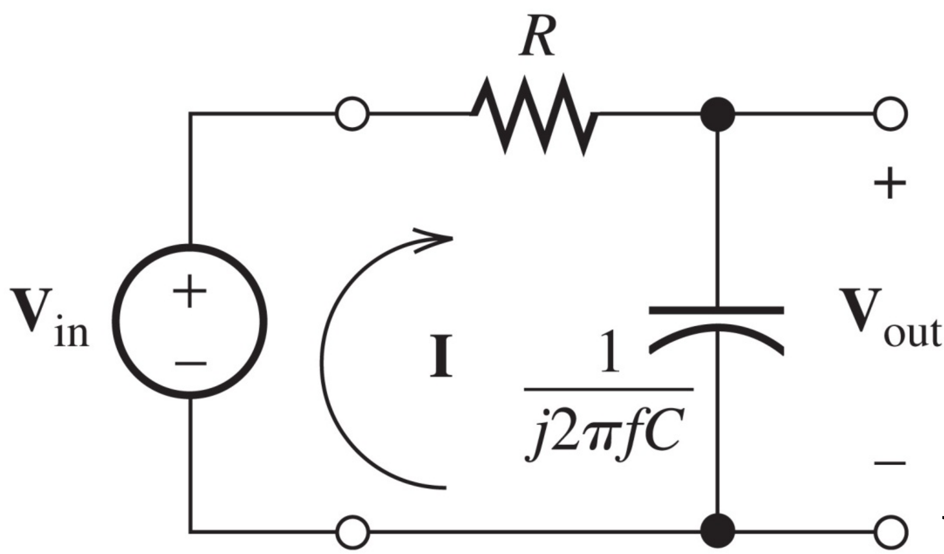
- ✓ At low frequencies, the capacitor has high impedance
- ✓ It therefore acts like an open circuit
- ✓ $\mathbf{V}_{out} \approx \mathbf{V}_{in}$

First Order RC Low-Pass Filter



- ✓ At high frequencies, the capacitor has low impedance
- ✓ It therefore acts like an short circuit
- ✓ $\mathbf{V}_{\text{out}} \approx 0$

First Order RC Low-Pass Filter



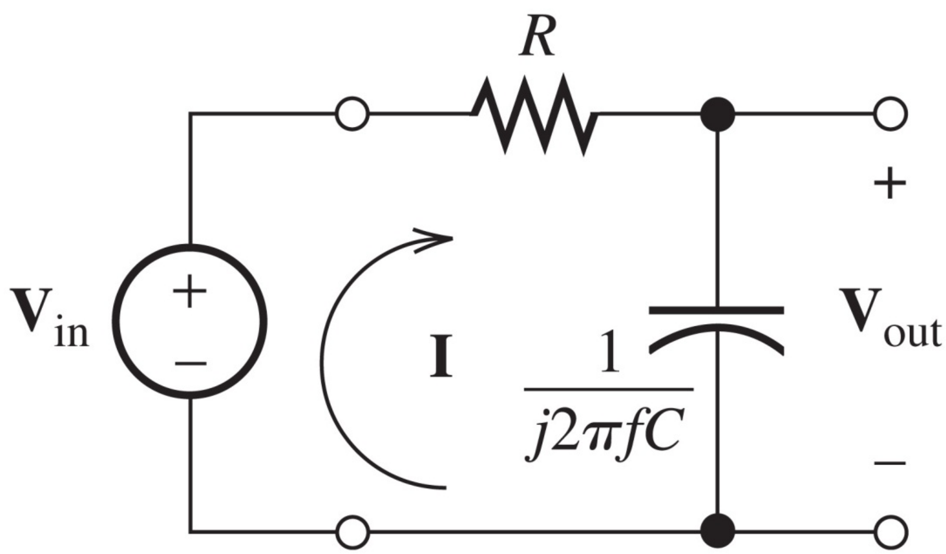
$$\mathbf{I} = \frac{\mathbf{V}_{\text{in}}}{R + 1/j2\pi fC}$$

$$\mathbf{V}_{\text{out}} = \frac{1}{j2\pi fC}\mathbf{I}$$

$$\mathbf{V}_{\text{out}} = \frac{1}{j2\pi fC} \times \frac{\mathbf{V}_{\text{in}}}{R + 1/j2\pi fC}$$

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{1 + j2\pi fRC}$$

First Order RC Low-Pass Filter

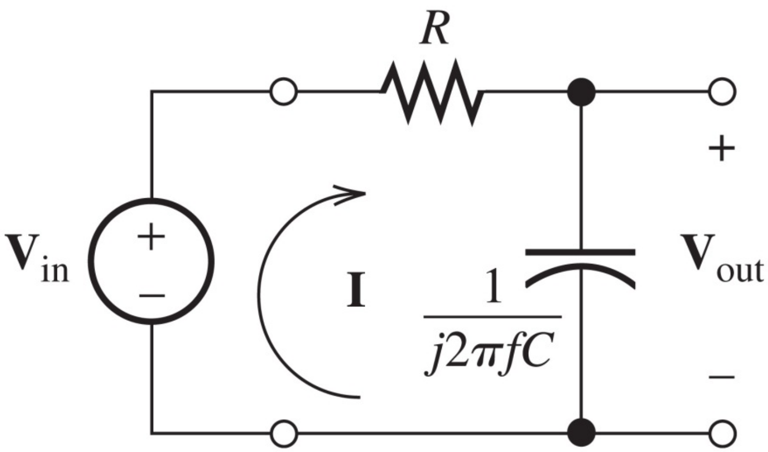


$$H(f) = \frac{\mathbf{V}_{\mathrm{out}}}{\mathbf{V}_{\mathrm{in}}} = \frac{1}{1 + j2\pi fRC}$$

$$f_B = \frac{1}{2\pi RC}$$

$$H(f) = \frac{1}{1 + j\left(f / f_B\right)}$$

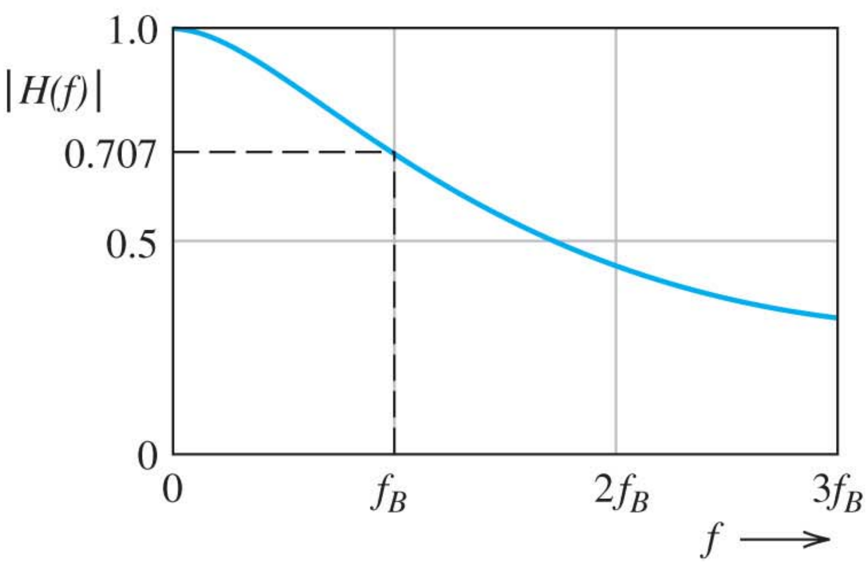
First Order RC Low-Pass Filter

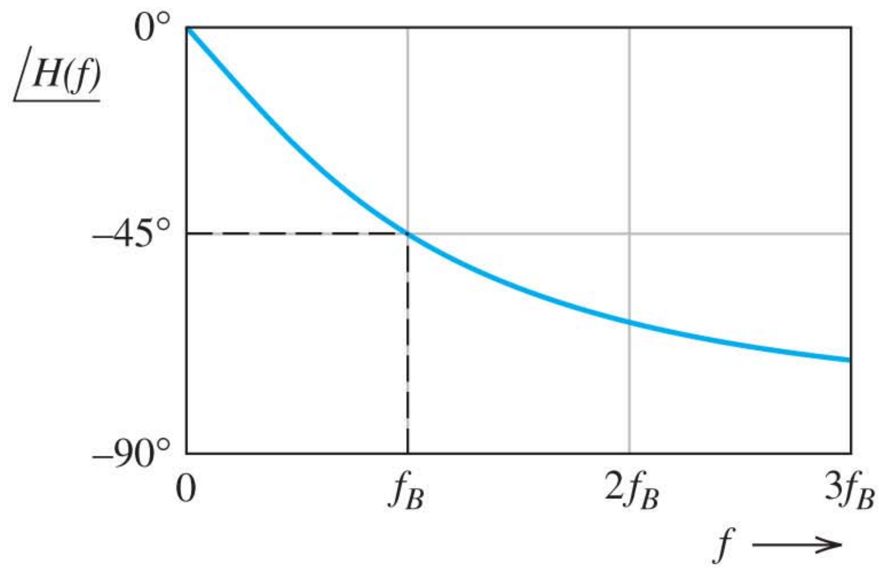


$$H(f) = \frac{1}{1 + j\left(f/f_B\right)}$$

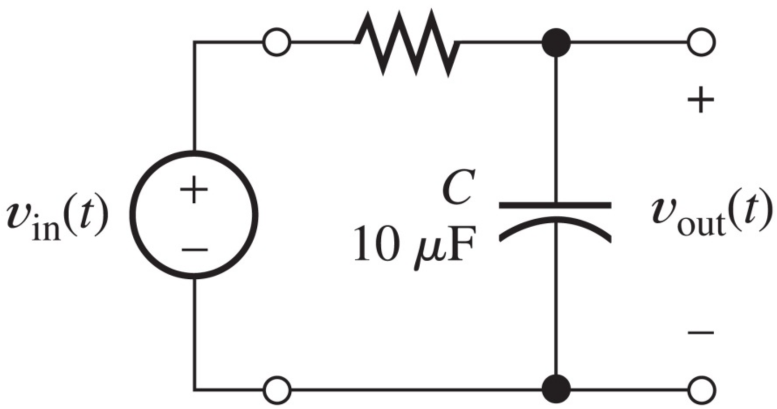
$$\left|H(f)\right| = \frac{1}{\sqrt{1 + \left(f/f_B\right)^2}}$$

$$\angle H(f) = -\arctan\left(f/f_B\right)$$





Example of RC Low Pass Filter



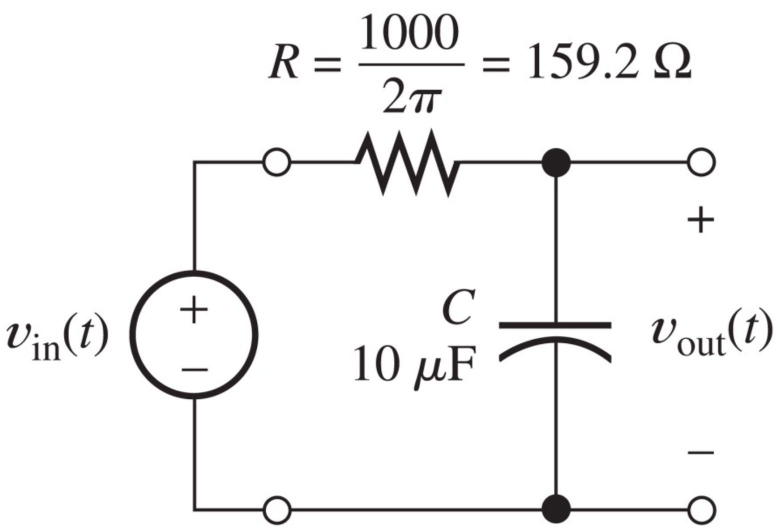
- ✓ We want $f_B = 100\text{ Hz}$

$$f_B = \frac{1}{2\pi RC} = 100\text{ Hz}$$

$$= \frac{1}{2\pi R \times \left(10 \times 10^{-6}\right)}$$

$$R = \frac{1000}{2\pi} = 159.2\Omega$$

Example of RC Low Pass Filter

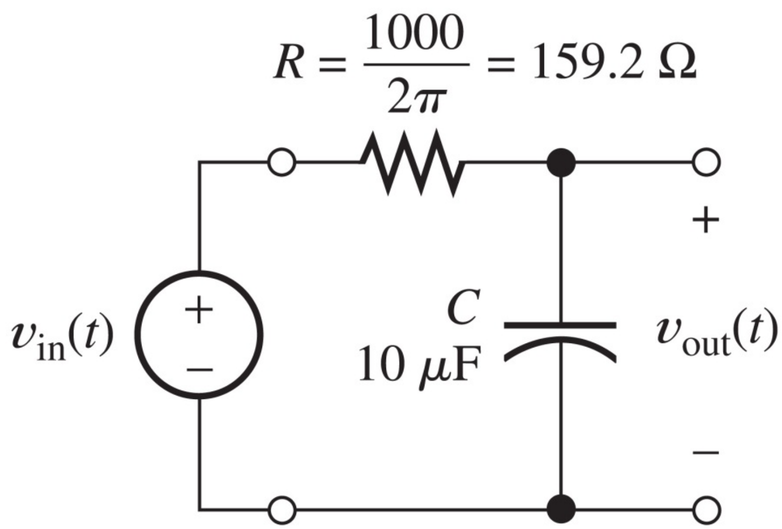


- ✓ Supose the input voltage is as follows:

$$v_{in}(t) = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t)$$

- ✓ Let $v_{in1}(t) = 5\cos(20\pi t)$
- ✓ Using phasor notation, this is $\mathbf{V}_{\text{in1}} = 5\angle 0^\circ$ with an angular frequency of $\omega = 20\pi; f = \frac{\omega}{2\pi} = 10\ \text{Hz}$

Example of RC Low Pass Filter



- ✓ We now apply the transfer function at f_1:

$$H(f_1) = \frac{1}{1 + j\left(f_1/f_B\right)}$$

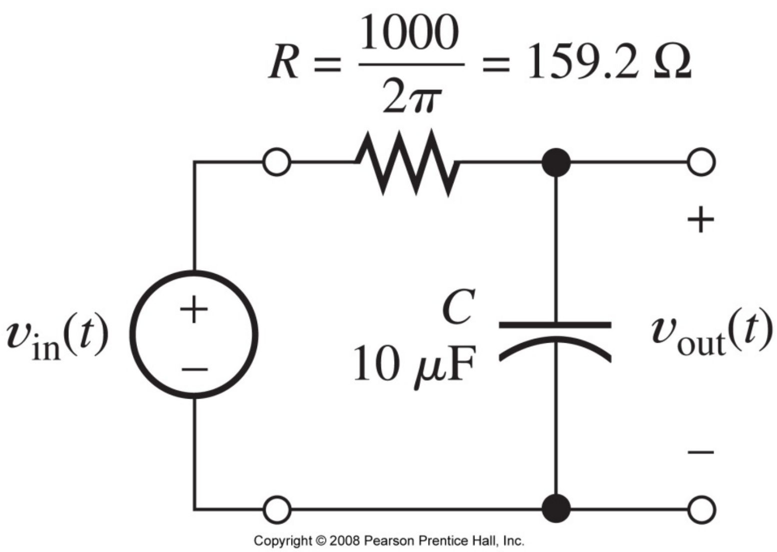
$$H(10) = \frac{1}{1 + j\left(10/100\right)}$$

$$= 0.9950\angle -5.71^\circ$$

$$\mathbf{V}_{\text{out1}} = H\left(10\right) \times \mathbf{V}_{\text{in1}}$$

$$= \left(0.9950\angle -5.71^\circ\right) \times \left(5\angle 0^\circ\right) = 4.975\angle -5.71^\circ$$

Example of RC Low Pass Filter



✓ Repeating the process for the 2nd and 3rd components of $\mathbf{V}_{in}$ at frequencies f_2 and f_3 we get the following:

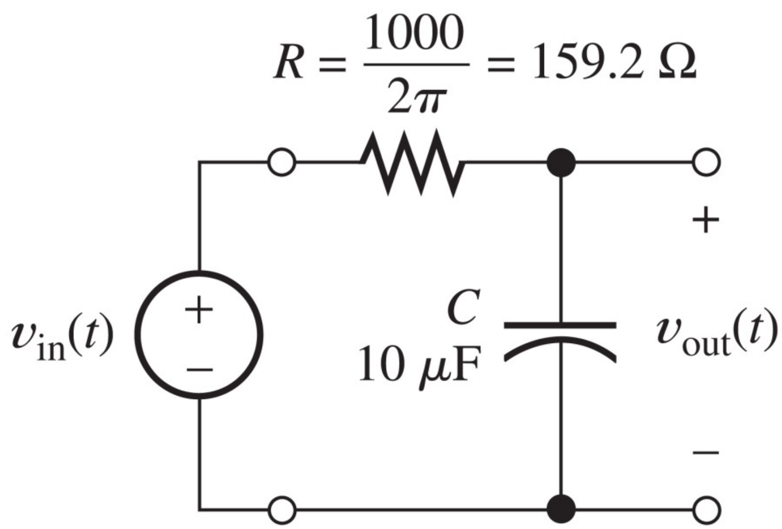
$$\mathbf{V}_{out2} = H(100) \times \mathbf{V}_{in2}$$

$$= \left(0.7071\angle -45^{\circ}\right) \times \left(5\angle 0^{\circ}\right) = 3.535\angle -45^{\circ}$$

$$\mathbf{V}_{out3} = H(1000) \times \mathbf{V}_{in3}$$

$$= \left(0.0995\angle -84.29^{\circ}\right) \times \left(5\angle 0^{\circ}\right) = 0.4975\angle -84.29^{\circ}$$

Example of RC Low Pass Filter



✓ Adding all of the components together gives us:

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{out1}} + \mathbf{V}_{\text{out2}} + \mathbf{V}_{\text{out3}}$$

$$= \left(4.975\angle -5.71^\circ\right)$$

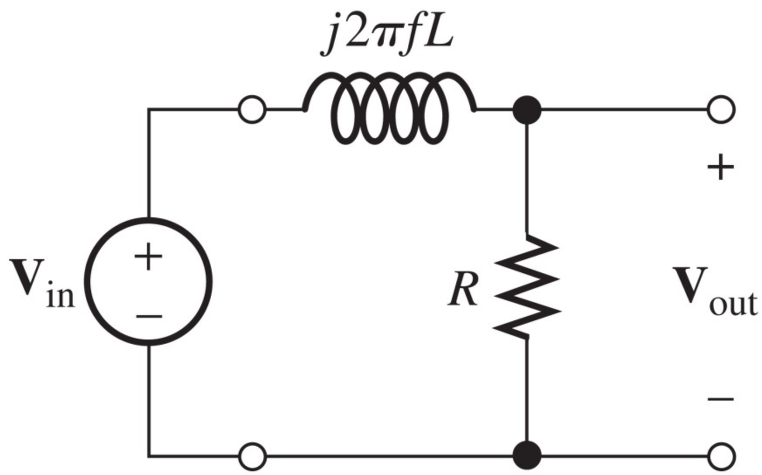
$$+\left(3.535\angle -45^\circ\right)$$

$$+\left(0.4975\angle -84.29^\circ\right)$$

$$\mathbf{V}_{\text{out}} = 4.975\cos\left(20\pi t - 5.71^\circ\right) + 3.535\cos\left(200\pi t - 45^\circ\right)$$

$$+0.4975\cos\left(2000\pi t - 84.29^\circ\right)$$

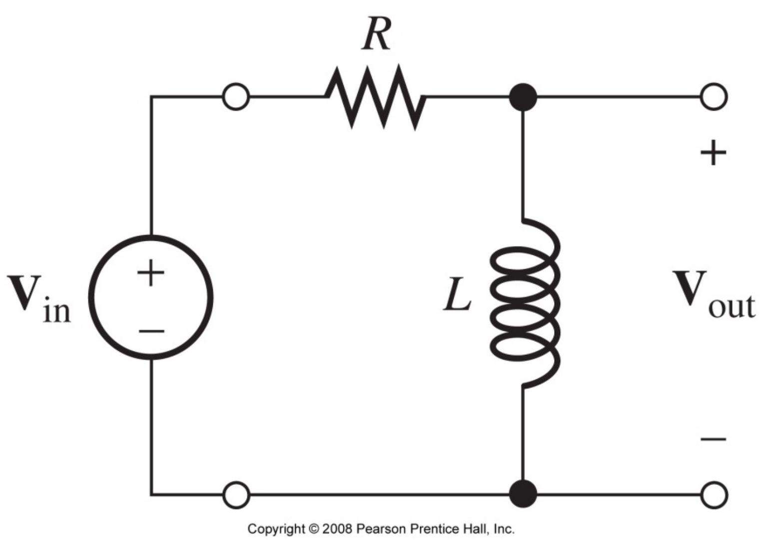
Alternative Low Pass Filter



- ✓ We've seen one way of creating a low pass filter using an RC network
- ✓ Are there any alternatives to that?
- ✓ In this case we are using the fact that, at low frequencies, the inductor acts as a short-circuit
- ✓ Conversely, at high frequencies, the inductor acts as an open-circuit

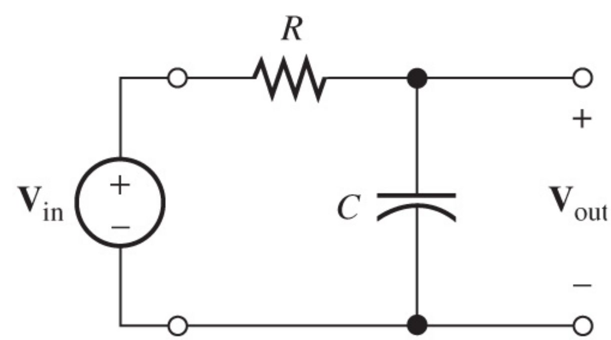
High Pass Filters

- ✓ We’ve seen how to create low pass filters using RC and RL networks
- ✓ Any ideas on how we might create a high pass filter?



Higher Order Filters

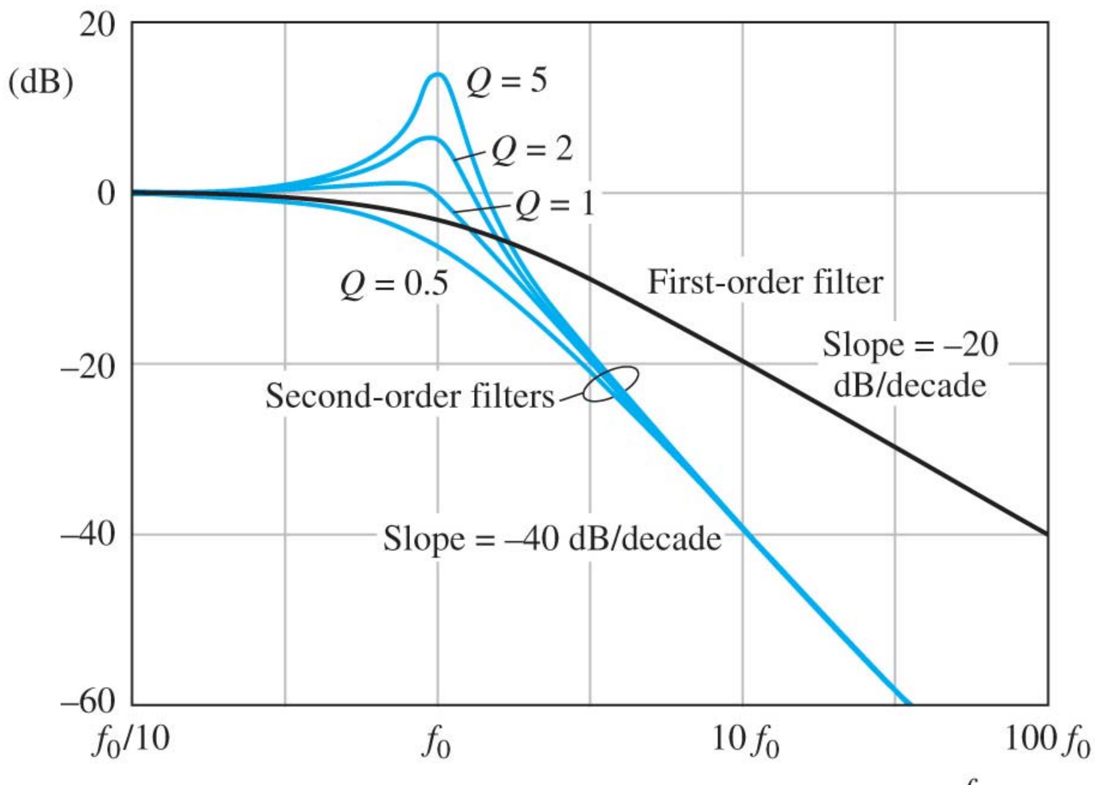
- ✓ As we've seen, we can get a certain amount of rejection per decade or octave with an RC or RL filter
- ✓ Suppose we need a greater amount of rejection per decade or octave – is there a solution?



(b) First-order lowpass filter

Higher Order Filters

✓ A second order filter gives us twice the slope (in dB/decade) as compared to a first order filter



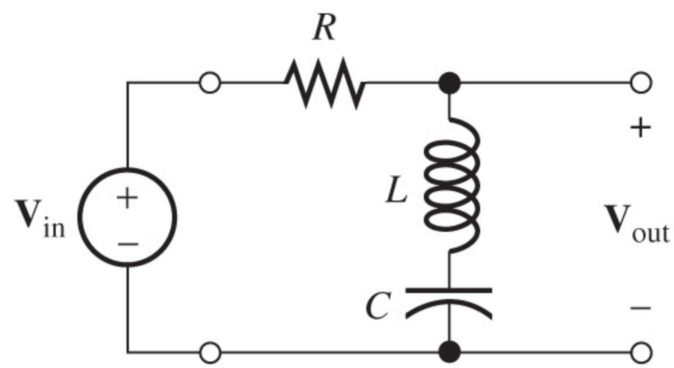
(c) Transfer-function magnitudes

$$\left|H(f)\right|_{\mathrm{dB}} = 20\log\left|H(f)\right|$$

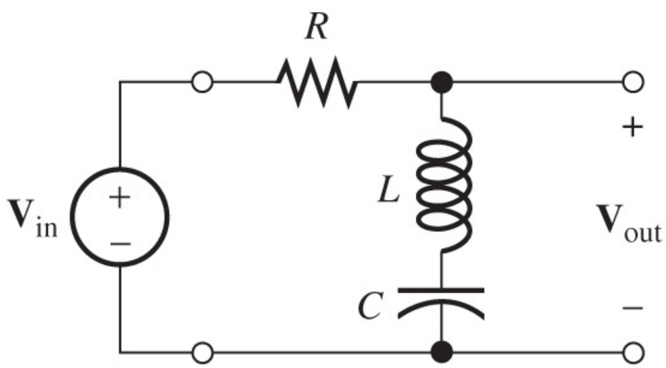
$\lvert H(f)\rvert$	$\lvert H(f)\rvert_{dB}$
100	40
10	20
2	6
√2	3
1	0
1/√2	-3
1/2	-6

Band Reject Filters

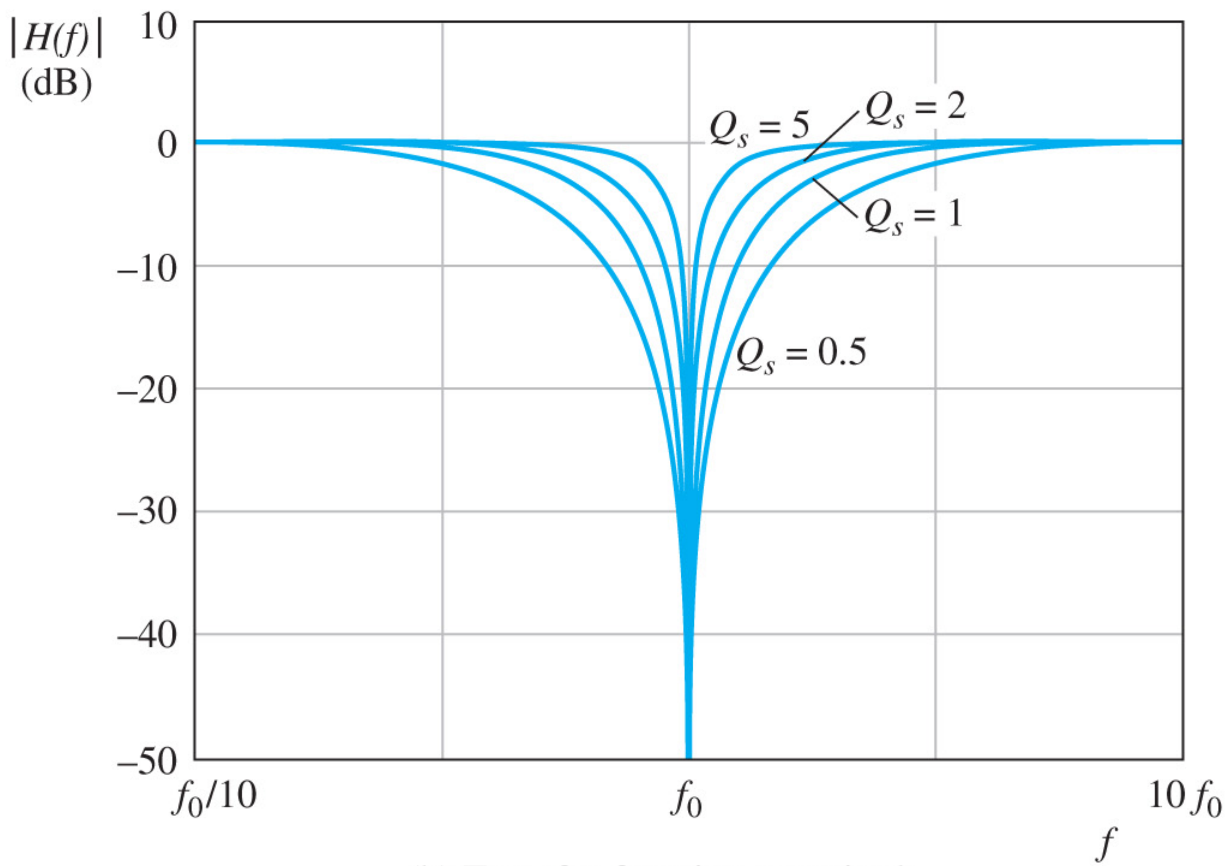
- ✓ Suppose we need a band reject filter – a filter that lets through low and high frequencies, but doesn't let through any frequencies in the middle
- ✓ How might we build such a filter?



Band Reject Filters

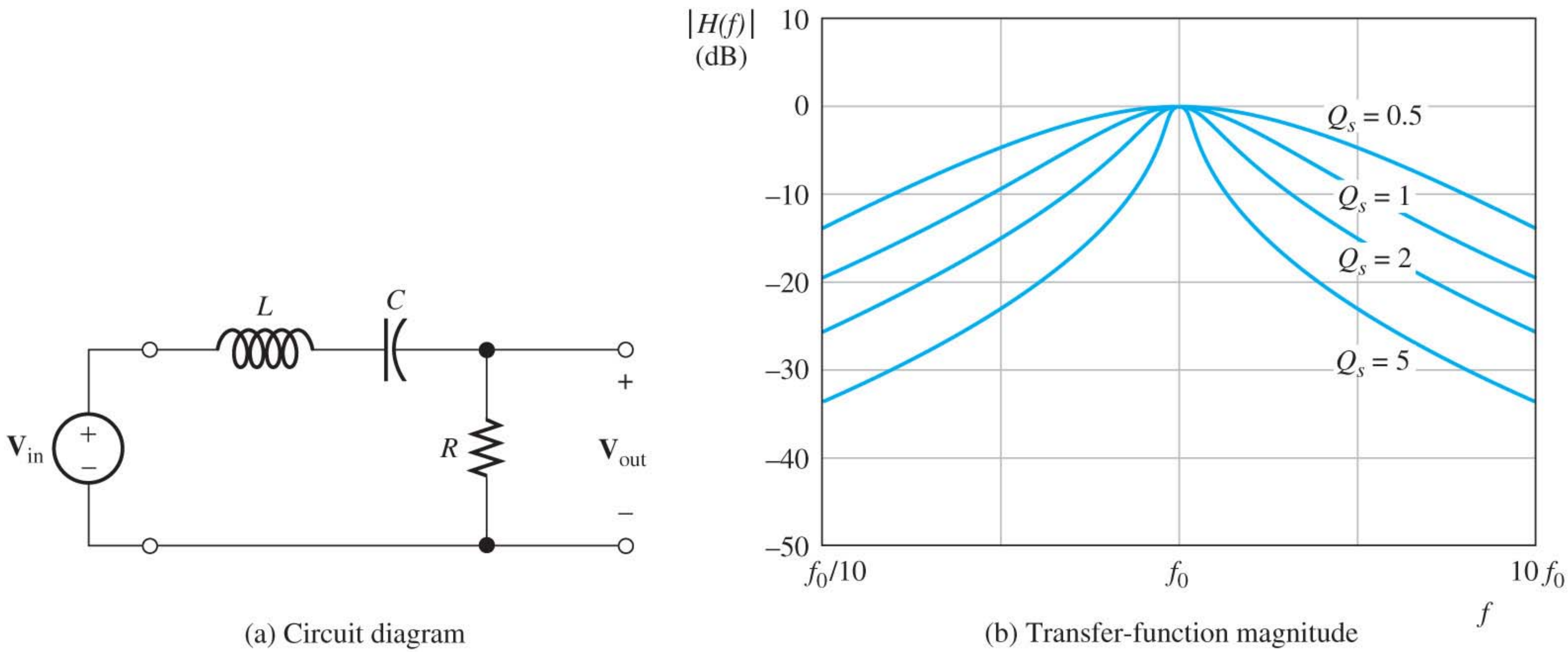


(a) Circuit diagram



(b) Transfer-function magnitude

Band Pass Filter

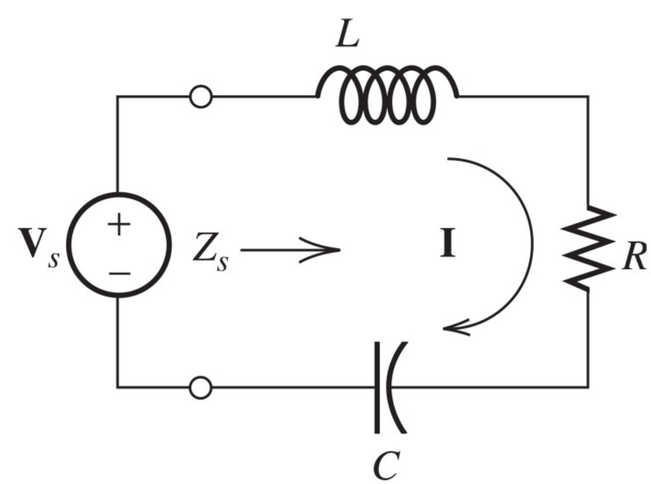


Quality Factor

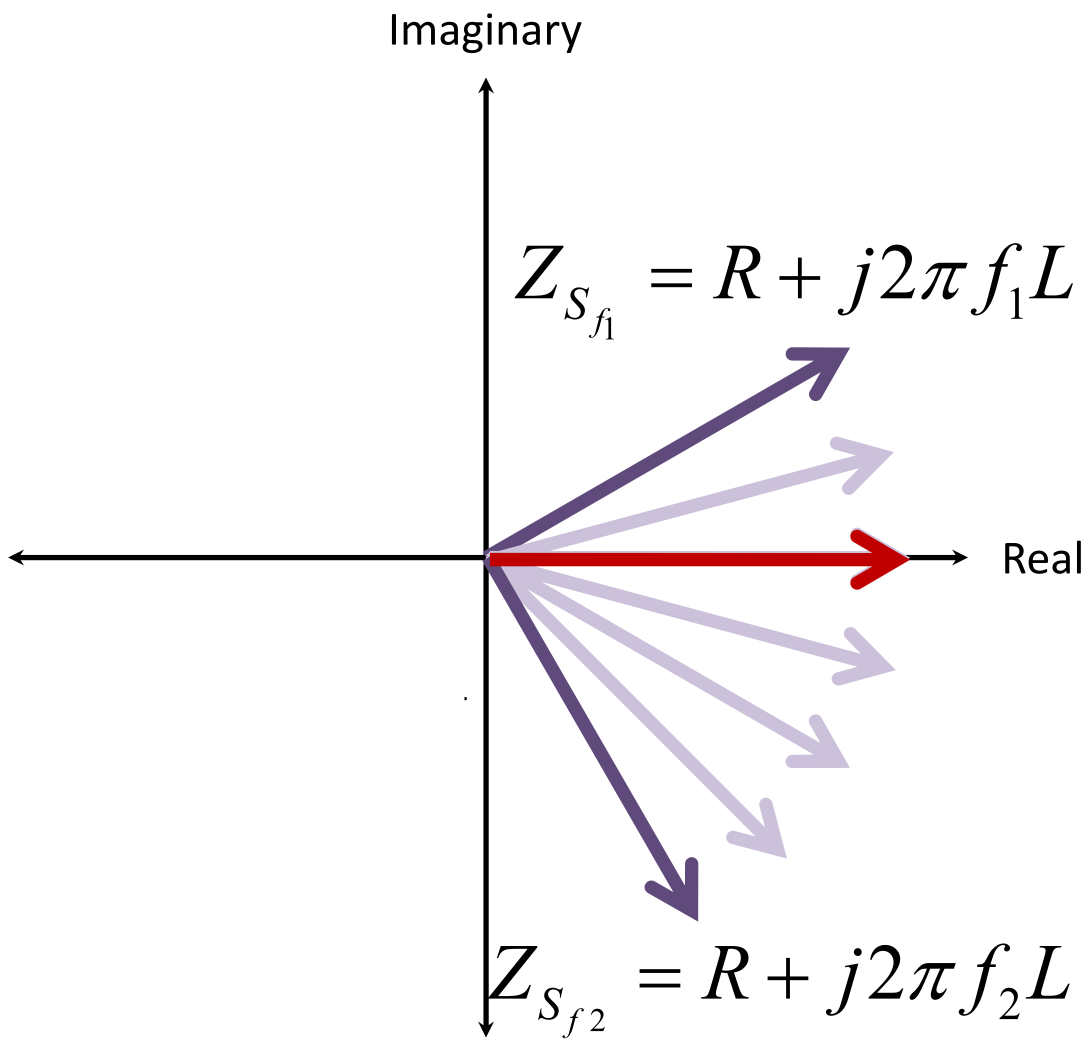
- ✓ What are the Q values all about?
- ✓ These reflect the *Quality Factor* for the circuit
- ✓ What is the impedance Z_S in the circuit below?

$$Z_S = R + j2\pi fL - j\frac{1}{2\pi fC}$$

- ✓ There will be a frequency f_0 where, for particular values of L and C, the two imaginary components cancel each other out



Quality Factor



$$Z_{S_{f_1}} = R + j2\pi f_1 L - j\frac{1}{2\pi f_1 C}$$

$$Z_{S_{f0}} = R$$

$$Z_{S_{f2}} = R + j2\pi f_2 L - j\frac{1}{2\pi f_2 C}$$

Quality Factor

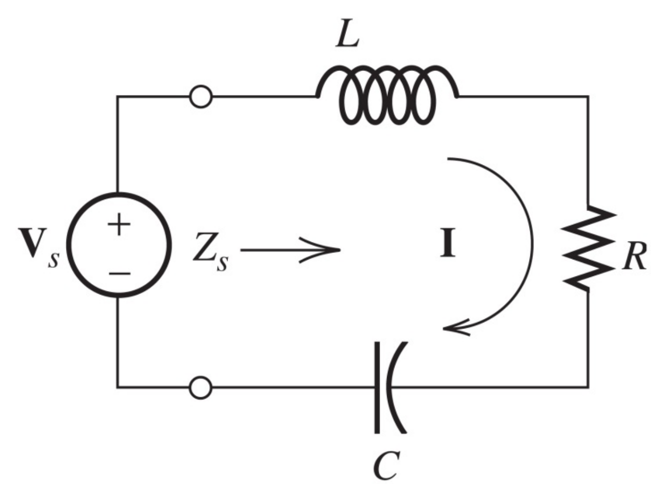
- ✓ That frequency will be when the following is true:

$$2\pi f_o L = \frac{1}{2\pi f_o C}$$

- ✓ Solving for f_0 we get:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

- ✓ This is called the ***resonant frequency*** of the circuit



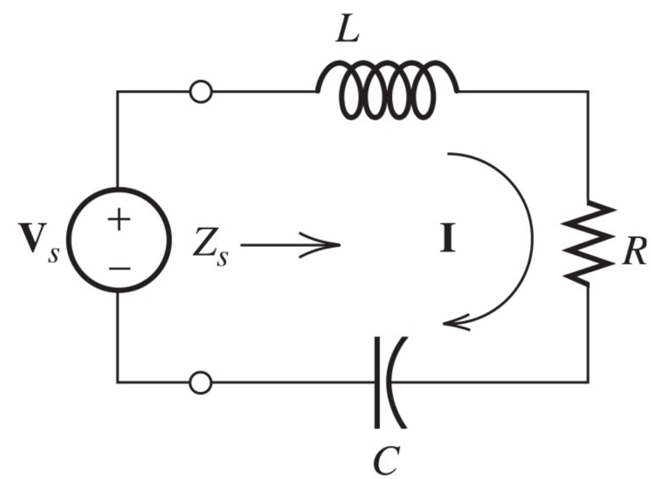
Quality Factor

- ✓ The Quality Factor for this (series) circuit is defined as:

$$Q_S = \frac{2\pi f_0 L}{R}$$

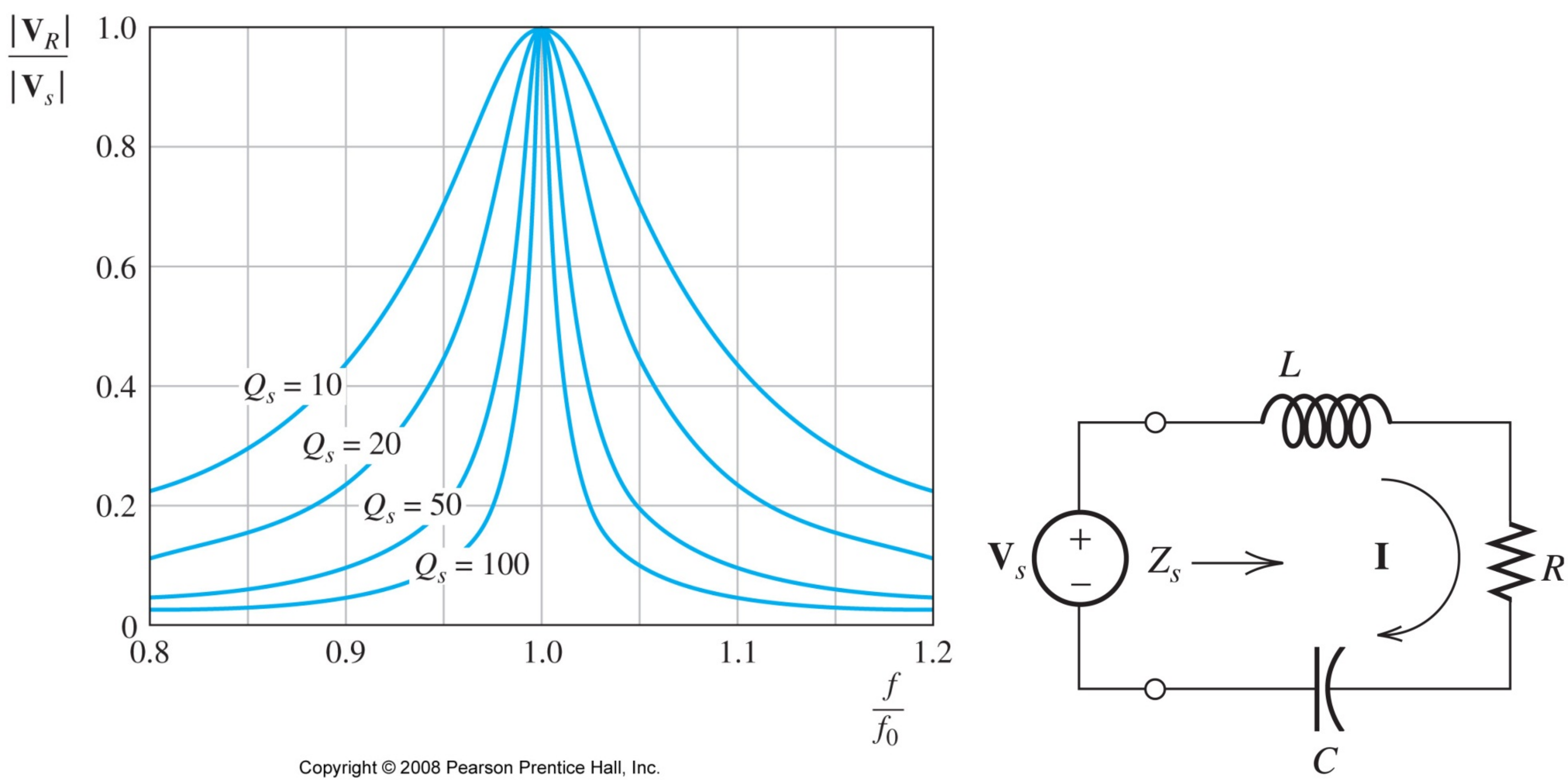
- ✓ Substituting in the previously derived equation for the resonant frequency f_0 and solving for L we can also get:

$$Q_S = \frac{1}{2\pi f_0 CR}$$



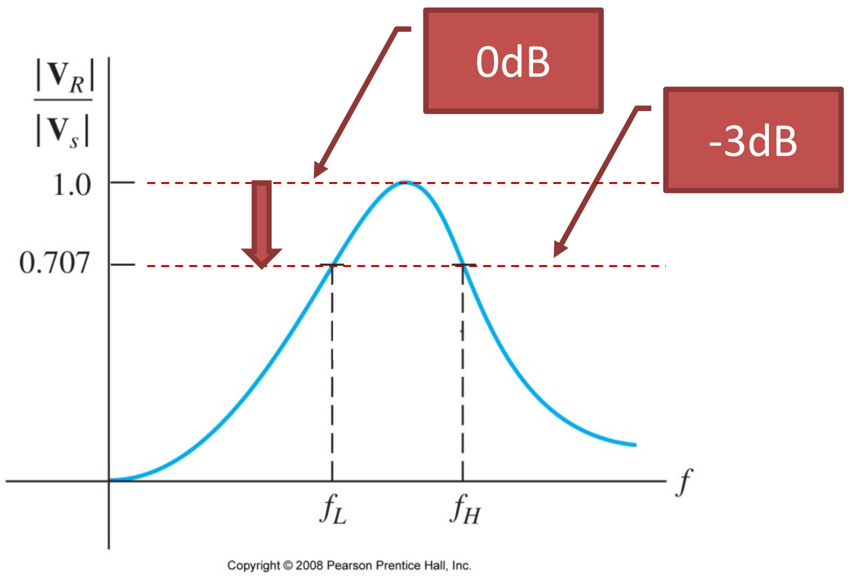
Quality Factor

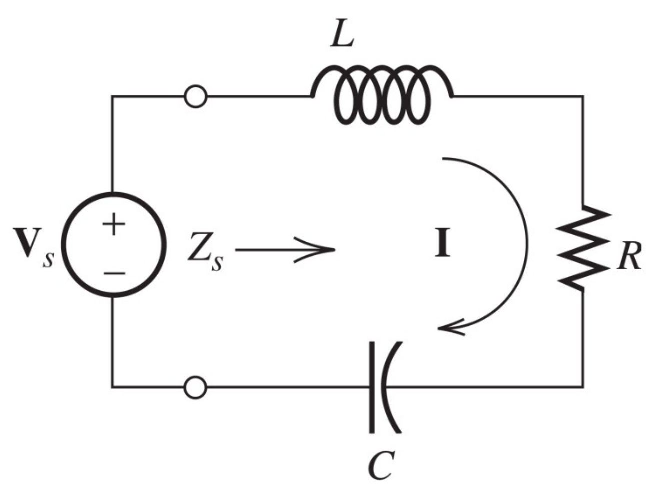
✓ For our series resonant band pass filter we can see how things change as we vary the value of Q_S



Bandwidth

- ✓ For a given Q_S, let's look at the half-power frequencies
- ✓ Recall this is where the value of the transfer function is at 0.707
- ✓ Call these frequencies f_L and f_H





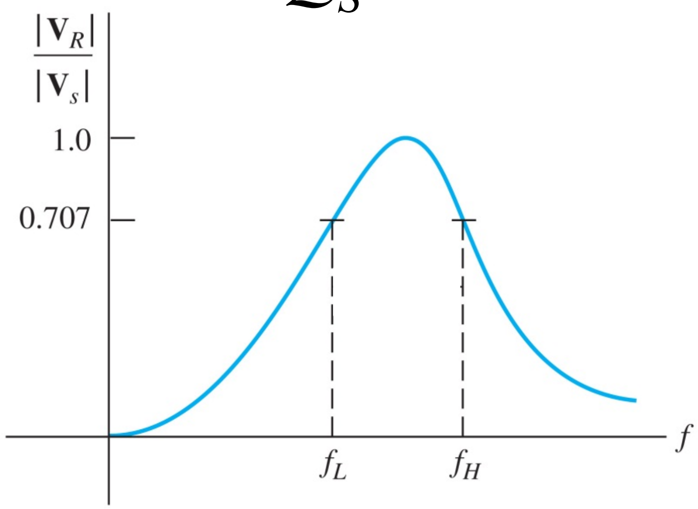
Bandwidth

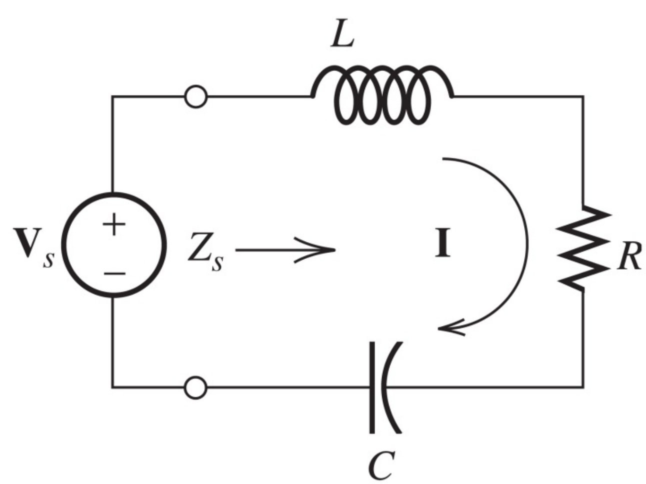
- ✓ The bandwidth of the filter is defined as:

$$B = f_H - f_L$$

- ✓ For the series resonant circuit, it can be shown that:

$$B = \frac{f_0}{Q_S}$$

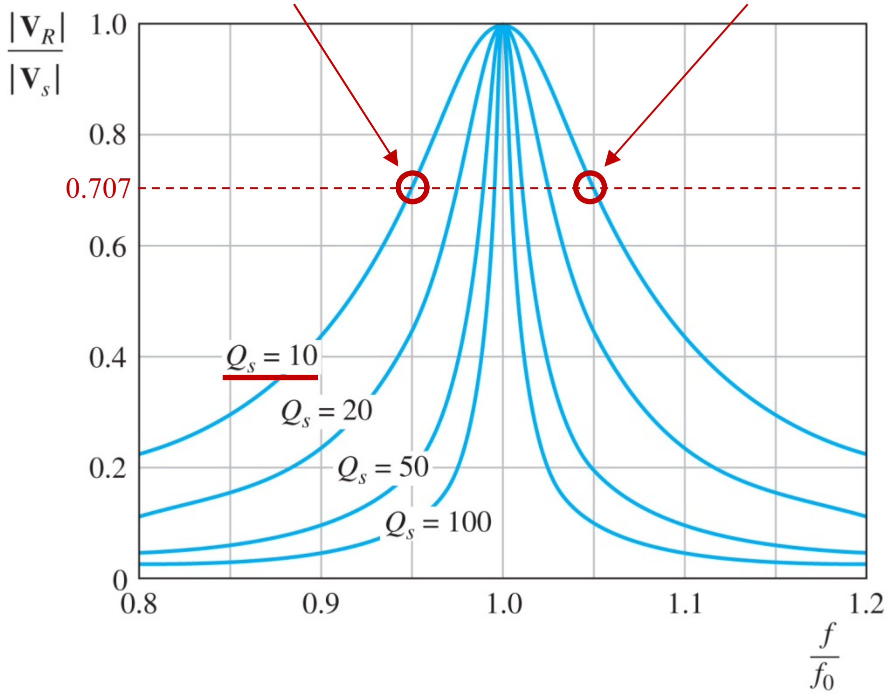


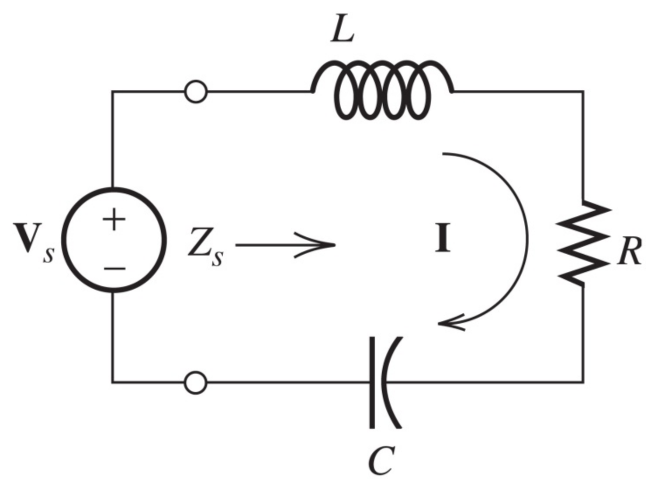


Bandwidth

✓ Furthermore, for $Q_S >> 1$, the half-power frequencies are given by:

$$f_L \cong f_0 - \frac{B}{2}; \quad f_H \cong f_0 + \frac{B}{2}$$





Bandwidth

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