
Isometry pursuit

Samson Koelle
Amazon
koelle@amazon.com

Marina Meila
Department of Statistics
University of Washington
mmp@uw.edu

Abstract

Isometry pursuit is a convex algorithm for identifying orthonormal column-submatrices of wide matrices. It consists of a vector normalization followed by multitask basis pursuit. Applied to Jacobians of putative coordinate functions, it helps identify locally isometric embeddings from within interpretable dictionaries. We provide theoretical and experimental results justifying this method, including a proof with realistic assumptions that such isometric submatrices, should they exist, are contained within the obtained support. For problems involving coordinate selection and diversification, it offers a synergistic alternative to greedy and brute force search.

1 Introduction

Many real-world problems may be abstracted as selecting a subset of the columns of a matrix representing stochastic observations or analytically exact data. This paper focuses on a simple such problem. Given a rank D matrix $X \in \mathbb{R}^{D \times P}$ with $P > D$, select a square submatrix X_S where subset $S \subset [P]$ satisfies $|S| = D$ that is as orthonormal as possible.

This problem arises in interpretable learning specifically because while the coordinate functions of a given feature space may have no intrinsic meaning, it is sometimes possible to generate a dictionary of interpretable features which may be considered as potential parametrizing coordinates. When this is the case, selection of candidate interpretable features as coordinates can take the above form. While implementations vary across data and algorithmic domains, identification of such coordinates generally aids mechanistic understanding, generative control, and statistical efficiency.

This paper shows that an adapted version of the algorithm in Koelle et al. [2024] leads to a convex procedure that can improve upon greedy approaches such as those in Cai and Wang [2011], Chen and Meila [2019], Kohli et al. [2021], Jones et al. [2007] for finding isometries. The insight leading to isometry pursuit is that multitask basis pursuit applied to an appropriately normalized X selects orthonormal submatrices. Given vectors in \mathbb{R}^D , the normalization log-symmetrizes length and favors those closer to unit length, while basis pursuit favors those which are orthogonal. Our results formalize this intuition within a limited setting, and show the usefulness of isometry pursuit as a trimming procedure prior to brute force search for diversification and interpretable coordinate selection. We also introduce a novel ground truth objective function against which we measure the success of our algorithm, and discuss the reasonableness of the trimming procedure.

2 Background

Our algorithm is motivated by spectral and convex analysis.

¹Work conducted outside of Amazon.

²Code is available at <https://github.com/sjkoelle/isometry-pursuit>.

2.1 Problem

Our goal is, given a matrix $X \in \mathbb{R}^{D \times P}$, to select a subset $S \subset [P]$ with $|S| = D$ such that $X_{\cdot S}$ is as orthonormal as possible in a computationally efficient way. To this end, we define a ground truth loss function that measures orthonormalness, and then introduce a surrogate loss function that convexifies the problem so that it may be efficiently solved. Figure 1a and 1b provide a visual depiction of this problem when $D = 2$ and $P = 30$.

2.2 Interpretability and isometry

Our motivating example is the selection of data representations from within sets of putative coordinates: the columns of a provided wide matrix. Compared with Sparse PCA [d’Aspremont et al., 2007, Dey et al., 2017, Bertsimas and Kitane, 2022, Bertsimas et al., 2022], we seek a low-dimensional representation from the set of these column vectors rather than their span.

This method applies to interpretability, for which parsimony is at a premium. Interpretability arises through comparison of data with what is known to be important in the domain of the problem. This knowledge often takes the form of a functional dictionary. Evaluation of independence of dictionary features arises in numerous scenarios [Chen and Meilä, 2019, Koelle et al., 2022, He et al., 2023]. The requirement that dictionary features be full rank has been called functional independence [Koelle et al., 2022] or feature decomposability [Templeton et al., 2024], with connection between dictionary rank and independence via the implicit function theorem. Besides independence, the metric properties of such dictionary elements are of natural interest. This is formalized through the notion of differential.

Definition 1 *The **differential** of a smooth map $\phi : \mathcal{M} \rightarrow \mathcal{N}$ between D dimensional manifolds $\mathcal{M} \subseteq \mathbb{R}^B$ and $\mathcal{N} \subseteq \mathbb{R}^P$ is a map in tangent bases $x_1 \dots x_D$ of $T_\xi \mathcal{M}$ and $y_1 \dots y_D$ of $T_{\phi(\xi)} \mathcal{N}$ consisting of entries*

$$D\phi(\xi) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}(\xi) & \dots & \frac{\partial \phi_1}{\partial x_D}(\xi) \\ \vdots & & \vdots \\ \frac{\partial \phi_D}{\partial x_1}(\xi) & \dots & \frac{\partial \phi_D}{\partial x_D}(\xi) \end{bmatrix}. \quad (1)$$

It is not always necessary to explicitly estimate tangent spaces when applying this definition. The most commonly encountered manifolds are vector spaces for which the tangent spaces are trivial. This is the case for full-rank tabular data, for which isometry has a natural interpretation as a type of diversification, and often for the latent spaces of deep learning models. In this case, $B = D$.

Definition 2 *A map ϕ between D dimensional submanifolds with inherited Euclidean metric $\mathcal{M} \subseteq \mathbb{R}^B$ and $\mathcal{N} \subseteq \mathbb{R}^P$ ϕ is an **isometry at a point** $\xi \in \mathcal{M}$ if*

$$D\phi(\xi)^T D\phi(\xi) = I_D. \quad (2)$$

That is, ϕ is an isometry at ξ if $D\phi(\xi)$ is orthonormal.

The applications of pointwise isometry are themselves manifold. Pointwise isometric embeddings faithfully preserve high-dimensional geometry. For example, Local Tangent Space Alignment [Zhang and Zha, 2004], Multidimensional Scaling [Chen and Buja, 2009] and Isomap [Tenenbaum et al., 2000] non-parametrically estimate embeddings that are as isometric as possible. Another approach stitches together pointwise isometries selected from a dictionary to form global embeddings [Kohli et al., 2021]. The method is particularly relevant since it constructs such isometries through greedy search, with putative dictionary features added one at a time.

That $D\phi$ is orthonormal has several equivalent formulations. The one motivating our ground truth loss function comes from spectral analysis.

Proposition 1 *The singular values $\sigma_1 \dots \sigma_D$ are equal to 1 if and only if $U \in \mathbb{R}^{D \times D}$ is orthonormal.*

On the other hand, the formulation that motivates our convex approach is that orthonormal matrices consist of D coordinate features whose gradients are orthogonal and of unit length.

Proposition 2 *The component vectors $u_1 \dots u_D \in \mathbb{R}^B$ form a orthonormal matrix if and only if, for*

$$\text{all } d_1, d_2 \in [D], \langle u_{d_1}, u_{d_2} \rangle = \begin{cases} 1 & d_1 = d_2 \\ 0 & d_1 \neq d_2 \end{cases}.$$

2.3 Best subset selection

Given a matrix $X \in \mathbb{R}^{D \times P}$, we compare algorithmic paradigms for solving problems of the form

$$\arg \min_{S \in \binom{[P]}{D}} l(X_{\cdot S}) \quad (3)$$

where $\binom{[P]}{D} = \{A \subseteq [P] : |A| = D\}$. Brute force algorithms consider all possible solutions. These algorithms are conceptually simple, but have the often prohibitive time complexity $O(C_l P^D)$ where C_l is the cost of evaluating l . Greedy algorithms consist of iteratively adding one element at a time to S . These algorithms have time complexity $O(C_l P D)$ and so are computationally more efficient than brute force algorithms, but can get stuck in local minima. Formal definitions are given in Section 6.1.

Sometimes, it is possible to introduce an objective which convexifies problems of the above form [Liberti, 2004, Abdi, 2013, Zhou and Low, 2021]. Solutions

$$\arg \min f(\beta) : Y = X\beta \quad (4)$$

to the overcomplete regression problem $Y = X\beta$ are a classic example [Scott Shaobing Chen and David L. Donoho and Michael A. Saunders, 2001, ?]. When $f(\beta) = \|\beta\|_0$, this problem is non-convex, and is thus suitable for greedy or brute algorithms, but when $f(\beta) = \|\beta\|_1$, the problem is convex, and may be solved efficiently via interior-point methods. When the equality constraint is relaxed, Lagrangian duality may be used to reformulate as a so-called Lasso problem, which leads to an even richer set of optimization algorithms [?].

The form of basis pursuit that we apply is inspired by the group basis pursuit approach in Koelle et al. [2022]. In group basis pursuit (which we call multitask basis pursuit when grouping is dependent only on the structure of matrix-valued response variable y) the objective function is $f(\beta) = \|\beta\|_{1,2} := \sum_{p=1}^P \|\beta_p\|_2$ [Yuan and Lin, 2006, Obozinski et al., 2006, Yeung and Zhang, 2011]. This objective creates joint sparsity across entire rows β_p , and was used in Koelle et al. [2022] to select between sets of interpretable features.

3 Method

We adapt the group lasso paradigm used to select independent dictionary elements in Koelle et al. [2022, 2024] to select pointwise isometries from a dictionary. We first define a ground truth objective computable via brute and greedy algorithms that is uniquely minimized by orthonormal matrices. We then define the combination of normalization and multitask basis pursuit that approximates this ground truth loss function. We finally give a brute post-processing method for ensuring that the solution is D sparse, and provide a theoretical result that the two stage approach will always result in recovery of an isometric solution from the dictionary, should one exist.

3.1 Ground truth

We'd like a ground truth objective to be minimized uniquely by orthonormal matrices, invariant under rotation, and depend on all changes in the matrix. Deformation [Kohli et al., 2021] and nuclear norm [Boyd and Vandenberghe, 2004] use only a subset of the differential's information and are not uniquely minimized at unitarity, respectively. We therefore introduce an alternative ground truth objective that satisfies the above desiderata and has convenient connections to isometry pursuit.

This ground truth objective is

$$l_c : \mathbb{R}^{D \times P} \rightarrow \mathbb{R}^+ \quad (5)$$

$$X \mapsto \sum_{d=1}^D g(\sigma_d(X), c) \quad (6)$$

where $\sigma_d(X)$ is the d -th singular value of X and

$$g : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad (7)$$

$$t, c \mapsto \frac{e^{t^c} + e^{t^{-c}}}{2e}. \quad (8)$$

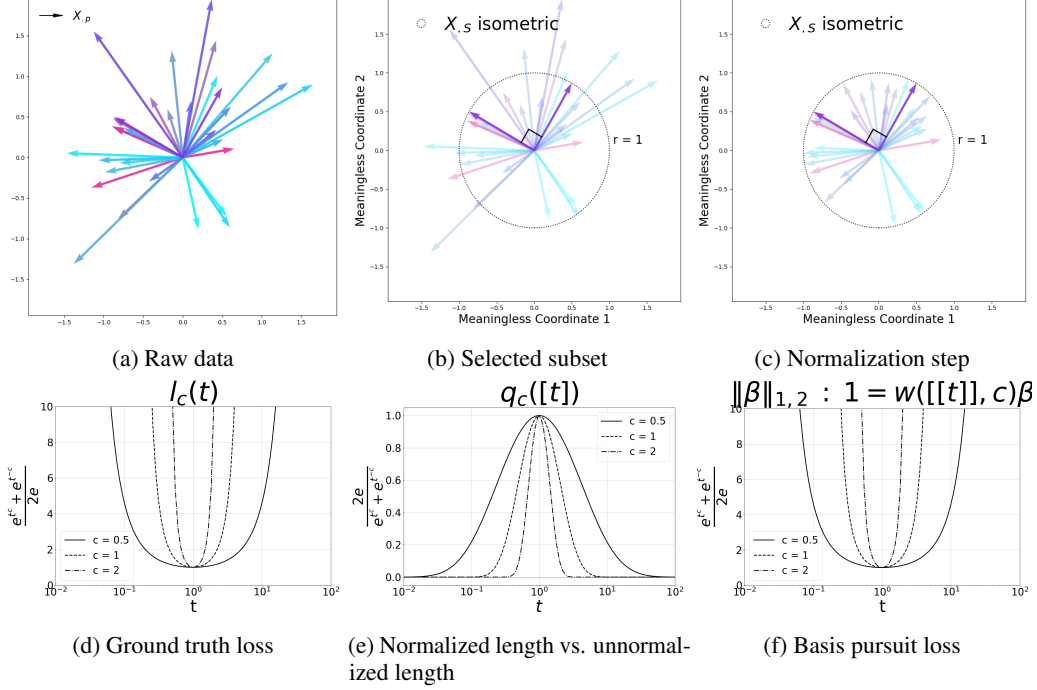


Figure 1: 1a Raw data X represented as column-vectors. 1b An isometric subset - the identification of which is our objective. 1c The vectors after normalization so that vectors of length 1 maintain longest length after normalization. 1d Our ground truth loss function. 1e Normalized length used to rescale Figure 1a to get vectors in Figure 1c. 1f Basis pursuit loss for different values of c in the one-dimensional case $D = 1$. This loss is equivalent to that shown in Figure 1d in the one-dimensional case.

By Proposition 1, we can see that ℓ_c is uniquely maximized by orthonormal matrices. Moreover, g is convex, and $\ell_c(X^{-1}) = \ell_c(X)$ when X is invertible. Figure 1d gives a graph of ℓ_c when $D = 1$.

Our ground truth objective is therefore the best subset selection problem

$$\arg \min_{S \in \binom{[P]}{d}} \ell_c(X_{\cdot S}). \quad (9)$$

Regardless of the convexity of ℓ_c , brute combinatorial search over $[P]$ is inherently non-convex. This motivates our alternative formulation.

3.2 Normalization

Since basis pursuit methods tend to select longer vectors, selection of orthonormal submatrices requires normalization such that both long and short candidate basis vectors are penalized in the subsequent regression. We introduce the following definition.

Definition 3 (Symmetric normalization) A function $q : \mathbb{R}^D \rightarrow \mathbb{R}^+$ is a symmetric normalization if

$$\arg \max_{v \in \mathbb{R}^D} q(v) = \{v : \|v\|_2 = 1\} \quad (10)$$

$$q(v) = q\left(\frac{v}{\|v\|_2}\right) \quad (11)$$

$$q(v_1) = q(v_2) \quad \forall v_1, v_2 \in \mathbb{R}^D : \|v_1\|_2 = \|v_2\|_2. \quad (12)$$

We use such functions to normalize column-vector length in such a way that vectors of length 1 prior to normalization have longest length after normalization and vectors are shrunk proportionately to

their deviation from 1. That is, we normalize vectors by

$$n : \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (13)$$

$$v \mapsto q(v)v \quad (14)$$

and matrices by

$$w : \mathbb{R}^{D \times P} \rightarrow \mathbb{R}^D \quad (15)$$

$$X_{\cdot p} \mapsto n(X_{\cdot p}) \quad \forall p \in [P]. \quad (16)$$

Given $c > 0$, we choose q as follows.

$$q_c : \mathbb{R}^D \rightarrow \mathbb{R}^+ \quad (17)$$

$$v \mapsto \frac{e^{\|v\|_2^c} + e^{\|v\|_2^{-c}}}{2e}. \quad (18)$$

Besides satisfying the conditions in Definition 3, this normalization has some additional nice properties. First, q is convex. Second, it grows asymptotically log-linearly. Third, while $\exp(-|\log t|) = \exp(-\max(t, 1/t))$ is a seemingly natural choice for normalization, it is non smooth, and the LogSumExp [Boyd and Vandenberghe, 2004] replacement of $\max(t, 1/t)$ with $\log(\exp(t) + \exp(1/t))$ simplifies to 18 upon exponentiation. Finally, the parameter c grants control over the width of the basin, which may be useful for avoiding numerical issues arising close to 0 and ∞ .

3.3 Isometry pursuit

Isometry pursuit is the application of multitask basis pursuit to the normalized design matrix $w(X, c)$ to identify submatrices of X that are as orthonormal as possible. Define the multitask basis pursuit penalty

$$\|\cdot\|_{1,2} : \mathbb{R}^{P \times D} \rightarrow \mathbb{R}^+ \quad (19)$$

$$\beta \mapsto \sum_{p=1}^P \|\beta_p\|_2. \quad (20)$$

Given a matrix $Y \in \mathbb{R}^{D \times D}$, the multitask basis pursuit solutions are

$$\hat{\beta}_{MBP}(X, Y) := \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_{1,2} : Y = X\beta. \quad (21)$$

Note that multitask basis pursuit solution is not unique under more relaxed conditions than for standard basis pursuit. Recall that, given a design matrix X , there exists a response variable y such that the lasso admits a non-unique solution if and only if the rowspan of the design matrix X contains a sufficient codimension edge of the p dimensional hypercube [Schneider and Tardivel, 2022]. This condition generalizes the previously introduced general position condition - that affinely independent columns of the design matrix guarantee non-uniqueness. However, for Isometry Pursuit, non-unique solutions

may occur even when this condition is satisfied. A simple example - $X = \begin{bmatrix} 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{-2}}{2} \end{bmatrix}$ results from the rotation invariance Proposition 4, but more subtle examples exist. We therefore define

$$\hat{\beta}_{MLP}^{\ell_2}(X, Y) = \arg \min \|\beta\|_F : \beta \in \beta_{MBP}(X, Y). \quad (22)$$

By the strong convexity of the ℓ_2 norm, $\hat{\beta}_{MLP}^{\ell_2}(X, Y)$ is well defined.

This implicit regularization assumption is reasonable. For example, in Tibshirani [2012a] the implicit selection of the minimum ℓ_2 -norm solution among the lasso solutions was proven for the Least Angle Regression Selection (LARS) algorithm for solving the lasso, while Mishkin and Pilanci [2022a], ? simply assumes that the group lasso solution is in fact min-norm prior to subsequent theoretical analyses. Empirically, the Splitting Conic Solver type method (which is itself a variant of the Alternating Direction Method of Multipliers) stably estimates the min-norm solution when

initialized at 0, but we leave theoretical and experimental proof of this feature of this feature our optimization approach for future work, and instead make the ability of our optimizer to obtain the min-norm solution an assumption of our eventual proposition.

Isometry pursuit is then given by

$$\hat{\beta}_c(X) := \hat{\beta}_{MBP}^{\ell_2}(w(X, c), I_D) \quad (23)$$

where I_D is the D dimensional identity matrix and recovered functions are the indices of the dictionary elements with non-zero coefficients. That is, they are given by $S(\beta)$ where

$$S : \mathbb{R}^{P \times D} \rightarrow \binom{[P]}{D} \quad (24)$$

$$\beta \mapsto \{p \in [P] : \|\beta_p\| > 0\}. \quad (25)$$

ISOMETRYPURSUIT(Matrix $X \in \mathbb{R}^{D \times P}$, scaling constant c)

- 1: Normalize $X_c = w(X, c)$
 - 2: Optimize $\hat{\beta} = \hat{\beta}_{MBP}^{\ell_2}(X_c, I_D)$
 - 3: **Output** $\hat{S} = S(\hat{\beta})$
-

3.4 Theory

The intuition behind our application of multitask basis pursuit is that submatrices consisting of vectors which are closer to 1 in length and more orthogonal will have smaller loss. A key theoretical assertion is that ISOMETRYPURSUIT is invariant to choice of basis for X .

Proposition 3 *Let $U \in \mathbb{R}^{D \times D}$ be orthonormal. Then $S(\hat{\beta}(UX)) = S(\hat{\beta}(X))$.*

A proof is given in Section 6.2.1. A corollary is that we may replace I_D in the constraint by any orthonormal $D \times D$ matrix.

Our main theoretical result is that when a rank D orthonormal column-submatrix $X_{.S}$ exists, the output of Program 23 will contain S .

Proposition 4 *Let $X \in \mathbb{R}^{D \times P}$ have a rank D orthonormal column submatrix $X_{.S}$. Then $S \subseteq \hat{\beta}_c(X)$.*

Proofs of these propositions are given in Section 6.2, with corresponding experimental results in Section 4.

3.5 Two-stage isometry pursuit

Proposition 4 suggests the following algorithm, which first uses the convex problem to prune and then apply brute search upon the substantially reduced feature set. Similar two-stage approaches are standard in the Lasso literature Hesterberg et al. [2008], Koelle et al. [2022], and Proposition 4 puts them on relatively strong footing in our application, since this approach is guaranteed to select an isometric submatrix should one exist. This method forms our practical isometry estimator. Note however that if an isometric column submatrix does not exist, then it will not identify the optimum of Program 9, since the lasso objective and the singular value objective do not necessarily share a minimizer for all choices of X .

TWOSTAGEISOMETRYPURSUIT(Matrix $X \in \mathbb{R}^{D \times P}$, scaling constant c)

- 1: $\hat{S}_{IP} = \text{ISOMETRYPURSUIT}(X, c)$
 - 2: $\hat{S} = \text{BRUTESEARCH}(X_{.\hat{S}_{IP}}, l_c)$
 - 3: **Output** \hat{S}
-

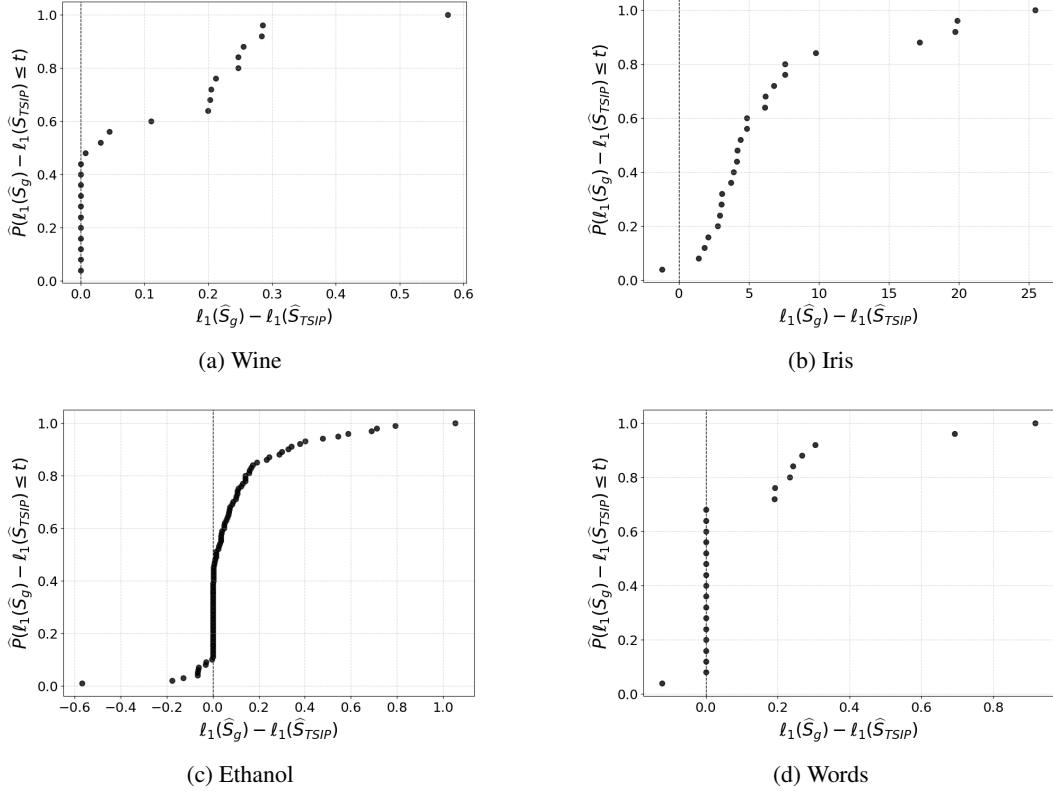


Figure 2: Empirical CDFs of isometry losses $\ell_1(\hat{S}_g) - \ell_1(\hat{S}_{TSIP})$ for Wine, Iris, and Ethanol datasets across R replicates. The CDF reflects the proportion of replicates where the greedy loss is less than or equal to the two-stage loss. As detailed in Table 1, losses are generally lower for two-stage isometry pursuit solutions.

4 Experiments

Say you are hosting an elegant dinner party, and wish to select a balanced set of wines for drinking and flowers for decoration. We demonstrate `TWOSTAGEISOMETRYPURSUIT` and `GREEDYSEARCH` on the Iris and Wine datasets [Fisher, 1988, Aeberhard and Forina, 1991, Pedregosa et al., 2011]. This has an intuitive interpretation as selecting diverse elements that reflects the peculiar structure of the diversification problem. Features like *petal width* are rows in X . They are features on the basis of which we may select among the flowers those which are most distinct from another. Thus, in diversification, $P = n$.

We also analyze the Ethanol dataset from Chmiela et al. [2018], Koelle et al. [2022], but rather than selecting between bourbon and scotch we evaluate a dictionary of interpretable features - bond torsions - for their ability to parameterize the molecular configuration space. In this interpretability use case, columns denote gradients of informative features. We compute Jacobian matrices of putative parametrization functions and project them onto estimated tangent spaces (see Koelle et al. [2022] for preprocessing details). Rather than selecting between data points, we are selecting between functions which parameterize the data. This dataset exemplifies our motivating example - the search for locally isometric embeddings.

Finally, we also construct a small word embedding dataset. Inspired by the linear representation hypothesis [Park et al., 2023, ?] and the construction over complete dictionaries of concepts in representation space, we apply isometric pursuit to prune down embeddings of concept dictionaries into their basis components. As with the other datasets, we measure success primarily numerically against the ground truth objective values obtained by greedy solutions.

For basis pursuit, we use the SCS interior point solver [O’Donoghue et al., 2016] from CVXPY [Diamond and Boyd, 2016, Agrawal et al., 2018], which is able to push sparse values arbitrarily close to 0 [CVXPY Developers]. Statistical replicas for Wine, Iris, and Words are created by resampling

Name	D	P	R	c	$\ell_1(X_{\cdot, \hat{S}_G})$	$ \hat{S} $	$\ell_1(X_{\cdot, \hat{S}})$	$\hat{P}(\ell_1(X_{\cdot, \hat{S}_G}) > \ell_1(X_{\cdot, \hat{S}}))$	$\hat{P}(\ell_1(X_{\cdot, \hat{S}_G}) = \ell_1(X_{\cdot, \hat{S}}))$	$\hat{P}(\bar{\ell}_1(X_{\cdot, \hat{S}_G}) > \bar{\ell}_1(X_{\cdot, \hat{S}}))$
Iris	4	75	25	1	14 ± 7.3	6.7 ± 1	6.9 ± 1.4	0.96	0	2.4e-05
Wine	6	89	25	1	7.7 ± 0.33	13 ± 1.5	7.6 ± 0.29	0.64	0.16	6.3e-04
Ethanol	2	756	100	1	2.6 ± 0.3	$90 \pm 1.7e+02$	2.5 ± 0.2	0.66	0.17	2.1e-05
Words	6	61	25	1	14 ± 1.3	11 ± 1.3	14 ± 1.2	0.52	0.12	2.1e-02

Table 1: Experimental parameters and results. For Iris and Wine, \hat{P} estimates result from random downsampling by a factor of 2 to create R replicates. Distributional probabilities $\hat{P}(\ell_1(X_{\cdot, \hat{S}_G}) > \ell_1(X_{\cdot, \hat{S}_{TSIP}}))$ and $\hat{P}(\ell_1(X_{\cdot, \hat{S}_G}) = \ell_1(X_{\cdot, \hat{S}_{TSIP}}))$ are empirical across replicates, while asymptotic probabilities $\hat{P}(\bar{\ell}_1(X_{\cdot, \hat{S}_G}) > \bar{\ell}_1(X_{\cdot, \hat{S}_{TSIP}}))$ is computed by paired two-sample T-test on $\ell_1(X_{\cdot, \hat{S}})$ and $\ell_1(X_{\cdot, \hat{S}_G})$. For brevity, in this table $\hat{S} := \hat{S}_{TSIP}$.

across $[P]$. Due to differences in scales between rows, these are first standardized. For the Wine dataset, even BRUTESEARCH on \hat{S}_{IP} is prohibitive in $D = 13$, and so we truncate our inputs to $D = 6$. For Ethanol, replicas are created by sampling from data points and their corresponding tangent spaces are estimated in $B = 252$.

Figure 2 and Table 1 show that the ℓ_1 accrued by the subset \hat{S}_G estimated using GREEDYSEARCH with objective ℓ_1 is higher than that for the subset estimated by TWOSTAGEISOMETRYPURSUIT. This effect is statistically significant, but varies across datapoints and datasets. Generally speaking, the ground truth loss obtained by greedy solutions is greater than obtained by two-stage Isometry Pursuit. Figure 4 details intermediate support recovery cardinalities from ISOMETRYPURSUIT. We also evaluated second stage BRUTESEARCH selection after random selection of \hat{S}_{IP} but do not report it since it often lead to catastrophic failure to satisfy the basis pursuit constraint. Wall-clock runtimes are given in Section 6.5.

Finally, we also analyze robustness of two-stage Isometry Pursuit to different choices of c and D on the Wine dataset in Figure 3. These show that the preference for the two-stage Isometry Pursuit solution is strongest around $c = 1$ and is consistent across feature truncation dimensions.

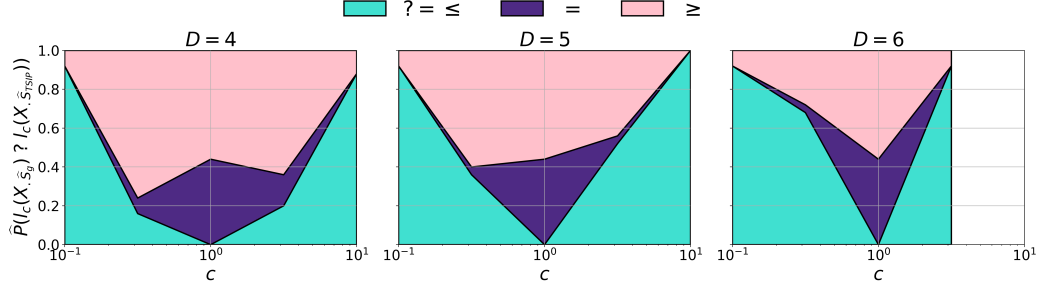


Figure 3: Proportions of support selection outcomes as power parameter c varies, for different values of D in the Wine dataset across various values of c . The turquoise area indicates when greedy solution \hat{S}_G outperforms \hat{S}_{TSP} , gray shows ties, and pink indicates \hat{S}_{TSP} is better. Solution at $c = 10$ is not plotted for $D = 6$ due to numerical overflow.

5 Discussion

We have shown that multitask basis pursuit can help select isometric submatrices from appropriately normalized wide matrices. This approach - isometry pursuit - is a convex alternative to greedy methods for selection of orthonormalized features from within a dictionary. Isometry pursuit can be applied to diversification and geometrically-faithful coordinate estimation. Our experiments exemplify these applications, but more can be done. One potential application is diversification in recommendation systems [Carbonell and Goldstein, 1998a, Wu et al., 2019, Lan] and other retrieval systems such as in RAG [Gao et al., 2023, Pickett et al., 2024, In et al., 2024, Weiss, 2024, Vec]. This is particularly relevant as maximum cosine similarity used in retrieval corresponds to the first coefficient of the Lasso regularization path [Koelle et al., 2022]. Another is decomposing interpretable yet overcomplete dictionaries in transformer residual streams, with each token considered as generating its own tangent space [Templeton et al., 2024, Makelov et al., 2024].

Compared with the greedy algorithms used in diversification [Carbonell and Goldstein, 1998b, Barioni et al., 2011, Drosou and Pitoura, 2010, Qin et al., 2012, Kunaver and Požrl, 2017, Guo and Sanner, 2010, Abdool et al., 2020, Yu et al., 2016, Huang et al., 2024, Pickett et al., 2024, ?, Zhu et al., 2020], the convex reformulation may add speed and convergence to a global minima. However, we note that the standard max-sum diversity formulation that seeks to identify datapoints that are as far apart as possible is substantially different from the notion of diversity selected for by Isometry Pursuit [Kuo et al., 1993, Ghosh, 1996, Alfonso Cevallos, Friedrich Eisenbrand, and Rico Zenklusen, 2016, Ashkan et al., 2015]. The comparison of greedy Mallat and Zhang [1993, 1992], Pati et al. [1993], Tropp et al. [2005] and convex [Scott Shaobing Chen and David L. Donoho and Michael A. Saunders, 2001, Tropp, 2006, Chen and Huo, 2006] basis pursuit formulations has a rich history, and theoretical understanding of the behavior of this approximation is evolving. That the solution of a lasso problem can sometimes be a non-singleton set is well-known [Osborne et al., 2000, Dossal, 2012, Chr tien and Darses, 2011, Tibshirani, 2012b, Ewald and Schneider, 2017, Ali and Tibshirani, 2018, Schneider and Tardivel, 2020, Mishkin and Pilanci, 2022b, Dupuis and Vaiter, 2019, Debarre et al., 2020, Everink et al., 2024].

The nature of the approximation to the ℓ_0 solution for Isometry Pursuit is different than for standard Lasso theory. The main theoretical question of isometry pursuit is how well the minimizer of a convex loss approximates the singular value loss, rather than how well the convex loss performs in statistical estimation. Conditions on solution uniqueness for lasso need to be understood for multitask regression. Related conditions have been discussed in Donoho [2006], Mishkin and Pilanci [2022b].

Algorithmic variants include the multitask lasso [Hastie et al., 2015] extension of our estimator, as well as characterization of D function selection within \mathbb{R}^B . Tangent-space specific variants have been studied in more detail in Koelle et al. [2022, 2024] with additional grouping across datapoints, and a corresponding variant of the isometry theorem that missed non-uniqueness was claimed in Koelle [2022]. Comparison of our loss with curvature - whose presence prohibits D element isometry - could prove fertile, as could comparison with the so-called restricted isometry property used to show guaranteed recovery at fast convergence rates in supervised learning [Candes and Tao, 2005, Hastie et al., 2015].

References

- Samson J Koelle, Hanyu Zhang, Octavian-Vlad Murad, and Marina Meila. Consistency of dictionary-based manifold learning. In Sanjoy Dasgupta, Stephan Mandt, and Yingzhen Li, editors, *Proceedings of The 27th International Conference on Artificial Intelligence and Statistics*, volume 238 of *Proceedings of Machine Learning Research*, pages 4348–4356. PMLR, 2024.
- T. Tony Cai and Lie Wang. Orthogonal matching pursuit for sparse signal recovery with noise. *IEEE Transactions on Information Theory*, 57(7):4680–4688, 2011. doi: 10.1109/TIT.2011.2146090.
- Yu-Chia Chen and Marina Meila. Selecting the independent coordinates of manifolds with large aspect ratios. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019. URL https://proceedings.neurips.cc/paper_files/paper/2019/file/6a10bbd480e4c5573d8f3af73ae0454b-Paper.pdf.
- Dhruv Kohli, Alexander Cloninger, and Gal Mishne. LDLE: Low distortion local eigenmaps. *J. Mach. Learn. Res.*, 22, 2021.
- Peter W Jones, Mauro Maggioni, and Raanan Schul. Universal local parametrizations via heat kernels and eigenfunctions of the laplacian. September 2007.
- Alexandre d’Aspremont, Laurent El Ghaoui, Michael I Jordan, and Gert R G Lanckriet. A direct formulation for sparse PCA using semidefinite programming. *SIAM Rev. Soc. Ind. Appl. Math.*, 49(3):434–448, January 2007.
- Santanu S Dey, R Mazumder, M Molinaro, and Guanyi Wang. Sparse principal component analysis and its l_1 -relaxation. *arXiv: Optimization and Control*, December 2017.
- D Bertsimas and Driss Lahlou Kitane. Sparse PCA: A geometric approach. *J. Mach. Learn. Res.*, 24: 32:1–32:33, October 2022.
- Dimitris Bertsimas, Ryan Cory-Wright, and Jean Pauphilet. Solving Large-Scale sparse PCA to certifiable (near) optimality. *J. Mach. Learn. Res.*, 23(13):1–35, 2022.
- Yu-Chia Chen and M Meilă. Selecting the independent coordinates of manifolds with large aspect ratios. *Adv. Neural Inf. Process. Syst.*, abs/1907.01651, July 2019.
- Samson J Koelle, Hanyu Zhang, Marina Meila, and Yu-Chia Chen. Manifold coordinates with physical meaning. *J. Mach. Learn. Res.*, 23(133):1–57, 2022.
- Jesse He, Tristan Brugère, and Gal Mishne. Product manifold learning with independent coordinate selection. In *Proceedings of the 2nd Annual Workshop on Topology, Algebra, and Geometry in Machine Learning (TAG-ML) at ICML*, June 2023.
- Adly Templeton, Tom Conerly, Jonathan Marcus, Jack Lindsey, Trenton Bricken, Brian Chen, Adam Pearce, Craig Citro, Emmanuel Ameisen, Andy Jones, Hoagy Cunningham, Nicholas L Turner, Callum McDougall, Monte MacDiarmid, C. Daniel Freeman, Theodore R. Sumers, Edward Rees, Joshua Batson, Adam Jermy, Shan Carter, Chris Olah, and Tom Henighan. Scaling monosemanticity: Extracting interpretable features from claude 3 sonnet. *Transformer Circuits Thread*, 2024. URL <https://transformer-circuits.pub/2024/scaling-monosemanticity/index.html>.
- Zhenyue Zhang and Hongyuan Zha. Principal manifolds and nonlinear dimensionality reduction via tangent space alignment. *SIAM J. Scientific Computing*, 26(1):313–338, 2004.
- Lisha Chen and Andreas Buja. Local Multidimensional Scaling for nonlinear dimension reduction, graph drawing and proximity analysis. *Journal of the American Statistical Association*, 104(485): 209–219, March 2009.
- J.B. Tenenbaum, V. Silva, and J.C. Langford. A global geometric framework for nonlinear dimensionality reduction. *Science*, 290(5500):2319–2323, 2000.
- Leo Sergio Liberti. *Reformulation and Convex Relation Techniques for Global Optimization*. PhD thesis, Imperial College London, October 2004.

- M Abdi. *Cardinality optimization problems*. PhD thesis, December 2013.
- Fengyu Zhou and Steven H Low. Conditions for exact convex relaxation and no spurious local optima. *arXiv [math.OC]*, February 2021.
- Scott Shaobing Chen and David L. Donoho and Michael A. Saunders. Atomic decomposition by basis pursuit. *SIAM REVIEW*, 43(1):129, February 2001.
- Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *J. R. Stat. Soc. Series B Stat. Methodol.*, 68(1):49–67, February 2006.
- G Obozinski, B Taskar, and Michael I Jordan. Multi-task feature selection. 2006.
- Dit-Yan Yeung and Yu Zhang. A probabilistic framework for learning task relationships in multi-task learning. 2011.
- Stephen P Boyd and Lieven Vandenbergh. *Convex Optimization*. Cambridge University Press, March 2004.
- Ulrike Schneider and Patrick Tardivel. The geometry of uniqueness, sparsity and clustering in penalized estimation. *J. Mach. Learn. Res.*, 23(331):1–36, 2022.
- Ryan J Tibshirani. The lasso problem and uniqueness. *arXiv [math.ST]*, June 2012a.
- Aaron Mishkin and Mert Pilanci. The solution path of the group lasso. 2022a.
- Tim Hesterberg, Nam Hee Choi, Lukas Meier, and Chris Fraley. Least angle and ℓ_1 penalized regression: A review. February 2008.
- R. A. Fisher. Iris. UCI Machine Learning Repository, 1988. DOI: <https://doi.org/10.24432/C56C76>.
- Stefan Aeberhard and M. Forina. Wine. UCI Machine Learning Repository, 1991. DOI: <https://doi.org/10.24432/C5PC7J>.
- F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-learn: Machine learning in Python. *Journal of Machine Learning Research*, 12:2825–2830, 2011.
- Stefan Chmiela, Huziel E Sauceda, Klaus-Robert Müller, and Alexandre Tkatchenko. Towards exact molecular dynamics simulations with machine-learned force fields. *Nat. Commun.*, 9(1):3887, September 2018.
- Kiho Park, Yo Joong Choe, and Victor Veitch. The linear representation hypothesis and the geometry of large language models. *arXiv [cs.CL]*, November 2023.
- Brendan O’Donoghue, Eric Chu, Neal Parikh, and Stephen Boyd. Conic optimization via operator splitting and homogeneous self-dual embedding. *Journal of Optimization Theory and Applications*, 169(3):1042–1068, June 2016. URL <http://stanford.edu/~boyd/papers/scs.html>.
- Steven Diamond and Stephen Boyd. CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research*, 17(83):1–5, 2016.
- Akshay Agrawal, Robin Verschueren, Steven Diamond, and Stephen Boyd. A rewriting system for convex optimization problems. *Journal of Control and Decision*, 5(1):42–60, 2018.
- CVXPY Developers. Sparse solution with cvxpy. https://www.cvxpy.org/examples/applications/sparse_solution.html. Accessed: 2024-07-11.
- Jaime Carbonell and Jade Goldstein. The use of MMR, diversity-based reranking for reordering documents and producing summaries. *SIGIR Forum*, 51(2):209–210, August 1998a.
- Qiong Wu, Yong Liu, Chunyan Miao, Yin Zhao, Lu Guan, and Haihong Tang. Recent advances in diversified recommendation. May 2019.

Select by maximal marginal relevance (MMR). https://python.langchain.com/docs/how_to/example_selectors_mmr/. Accessed: 2024-11-22.

Yunfan Gao, Yun Xiong, Xinyu Gao, Kangxiang Jia, Jinliu Pan, Yuxi Bi, Yi Dai, Jiawei Sun, Meng Wang, and Haofen Wang. Retrieval-augmented generation for large language models: A survey. *arXiv [cs.CL]*, December 2023.

Marc Pickett, Jeremy Hartman, Ayan Kumar Bhowmick, Raquib-Ul Alam, and Aditya Vempaty. Better RAG using relevant information gain. *arXiv [cs.CL]*, July 2024.

Yeonjun In, Sungchul Kim, Ryan A Rossi, Md Mehrab Tanjim, Tong Yu, Ritwik Sinha, and Chanyoung Park. Diversify-verify-adapt: Efficient and robust retrieval-augmented ambiguous question answering. *arXiv [cs.CL]*, September 2024.

Sam Weiss. Enhancing diversity in RAG document retrieval using projection-based techniques. https://medium.com/@samcarlos_14058/enhancing-diversity-in-rag-document-retrieval-using-projection-based-techniques-9fef5422e043, August 2024. Accessed: 2024-11-22.

Get diverse results and comprehensive summaries with vectara’s MMR reranker. URL <https://www.vectara.com/blog/get-diverse-results-and-comprehensive-summaries-with-vectaras-mmr-reranker>. Accessed: 2024-11-22.

Aleksandar Makelov, George Lange, and Neel Nanda. Towards principled evaluations of sparse autoencoders for interpretability and control. *arXiv [cs.LG]*, May 2024.

Jaime Carbonell and Jade Goldstein. The use of MMR, diversity-based reranking for reordering documents and producing summaries. In *Proceedings of the 21st annual international ACM SIGIR conference on Research and development in information retrieval*, New York, NY, USA, August 1998b. ACM.

Maria C. N. Barioni, Marios Hadjieleftheriou, Marcos R. Vieira, Caetano Traina, Vassilis J. Tsotras, Humberto L. Razente, and Divesh Srivastava. On query result diversification. In *2011 27th IEEE International Conference on Data Engineering (ICDE 2011)*, pages 1163–1174, Los Alamitos, CA, USA, April 2011. IEEE Computer Society. doi: 10.1109/ICDE.2011.5767846. URL <https://doi.ieeecomputersociety.org/10.1109/ICDE.2011.5767846>.

Marina Drosou and Evaggelia Pitoura. Search result diversification. *SIGMOD Rec.*, 39(1):41–47, September 2010. ISSN 0163-5808. doi: 10.1145/1860702.1860709. URL <https://doi.org/10.1145/1860702.1860709>.

Lu Qin, Jeffrey Xu Yu, and Lijun Chang. Diversifying top-k results. *Proceedings VLDB Endowment*, 5(11):1124–1135, July 2012.

Matevž Kunaver and Tomaž Požrl. Diversity in recommender systems – a survey. *Knowledge-Based Systems*, 123:154–162, 2017. ISSN 0950-7051. doi: <https://doi.org/10.1016/j.knosys.2017.02.009>. URL <https://www.sciencedirect.com/science/article/pii/S0950705117300680>.

Shengbo Guo and Scott Sanner. Probabilistic latent maximal marginal relevance. In *Proceedings of the 33rd International ACM SIGIR Conference on Research and Development in Information Retrieval, SIGIR ’10*, page 833–834, New York, NY, USA, 2010. Association for Computing Machinery. ISBN 9781450301534. doi: 10.1145/1835449.1835639. URL <https://doi.org/10.1145/1835449.1835639>.

Mustafa Abdool, Malay Haldar, Prashant Ramanathan, Tyler Sax, Lanbo Zhang, Aamir Manaswala, Lynn Yang, Bradley Turnbull, Qing Zhang, and Thomas LeGrand. Managing diversity in airbnb search. In *Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, KDD ’20*, page 2952–2960, New York, NY, USA, 2020. Association for Computing Machinery. ISBN 9781450379984. doi: 10.1145/3394486.3403345. URL <https://doi.org/10.1145/3394486.3403345>.

- Hsiang-Fu Yu, Cho-Jui Hsieh, Qi Lei, and Inderjit S. Dhillon. A greedy approach for budgeted maximum inner product search. In *Neural Information Processing Systems*, 2016. URL <https://api.semanticscholar.org/CorpusID:7076785>.
- Qiang Huang, Yanhao Wang, Yiqun Sun, and Anthony K H Tung. Diversity-aware k -maximum inner product search revisited. *arXiv [cs.IR]*, February 2024.
- Jianchen Zhu, Shengjie Zhao, and Rongqing Zhang. A match pursuit based method adapted to overcomplete dictionary for compressive spectral imaging. In *2020 Asia-Pacific Signal and Information Processing Association Annual Summit and Conference (APSIPA ASC)*, pages 170–173. IEEE, December 2020.
- Ching-Chung Kuo, Fred Glover, and Krishna S Dhir. Analyzing and modeling the maximum diversity problem by zero-one programming. *Decis. Sci.*, 24(6):1171–1185, November 1993.
- Jay B Ghosh. Computational aspects of the maximum diversity problem. *Operations Research Letters*, 19(4):175–181, 1996.
- Alfonso Cevallos, Friedrich Eisenbrand, and Rico Zenklusen. Max-sum diversity via convex programming, June 2016.
- Azin Ashkan, B Kveton, S Berkovsky, and Zheng Wen. Optimal greedy diversity for recommendation. *Int Jt Conf Artif Intell*, pages 1742–1748, July 2015.
- S.G. Mallat and Zhifeng Zhang. Matching pursuits with time-frequency dictionaries. *IEEE Transactions on Signal Processing*, 41(12):3397–3415, 1993. doi: 10.1109/78.258082.
- S. Mallat and Z. Zhang. Adaptive time-frequency decomposition with matching pursuits. In *[1992] Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis*, pages 7–10, 1992. doi: 10.1109/TFTSA.1992.274245.
- Y.C. Pati, R. Rezaifar, and P.S. Krishnaprasad. Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition. In *Proceedings of 27th Asilomar Conference on Signals, Systems and Computers*, pages 40–44 vol.1, 1993. doi: 10.1109/ACSSC.1993.342465.
- J.A. Tropp, A.C. Gilbert, and M.J. Strauss. Simultaneous sparse approximation via greedy pursuit. In *Proceedings. (ICASSP '05). IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005.*, volume 5, pages v/721–v/724 Vol. 5, 2005. doi: 10.1109/ICASSP.2005.1416405.
- Joel A. Tropp. Algorithms for simultaneous sparse approximation. part ii: Convex relaxation. *Signal Processing*, 86(3):589–602, 2006. ISSN 0165-1684. doi: <https://doi.org/10.1016/j.sigpro.2005.05.031>. URL <https://www.sciencedirect.com/science/article/pii/S0165168405002239>. Sparse Approximations in Signal and Image Processing.
- Jie Chen and Xiaoming Huo. Theoretical results on sparse representations of multiple-measurement vectors. *IEEE Transactions on Signal Processing*, 54:4634–4643, 2006. URL <https://api.semanticscholar.org/CorpusID:17333301>.
- Michael R. Osborne, Brett Presnell, and Berwin A. Turlach. On the lasso and its dual. *Journal of Computational and Graphical Statistics*, 9:319 – 337, 2000. URL <https://api.semanticscholar.org/CorpusID:14422381>.
- Charles Dossal. A necessary and sufficient condition for exact sparse recovery by ℓ_1 minimization. *Comptes Rendus Mathématique*, 350(1):117–120, 2012. ISSN 1631-073X. doi: <https://doi.org/10.1016/j.crma.2011.12.014>. URL <https://www.sciencedirect.com/science/article/pii/S1631073X11003694>.
- Stéphane Chrétien and Sébastien Darses. On the generic uniform uniqueness of the lasso estimator. *arXiv: Statistics Theory*, 2011. URL <https://api.semanticscholar.org/CorpusID:88518316>.
- Ryan J. Tibshirani. The lasso problem and uniqueness. *Electronic Journal of Statistics*, 7:1456–1490, 2012b. URL <https://api.semanticscholar.org/CorpusID:5849668>.

- Karl Ewald and Ulrike Schneider. On the distribution, model selection properties and uniqueness of the lasso estimator in low and high dimensions. *Electronic Journal of Statistics*, 2017. URL <https://api.semanticscholar.org/CorpusID:54044415>.
- Alnur Ali and Ryan J. Tibshirani. The generalized lasso problem and uniqueness. *Electronic Journal of Statistics*, 2018. URL <https://api.semanticscholar.org/CorpusID:51755233>.
- Ulrike Schneider and Patrick Tardivel. The geometry of uniqueness, sparsity and clustering in penalized estimation. *arXiv [math.ST]*, April 2020.
- Aaron Mishkin and Mert Pilanci. The solution path of the group lasso. 2022b. URL <https://api.semanticscholar.org/CorpusID:259504228>.
- Xavier Dupuis and Samuel Vaïter. The geometry of sparse analysis regularization. *SIAM J. Optim.*, 33:842–867, 2019. URL <https://api.semanticscholar.org/CorpusID:195791526>.
- Thomas Debarre, Quentin Denoyelle, and Julien Fageot. On the uniqueness of solutions for the basis pursuit in the continuum. *Inverse Problems*, 38, 2020. URL <https://api.semanticscholar.org/CorpusID:246473440>.
- Jasper Marijn Everink, Yiqiu Dong, and Martin Skovgaard Andersen. The geometry and well-posedness of sparse regularized linear regression. 2024. URL <https://api.semanticscholar.org/CorpusID:272424099>.
- David L. Donoho. For most large underdetermined systems of linear equations the minimal ℓ_1 -norm solution is also the sparsest solution. *Communications on Pure and Applied Mathematics*, 59, 2006. URL <https://api.semanticscholar.org/CorpusID:8510060>.
- T Hastie, R Tibshirani, and M Wainwright. Statistical learning with sparsity: The lasso and generalizations. May 2015.
- Samson Jonathan Koelle. *Geometric algorithms for interpretable manifold learning*. Phd thesis, University of Washington, 2022. URL <http://hdl.handle.net/1773/48559>. Statistics [108].
- Emmanuel Candes and Terence Tao. Decoding by linear programming. February 2005.
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, Third Edition*. The MIT Press, 3rd edition, 2009. ISBN 0262033844.
- Stuart Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall Press, USA, 3rd edition, 2009. ISBN 0136042597.
- Travis Dick, Eric Wong, and Christoph Dann. How many random restarts are enough. 2014. URL <https://api.semanticscholar.org/CorpusID:9473630>.

6 Supplement

This section contains algorithms, proofs, and experiments in support of the main text.

6.1 Algorithms

We give definitions of the brute and greedy algorithms for the combinatorial problem studied in this paper. The brute force algorithm is computationally intractable for all but the smallest problems, but always finds the global minima.

BRUTESEARCH(Matrix $X \in \mathbb{R}^{D \times P}$, objective f)

```

1: for each combination  $S \subseteq \{1, 2, \dots, P\}$  with  $|S| = D$  do
2:   Evaluate  $f(X_{\cdot S})$ 
3: end for
4: Output the combination  $S^*$  that minimizes  $f(X_{\cdot S})$ 

```

Greedy algorithms are computationally expedient but can get stuck in local optima [Cormen et al., 2009, Russell and Norvig, 2009], even with randomized restarts [Dick et al., 2014].

GREEDYSEARCH(Matrix $X \in \mathbb{R}^{D \times P}$, objective f , selected set $S = \emptyset$, current size $d = 0$)

```

1: if  $d = D$  then
2:   Return  $S$ 
3: else
4:   Initialize  $S_{\text{best}} = S$ 
5:   Initialize  $f_{\text{best}} = \infty$ 
6:   for each  $p \in \{1, 2, \dots, P\} \setminus S$  do
7:     Evaluate  $f(X_{\cdot(S \cup \{p\})})$ 
8:     if  $f(X_{\cdot(S \cup \{p\})}) < f_{\text{best}}$  then
9:       Update  $S_{\text{best}} = S \cup \{p\}$ 
10:      Update  $f_{\text{best}} = f(X_{\cdot(S \cup \{p\})})$ 
11:    end if
12:  end for
13:  Return GREEDYSEARCH( $X, f, S_{\text{best}}, d + 1$ )
14: end if

```

6.2 Proofs

6.2.1 Proof of Proposition 3

In this proof we first show that the penalty $\|\beta\|_{1,2}$ is unchanged by unitary transformation of β .

Proposition 5 *Let $U \in \mathbb{R}^{D \times D}$ be unitary. Then $\|\beta\|_{1,2} = \|\beta U\|$.*

Proof:

$$\|\beta U\|_{1,2} = \sum_{p=1}^P \|\beta_p \cdot U\| \quad (26)$$

$$= \sum_{p=1}^P \|\beta_p \cdot\| \quad (27)$$

$$= \|\beta\|_{1,2} \quad (28)$$

□

We then show that this implies that the resultant loss is unchanged by unitary transformation of X .

Proposition 6 *Let $U \in \mathbb{R}^{D \times D}$ be unitary. Then $\hat{\beta}(UX) = \hat{\beta}(X)U$.*

Proof:

$$\widehat{\beta}(UX) = \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_{1,2} : I_D = UX\beta \quad (29)$$

$$= \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_{1,2} : U^{-1}U = U^{-1}UX\beta U \quad (30)$$

$$= \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_{1,2} : I_D = X\beta U \quad (31)$$

$$= \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta U\|_{1,2} : I_D = X\beta U \quad (32)$$

$$= \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_{1,2} : I_D = X\beta. \quad (33)$$

□

6.2.2 Proof of Proposition 4

In this proof we first show that the orthonormal submatrix $X_{\cdot S}$ is contained within the set $\widehat{\beta}_{MBP}(w(X, c), I_D)$.

Proposition 7 *Let $X_{\cdot S}$ be a rank D orthonormal submatrix of X . Then $S \in S(\beta) : \beta \in \widehat{\beta}_{MBP}(w(X, c), I_D)$.*

Proof: By Proposition 3 without loss of generality, let $X_{\cdot S} = I_D$. Also, without loss of generality let $S = \{1 \dots D\}$. We show that $\beta = [I_D 0]$ satisfies the KKT conditions for $X = [I_D X_{\cdot S}]$.

Recall that the KKT conditions for a convex optimization problem are a set of conditions on an element of the domain of the optimization algorithm that, if satisfied, certify that the element is a solution. To write out the KKT conditions, we define the Lagrangian

$$\mathcal{L}(X, \beta, \nu) = \|\beta\|_{1,2} + \nu^T (I_D - w(X, c)\beta) \quad (34)$$

where $\nu \in \mathbb{R}^{D \times D}$ is the dual variable.

The KKT conditions are then

- Primal feasibility: $w(X, c)\beta = I_D$
- Stationarity: there exists a dual variable ν such that $0 \in \partial \|\beta\|_{1,2} - w(X, c)^T \nu$ where ∂ is the subdifferential operator.

where dual feasibility and complementary slackness are ignorable by virtue of the absence of inequality constraints.

First note that in our case, primal feasibility is satisfied by X being rank D and w not effecting the rank of the transformed matrix since it only rescales length and not direction.

For stationarity, recall that

$$\|\beta\|_{1,2} = \sum_{p=1}^P \|\beta_p\|_2. \quad (35)$$

Then, by the definition of subdifferential

$$\partial \|\beta\|_{1,2} = \begin{bmatrix} I_D v_{d+1} \\ \dots \\ v_P \end{bmatrix} \quad (36)$$

where $v_p \in \mathbb{R}^D$ can be any vector satisfying $\|v_p\|_2 \leq 1$.

We therefore must show that there exists a ν that satisfies

$$\begin{bmatrix} I_D \\ w(X_{\cdot S}, c)^T \end{bmatrix} \nu = \begin{bmatrix} I_D \\ V_{\cdot S} \end{bmatrix}. \quad (37)$$

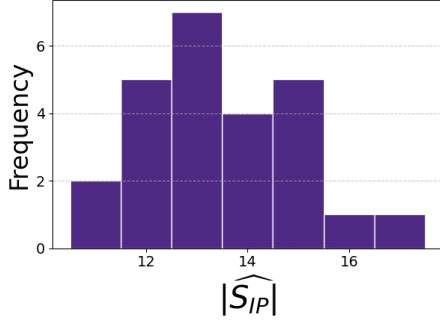
At this point we can see that in fact I_D is an appropriate choice of ν since normalization by w leads to vectors satisfying $\|w(X, c)\|_2 \leq 1$.

Since for this β , $S(\beta) = S$, we have shown that S is the support of a solution to $\beta_{MBP}(w(X, c), I_D)$.
 \square

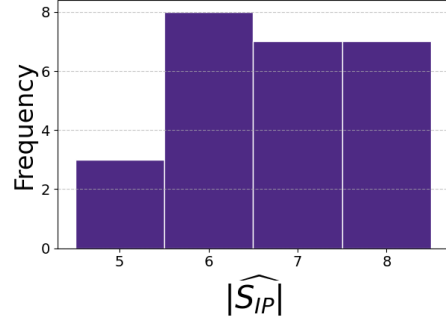
The next part of the proof handles the presence of non-unique solutions.

Proposition 8 $S(\widehat{\beta}_c(X)) = \cup S(\beta) : \beta \in \beta_{MBP}(w(X, c), I_D)$

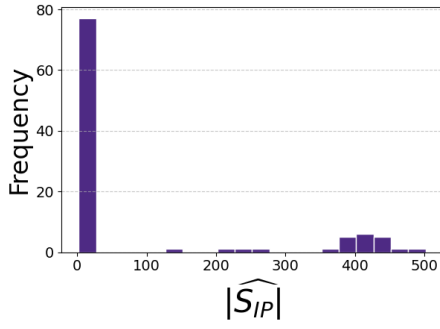
Proof: The proposition follows from geometric properties of the solution set. We ignore the case for singleton solutions, for which the proposition is self-evident. Recall that $\widehat{\beta}_c(X) := \widehat{\beta}_{MBP}^{\ell_2}(w(X, c), I_D) = \arg \min_{\beta \in \mathbb{R}^{P \times D}} \|\beta\|_F : \beta \in \widehat{\beta}_{MBP}(X, I_D)$. By definition of $\widehat{\beta}_{MBP}(X, I_D)$, $\|\beta_p\|_2 : \beta \in \widehat{\beta}_{MBP}(X, I_D)$ is an affine linear subspace of \mathbb{R}^P . The projection of the vector of minimum ℓ_2 distance from the origin to a affine space intersects that space perpendicularly. However, for all $\|\beta_p\|_2$ such that β is on the intersection of this affine space with the coordinate hyperplanes formed by fixing any choices of $\|\beta_p\|_2 =$, this vector doesn't intersect this space perpendicularly. Thus, $\widehat{\beta}_c(X)$ is not on the intersection of the solution polytope $\widehat{\beta}_{MBP}(w(X, c), I_D)$ with the coordinate hyperplanes. It is therefore maximally non-sparse with respect to any of the minimizing solutions. \square



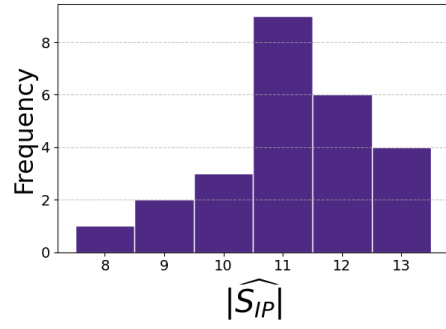
(a) Wine Dataset



(b) Iris Dataset



(c) Ethanol Dataset



(d) Words Dataset

Figure 4: Support cardinalities for each dataset under Basis Pursuit.

6.3 Support cardinalities

Figure 4 plots the distribution of $|\hat{S}_{IP}|$ from Table 1 in order to contextualize the reported means. While typically $|\hat{S}_{IP}| \ll P$, there are cases for Ethanol where this is not the case that drive up the means.

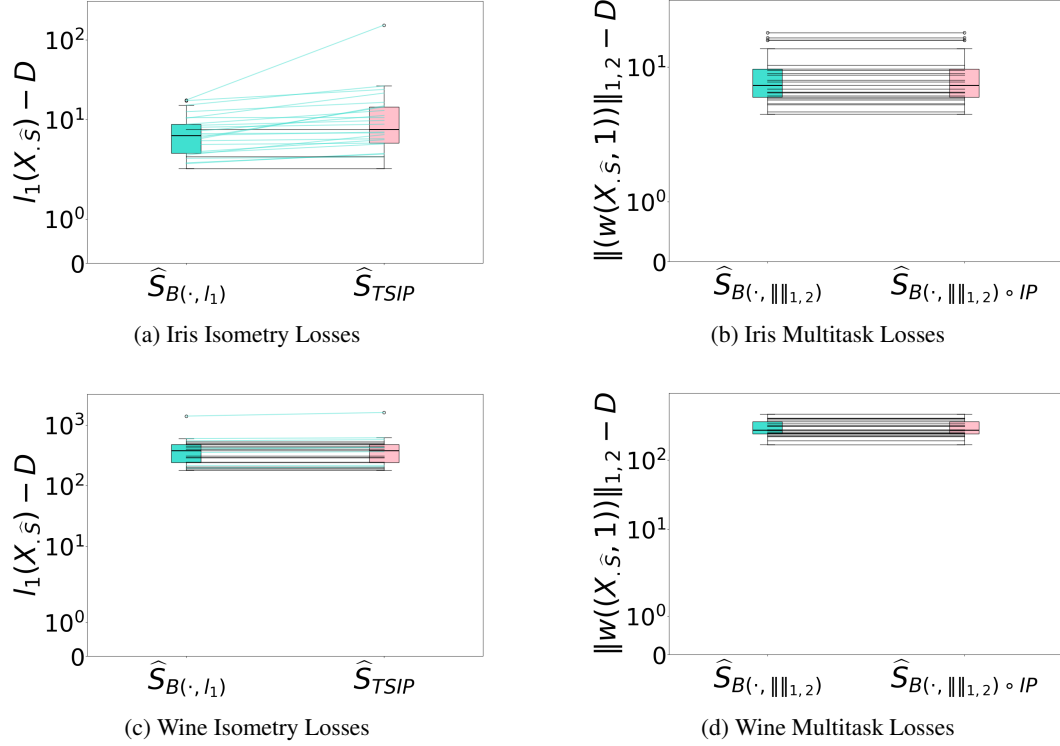


Figure 5: Comparison of Isometry and Group Lasso Losses across 25 replicates for randomly downsampled Iris and Wine Datasets with $(P, D) = (4, 15)$ and $(13, 18)$, respectively. Note that this further downsampling compared with Section 4 was necessary to compute global minimizers of BRUTESEARCH. Lower brute losses are shown with turquoise, while lower two stage losses are shown with pink. Equal losses are shown with black lines.

6.4 Proposition 4 deep dive

The results in Section 4 show that there are circumstances in which the GREEDYSEARCH performs better than TWOSTAGEISOMETRYPURSUIT, so clearly TWOSTAGEISOMETRYPURSUIT does not always achieve a global optimum. Figure 5 gives results on the line of inquiry about why this is the case based on the reasoning presented in Section 5. In these results a two-stage algorithm achieves the global optimum of a slightly different brute problem, namely brute optimization of the multitask basis pursuit penalty $\|\cdot\|_{1,2}$. That is, brute search on $\|\cdot\|_{1,2}$ gives the same result as the two stage algorithm with brute search on $\|\cdot\|_{1,2}$ subsequent to isometry pursuit. This suggests that failure to select the global optimum by TWOSTAGEISOMETRYPURSUIT is in fact only due to the mismatch between global optimums of brute optimization of the multitask penalty and the isometry loss given certain data.

6.5 Timing

While wall-time of algorithms is a non-theoretical quantity that depends on implementation details, it provides valuable context for practitioners. We therefore report the following runtimes on a 2021 Macbook Pro. The particularly high variance for brute force search in the second step of `TWOSTAGEISOMETRYPURSUIT` is likely due to the large cardinalities reported in Figure 4.

Name	IP	2nd stage	Greedy
Iris	1.24 ± 0.02	0.00 ± 0.00	0.02 ± 0.00
Wine	2.32 ± 0.17	0.13 ± 0.12	0.03 ± 0.00
Ethanol	8.38 ± 0.57	0.55 ± 1.08	0.07 ± 0.01
Words	0.98 ± 0.20	0.03 ± 0.02	0.01 ± 0.00

Table 2: Algorithm runtimes in seconds across replicates.