Asymptotic Analysis of Quasi-limiting Behavior for Drifted Brownian Motion Conditioned to Stay Positive

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- Stochastic processes can be used to model different phenomena in real life.
- Some of them exhibit an *absorbing state* in the model.
  - In population dynamics, a species that go extinct cannot naturally regenerate (e.g. modeled through birth and death or branching processes)
  - For a single-parameter human health model, an individual that passes away cannot be resurrected.
  - A freely moving particle near a black hole cannot return once it crosses the event horizon.

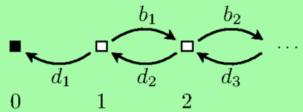

Figure: Birth-death chain

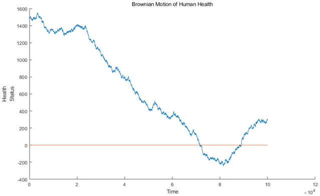

Figure: Brownian motion modeling human health status, 0 means death.

Markov process, conditioned not to be absorbed

The model:

- $\mathbf{X} = (X_t : t \geq 0)$, a Markov process on $\mathbb{R}_+ = [0, \infty)$.
- Let $\tau = \inf\{t \geq 0 : X_t = 0\}$ be the time the process first hits 0.
- We will work under the assumption

$$P_x(\tau < \infty) = 1, \text{ for all } x \in \mathbf{R}_+$$

  The notation P_x is a shorthand for distribution of the process, conditioned on $X_0 = x$.
- We will assume 0 is an absorbing state, that is,

$$P_0(X_t \in \cdot) = \delta_0, \ t \geq 0.$$

Observation:

- If π is a stationary distribution for $\mathbf{X}$, then the above assumption guarantees that $\pi = \delta_0$.
- However, the distribution of the process and particularly of X_t *conditioned* on $\{\tau > t\}$, is in general far from trivial.
- This naturally leads to the "conditional" analog for a stationary distribution.

Example: Unconditioned Brownian motion with drift

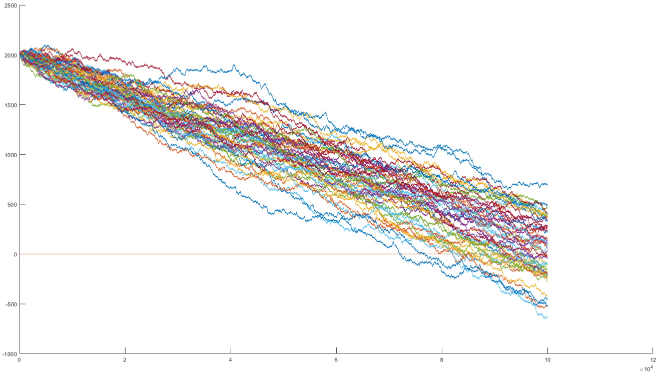

Figure: Sample paths of 1-dimensional Brownian Motion with constant drift -0.02, up to $t = 100000$, with fixed initial state $X_0 = 2000$.

Example: Unconditioned Brownian motion with drift

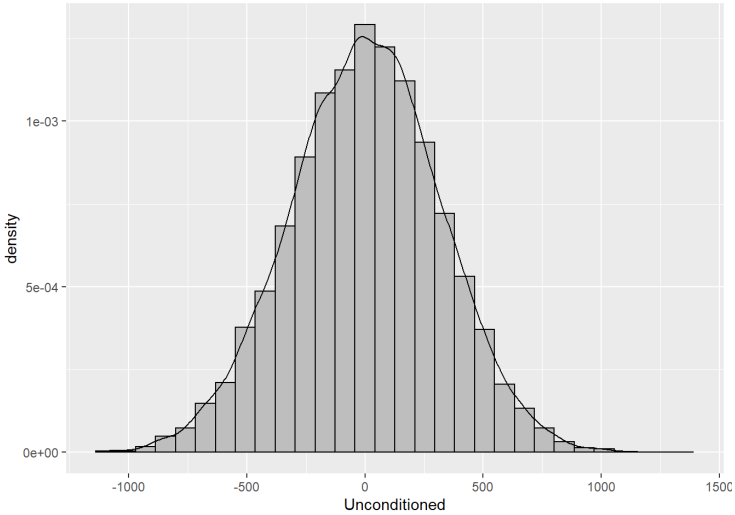

Figure: Histogram and PDF plot of sample paths for 1-dimensional Brownian Motion with constant negative drift, with fixed initial state $X_0 = 2000$ at $t = 100000$. Sample size is 10000. X_{10000} follows a Gaussian distribution, which we can verify from the simulation.

Example: Brownian motion with drift, conditioned on $\tau > t$

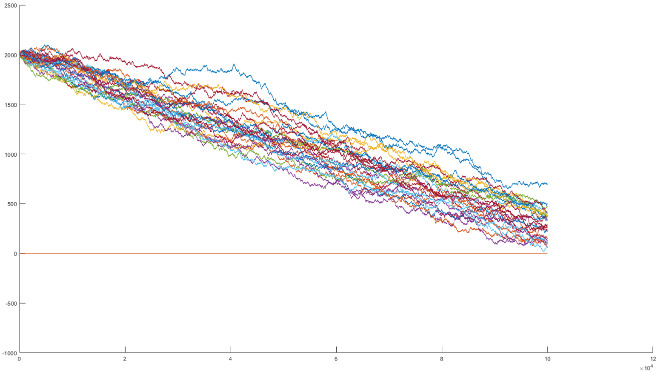

Figure: Remaining paths of same processes, conditioned not to be absorbed by $t = 100000$.

Example: Brownian motion with drift, conditioned on $\tau > t$

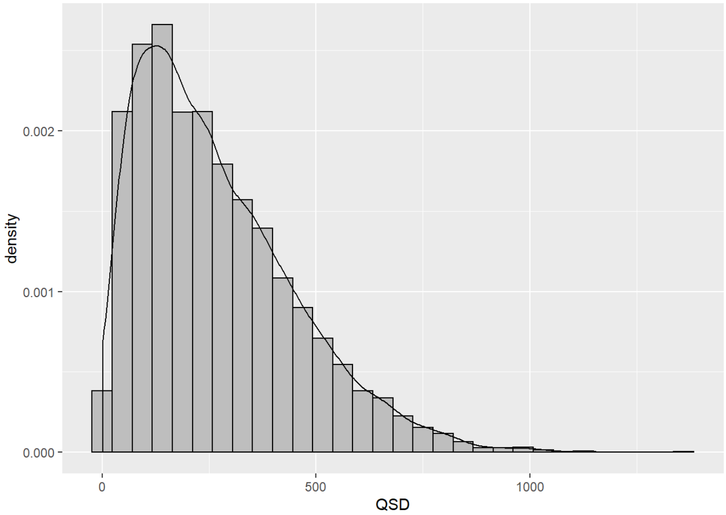

Figure: Histogram and PDF plot of same sample processes as before, with the condition $\tau > 10000$. Unlike before, this distribution has exponential tail. Also, the density near 0 drop in this setting.

Definitions:

- A probability distribution π is a Quasi-Stationary Distribution (QSD) if for each $t \geq 0$
$$P_\pi(X_t \in \cdot \mid \tau > t) = \pi(\cdot)$$
- A probability distribution π is a Quasi-Limiting Distribution (QLD) of an initial distribution μ if
$$\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi(\cdot)$$

Proposition

π is a QSD if and only if π is a QLD for some μ.

Theorem (Ferrari, Martinez, Picco, 1995)

If π be a QSD, then for each $t \geq 0$

$$P_\pi(\tau > t) = e^{-\lambda t}$$

for some $\lambda > 0$ that depends on π.

Basic Setting:

- $\mathbf{B} = (B_t : t \geq 0)$, standard BM on $\mathbf{R}$.
- $Y_t = x + B_t - \alpha t$, $\alpha > 0$, Brownian motion with drift $-\alpha$, starting from $x > 0$.
- $\tau = \inf\{t \geq 0 : Y_t = 0\}$.
- $X_t = \begin{cases} Y_t, & \tau > t, \\ 0, & \tau \leq t. \end{cases}$
- $\mathbf{X} = (X_t : t \in \mathbf{R}_+)$ is BM with drift $-\alpha$ on $\mathbf{R}_+$, absorbed at 0.
- The associated semigroup has generator $\mathcal{L} = \frac{1}{2}\partial_{xx} - \alpha\partial_x$ on functions $C^2(\mathbf{R}_+)$ vanishing at 0.

Existence and Uniqueness of QSD:

Theorem (Martinez, San Martin, 1994)

For each $\gamma \in [0, \alpha)$ there is a QSD π_γ which has the following density:

$$\pi_\gamma(x) = \begin{cases} \frac{\alpha^2-\gamma^2}{\gamma} e^{-\alpha x} \sinh(\gamma x) dx & \text{if } \gamma > 0 \\ \alpha^2 x e^{-\alpha x} dx & \text{if } \gamma = 0 \end{cases}$$

Remark. π_γ satisfies $\mathcal{L}^*\pi_\gamma = -\lambda\pi_\gamma$, where $\mathcal{L}^*$ is the formal adjoint (WRT integration by parts) of $\mathcal{L}$ and $\lambda = \dfrac{\alpha^2 - \gamma^2}{2}$.

Domain of Attraction:
A distribution μ is in the domain of attraction of a QSD π if π is the QLD of μ.

Question:
What is the characterization of the domain of attraction in our model?

Theorem (Martinez, Picco, San Martin, 1998)

If μ satisfies

$$\rho := \liminf_{x\to\infty} -\frac{\log \mu[x,\infty)}{x} \geq \alpha$$

then μ is in the domain of attraction of π_0.
If μ has a density $\mu(x)$ that satisfies

$$\rho := \lim_{x\to\infty} -\frac{\log \mu(x)}{x} \in (0,\alpha)$$

then μ is in the domain of attraction of π_γ where $\gamma = \alpha - \rho$.

The theorem was originally proved through spectral analysis.

Our method

- We seek to implement a more probabilistic approach, to achieve following goals:
    1. We believe our approach is more intuitive, simpler,and less specific, using "concentration of measure" to get the results.
    2. Specifically, our method is well equipped to deal with cases like alternating exponential tails and heavier tails.

- The general principle we develop appears to be applicable in a wider class of processes.

Our method

Proposition

$$
\begin{aligned}
&P_\mu(X_t \in dy, \tau > t) \\
&= \frac{1}{\sqrt{2\pi t}} \int \exp\left(\alpha x - \frac{\alpha^2 t}{2} - \alpha y\right)\left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) d\mu(x) \\
&= \frac{t e^{-\frac{t\alpha^2}{2}}}{\sqrt{2\pi t}} \int e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) d\nu_t(x)
\end{aligned}
$$

where the CDF of ν_t is $F_{\nu_t}(z) = C_t \int_{[0,z]} e^{-\frac{tx^2}{2}} e^{\alpha tx} d\mu(tx)$.

(C_t is the normalization constant)

The idea is to follow ν_t:

Principle

$$
\boxed{\lim_{t\to\infty} \nu_t = \delta_\gamma} \implies \boxed{\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi_\gamma}
$$

ρ	$\lim \nu_t$	QLD ($=$ QSD)	Example distributions
$\rho \geq \alpha$	δ_0	π_0	Half-normal distribution Delta distribution
$\alpha > \rho > 0$	$\delta_{\alpha-\rho}$	$\pi_{\alpha-\rho}$	Exponential distribution with rate $\lambda < \alpha$
$\rho = 0$	δ_α	QLD does not exist: **scaling is necessary**.	Pareto distribution Half-Cauchy distribution

Table: Domain of attraction classified by parameter $\rho = \lim_{x\to\infty} -\frac{\log \mu([x,\infty))}{x}$

Theorem

If $\rho := \lim_{x\to\infty} -\frac{\log \mu([x,\infty))}{x} \in (0,\alpha)$,

$$P_\mu(X_t \in \cdot \mid \tau > t) \to \pi_{\alpha-\rho}(\cdot)$$

In particular, μ does not need to have a density.

Sketch of proof:
Define upper and lower measures ν_t^+, ν_t^- and use the squeeze theorem.

$$d\nu_t^+(x) = e^{-\frac{tx^2}{2}} e^{\alpha tx} e^{-(\rho-\epsilon)tx}, \quad \nu_t^+ \rightharpoonup \delta_{\alpha-\rho+\epsilon}$$

$$d\nu_t^-(x) = e^{-\frac{tx^2}{2}} e^{\alpha tx} e^{-(\rho+\epsilon)tx}, \quad \nu_t^- \rightharpoonup \delta_{\alpha-\rho-\epsilon}$$

Then

$$P_\mu(X_t \in \cdot \mid \tau > t) = \frac{\int e^{-\alpha y} e^{-\frac{y^2}{2t}} \sinh(xy) d\nu_t(x)}{\int \int_0^\infty e^{-\alpha y} e^{-\frac{y^2}{2t}} \sinh(xy) dy d\nu_t(x)}$$

$$\lim_{t\to\infty} P_\mu(X_t \in dy \mid \tau > t) = \frac{e^{-\alpha y} \sinh((\alpha-\rho)y)}{\int_0^\infty e^{-\alpha y} \sinh((\alpha-\rho)y) dy} = \pi_{\alpha-\rho}(y)$$

Theorem

If $\rho := \liminf_{x\to\infty} -\frac{\log \mu([x,\infty))}{x} \geq \alpha$,

$$P_\mu(X_t \in \cdot \mid \tau > t) \to \pi_0(\cdot)$$

Sketch of proof:

Define ν_t (supported on $\mathbf{R}_+$) as follows.

$$d\nu_t(x) = xe^{\alpha tx} e^{-\frac{tx^2}{2}} d\mu(tx), \quad \nu_t \rightharpoonup \delta_{0+}$$

Then

$$P_\mu(X_t \in dy \mid \tau > t) = \frac{\int ye^{-\alpha y} e^{-\frac{y^2}{2t}} \frac{\sinh(xy)}{xy} d\nu_t(x)}{\int \int_0^\infty ye^{-\alpha y} e^{-\frac{y^2}{2t}} \frac{\sinh(xy)}{xy} dy d\nu_t(x)}$$

$$\lim_{t\to\infty} P_\mu(X_t \in dy \mid \tau > t) = \frac{ye^{-\alpha y}}{\int_0^\infty ye^{-\alpha y} dy} = \pi_0(y)$$

Question:
What happens if initial distribution μ is NOT in any domain of attraction?

- μ can be a heavy-tailed distribution.
- μ may have alternating tail behavior; for example, μ can have compound exponential tail. (disjoint combination of multiple different exponential distributions)

Proposition

Suppose μ is in the domain of attraction of a QSD π. Then $P_\mu(\tau > t) = O(c^t)$ for some $0 < c < 1$.

Lemma

Suppose

$$\lim_{x\to\infty} \frac{\log \mu([x,\infty))}{x} = 0.$$

Then $(P_\mu(X_t \in \cdot \mid \tau > t) : t \geq 0)$ is not tight.

Therefore in order to obtain a non-trivial limit, one has to scale X_t as $t \to \infty$.

Assumption $(*)$

1. $\mu([x,\infty)) = e^{-F(x)}$, with F smoothly varying (regularly varying and smooth) with index parameter β.
2. There exists a positive function $R(x,c)$ on $\mathbf{R}_+ \times \mathbf{R}_+$ increasing in c, such that for all $c > 0$

$$\lim_{x\to\infty} F(x + R(x,c)) - F(x) = c.$$

A function F is regularly varying with index parameter β if for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{F(\lambda x)}{F(x)} = \lambda^{\beta}.$$

Some examples include:

- Weibull distribution with shape parameter $k < 1/2$ ($\beta = k$)
  - $\mu([x,\infty)) = e^{-x^k}$
- Pareto distribution with shape parameter κ ($\beta = 0$)
  - $\mu([x,\infty)) = \left(\frac{c}{x}\right)^{\kappa} = \exp\left(-\kappa \ln(x/c)\right)$ for $x > c$
- Half-Cauchy distribution ($\beta = 0$)
  - $\mu([x,\infty)) = 1 - \dfrac{2\arctan(x)}{\pi}$

Comments:

- If $\beta \geq 1$ then μ is in the domain of attraction of some QSD.
  - If $\beta > 1$ then μ is in the domain of attraction of π_0.
  - If $\beta = 1$ then the QLD of μ depends on $\rho := \lim -\frac{\ln \mu([x,\infty))}{x}$.

- If F is assumed to be smooth enough, Taylor expansion yields the natural choice for R to be $R(x,c) = \dfrac{c}{F'(x)}$.

- In particular, if $\beta = 0$, then $\mu([x,\infty)) = G(x)$ is regularly varying with index parameter $-\kappa \leq 0$.
  - When $\kappa = 0$, μ has a super-heavy tail.

Proposition

Under Assumption $(*)$, if $0 \leq \beta < 1/2$, then

$$P_\mu(X_t > a_t, \tau > t) \sim \mu([t\alpha + a_t, \infty))$$

for any $a_t \gg \sqrt{t}$.

Theorem

Suppose $\mu([x,\infty)) = \exp(-F(x))$ where $F(x)$ is strictly increasing smoothly varying function with index $\beta < 1/2$. Then

$$\lim_{t\to\infty} P_\mu\left(X_t > \frac{c}{F'(\alpha t)} \,\middle|\, \tau > t\right) = e^{-c}$$

Corollary

Suppose $\mu([x,\infty)) = G(x)$, where G is smoothly varying function with index $-\kappa < 0$. Then

$$\lim_{t\to\infty} P_\mu\left(X_t > ct \mid \tau > t\right) = \left(\frac{\alpha + c}{\alpha}\right)^{-\kappa}$$

that is, the limiting distribution is Lomax (shifted Pareto) distribution with shape parameter κ and scale parameter α.

Example

Suppose μ is Weibull distribution with shape parameter $k < 1/2$ and scale parameter $\lambda > 0$. Then

$$\lim_{t\to\infty} P_\mu\left(X_t > ct^{1-k} \mid \tau > t\right) = \exp\left(-k(\alpha/\lambda)^{k-1}c\right)$$

Example

If μ is a Half-Cauchy distribution (Cauchy distribution supported on $\mathbf{R}_+$), the limiting distribution of $P_\mu(X_t > ct \mid \tau > t)$ is Lomax distribution with shape parameter 1 and scale parameter α.

Example

Suppose $\mu([x,\infty)) \sim \dfrac{1}{\ln x}$ as $x \to \infty$. Then

$$\lim_{t\to\infty} P_\mu\left(X_t > t^c \mid \tau > t\right) = \begin{cases} 1 & c \leq 1, \\ \frac{1}{c} & c > 1. \end{cases}$$

That is, the limiting distribution is Pareto distribution with both shape and scale parameter 1. (no dependence on α)

The limit condition $\rho := \lim_{x\to\infty} -\frac{\log \mu([x,\infty))}{x} \in (0,\alpha)$ for sub-critical exponential tail case is important.

Define constants $A, B, \rho_1, \rho_2, \theta$ and sequences $(a_k), (t_n)$ as follows.

$$0 < A < \rho_1 < \rho_2 < B < \alpha, \quad \theta = \frac{B}{A} > 1, \quad a_k = A\theta^k, \quad t_n = \theta^n$$

Define initial measure μ as follows. (c is normalization constant)

$$\mu(x) = \begin{cases} c1_{[a_k,a_{k+1})}\rho_1 e^{-\rho_1 x}, & k \text{ odd}, \\ c1_{[a_k,a_{k+1})}\rho_2 e^{-\rho_2 x}, & k \text{ even}. \end{cases}$$

Proposition

Under the above settings,

$$\lim_{t_{n_1}\to\infty} P_\mu(X_{t_{n_1}} \in \cdot \mid \tau > t_{n_1}) = \pi_{\alpha-\rho_1}(\cdot) \qquad (\nu_{t_{n_1}} \rightharpoonup \delta_{\alpha-\rho_1})$$
$$\lim_{t_{n_2}\to\infty} P_\mu(X_{t_{n_2}} \in \cdot \mid \tau > t_{n_2}) = \pi_{\alpha-\rho_2}(\cdot) \qquad (\nu_{t_{n_2}} \rightharpoonup \delta_{\alpha-\rho_2})$$

where (n_1) is the sequence of odd integer and (n_2) is the sequence of even integer.

Question:
What happens if μ follows Assumption $(*)$ (i.e. heavy-tailed distribution) but $1/2 \leq \beta < 1$?

- We see a clear relation between the index φ of the scaling factor $R(x, c)$ and the index β of the initial distribution exponent $F(x)$, which is that $\varphi = 1 - \beta$ when $\beta \neq 0$.
- Intuitively, we expect that the relation $\varphi = 1 - \beta$ should extend to any $\beta < 1$.

Prediction

Suppose μ satisfies Assumption $(*)$ with $1/2 \leq \beta < 1$. Then

$$P_\mu(X_t > a_t, \tau > t) \sim \sqrt{t}\mu(t\alpha + a_t)$$

Future work

Question:
Can we apply this idea to other models?

- Ornstein-Uhlenbeck process defined by $dX_t = dB_t - \alpha X_t dt$ is one of the few other non-discrete state space models which QSDs and domain of attraction is known (Lldaser and San Martin, 2000).
- Transition density of this model can be explicitly expressed.

$$P(X_0 \in dx, X_t \in dy, \tau > t)$$
$$= \sqrt{\frac{4\alpha}{\pi(1-e^{-2\alpha t})}} \exp\left(-\frac{\alpha(e^{-\alpha t}x)^2}{1-e^{-2\alpha t}} - \frac{\alpha y^2}{1-e^{-2\alpha t}}\right) \sinh\left(\frac{2\alpha e^{-\alpha t}xy}{1-e^{-2\alpha t}}\right)$$

- We can define ν_t as follows and study the limiting behavior.

$$d\nu_t(x) = \exp\left(-\frac{\alpha(e^{-\alpha t}x)^2}{1-e^{-2\alpha t}}\right) d\mu(x)$$

Question:
What is the rate of convergence to a QSD?

- We can study the rate of convergence through the total variation distance.

Thank you!

Any questions?

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- Manuel Lladser and Jaime San Martín, *Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process*, J. Appl. Probab. **37** (2000), no. 2, 511–520. MR 1781008
- Servet Martinez, Pierre Picco, and Jaime San Martin, *Domain of attraction of quasi-stationary distributions for the Brownian motion with drift*, Adv. in Appl. Probab. **30** (1998), no. 2, 385–408. MR 1642845
- Servet Martínez and Jaime San Martín, *Quasi-stationary distributions for a Brownian motion with drift and associated limit laws*, J. Appl. Probab. **31** (1994), no. 4, 911–920. MR 1303922
- Denis Villemonais, *Distributions quasi-stationnaires et méthodes particulaires pour l'approximation de processus conditionnés*, Ph.D. thesis, Ecole Polytechnique X, 2011.

Define $L(y) = \int_y^\infty e^{-\frac{w^2}{2}} dw$.

$$
\begin{aligned}
&P_\mu(X_t > a_t, \tau > t) \\
&= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty e^{-\frac{x^2}{2t}} e^{\alpha x} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}} \right) dy d\mu(x) \right) \\
&= \frac{e^{-\frac{\alpha^2 t}{2}} t}{\sqrt{2\pi t}} \int_0^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} (e^{xy} - e^{-xy}) dy dx \\
&= \frac{t}{\sqrt{2\pi}} \left(\underbrace{\int_0^\infty \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x \right) dx}_{=J_1(t)} \right. \\
&\qquad \left. - \underbrace{\int_0^\infty \mu(tx) e^{2\alpha tx} L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} + \sqrt{t}x \right) dx}_{=J_2(t)} \right)
\end{aligned}
$$

Pick

$$
t^{\beta-1} \ll \eta_t \ll a_t/t, \qquad \epsilon_t = t^{-b}, \qquad \beta < b < 0.5
$$

Sketch of proof (of heavy-tail proposition)

$$J_1(t) = \underbrace{\int_0^{\alpha+a_t/t-\eta_t} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,1}(t)}$$

$$+ \underbrace{\int_{\alpha+a_t/t-\eta_t}^{\alpha+a_t/t+\epsilon_t} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,2}(t)}$$

$$+ \underbrace{\int_{\alpha+a_t/t+\epsilon_t}^{\infty} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,3}(t)}$$

$$J_{1,3}(t) \sim \frac{\mu([t\alpha + a_t, \infty))}{t}$$

$$J_{1,2}(t) \leq \epsilon_t F'(t\alpha + a_t)\mu([t\alpha + a_t, \infty)) = o(J_{1,3}(t))$$

$$\log J_{1,1}(t) \ll \log \mu([t\alpha + a_t, \infty)) \Rightarrow J_{1,1}(t) = o(J_{1,3}(t))$$

$$J_2(t) \leq e^{-\frac{t\epsilon_t^2}{2}} \mu([t\alpha + a_t, \infty)) + O(J_{1,1}(t)) + O(J_{1,2}(t)) = o(J_{1,3}(t))$$