A. Proofs

Lemma 4.1. *For a total partition* (T, F)*, the set* $H_{\mathcal{K}}^{(T,F)}$ *is empty if and only if there exists a rule* $r \in \mathcal{P}$ *where* $body(r) \sqsubseteq (T, F)$ *and* $head(r) \cap T = \emptyset$*.*

Proof. ($\Rightarrow$) We prove the contrapositive by constructing a head-cut R: For each rule $r \in \mathcal{P}$ where $body(r) \sqsubseteq (T, F)$, include a pair (r, h) in R where h is selected arbitrarily from $head(r) \cap T$. If $body(r) \not\sqsubseteq (T, F)$, then $r \notin rule(R)$, thus $R \in H_{\mathcal{K}}^{(T,F)}$; $H_{\mathcal{K}}^{(T,F)}$ is nonempty.

($\Leftarrow$) Let $r \in \mathcal{P}$ be a rule where $body(r) \sqsubseteq (T, F)$ and $head(r) \cap T = \emptyset$. Any head-cut $R \in H_{\mathcal{K}}^{(T,F)}$ must contain a pair with r. For R to be a head-cut, h must come from $head(r)$, and $H_{\mathcal{K}}^{(T,F)}$ requires that $h \in T$. No head-cut can satisfy both of these requirements, thus, $H_{\mathcal{K}}^{(T,F)}$ is empty. $\square$

Proposition 4.1. *The set* $H_{\mathcal{K}}^{(T,F)}$ *is a supporting set of the dependable partition* (T, F)*.*

Proof. Let (T, F) be a total dependable partition. We show that an MKNF model M that induces (T, F) exists if and only if $H_{\mathcal{K}}^{(T,F)}$ is nonempty and for each $R \in H_{\mathcal{K}}^{(T,F)}$, $\mathbf{lfp}\ Q^R = T$.

($\Leftarrow$) Assume $H_{\mathcal{K}}^{(T,F)}$ is nonempty and for each $R \in H_{\mathcal{K}}^{(T,F)}$, $\mathbf{lfp}\ Q^R$ is precisely T. We construct an MKNF model of $\mathcal{K}$ that induces (T, F). Let M be an MKNF interpretation containing all first-order interpretations that satisfy both $\mathcal{O}$ and T, i.e.,

$$M = \{I \mid I \models \pi(\mathcal{O})\} \cap \{I \mid I \models t \text{ for each } \mathbf{K}\, t \in T\}$$

We have $M \models_{MKNF} \neg\mathbf{K}\, a$ for each $\mathbf{K}\, a \in F$ and $M \models_{MKNF} \mathbf{K}\, a$ for each $\mathbf{K}\, a \in T$, thus M induces the dependable partition (T, F). By construction, we have $M \models_{MKNF} \mathbf{K}\, \pi(\mathcal{O})$; Applying Lemma 4.1 while knowing $H_{\mathcal{K}}^{(T,F)} \neq \emptyset$, we get $M \models_{MKNF} \pi(\mathcal{P})$. It remains to be shown that M is maximal, i.e., there does not exist a larger MKNF interpretation M' such that $M' \supset M$ and $(I, M', M) \models \pi(\mathcal{K})$ for each $I \in M'$.

Assume for the sake of contradiction, that such an interpretation M' exists and let (T', F') be the dependable partition induced by M' (Note that because we use M' to evaluate positive rule bodies only, we rely on the dependable partition (T', F) rather than (T', F')). We will locate a head-cut $R \in H_{\mathcal{K}}^{(T,F)}$ such that $\mathbf{lfp}\ Q^R \subset T$ and contradict the assumption that $\mathbf{lfp}\ Q^R = T$. With $M' \models_{MKNF} \pi(\mathcal{P})$, we have for each rule $r \in \mathcal{P}$ where $body(r) \sqsubseteq (T, F)$ (satisfied by M) if $head(r) \cap T' = \emptyset$, then $body^+(r) \subseteq T \setminus T'$. Let R be a head-cut $R \in H_{\mathcal{K}}^{(T,F)}$ where for each $(r, h) \in R$, either $h \in T'$ or $body^+(r) \subseteq T \setminus T'$. Clearly, such a head-cut exists. Intuitively, we only constrain the pairs whose positive rule bodies are satisfied by M'; A pair $(r, h) \in R$ containing a rule r whose positive body is not satisfied by M' may use any $h \in head(r) \cap T$. Suppose for the sake of contradiction that $\mathbf{lfp}\ Q^R \not\subset T$. Because $T' \subset T$, the Q^R operator has computed atoms in $T \setminus T'$ through either the ontology or a pair in R. If it was through a pair (r, h) in R, then we have $body^+(r) \subseteq T \setminus T'$. We must have that $\text{OB}_{\mathcal{O},T'} \models t$ for some $t \in T \setminus T'$ which contradicts the assumption that $(I, M', M) \models \pi(\mathcal{O})$ for each $I \in M'$. This gives us that $\mathbf{lfp}\ Q^R \subset T$ which contradicts the assumption that $\mathbf{lfp}\ Q^R = T$. We conclude that M is an MKNF model of $\mathcal{K}$.

($\Rightarrow$) Assume that either $H_{\mathcal{K}}^{(T,F)}$ is empty or there exists a head-cut $R \in H_{\mathcal{K}}^{(T,F)}$ such that $\textbf{lfp}\ Q^R \neq T$. Let M be an MKNF interpretation induced by (T, F). We show that either $M \not\models_{MKNF} \pi(\mathcal{K})$ or there exists an interpretation $M' \supset M$ such that $(I, M', M) \models \pi(\mathcal{K})$ for each $I \in M'$. Clearly if $H_{\mathcal{K}}^{(T,F)} = \emptyset$, then $M \not\models_{MKNF} \pi(\mathcal{K})$ because there exists a rule $r \in \mathcal{P}$ where $M \not\models_{MKNF} \pi(r)$ (Lemma 4.1). Assume there exists a head-cut $R \in H_{\mathcal{K}}^{(T,F)}$ such that $\textbf{lfp}\ Q^R \neq T$. Clearly if $\textbf{lfp}\ Q^R \setminus T \neq \emptyset$, then T is missing **K**-atoms derived from either a rule or the ontology and thus $M \not\models_{MKNF} \pi(\mathcal{K})$. Assuming $\textbf{lfp}\ Q^R \subset T$, let T' be the set of **K**-atoms that R fails to compute, i.e., $T' = T \setminus \textbf{lfp}\ Q^R$. Construct the MKNF interpretation M' that both induces $(T', \mathsf{KA}(\mathcal{K}) \setminus T')$ and satisfies $\mathcal{O}$.

$$M' = \{I \mid I \models \pi(\mathcal{O})\} \cap \{I \mid I \models t \text{ for each } \mathbf{K}\, t \in T'\}$$

We have $M' \supset M$ and $M' \models_{MKNF} \pi(\mathcal{K})$, thus M is not an MKNF model. We conclude that if given an MKNF model M that induces a total dependable partition (T, F), then $\textbf{lfp}\ Q^R = T$ for each $R \in H_{\mathcal{K}}^{(T,F)}$. $\square$

Proposition 4.2. *$HM_{\mathcal{K}}^{(T,F)}$ is a supporting set of (T, F).*

Proof. It is trivial to show that the property demonstrated in Lemma 4.1 also applies to $HM_{\mathcal{K}}^{(T,F)}$, thus $HM_{\mathcal{K}}^{(T,F)} = H_{\mathcal{K}}^{(T,F)}$ if and only if either $H_{\mathcal{K}}^{(T,F)}$ or $HM_{\mathcal{K}}^{(T,F)}$ is empty. We build upon the proof in Proposition 4.1 by showing $\textbf{lfp}\ Q^{R'} = T$ for each $R' \in HM_{\mathcal{K}}^{(T,F)}$ if and only if $\textbf{lfp}\ Q^R = T$ for each $R \in H_{\mathcal{K}}^{(T,F)}$. It follows that $HM_{\mathcal{K}}^{(T,F)}$ is a supporting set of (T, F).

($\Rightarrow$) Let $R \in H_{\mathcal{K}}^{(T,F)}$ and have $R' \in HM_{\mathcal{K}}^{(T,F)}$ be the head-cut where $R' \subseteq R$. We assume $\textbf{lfp}\ Q^{R'} = T$ and show $\textbf{lfp}\ Q^R = T$. Apply Lemma A.1 to obtain $\textbf{lfp}\ Q^R \supseteq T$. By $\textbf{lfp}\ Q^{R'} = T$, we have $\mathsf{OB}_{\mathcal{O},T} \not\models a$ for each $\mathbf{K}\, a \in \mathsf{KA}(\mathcal{K}) \setminus T$. We have *head*$(R) \subseteq T$, thus it would be absurd for a **K**-atom $\mathbf{K}\, a \in \mathsf{KA}(\mathcal{K}) \setminus T$ to be in $\textbf{lfp}\ Q^R$ because it would imply $\mathsf{OB}_{\mathcal{O},\mathit{head}(R)} \models a$.

($\Leftarrow$) Let $R' \in HM_{\mathcal{K}}^{(T,F)}$ and assume for each $R \in H_{\mathcal{K}}^{(T,F)}$, $\textbf{lfp}\ Q^R = T$. We show $\textbf{lfp}\ Q^{R'} = T$ by adding a set of rule pairs S to R' allowing $(R' \cup S)$ to be a head-cut in $H_{\mathcal{K}}^{(T,F)}$ while keeping $\textbf{lfp}\ Q^{R' \cup S} = \textbf{lfp}\ Q^{R'}$. For any rule $R \in H_{\mathcal{K}}^{(T,F)}$, the set *rule*$(R)$ is the same, i.e. $\{\mathit{rule}(R)\} = \bigcap\{\mathit{rule}(R) \mid R \in H_{\mathcal{K}}^{(T,F)}\}$. The set *rule*$(R) \setminus$ *rule*(R') contains precisely the rules that need to be added to R' via S to have $(R' \cup S) \in H_{\mathcal{K}}^{(T,F)}$. From the construction of $HM_{\mathcal{K}}^{(T,F)}$: For each rule r in *rule*$(R) \setminus$ *rule*(R'), we have a **K**-atom $\mathbf{K}\, h \in \mathit{head}(r)$ such that $h \in Q_i^{R'}$ where $i + 1$ is the smallest integer such that $Q_{i+1}^{R'} \supseteq \mathit{body}^+(r)$. Have S be the set of pairs (r, h) where $r \in \mathit{rule}(R) \setminus \mathit{rule}(R')$ and $\mathbf{K}\, h$ is such an **K**-atom from *head*(r). It follows that $\textbf{lfp}\ Q^{R'} = \textbf{lfp}\ Q^{R' \cup S}$, thus $\textbf{lfp}\ Q^{R'} = T$. $\square$

Lemma A.1. *Given two head-cuts $R, R' \in H_{\mathcal{K}}^{(T,F)}$ where $R \subset R'$ we have $\textbf{lfp}\ Q^R \subseteq \textbf{lfp}\ Q^{R'}$*

Proof. Assume the contrary and let i be the smallest integer such that $Q_{i+1}^R \not\subseteq Q_{i+1}^{R'}$. Let $X = Q_i^R = Q_i^{R'}$ and let a be an atom in Q_{i+1}^R but not in $Q_{i+1}^{R'}$. From $a \notin Q_{i+1}^{R'}$, we have $\mathsf{OB}_{\mathcal{O},X} \not\models a$ (otherwise $Q_{i+1}^{R'}$ would compute a). Therefore there must exist a pair $(r, a) \in R$

where $\mathit{body}(r) \subseteq X$ for Q^R_{i+1} to compute a. However, this pair also exists in R' ($R \subseteq R'$), thus $a \in Q^R_{i+1}$ a contradiction. $\square$

Proposition 4.3. *$HP^{(T,F)}_{\mathcal{K}}$ is a supporting set of (T, F).*

Proof. We have $HP^{(T,F)}_{\mathcal{K}} \subseteq HM^{(T,F)}_{\mathcal{K}}$, and if $HM^{(T,F)}_{\mathcal{K}}$ is nonempty, then there exists a least head-cut w.r.t. the branch-minimal relation. Thus, $HP^{(T,F)}_{\mathcal{K}}$ is empty if and only if $HM^{(T,F)}_{\mathcal{K}}$ is empty. We show

$$(\forall R \in HM^{(T,F)}_{\mathcal{K}}, \mathbf{lfp}\ Q^R = T) \iff (\forall R' \in HP^{(T,F)}_{\mathcal{K}}, \mathbf{lfp}\ Q^{R'} = T)$$

and it follows from Proposition 4.2 that $HP^{(T,F)}_{\mathcal{K}}$ is a supporting set of (T, F).

($\Rightarrow$) Trivial because $HP^{(T,F)}_{\mathcal{K}} \subseteq HM^{(T,F)}_{\mathcal{K}}$ by definition.

($\Leftarrow$) Assume that for each $R' \in HP^{(T,F)}_{\mathcal{K}}$, we have $\mathbf{lfp}\ Q^{R'} = T$. Let $R \in HM^{(T,F)}_{\mathcal{K}} \setminus HP^{(T,F)}_{\mathcal{K}}$ We know $\mathbf{lfp}\ Q^R \subseteq T$ and show $\mathbf{lfp}\ Q^R \subseteq T$. Because $R \notin HP^{(T,F)}_{\mathcal{K}}$, we have an $R' \in HP^{(T,F)}_{\mathcal{K}}$ such that $\mathit{head}(R) \supset \mathit{head}(R')$. By the initial assumption, $\mathrm{OB}_{\mathcal{O},\mathit{head}(R')} \models t$ for each $\mathbf{K}\, t \in T$, thus $\mathbf{lfp}\ Q^R \supseteq \mathbf{lfp}\ Q^{R'}$ because $\mathcal{O}$ is monotonic. It follows that $\mathbf{lfp}\ Q^{R'} = T$. $\square$

Lemma 4.2. *If there exists a head-cut in $R \in HP^{(T,F)}_{\mathcal{K}}$ such that **lfp** $Q^R \subset T$, then there is a head-cut $R' \in HP^{(T,F)}_{\mathcal{K}} \cap HG^{(T,F)}_{\mathcal{K}}$ such that **lfp** $Q^{R'} \subset T$*

Proof. We demonstrate this property by describing an algorithm to convert an arbitrary head-cut R from $HP^{(T,F)}_{\mathcal{K}}$ into a head-cut R' in $HP^{(T,F)}_{\mathcal{K}} \cap HG^{(T,F)}_{\mathcal{K}}$. This new head-cut has the property that $\mathbf{lfp}\ Q^{R'} \subseteq \mathbf{lfp}\ Q^R$. Thus, if there exists a head-cut $R \in HP^{(T,F)}_{\mathcal{K}}$ such that $\mathbf{lfp}\ Q^R \subset T$, then we can apply this algorithm to obtain a head-cut $R' \in HP^{(T,F)}_{\mathcal{K}} \cap HG^{(T,F)}_{\mathcal{K}}$ such that $\mathbf{lfp}\ Q^{R'} \subset T$.

Algorithm 5: hp-to-hm($R_{initial}$)

$R \leftarrow R_{initial}$;
$S_1 \leftarrow \{(r, \mathbf{select}\ h \in \mathbf{lfp}\ Q^{R_{initial}} \cap \mathit{head}(r)) \mid r \in \mathcal{P} \setminus \mathit{rule}(R_{initial}), \mathit{body}^+(r) \subseteq \mathbf{lfp}\ Q^{R_{initial}}\}$;
$S_2 \leftarrow \{(r, \mathbf{select}\ h \in \mathit{head}(r)) \mid r \in \mathcal{P} \setminus \mathit{rule}(R \cup S_1)\}$;
$R \leftarrow R \cup S_1 \cup S_2$;
for $i \leftarrow 0$; $i = 0$ *or* $Q^R_{i-1} \neq Q^R_i$; $i \leftarrow i + 1$ **do**
    $R^* \leftarrow R \setminus \{(r, h) \mid \neg\exists j, Q^R_j \supseteq \mathit{body}(r), Q^R_j \cap \mathit{head}(r) = \emptyset\}$;
    if $\exists R' \in HG^{(T,F)}_{\mathcal{K}} \setminus HP^{(T,F)}_{\mathcal{K}}$ *s.t.* $R'[0..(i-1)] = R^*[0..(i-1)]$ *and* $\mathit{head}(R'[i]) \subset \mathit{head}(R^*[i])$ **then**
        $R \leftarrow (R \setminus R^*[i]) \cup R'[i]$;
$R^* \leftarrow R \setminus \{(r, h) \mid \neg\exists j, Q^R_j \supseteq \mathit{body}(r), Q^R_j \cap \mathit{head}(r) = \emptyset\}$;
return R^*;

We use **select** $h \in S$ to denote that an atom h may be selected from a set S arbitrarily. We give an informal overview of the above algorithm and follow it with a formal proof. The algorithm begins by converting the head-cut R into a head-cut in $H_{\mathcal{K}}^{(T,F)}$ by adding missing rules appropriately. At the beginning of each iteration of the loop, we create a copy of R, named R^*, that has had these additions removed so that $R^* \in HG_{\mathcal{K}}^{(T,F)}$. We check whether R^* is semi-branch-minimal, and if it is not, we make it so. The body of the loop is repeated for each iteration of the Q^R operator. The end result is a head-cut $R \in HP_{\mathcal{K}}^{(T,F)} \setminus HM_{\mathcal{K}}^{(T,F)}$ such that **lfp** Q^R comptues fewer atoms than **lfp** $Q^{R_{initial}}$

We show formally that the head-cut returned by the algorithm has this property. Assume that Algorithm 4.2 is invoked with a head-cut $R_{initial} \in HP_{\mathcal{K}}^{(T,F)}$. Let $R^* = \{(r,h) \mid \neg\exists j, Q_j^R \supseteq body(r), Q_j^R \cap head(r) = \emptyset\}$ (the expression at line 9). The following invariants hold at the beginning of every iteration of the loop on line 5:

1. $R \in H_{\mathcal{K}}^{(T,F)}$
2. **lfp** $Q^{R^*} \subseteq$ **lfp** $Q^{R_{initial}}$
3. $R^* \in HM_{\mathcal{K}}^{(T,F)} \cap HP_{\mathcal{K}}^{(T,F)}$
4. For each j such that $0 \leq j < i$ there does not exist a head-cut $R' \in HG_{\mathcal{K}}^{(T,F)} \setminus HP_{\mathcal{K}}^{(T,F)}$ such that $R'[0..(i-1)] = R[0..(i-1)]$ and $head(R'[i]) \subset head(R[i])$ (the condition on line 7) is satisfied.

We first show that the invariants 1 through 4 hold for the first iteration of the loop. Let $R = R_{initial} \cup S_1 \cup S_2$.

(1) We have $R_{initial} \in HM_{\mathcal{K}}^{(T,F)}$, thus there is some head-cut $J \in H_{\mathcal{K}}^{(T,F)}$ such that $R = J \setminus (S_1 \cup S_2)$. Note that for every pair $(r,h) \in S_1$, there exists a head atom $h \in head(r) \cap$ **lfp** $Q^{R_{initial}}$ by $R_{initial} \in HM_{\mathcal{K}}^{(T,F)}$.

(2) We have **lfp** $Q^R \subseteq$ **lfp** $Q^{R_{initial}}$ because for every pair (r,h) added to $R_{initial}$ via S_1 and S_2 we have either $(r,h) \in S_2$ and $body^+(r) \not\subseteq$ **lfp** $Q^{R_{initial}}$ (thus $h \in$ **lfp** Q^R only if h is computed via a different pair) or $(r,h) \in S_1$ and $h \in$ **lfp** $Q^{R_{initial}}$. Because R^* is obtained by removing pairs from $R \cup S_1 \cup S_2$, we have **lfp** $Q^{R^*} \subseteq$ **lfp** $Q^{R_{initial}}$.

(3) From (1) we have $R \in H_{\mathcal{K}}^{(T,F)}$. The pairs R^* removes from R are precisely the pairs to have $R^* \in HM_{\mathcal{K}}^{(T,F)}$. From $R_{initial} \in HP_{\mathcal{K}}^{(T,F)}$ and 2, we have $R^* \in HP_{\mathcal{K}}^{(T,F)}$.

(4) We have $i = 0$, thus there does not exist an integer j, $0 \leq j < 0$.

Assuming each invariant holds at the beginning of an iteration, we show they each hold at the end of an iteration.

(1) We have $rule(R'[i]) = rule(R^*[i])$, thus line 8 does not affect the set $rule(R)$ and we still have $R \in H_{\mathcal{K}}^{(T,F)}$.

(2) Under the condition on line 7, we have $head(R'[i]) \subseteq head(R^*[i])$, thus line 8 can only shrink the set **lfp** Q^{R^*}.

(3) From (1) we have $R \in H_{\mathcal{K}}^{(T,F)}$. The pairs R^* removes from R are precisely the pairs to have $R^* \in HM_{\mathcal{K}}^{(T,F)}$. From $R_{initial} \in HP_{\mathcal{K}}^{(T,F)}$ and 2, we have $R^* \in HP_{\mathcal{K}}^{(T,F)}$.

(4) Because $R' \in HG_{\mathcal{K}}^{(T,F)}$, there does not exist another set $R'' \in HG$ such that $R'[0..(i-1)] = R''[0..(i-1)]$ and *head*$(R''[i]) \subset$ *head*$(R'[i])$. It is minimal in this respect. Because line 8 only effects pairs in $R[i]$, this invariant continues to hold after i is incremented.

Finally, we return a head-cut which is in $HM_{\mathcal{K}}^{(T,F)} \cap HP_{\mathcal{K}}^{(T,F)} \cap HG_{\mathcal{K}}^{(T,F)}$ where $\textbf{lfp } Q^{R^*} \subseteq \textbf{lfp } Q^{R_{initial}}$. □

Proposition 4.4. *$HG_{\mathcal{K}}^{(T,F)}$ is a supporting set of (T,F).*

Proof. $HG_{\mathcal{K}}^{(T,F)}$ is empty if and only if $HP_{\mathcal{K}}^{(T,F)}$ is empty. It is sufficient to show

$$(\forall R \in HM_{\mathcal{K}}^{(T,F)}, \textbf{lfp } Q^R = T) \iff (\forall R' \in HG_{\mathcal{K}}^{(T,F)}, \textbf{lfp } Q^{R'} = T)$$

. ($\Rightarrow$) Trivial since $HG_{\mathcal{K}}^{(T,F)} \subseteq HM_{\mathcal{K}}^{(T,F)}$.

($\Leftarrow$) We show the contrapositive. Let $R \in HM_{\mathcal{K}}^{(T,F)}$ such that $\textbf{lfp } Q^R \subset T$. By Proposition 4.3, we have a head-cut $R^p \in HP_{\mathcal{K}}^{(T,F)}$ such that $\textbf{lfp } Q^{R^p} \subset T$. Apply Lemma 4.2 to R^p to obtain a head-cut $R' \in HG_{\mathcal{K}}^{(T,F)}$ such that $\textbf{lfp } Q^{R'} \subset T$. We conclude

$$\neg(\forall R \in HM_{\mathcal{K}}^{(T,F)}, \textbf{lfp } Q^{R'} = T) \implies \neg(\forall R' \in HG_{\mathcal{K}}^{(T,F)}, \textbf{lfp } Q^{R'} = T)$$

.

It follows that $HG_{\mathcal{K}}^{(T,F)}$ is a supporting set of (T,F). □

Lemma 5.1. *For a solver state R and any head-cut $R^* \in \bigcup D_{\mathcal{K}}^R$ where $R = R^*[0..i]$, we either have $R^*[0..(i+1)] = T(R)$ or $R = T(R)$.*

Proof. We show that the assertion on line 5 holds and it directly follows that either $R = T(R)$ (when B is empty or the assert was not reached) or $T(R) = R^*[0..(i+1)]$ because $R^*[0..(i+1)]$ is the same for any R^*. Consider for the sake of contradiction that $|B| > 1$ at line 5. We have two head-cuts $R_1^*, R_2^* \in \bigcup D_{\mathcal{K}}^R$ such that $R_1^*[0..i] = R_2^*[0..i] = R$ and $R_1^*[i+1] \neq R_2^*[i+1]$. Let $(r, h_1) \in R_1^*[i+1] \setminus R_2^*[i+1]$. If there is a pair $(r, h_2) \in R_2^*[i+1]$ such that $h_1 \neq h_2$, then $|$*head*$(r) \setminus F| \neq 1$, otherwise, *body*$(r) \not\sqsubseteq (T,F)$. In either case, the algorithm would have met the condition on line 3 and returned before reaching line 5, which contradictions the assumption that $|B| > 1$ at line 5. □

Lemma 5.2. *Given a solver state R and an MKNF model M that induces $\mathcal{S}(R)$, there is either a solver state $R' \in decisions(R)$ such that M induces $\mathcal{S}(R')$ or $decisions(R)$ is empty.*

Proof. Let (T^*, F^*) be the total dependable partial induced by M. If the condition does not hold on line 2, i.e. $R \notin \bigcup D_{\mathcal{K}}^R$, then there is at least one head-cut R' that meets the condition of the loop. Because M is a model, for at least one R', $\mathcal{S}(R)$ is dependable. The set $HG_{\mathcal{K}}^{(T^*,F^*)}$ is in $D_{\mathcal{K}}^R$, and there is a head-cut $R^* \in HG_{\mathcal{K}}^{(T^*,F^*)}$ such that $R = R^*[0..i]$. Thus, the head-cut $R^*[0..(i+1)]$ is in $decisions(R)$. We have $\mathcal{S}(R^*[0..(i+1)]) \sqsubseteq (T^*, F^*)$, it follows that M induces $\mathcal{S}(R^*[0..(i+1)])$. □

Definition A.1. *The definition of check-model*(R)*:*

Algorithm 6: *check-model*(R)

if $HG_{\mathcal{K}}^{\mathcal{S}^*(R)} = \emptyset$ **then**
  return $false$;

if $\mathcal{S}^*(R)$ *is dependable and for each* $R \in HG_{\mathcal{K}}^{\mathcal{S}^*(R)}$, ***lfp*** $Q^R = T$ *where* $(T, F) = \mathcal{S}^*(R)$ **then**
  return $\mathcal{S}^*(R)$;

else
  return $false$;

Lemma 5.3. *Given a solver state* R*, if there exists a model that induces* $\mathcal{S}(R)$*, then* $solver(R)$ *will a total partition induced by a model.*

Proof. Let (T^*, F^*) be the total dependable partition induced by M. The solver algorithm is simply repeated application of $T(R)$ and $decisions(R)$. From Lemma 5.1 we know that $T(R)$ will preserve $\mathcal{S}(R) \sqsubseteq (T^*, F^*)$. There is either a head-cut $R' \in decisions(R)$ such that $\mathcal{S}(R') \sqsubseteq (T^*, F^*)$ or $decisions(R)$ is empty (Lemma 5.2). In the case that $decisions(R)$ is empty, we check whether $\mathcal{S}^*(R)$, a total partition, can be extended to a model and return it. Clearly, if this is the case then $\mathcal{S}^*(R) = (T^*, F^*)$. □

Lemma A.2. *Given a dependable partition* (T, F)*, let* $R_1, R_2 \in HG_{\mathcal{K}}^{(T,F)}$ *s.t. for some* i*,* ***lfp*** $Q^{R_1} \supseteq Q_i^{R_2}$*. There exists a head-cut* $R_3 \in HG_{\mathcal{K}}^{(T,F)}$ *such that* $R_3[0..i] = R_2[0..i]$ *and* ***lfp*** $Q^{R_1} \supseteq$ ***lfp*** Q^{R_3}*.*

Proof. We construct R_3 by iteratively adding pairs its base, $R_2[0..i]$. For each $r \in \mathcal{P}$ s.t. *body*$(r) \sqsubseteq (Q_j^{R_3}, F)$ and *head*$(r) \cap Q_j^{R_3} = \emptyset$, we have *body*$(r) \sqsubseteq$ (**lfp** $Q^{R_1}, F)$ and thus some $h \in$ *head*$(r) \cap$ **lfp** Q^{R_1}. We add these pairs to R_3 so that **lfp** $Q^{R_1} \supseteq Q_{i+1}^{R_3}$ is maintained. Let $R[i+1] = \{(r, h) \mid$ *body*$(r) \sqsubseteq (Q_j^{R_3}, F)$ and $h \in$ *head*$(r) \cap$ **lfp** $Q^{R_1}\}$. [1]. This process can be repeated until there are no pairs to add and the result is a head-cut $R_3 \in HG_{\mathcal{K}}^{(T,F)}$ such that **lfp** $Q^{R_3} \subseteq$ **lfp** Q^{R_1}. □

Proposition 5.1. *If a solver state* R *is P-verifiable and* $\mathcal{S}^*(R)$ *is dependable, then* ***lfp*** $Q^R = T$ *if and only if* ***lfp*** $Q^{R'} = T$ *for each* $R' \in HG_{\mathcal{K}}^{\mathcal{S}^*(R)}$*.*

Proof. Because R is a solver state, $HG_{\mathcal{K}}^{(T,F)}$ is nonempty. Given that $R \in HG_{\mathcal{K}}^{\mathcal{S}^*(R)}$, we need only show that **lfp** $Q^R = T$ implies $\forall R^* \in HG_{\mathcal{K}}^{\mathcal{S}^* R}$, **lfp** Q^{R^*}.

We begin by defining a mapping $m(x) : y$:

[1] The pairs in $R[i+1]$ must also be minimal w.r.t. *head*$(R[i+1])$ but this does not affect the proof.

$$
\begin{cases}
\text{the smallest positive integer } y \text{ s.t. } Q^R_{y+1} = Q^R_y & x = 0 \\
\text{the largest positive integer } y \text{ s.t. } y < m(x-1) \text{ and} & x > 0 \\
\exists R^* \in HG^{(T,F)}_{\mathcal{K}}, R^*[0..y] = R[0..y], \text{ and} & \\
R^*[y+1] \neq R[y+1] & \\
0 & \text{there does not exist such an integer } y
\end{cases}
$$

We show that for any $R^* \in HG^{(T,F)}_{\mathcal{K}}$, if for some i, $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i)}$, then $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i+1)}$.

Let $(r,h) \in R[m(i)+1]$. From $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i)}$, we have $\mathit{body}^+(r) \subseteq \mathbf{lfp}\ Q^{R^*}$, thus we have $\mathbf{K}\,h' \in \mathit{head}(r)$ such that either $(r,h) \in R^*$ or we have that if $Q^{R^*}_j \supseteq \mathit{body}^+(r)$, then $h' \in Q^{R^*}_j$. In either case, we have $h' \in \mathbf{lfp}\ Q^{R^*}$ and with the definition of m, we have a head-cut R' such that

$$
\begin{aligned}
R[0..m(i)] &= R'[0..m(i)] \\
R[m(i)+1] &\neq R'[m(i)+1] \\
\mathbf{lfp}\ Q^{R^*} &\supseteq R'[m(i)+1]
\end{aligned}
$$

(Although the definition of m does not say that R' with $\mathbf{lfp}\ Q^{R^*} \supseteq R[m(i)+1]$ exists, it is clear that it must by the definition of $HG^{(T,F)}_{\mathcal{K}}$). Because R is P-verifiable and $R[0..m(i)] = R'[0..m(i)]$, we have for each $R'' \in HG^{(T,F)}_{\mathcal{K}}$, $R''[m(i)+1] = R'[m(i)+1]$ that $\mathbf{lfp}\ Q^{R''} \supseteq \mathit{head}(R[m(i)+1])$. Applying Lemma A.2 with R^* and R', we obtain a head-cut R'' such that $R''[m(i)+1] = R'[m(i)+1]$ and $\mathbf{lfp}\ Q^{R^*} \supseteq \mathbf{lfp}\ Q^{R''}$. Thus, we have $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i)+1}$.

For each j s.t. $m(i-1)+1 < j < m(i)+1$, we have $\mathbf{lfp}\ Q^{R^*} \supseteq \mathit{head}(R[j])$. Note that if this were not the case, there would be multiple true atoms in the head of some $r \in \mathit{rule}(R[j])$ and thus $m(i-1)$ would be equal to $j+1$.

We have shown that for any $R^* \in HG^{(T,F)}_{\mathcal{K}}$, if for some i, $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i)}$, then $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i+1)}$. We now show inductively that $\mathbf{lfp}\ Q^{R^*} \supseteq T$. Let e be the integer such that $m(e-1) \neq 0$ and $m(e) = 0$. Clearly, $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(e)}$, since $Q^R_{m(e)} = \emptyset$. Assume for some k, $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(i)}$, we have $\mathbf{lfp}\ Q^{R^*} \supseteq Q^R_{m(k+1)}$ (shown above).

By the initial assumption, we have $Q^R_{m(0)} = T$, thus $\mathbf{lfp}\ Q^{R^*} \subseteq T$. We can not have $\mathbf{lfp}\ Q^{R^*} \subset T$, thus $\mathbf{lfp}\ Q^{R^*} = T$ □

Proof. (⇒) See Proposition 5.1. (⇐) Under the definition of solver states, we have that $HG^{\mathcal{S}^*(R)}_{\mathcal{K}}$ is nonempty. If R were not P-verifiable, then we have a head-cut $R^* \in HG^{\mathcal{S}^*(R)}_{\mathcal{K}}$ such that $\mathbf{lfp}\ Q^{R^*} \neq T$ where $(T,F) = \mathcal{S}^*(R)$. With Proposition 4.4, $\mathcal{S}^*(R)$ can not be extended to an MKNF model of $\mathcal{K}$. □