Asymptotic analysis of quasi-limiting behavior for drifted Brownian motion conditioned to stay positive

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ABSTRACT

A Quasi-Stationary Distribution for a Markov process with an almost surely reached absorbing state is a conditionally time-invariant distribution on the state space, which the condition is that the process is not absorbed by the given time. Previous works of Martinez *et al.* [9] [8] identify the family of Quasi-Stationary Distribution for Brownian motion with negative drift, and characterize the domain of attraction for each of them.

This paper will mainly focus on two subjects.

1. We provide a new approach simplifying the existing results, which explains the direct relation between a QSD and an initial distribution in the domain of attraction of the QSD.

2. We will discuss the quasi-limiting behavior of initial distributions that are not in the domain of attraction of any QSD, by finding the right scaling factor and scaling limit of such distributions.

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Asymptotic analysis of quasi-limiting behavior for drifted Brownian motion conditioned to stay positive

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Contents

Chapter 1

Introduction

1.1 General Theory

Consider $\mathbf{X} = (X_t : t \geq 0)$, a Markov process on $\mathbb{R}_+ = [0, \infty)$ with 0 as a unique absorbing state. Let

$$\tau = \inf\{t \geq 0 : X_t = 0\}.$$

We will work under the assumption

$$P_x(\tau < \infty) = 1, \text{ for all } x \in \mathbb{R}_+. \tag{1.1.1}$$

The notation P_x is a shorthand for $X_0 = x$.

Though the theory presented in this section is applicable to processes satisfying (1.1.1), the primary example and the main object of this paper is of one-dimensional Brownian Motion (BM) with negative constant drift absorbed at 0, and restricted to

$\mathbb{R}_+$.

If π is a stationary distribution for $\mathbf{X}$, then (1.1.1) guarantees that $\pi = \delta_0$, [5, Section 2.2]. While this result is not very interesting, the distribution of the process and particularly of X_t *conditioned* on $\{\tau > t\}$, is in general far from trivial. This naturally leads to the following "conditional" analog for a stationary distribution:

Definition 1.1.1. The probability distribution π is a Quasi-Stationary Distribution (QSD) for $\mathbf{X}$ if

$$P_\pi(X_t \in \cdot \mid \tau > t) = \pi \text{ for all } t > 0.$$

A seemingly more relaxed definition, in the spirit of ergodic theorems for Markov Chains, is the following:

Definition 1.1.2. A probability distribution π is a Quasi-Limiting Distribution (QLD) for $\mathbf{X}$ if for some μ,

$$\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi.$$

The notations P_π and P_μ are shorthand for X_0 following respective distributions. Limits in Definition 1.1.1 and 1.1.2 are in distribution. Also, some literature call π a Yaglom limit if it is a QLD of a Dirac-delta measure. (That is, X_0 is a fixed point)

Figures 1.1.1 through 1.1.4 illustrate the difference between the unconditioned process, the process that is required to be positive only at the given time, and the process that is required to never hit 0 up to the given time.

Unlike uniqueness of stationary distribution under irreducibility assumptions, QSDs are in general not unique, and typically a continuum of QSDs exists. Notable exceptions of this are Markov chains on finite state spaces (with a unique absorbing state).

In fact:

Proposition 1.1.3. *[13, Proposition 1.1] A distribution π is a QSD for* $\mathbf{X}$ *if and only if it is a QLD for* $\mathbf{X}$*.*

Whenever μ and π are as in Definition 1.1.2, then μ is in the *domain of attraction of* π. The domain of attraction of any QSD clearly contains itself.

One strategy of finding QSDs is to study the quasi-limiting behavior under different initial distributions. When the class of QSD is known it is natural to ask what is the domain of attraction of each.

The concept of QSD is fairly intuitive and straightforward, as the idea was first introduced as early as 1931 by Wright [14], and the terms related to QSD have been crystallized in 1950s by Bartlett [1] [2]. Mathematically, Yaglom [15] first showed an explicit solution to a limiting conditional distribution for the for the subcritical Bienaymé-Galton-Watson branching process. However, there is no well-developed general theory concerning classification, domain of attraction, or convergence rates for QSDs.

For example, there is no known general condition which guarantees the existence of a QSD, though a known necessary condition is the finiteness of the moment generating function of τ on some open interval. Regarding uniqueness, the situation is similar. Most work in the field are restricted to specific models; for example, explicit description of QSDs are known for certain birth-and-death processes [5, Theorem 5.4]. As for uniqueness, a necessary and sufficient conditions for birth-and-death processes were obtained by van Doorn [12] Martinez *et al.* later generalized the result to countable state processes [10]. For other discrete state space models, Buiculescu studied QSDs for multi-type Galton-Watson processes [4], and Ferrari discussed QSDs for Fleming-Viot processes [6].

As for non-discrete state space models, fewer results are known. The model we study is the Brownian Motion with constant drift, which is one among few in which all QSDs are explicitly known. Also, Lladser and San Martin [7] classified the class of QSDs and their domain of attraction for the Ornstein-Uhlenbeck model. Later, Ye [16] studied the radial Ornstein-Uhlenbeck to get partial results on the Yaglom limit of such models. Rate of convergence in continuous state space models are largely unknown.

We close this section with the following well-known nice properties related to QSDs.

Theorem 1.1.4. *[5, Theorem 2.2] Suppose that π is a QSD. Then under P_π, τ is exponentially distributed.*

1.2 Quasi Stationarity for Drifted BM

In this section and the sequel we will work under the following:

Assumption 1.2.1. **X** is Brownian Motion (BM) with constant negative drift $-\alpha$, $\alpha > 0$, on $\mathbb{R}_+$ absorbed at 0.

Analytically, BM with constant drift $-\alpha$ on $\mathbb{R}_+$ absorbed at 0 is the sub-Markovian process generated by $\mathcal{L}_\alpha$, which for each u satisfying $u \in C^2(\mathbb{R}_+)$ and $u(0) = 0$,

$$\mathcal{L}_\alpha u = \frac{1}{2}u'' - \alpha u'.$$

The works by Martinez, Picco and San Martin [9][8] studied QSDs for this class of models.

The formal adjoint $\mathcal{L}_\alpha^*$ of $\mathcal{L}_\alpha$, with respect to integration by parts, is given by

$$\mathcal{L}^* v = \frac{1}{2}v'' + \alpha v', \ v \in C^2(\mathbb{R}_+), v(0) = 0.$$

Observe that for any f in the domain of $\mathcal{L}_\alpha$,

$$\begin{aligned} &\frac{d}{dt}P_x(f(X_t), \tau > t) = \mathcal{L}_\alpha P_x(f(X_t), \tau > t) \\ &\Rightarrow P_x(f(X_t), \tau > t) = f(x) + \int_0^t \mathcal{L}_\alpha P_x(f(X_s), \tau > s)ds \end{aligned} \tag{1.2.1}$$

Suppose a probability density function π satisfies $\mathcal{L}_\alpha^* \pi = -\lambda \pi$ for some $\lambda > 0$. Notice that every QSD must be smooth, since if π is a QSD then by definition we have the following density.

$$\begin{aligned} \pi(y) &= P_\pi(X_s = y \mid \tau > s) \\ &= \frac{P_\pi(X_s = y, \tau > s)}{P_\pi(\tau > s)} \end{aligned} \tag{1.2.2}$$

Then we have the following computation.

$$
\begin{aligned}
E_\pi(f(X_t), \tau > t) &= \int E_x(f(X_t), \tau > t)\pi(x)dx \\
&= \int f(x)\pi(x)dx + \int \int_0^t \mathcal{L}_\alpha \left(E_x(f(X_s), \tau_s)\right) ds\pi(x)dx \\
&= \int f(x)\pi(x)dx + \int_0^t \int \mathcal{L}_\alpha \left(E_x(f(X_s), \tau_s)\right) \pi(x)dxds \\
&= \int f(x)\pi(x)dx + \int_0^t \int E_x(f(X_s), \tau > s)\mathcal{L}_\alpha^*\pi(x)dxds \\
&= \int f(x)\pi(x)dx - \lambda \int_0^t E_\pi(f(X_s), \tau > s)ds
\end{aligned}
\tag{1.2.3}
$$

Setting $h(t) = E_\pi(f(X_t), \tau > t)$, (1.2.3) gives

$$
\begin{aligned}
& h(t) = h(0) - \lambda \int_0^t h(s)ds \\
& \Rightarrow h'(t) = -\lambda h(t) \\
& \Rightarrow h(t) = h(0)e^{-\lambda t} \\
& \Rightarrow E_\pi(f(X_t), \tau > t) = e^{-\lambda t} \int f(x)\pi(x)dx
\end{aligned}
\tag{1.2.4}
$$

Therefore by monotone convergence,

$$P_\pi(\tau > t) = e^{-\lambda t}$$

$$E_\pi(f(X_t)|\tau > t) = \int f(x)\pi(x)dx$$

That is, π is a QSD if and only if $\mathcal{L}_\alpha^*\pi = -\lambda\pi$.

We can see that a QSD π is a solution to standard ODE and depends on the parameter λ. For $\lambda \in (0, \alpha^2/2]$, let $\gamma = \sqrt{\alpha^2 - 2\lambda}$, and π_γ be the probability measure on $\mathbb{R}_+$ with density which we also denote by π_γ

$$\pi_\gamma(x) = \begin{cases} \frac{\alpha^2-\gamma^2}{\gamma} e^{-\alpha x} \sinh(\gamma x) & \gamma > 0 \\ \alpha^2 x e^{-\alpha x} & \gamma = 0. \end{cases} \tag{1.2.5}$$

Theorem 1.2.2. *[9, Proposition 1] Every QSD for* $\mathbf{X}$ *is of the form* π_γ *for some* $\gamma \in [0, \alpha)$.

Theorem 1.2.3. *[8, Theorem 1.3] The probability measure* μ *is in the domain of attraction of* π_0 *if*

$$\liminf_{x\to\infty} \frac{\ln \mu([x,\infty))}{x} \leq -\alpha.$$

Theorem 1.2.4. *[8, Theorem 1.1] Let* $\rho \in (0, \alpha)$. *The probability measure* μ *is in the domain of attraction of* $\pi_{\alpha-\rho}$ *if*

$$\lim_{x\to\infty} \frac{\ln \mu([x,\infty))}{x} = -\rho.$$

We note the following:

1. Theorem 1.2.4 was proved under the assumption that μ has a smooth density.

2. The limit condition in Theorem 1.2.4 is not merely technical. The authors constructed an example [8, Theorem 1.4] with initial distribution with tail which alternates between two exponential decay rates and which is not in the domain of attraction of any QSDs. In section 6.1 we will provide a simpler construction of such initial distribution using the method we develop in this paper.

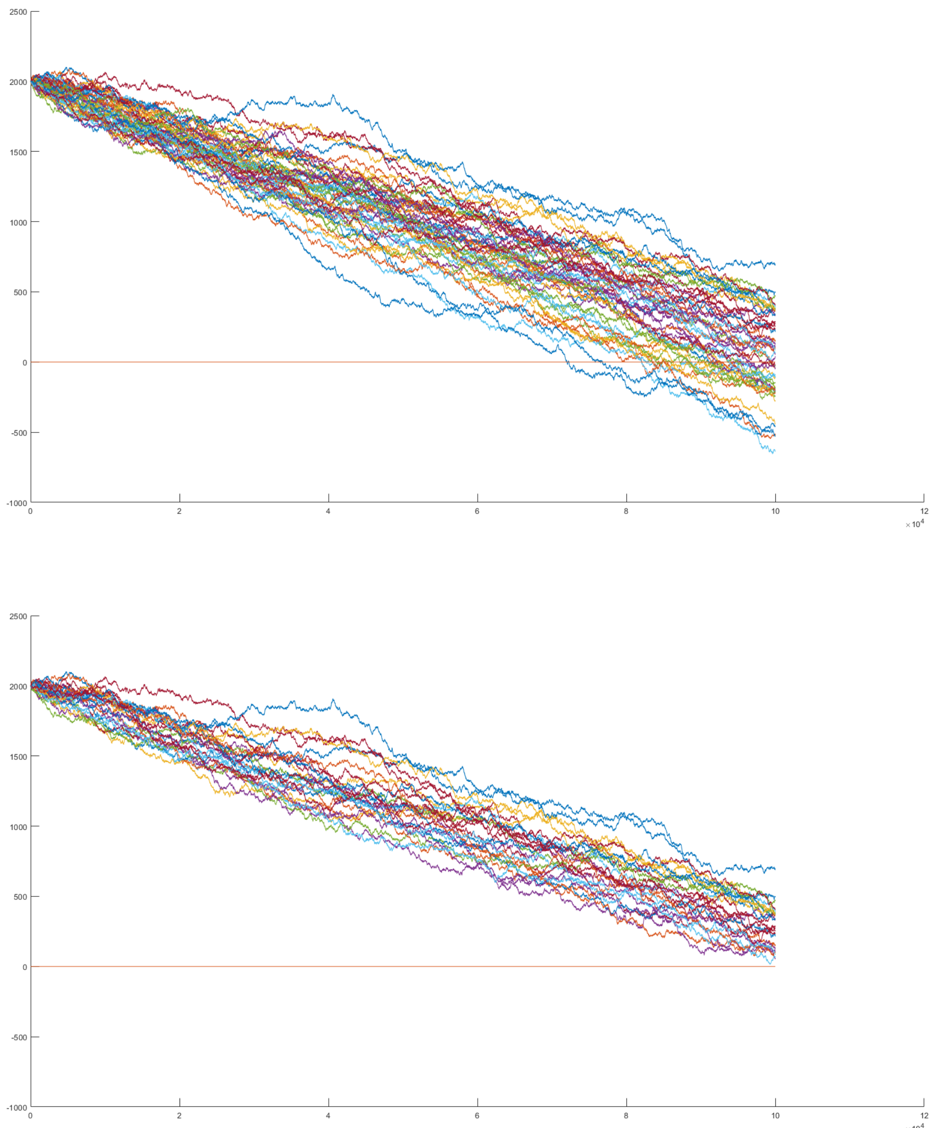

FIGURE 1.1.1: Top: Sample paths of 1-dimensional Brownian Motion with constant negative drift up to $t = 100000$, with fixed initial state $X_0 = 2000$
Bottom: Remaining paths of same processes, conditioned not to be absorbed by $t = 100000$

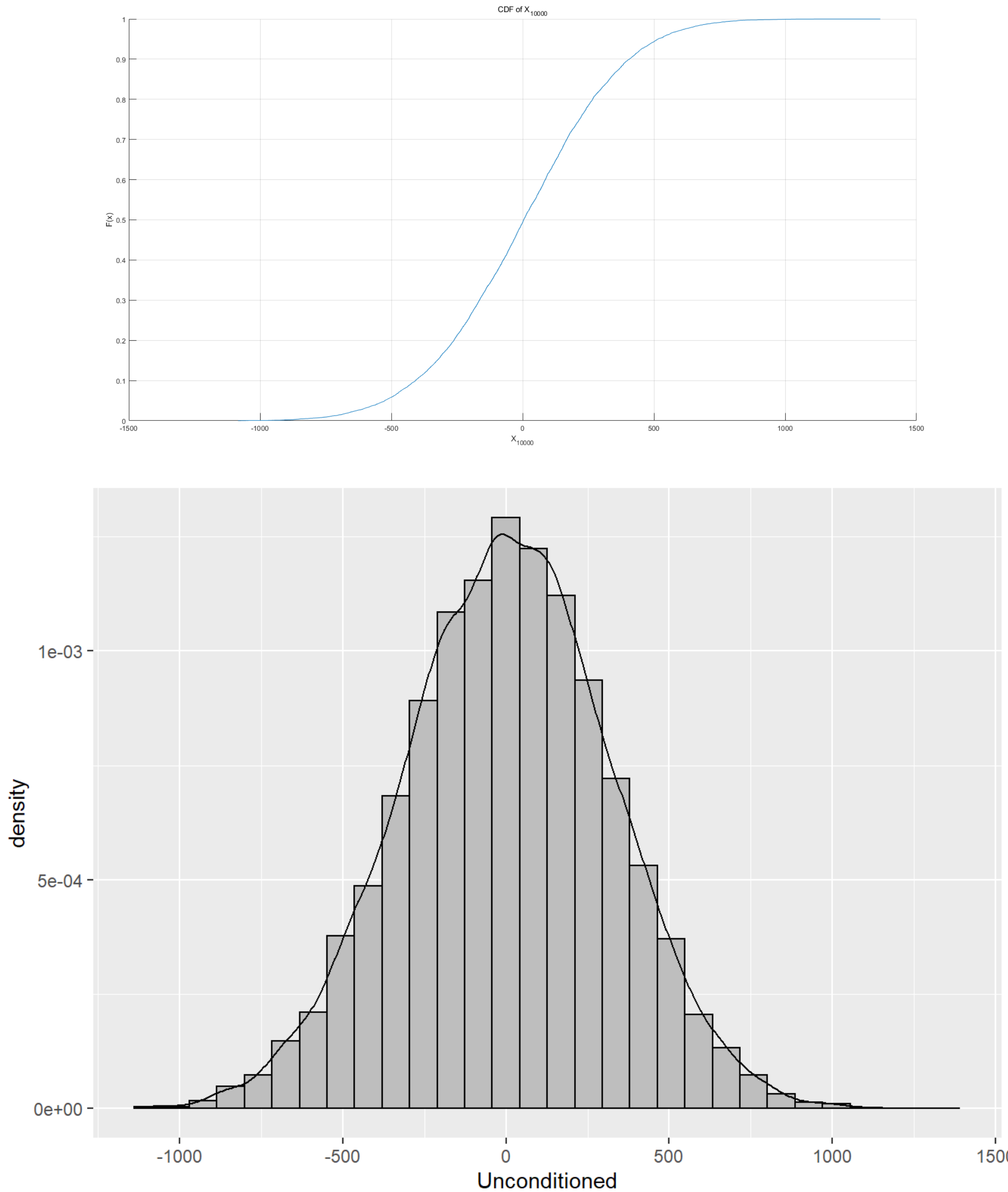

FIGURE 1.1.2: CDF and PDF plots of 1-dimensional Brownian Motion with constant negative drift, with fixed initial state $X_0 = 2000$ at $t = 100000$. Sample size is 10000. As expected, X_{10000} follows a Gaussian distribution.

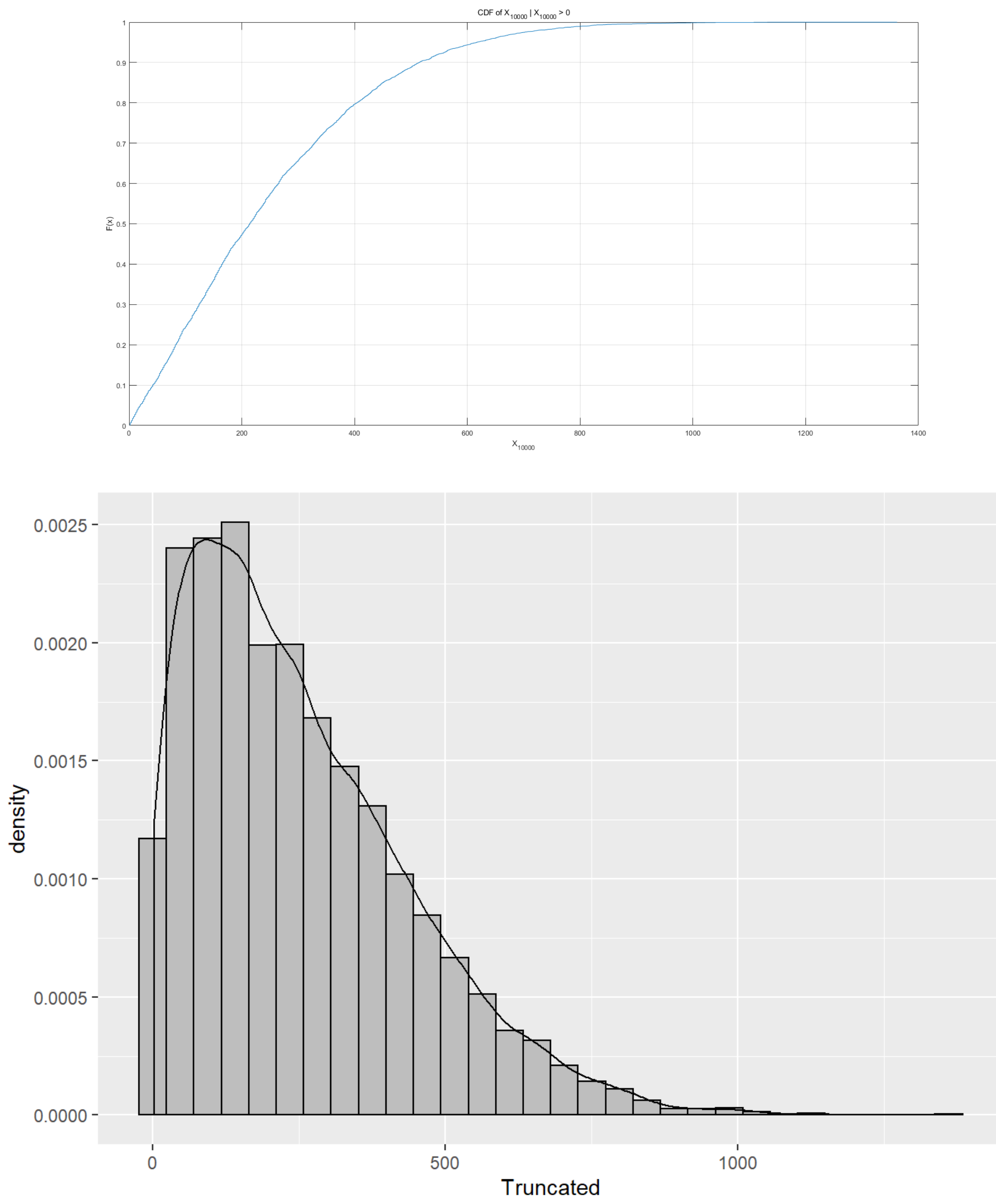

FIGURE 1.1.3: CDF and PDF plots of same sample processes as Figure 1.1.2, with the condition $X_{10000} > 0$. The result is a Half-Gaussian distribution.

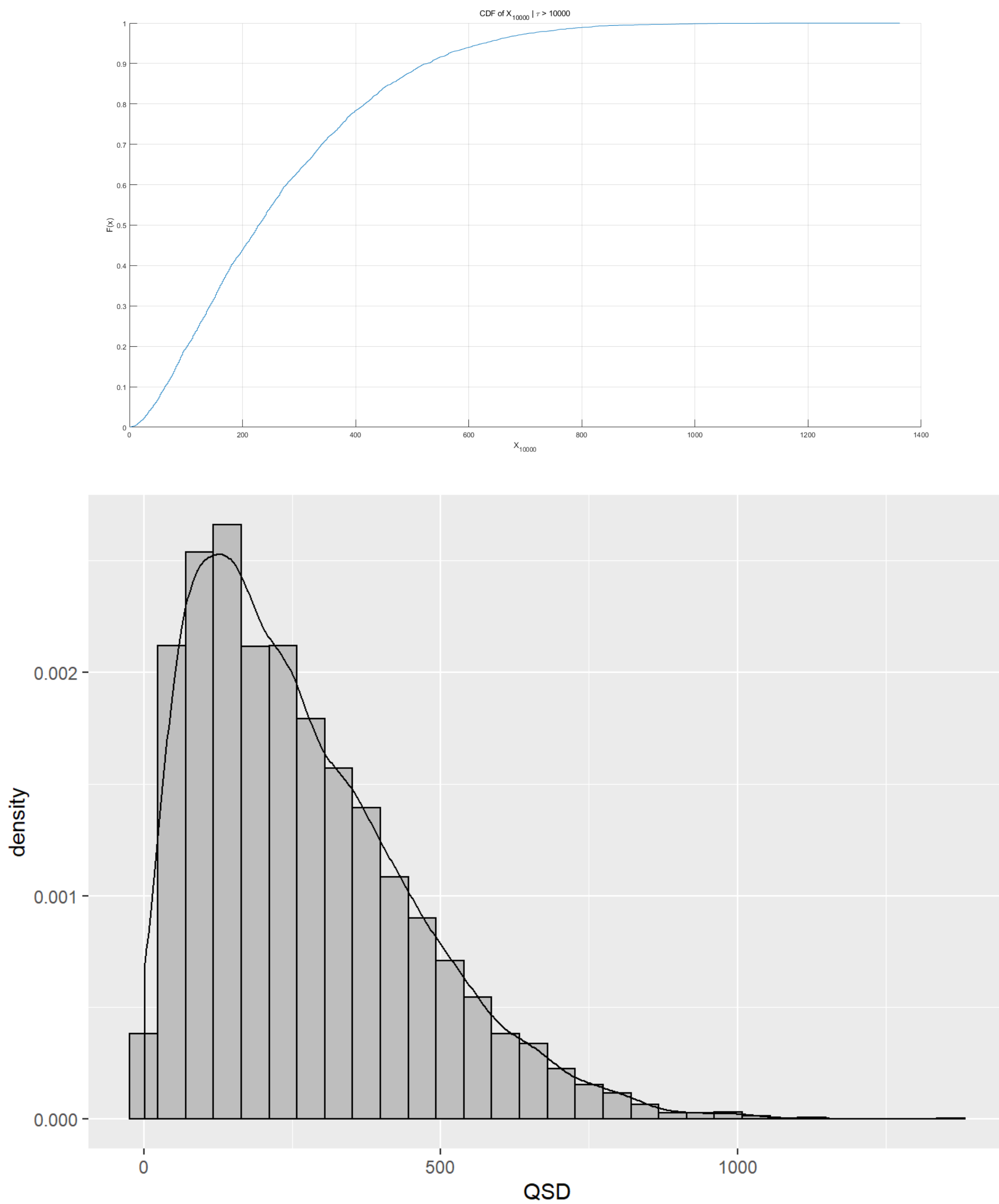

FIGURE 1.1.4: CDF and PDF plots of same sample processes as Figure 1.1.2, with the condition $\tau > 10000$. Unlike the other two figures, this distribution has exponential tail. Also, the density near 0 drops significantly in this setting.

Chapter 2

Main Results

Recall that we are working under Assumption 1.2.1.

Assumption 1.2.1. $\mathbf{X}$ is Brownian Motion (BM) with constant negative drift $-\alpha$, $\alpha > 0$, on $\mathbb{R}_+$ absorbed at 0.

Our goals are twofold:

1. Develop a method that would yield alternate proof to Theorems 1.2.3 and 1.2.4, which can be generalized to other models, as well as leading to complete characterization of the domain of attraction of every QSD. We discuss this in Section 2.1.

2. Characterize the asymptotic behavior when the initial distribution has tails which are heavier than exponential. It is not hard to show, see Lemma 2.2.1, that this class of initial distributions is not in the domain of attraction of any QSD. We discuss this in Section 2.2.

2.1 Domain of Attraction of QSDs

As at its core, the concept of quasi-stationarity concerns conditional probabilities under events with diminishing probabilities, namely the events $\{\tau > t\}$. It is therefore natural to study the rate at their probabilities, $P_\mu(\tau > t)$, tend to zero. One of the nice properties of our model is that through Girsanov theorem and the reflection principle (or formulas for Brownian bridges) a closed form formula for these probabilities is readily available. We have:

Proposition 2.1.1.

$$P_\mu(X_t \in dy, \tau > t) = \frac{1}{\sqrt{2\pi t}} \int \exp\left(\alpha x - \frac{\alpha^2 t}{2} - \alpha y\right)\left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) d\mu(x). \tag{2.1.1}$$

Our approach to the problem is to obtain for each initial distribution μ a family of probability measures $(\nu_t : t \geq 0)$, such that

Principle 2.1.2.

$$\boxed{\lim_{t\to\infty} \nu_t = \delta_\gamma} \implies \boxed{\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi_\gamma} \tag{2.1.2}$$

The measure ν_t is defined through its cumulative distribution function F_{ν_t}:

$$F_{\nu_t}(z) = C_t \int_{[0,zt]} e^{-x^2/(2t)} e^{\alpha x} d\mu(x) \tag{2.1.3}$$

where C_t is the normalization constant. Table 2.1.1 shows the relation between μ, ν_t and the QLD of μ.

ρ	$\lim \nu_t$	QLD (= QSD)	Example distributions
$\rho \geq \alpha$	δ_0	π_0	Half-normal distribution Delta distribution
$\alpha > \rho > 0$	$\delta_{\alpha-\rho}$	$\pi_{\alpha-\rho}$	Exponential distribution with rate $\lambda < \alpha$
$\rho = 0$	δ_α	QLD does not exist: scaling is necessary. See Table 2.2.1	Pareto distribution Half-Cauchy distribution

TABLE 2.1.1: Domain of attraction classified by parameter $\rho = \lim_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x}$

The key idea in the method is to "decouple" the initial distribution from the asymptotic distribution, then identifying the relevant QSD as a member of a one-parameter family selected according to the value of γ. Indeed, in our model, observe that the mapping $\gamma \to \pi_\gamma,\ \gamma \in [0, \alpha)$ as given in (1.2.5) is an explicit function, with the case $\gamma = 0$ is merely a removable singularity and is defined as $\lim_{\gamma\to 0+} \pi_\gamma$.

We believe that this method has a number of advantages:

1. It is more intuitive, simpler and elementary than the previous approach. It lets us understand how the initial distribution actually evolves over time, and at a specific time, which part of the initial distribution have evolved to consist the absolute majority of the process not absorbed.

2. The method allows for expanded characterization of the domain of attraction of QSDs.

3. Our approach simplifies the analysis for the case of a distribution with with alternating exponential tails, given in [8], and opens the possibility of studying general compound-tail distributions.

We expect this method to be applicable to other models and we hope it can be adopted as a general framework for classifying domain of attraction of QSDs.

Our Principle 2.1.2 will be employed in two ways. We first observe that

$$\lim_{t\to\infty} \nu_t = \begin{cases} \delta_0 & \iff \limsup_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x} \geq \alpha \\ \delta_{\alpha-\rho} & \iff \lim_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x} = \rho < \alpha \end{cases} \tag{2.1.4}$$

We will call the distributions μ that satisfy the first condition possess "Critical and Super-critical" tails (with critical being the case which $-\ln \mu(x,\infty)/x \to \alpha$) and such cases will be dealt in section 4.1. We will call the distributions μ that satisfy the second condition possess "Sub-critical Exponential" tails and such cases will be dealt in section 4.2. Finally, in section 6.1 we will provide a simple construction of an initial distribution μ that will satisfy similar result to [8, Theorem 1.4].

2.2 Tails Heavier than Exponential

A natural question to ask from [8] would be the following: what happens if the initial distribution is too heavy to be in the domain of attraction of any QSDs? A first step in this direction is to look for such initial distributions. In light of Theorems 1.2.3 and 1.2.4, the following is not surprising:

Lemma 2.2.1. *Suppose*

$$\lim_{x\to\infty}\frac{\ln\mu([x,\infty))}{x}=0.$$

Then $P_\mu(\tau>t)$ does not decay exponentially. As a consequence $(P_\mu(X_t\in\cdot\mid\tau>t):t\geq 0)$ is not tight.

Thus, in order to obtain a non-trivial limit, one has to scale X_t as $t\to\infty$. As we will see, the scaling itself depends on μ. We comment that all of the cases covered in this section correspond to $\nu_t\to\delta_\alpha$ in (2.1.2).

The next step is to study long-time behavior under such heavier-tailed distributions, and this is the main topic of this part of the project. In order to do so, we mainly rely on the theory of regularly varying functions [3].

Assumption 2.2.2. Suppose μ is a probability measure satisfying the following:

1. $\mu([x,\infty))=e^{-F(x)}$, with F smoothly varying [3, Section 1.8] with index parameter $\beta<1/2$.

2. There exists a positive function $R(x,c)$ on $\mathbb{R}_+\times\mathbb{R}_+$ increasing in c, such that for all $c>0$

$$\lim_{x\to\infty}F(x+R(x,c))-F(x)=c. \tag{2.2.1}$$

Some comments are in order:

1. Probability measures with regularly varying tails falls into the category $\beta=0$. Some distinguished cases are the Weibull distribution with $0<k<1$, which has a uniform decay rate with $\beta=k$, and the Pareto and Cauchy distributions, both having uniform decay rate with $\beta=0$.

2. If F is smooth enough, then

$$R(x,c) = \frac{c}{F'(x)} \tag{2.2.2}$$

So when $\beta \neq 0$, $R(x,c)$ is a regular varying function with index $\varphi = 1-\beta$.

3. When $\beta = 0$ it is more natural to replace the identity function on the right-hand side of (2.2.1) with a strictly increasing continuous and nonnegative function H satisfying $H(0) = 0$.

The main principle we developed to obtain results under Assumption 2.2.2 is the following.

Principle 2.2.3.

$$\boxed{\textit{Assumption 2.2.2}} \quad \Rightarrow \quad \boxed{\lim_{t\to\infty} P_\mu\left(X_t > R(t,c) \mid \tau > t\right) = e^{-c}} \tag{2.2.3}$$

We note that the assumption $\beta < 1$ is vital for this to work, as otherwise the conclusion contradicts the results of previous sections. This is due to the fact that $\beta = 1$ is the critical border where the relation between the survival rate $P_\mu(\tau > t)$ and the initial distribution μ changes. Also, although Lemma 2.2.1 applies whenever $0 \leq \beta < 1$, Principle 2.2.3 only applies to $0 < \beta < 1/2$. The remaining half $1/2 \leq \beta < 1$ is left open and is briefly discussed in section 6.2. Table 2.2.1 shows how β can lead to quasi-limiting behavior of such initial distribution. We also note that in the future we would like to expand this idea to other continuous state space models such as Ornstein-Uhlenbeck process [7].

In Chapter 5, we will prove Lemma 2.2.1 and Principle 2.2.3. In addition we will present some concrete results obtained through this principle.

β	Related theorem	Example distributions
$\beta > 1$	Theorem 4.0.1	Half-normal distribution Delta distribution Weibull distribution with shape parameter $k > 1$
$\beta = 1$	Theorem 4.0.1 if $\rho^a \geq \alpha$ Theorem 4.0.2 if $\rho < \alpha$	Exponential distribution Erlang distribution
$\frac{1}{2} \leq \beta < 1$	Open problem; see section 6.2	Weibull distribution with shape parameter $\frac{1}{2} \leq k < 1$
$0 < \beta < \frac{1}{2}$	Theorem 5.0.1	Weibull distribution with shape parameter $k < \frac{1}{2}$
$\beta = 0$	Corollary 5.3.3 if $\kappa^b \neq 0$ Corollary 5.3.5 if $\kappa = 0$	Pareto distribution Half-Cauchy distribution Log-Cauchy distribution

TABLE 2.2.1: Distributions classified by index parameter β of $F(x) = -\ln \mu([x, \infty))$

[a] $\rho = \lim_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x}$; see Table 2.1.1
[b] $\mu([x, \infty))$ is regularly varying with index $-\kappa$

Chapter 3

Base Formula

In this section, we prove Proposition 2.1.1, which is the master formula we use throughout this paper. We will also further explain the intuition behind the sequence of new measure ν_t. Finally, we will introduce the variations of Scheffe's lemma [11], which is one of the tool for Chapter 4.

3.1 Conditional transition density

When X_t is a drifted Brownian Motion with negative drift α, (such that $X_t + \alpha t$ is a standard BM B_t)

$$\begin{aligned}
P_x(X_t \in dy) &= P(X_0 \in dx, X_t \in dy) \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-(y+\alpha t))^2}{2t}\right) \\
&= \exp\left(\frac{-\alpha^2 t}{2} + (x-y)\alpha\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \\
&= \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) P_x(B_t \in dy)
\end{aligned} \tag{3.1.1}$$

We also want to enforce the condition $\tau > t$, where τ is the hitting time at 0. We can apply the reflection principle to compute $P_x(X_t \in dv, \tau > t)$.

$$\begin{aligned}
P_x(X_t \in dy, \tau > t) &= \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) P_x(B_t \in dy, \tau > t) \\
&= \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) (P_x(B_t \in dy) - P_x(B_t \in d(-y))) \\
&= \underbrace{\exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right)}_{=f(t,x,y)}
\end{aligned} \tag{3.1.2}$$

Integrating $f(t,x,y)$ with respect to μ gives (2.1.1). Furthermore, we can get the survival rate from the above formula as well.

$$P_\mu(\tau > t) = \int_0^\infty \int_0^\infty \mu(x) f(t,x,y) dy dx \tag{3.1.3}$$

We wrap this section with the principle behind finding the family of probability measures $(\nu_t : t \geq 0)$ in (2.1.2). From (3.1.3),

$$\begin{aligned} P_\mu(\tau > t) &= \int_0^\infty \int_0^\infty \mu(x) \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) dydx \\ &= \frac{1}{\sqrt{2\pi t}} \left(\int_0^\infty \int_0^\infty \mu(x) e^{-\frac{\alpha^2 t}{2}} e^{\alpha x} e^{-\alpha y} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) dydx\right) \\ &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty \mu(x) e^{-\frac{x^2}{2t}} e^{\alpha x} \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}}\right) dydx\right) \end{aligned} \tag{3.1.4}$$

We substitute $z = tx$.

$$(3.1.4) = \frac{e^{-\frac{\alpha^2 t}{2}} t}{\sqrt{2\pi t}} \left(\int_0^\infty \mu(tz) e^{-\frac{tz^2}{2}} e^{\alpha tz} \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{tz} - e^{-tz}\right) dydz\right) \tag{3.1.5}$$

For convenience, we will use x instead of z for (3.1.5) in later parts.

From the above equations, the natural construction of ν_t would come from the terms that consist the outer integral. Indeed, we will use $\nu_t(x) = \mu(tx)e^{-\frac{tx^2}{2}}e^{\alpha tx}$ in section 4.2. In section 4.1, (3.1.4) will be use with some modification.

3.2 Scheffe's Lemma

Scheffe's Lemma[11] suggests that under mild conditions, convergence in pdf implies convergence in distribution.

Lemma 3.2.1. *[11] Suppose* $\lim_{t\to\infty} \dfrac{P_\mu(X_t \in dy, \tau > t)}{P_\mu(\tau > t)} = f(y)$ *for some function* f*, and* $\int_0^\infty f(y)dy = 1$*. Then* $\lim_{t\to\infty} \mu_t$ *exists, and* f *is its density function.*

From (2.1.1) and (3.1.3), we can consider the conditional density

$$\begin{aligned} P_\mu(X_t \in dy \mid \tau > t) &= \frac{P_\mu(X_t \in dy, \tau > t)}{P_\mu(\tau > t)} \\ &= \frac{\int_0^\infty \mu(x) f(t,x,y)dx}{\int_0^\infty \int_0^\infty \mu(x) f(t,x,y)dydx} \end{aligned} \tag{3.2.1}$$

When t is fixed, this is clearly a probability density which we will call $\mu_t(y)$. Moreover, if $\lim_{t\to\infty} \mu_t(y)$ exists and is a probability density, then Scheffe's lemma allows us to claim that the limit is the density of the desired QLD π.

Lemma 3.2.1 will play an important role in section 4.2. Also, we present here a weaker version of Scheffe's lemma, which we will use in Section 4.1.

Lemma 3.2.2. *Suppose that f_n, f are probability densities on $\mathbb{R}_+$ satisfying* $\liminf f_n \geq f$*, a.e. Then $\int_A f_n dx \to \int_A f dx$ for any A.*

Proof. Let $dm_n = f_n dx$, and $dm_\infty = f dx$. By Fatou's lemma, for every A,

$$\liminf m_n(A) \geq m_\infty(A) \tag{3.2.2}$$

Now

$$1 - \limsup m_n(A) = \liminf(1 - m_n(A)) = \liminf m_n(A^c),$$

Thus, by (3.2.2) applied to A^c,

$$1 - \limsup m_n(A) = \liminf m_n(A^c) \geq m_\infty(A^c) = 1 - m_\infty(A).$$

In other words $\limsup m_n(A) \leq m_\infty(A)$ and the first statement follows. $\square$

Chapter 4

QSD of exponential or lighter tail distributions

In this chapter we will prove Principle 2.1.2. Recall that the family of QSDs under Assumption 1.2.1 is categorized into two cases as shown in Table 2.1.1. As a result our principle is proved by the following two theorem.

Theorem 4.0.1. *Suppose μ satisfies the following assumption.*

$$\rho := \liminf_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x} \geq \alpha. \tag{4.0.1}$$

Then

$$P_\mu(X_t \in \cdot | \tau > t) \to \pi_0.$$

Theorem 4.0.2. *Suppose μ satisfies the following assumption,*

$$\rho := \lim_{x\to\infty} -\frac{\ln \mu([x,\infty))}{x} \in (0,\alpha) \tag{4.0.2}$$

and let the sequence of measure $(\nu_t : t \geq 0)$ *defined as* (2.1.3). *Then*

$$\lim_{t\to\infty} \nu_t = \delta_{\alpha-\rho} \tag{4.0.3}$$

and moreover,

$$\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi_{\alpha-\rho} \tag{4.0.4}$$

Theorem 4.0.1 applies to μ that have critical or super-critical tails, which includes any tails that are lighter than exponential. Theorem 4.0.2 applies to μ that have sub-critical, yet still exponential tails. For both theorems, μ does not need to have a smooth density. One should also notice that Theorem 4.0.2 requires a stronger limit condition than the one in Theorem 4.0.1, as if the limit does not exist in Theorem 4.0.2 there can be a problem in the quasi-limiting behavior. Such cases are discussed in Chapter 6.

Throughout the rest of the paper, we will be using some asymptotic notations; $f(t) \sim g(t)$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} \in (0, \infty)$, and $f(t) \ll g(t)$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$.

4.1 Critical and Super-critical Tails

In this section we work under the assumption (4.0.1)

The main justification for our work is in obtaining a simple and unified argument for both the critical as well as the lighter case.

Define

$$f(t, x, y) = ye^{-\alpha y} e^{-\frac{y^2}{2t}} \frac{\sinh(xy)}{xy}$$

and

$$h(t,x) = \int_0^\infty f(t,x,y)dy$$

and let

$$h(x) = \lim_{t\to\infty} h(t,x) = \int_0^\infty e^{-\alpha y}\frac{\sinh(xy)}{x}dy$$

Note that $h(x)$ is increasing,

$$h(0) := \lim_{x\searrow 0} h(x) = \int_0^\infty ye^{-\alpha y}dy = \frac{1}{\alpha^2}$$

and $h(x) = \infty$ if and only if $x \geq \alpha$.

For every t, we define two measures on $[0,\infty)$:

$$\begin{aligned} d\gamma(x) &= xe^{\alpha x}d\mu(x) \\ d\nu_t(x) &= e^{-\frac{x^2}{2t}}d\gamma(x) \end{aligned} \tag{4.1.1}$$

By assumption, there exists a function $\delta(x) \to 0$ such that

$$\gamma([0,x]) \leq e^{\delta(x)x}$$

without loss of generality, we may also assume δ is decreasing.

Observe that

$$P(X_t \in dy | \tau > t) = \frac{\int f(t,x/t,y)d\nu_t(x)}{\int h(t,x/t)d\nu_t(x)}. \tag{4.1.2}$$

We will now prove the theorem through the application of Lemma 3.2.2, where

$$f_t(v) = \frac{\int f(t, x/t, y) d\nu_t(x)}{\int h(t, x/t) d\nu_t(x)}$$

and $f(v) = \alpha^2 y e^{-\alpha y}$

Proof of Theorem 4.0.1. Let $\epsilon \in (0, 1)$ and let $\eta_t = \epsilon \alpha t$. We begin by analyzing the behavior of the denominator in the right-hand side of (4.1.2).

Observe that $h(t, y)$ is bounded on $[0, M] \times \mathbb{R}_+$ and increases as $t \to \infty$ to

$$h(x) = \int_0^\infty y e^{-\alpha y} \frac{\sinh(xy)}{xy} dy$$

As a result, the convergence is uniform. From this it follows that

$$\limsup_{t \to \infty} \frac{\int_{[0,\eta_t]} h(t, x/t) d\nu_t(x)}{\nu_t([0, \eta_t])} \leq h(\epsilon\alpha). \tag{4.1.3}$$

We turn to evaluation of the interval on $[\eta_t, 0.9\alpha t]$. Since here $\frac{x}{t} \leq 0.9\alpha < \alpha$, $h\left(t, \frac{x}{t}\right)$ is uniformly bounded by a constant depending only on α. Below C denotes a positive constant depending only on α, ϵ, and whose value may change from line to line.

Integrating by parts,

$$\int_{[\eta_t, 0.9\alpha t]} h\left(t, \frac{x}{t}\right) d\nu_t(x) \leq C \frac{1}{t} \int_{[\eta_t, 0.9\alpha t]} x e^{-\frac{x^2}{2t}} \gamma([\eta_t, x]) dx.$$

Changing variables to $z = \dfrac{x}{\sqrt{t}}$, the last expression becomes

$$\int_{\sqrt{t}\alpha[\epsilon,0.9]} ze^{-\frac{z^2}{2}}\gamma([\eta_t, \sqrt{t}z])dz$$

Now

$$\gamma([\eta_t, \sqrt{t}z]) \leq \gamma([0, \sqrt{t}z]) \leq \gamma([0, \eta_t])e^{\delta(\eta_t)(\sqrt{t}z - \sqrt{t}\epsilon)} \leq \gamma([0, \eta_t])e^{\delta(\eta_t)\sqrt{t}z}$$

Putting this back in the integral gives an upper bound of the form

$$\gamma([0, \eta_t]) \int_{\sqrt{t}\alpha[\epsilon,0.9]} ze^{-\frac{z^2}{2}} e^{\delta(\eta_t)\sqrt{t}z}dz$$

Since $\delta(\eta_t) \to 0$ as $t \to \infty$, for all t large enough, we have

$$\delta(\eta_t) \leq \min\left(\frac{\alpha^2\epsilon^2}{4}, \alpha\epsilon\right) \tag{4.1.4}$$

To obtain an upper bound on the integral, observe that as a function of z,

$$-\frac{z^2}{2} + \delta(\eta_t)\sqrt{t}z = -\frac{z}{2}(z - 2\delta(\sqrt{t}))$$

is decreasing on $[\delta(\eta_t)\sqrt{t}, \infty)$, and by (4.1.4), if $z > \dfrac{\eta_t}{\sqrt{t}} = \epsilon\alpha\sqrt{t}$, then $z > \delta(\eta_t)\sqrt{t}$.

Therefore we have

$$\begin{aligned} -\frac{z^2}{2} + \delta(\eta_t)\sqrt{t}z &\leq -\frac{(\eta_t/\sqrt{t})^2}{2} + \delta(\eta_t)\sqrt{t}\left(\frac{\eta_t}{\sqrt{t}}\right) \\ &\leq -\frac{\alpha^2\epsilon^2 t}{2} + \frac{\alpha^2\epsilon^2 t}{4} \\ &= -\frac{(\alpha\epsilon)^2 t}{4} \end{aligned} \tag{4.1.5}$$

Thus,

$$\begin{aligned} \int_{[\eta_t, 0.9\alpha t]} h\left(t, \frac{x}{t}\right) d\nu_t(x) &\leq Ce^{-\frac{(\alpha\epsilon)^2 t}{4}} t^{\frac{3}{2}} \gamma([0, \eta_t]) \\ &\leq Ce^{\left(-\frac{(\alpha\epsilon)^2}{4} + \delta(\eta_t)\epsilon\alpha\right)t} t^{\frac{3}{2}} \to 0 \end{aligned} \tag{4.1.6}$$

Next we consider the behavior over the interval $[0.9\alpha t, \infty)$. Observe that

$$h(t, x) \leq \frac{\sqrt{2\pi t}}{x} E\left[e^{(x-\alpha)\sqrt{t}Z}\right]$$

where Z is standard Gaussian, and therefore

$$h\left(t, \frac{x}{t}\right) \leq \frac{\sqrt{2\pi t}}{x/t} e^{\frac{x^2}{2t}} e^{\frac{\alpha^2 t}{2}} e^{-\alpha x}$$

Hence

$$\int_{[0.9\alpha t, \infty)} h\left(t, \frac{x}{t}\right) d\nu_t(x) \leq \sqrt{2\pi t^3} e^{\frac{\alpha^2 t}{2}} \int_{[0.9\alpha t, \infty)} d\mu(x)$$

But $\mu([0.9\alpha t, \infty)) = e^{-0.9\alpha^2 t(1+o(1))}$, and as a result

$$\int_{[0.9\alpha t, \infty)} h\left(t, \frac{x}{t}\right) d\nu_t(x) \to 0. \tag{4.1.7}$$

Since $\liminf_{t\to\infty} \nu_t([0,\eta_t]) > 0$, it follows from (4.1.3), (4.1.6) and (4.1.7), that

$$\limsup_{t\to\infty} \frac{\int h(t,x/t)d\nu_t(x)}{\nu_t([0,\eta_t])} \le h(\epsilon\alpha). \tag{4.1.8}$$

Repeating the argument leading to that gave (4.1.3) mutatis mutandis, we obtain

$$\begin{aligned}\liminf_{t\to\infty} \frac{\int_{[0,\eta_t]} f(t,x/t,y)d\nu_t(x)}{\nu_t([0,\eta_t])} &\ge ye^{-\alpha y} \inf_{x\le\epsilon\alpha} \frac{\sinh(xy)}{xy} \\ &= ye^{-\alpha y}\end{aligned} \tag{4.1.9}$$

It therefore follows from (4.1.8) and (4.1.9), that

$$\liminf_{t\to\infty} \frac{\int f(t,x/t,y)d\nu_t(x)}{\int h(t,x/t)d\nu_t(x)} \ge \frac{ye^{-\alpha y}}{h(\epsilon\alpha)}$$

and this holds for every $\epsilon \in (0, 0.9)$.

Therefore since $\lim_{\epsilon\to 0} h(\epsilon\alpha) = \int_0^\infty ye^{-\alpha y}dy$, we obtain

$$\liminf_{t\to\infty} \frac{\int f(t,x/t,y)d\nu_t(x)}{\int h(t,x/t)d\nu_t(x)} \ge \frac{ye^{-\alpha y}}{\int_0^\infty ye^{-\alpha y}dy}$$

and the result follows from Lemma 3.2.2. $\square$

4.2 Sub-critical Exponential Tails

In this section we work under the assumption (4.0.2). We first split (3.1.4) into three parts.

$$
\begin{aligned}
P_\mu(\tau > t) = \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \Bigg(& \underbrace{\int_0^M e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_3(t)} \\
& + \underbrace{\int_M^{st} e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_1(t)} + \underbrace{\int_{st}^{\infty} e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_2(t)} \Bigg)
\end{aligned}
\tag{4.2.1}
$$

Where $h(t, x) = \displaystyle\int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) dy$.

Here, M is chosen such that we have the following inequality

$$
\frac{e^{-(\rho+\epsilon)x}}{\rho+\epsilon} \leq \frac{\mu([x,\infty))}{c} \leq \frac{e^{-(\rho-\epsilon)x}}{\rho-\epsilon} \tag{4.2.2}
$$

For each $x > M$ and some arbitrary $\epsilon > 0$. (c is the normalizing constant of μ) Also, we choose s such that $s = \alpha - \eta$ for some $\alpha > \eta > 0$ that depend on μ. Finally, since we are only interested in the limiting behavior with respect to t, we write $M < st$ which is always true for large enough t.

Proposition 4.2.1. *Under assumption* (4.0.2)

$$
\begin{aligned}
P_\mu(\tau > t) &\sim \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} J_1(t) \\
&\sim c e^{-\frac{(2\alpha\rho-\rho^2)t}{2}} \left(\frac{1}{\rho} - \frac{1}{2\alpha-\rho}\right)
\end{aligned}
\tag{4.2.3}
$$

where c is the constant in (4.2.2) *which only depend on* μ.

Proof. We first look at the region for $J_1(t)$. In this interval we have the following.

$$\begin{aligned} J_1(t) &= \int_M^{st} e^{-x^2/(2t)} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x) \\ &= t \int_{M/t}^{s} e^{-tx^2/2} e^{\alpha t x} h(t,x) d\mu(tx) \end{aligned} \tag{4.2.4}$$

Some observations on $h(t,x)$:

1. $h(t,x)$ is bounded in $\mathbb{R}_+ \times [0,s]$ since $s < \alpha$.
2. $h(x) = \lim_{t\to\infty} h(t,x) = \dfrac{1}{\alpha - x} - \dfrac{1}{\alpha + x}$ by dominated convergence theorem. Moreover, $h(x)$ is also bounded in $[0,s]$.

We introduce a new sequence of measures $(\nu_t^+, \nu_t^-, t \geq 0)$ defined as

$$\begin{aligned} d\nu_t^+(x) &= e^{-\frac{tx^2}{2}} e^{\alpha t x} e^{-(\rho-\epsilon)tx} = \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}} \sqrt{\frac{t}{2\pi}} e^{-\frac{t(x-(\alpha-\rho+\epsilon))^2}{2}} \\ d\nu_t^-(x) &= e^{-\frac{tx^2}{2}} e^{\alpha t x} e^{-(\rho+\epsilon)tx} = \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho-\epsilon)^2 t}{2}} \sqrt{\frac{t}{2\pi}} e^{-\frac{t(x-(\alpha-\rho-\epsilon))^2}{2}} \end{aligned} \tag{4.2.5}$$

For both case notice that the latter part is a Gaussian density with mean $\alpha - \rho \pm \epsilon$ and variance $1/t$, therefore we have the following convergence of measure:

$$\begin{aligned} \nu_t^+ &\rightharpoonup \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}} \delta_{\alpha-\rho+\epsilon} \\ \nu_t^- &\rightharpoonup \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho-\epsilon)^2 t}{2}} \delta_{\alpha-\rho-\epsilon} \end{aligned} \tag{4.2.6}$$

Therefore,

$$\begin{aligned}\limsup_{t\to\infty} J_1(t) &= \limsup_{t\to\infty} c\sqrt{2\pi t}\int_{M/t}^{s} h(t,x)d\nu_t^+(x) \\ &= c\sqrt{2\pi t}e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}}\left(\frac{1}{\rho-\epsilon}-\frac{1}{2\alpha-\rho+\epsilon}\right)\end{aligned} \tag{4.2.7}$$

$$\begin{aligned}\liminf_{t\to\infty} J_1(t) &= \liminf_{t\to\infty} c\sqrt{2\pi t}\int_{M/t}^{s} h(t,x)d\nu_t^-(x) \\ &= c\sqrt{2\pi t}e^{\frac{(\alpha-\rho-\epsilon)^2 t}{2}}\left(\frac{1}{\rho+\epsilon}-\frac{1}{2\alpha-\rho-\epsilon}\right)\end{aligned} \tag{4.2.8}$$

and since ϵ is arbitrary, we conclude that

$$J_1(t) \sim c\sqrt{2\pi t}e^{\frac{(\alpha-\rho)^2 t}{2}}\left(\frac{1}{\rho}-\frac{1}{2\alpha-\rho}\right) \tag{4.2.9}$$

For the second interval $x \in (st, \infty)$ we first study some bound for $h(t, x/t)$. we start from the obvious.

$$h\left(t, \frac{x}{t}\right) \leq \int_0^\infty \exp\left(-\frac{y^2}{2t} + \alpha y + \frac{xy}{t}\right) \tag{4.2.10}$$

We can rewrite the exponent as

$$\begin{aligned}-\frac{y}{2\sqrt{t}}\left(\frac{y}{\sqrt{t}} + 2\alpha\sqrt{t} - \frac{2x}{\sqrt{t}}\right) &= -\frac{1}{2}\frac{y}{\sqrt{t}}\left(\frac{y}{\sqrt{t}} + 2\varphi\right) \\ &= -\frac{1}{2}(w-\varphi)(w+\varphi)\end{aligned} \tag{4.2.11}$$

where $\varphi = \left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right)$, and $w = \frac{y}{\sqrt{t}} + \varphi$. Therefore, after changing variables

$y \to w$, we obtain

$$
\begin{aligned}
h(t,x) &\le \sqrt{t} e^{\frac{\varphi^2}{2}} \int_{\varphi}^{\infty} e^{-\frac{w^2}{2}} dw \\
&= \sqrt{t} e^{\frac{\alpha^2 t}{2}} e^{\frac{x^2}{2t}} e^{-\alpha x} L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right),
\end{aligned}
\tag{4.2.12}
$$

where $L(z) = \displaystyle\int_{z}^{\infty} e^{-\frac{w^2}{2}} dw.$

L has some nice properties:

1. $L(z)$ is strictly decreasing and bounded above by $\sqrt{2\pi}$.
2. When z is negative, $L(z) < \sqrt{2\pi}$.
3. When z is positive,
$$L(z) \le \min\left(\frac{e^{-\frac{z^2}{2}}}{z}, \sqrt{\frac{\pi}{2}}\right) \tag{4.2.13}$$
4. More specifically, if $z \ge 1$ then
$$L(z) \le e^{-\frac{z^2}{2}} \tag{4.2.14}$$

Using the bound above we get the following.

$$
\begin{aligned}
J_2(t) &\le \sqrt{t} \int_{st}^{\infty} e^{\frac{\alpha^2 t}{2}} L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right) d\mu(x) \\
&\le c\sqrt{2\pi t} e^{\frac{\alpha^2 t}{2}} e^{-\rho st} \\
&= c\sqrt{2\pi t} e^{t\left(\frac{\alpha^2}{2} - \rho(\alpha - \eta)\right)}
\end{aligned}
\tag{4.2.15}
$$

We want $J_2(t) = o(J_1(t)) = o\left(\sqrt{t}e^{\frac{(\alpha-\rho)^2 t}{2}}\right)$. Indeed, if we pick $\eta = \rho/4$,

$$\begin{aligned}\frac{(\alpha-\rho)^2}{2} - \left(\frac{\alpha^2}{2} - \gamma(\alpha-\eta)\right) &= \frac{\rho^2}{2} - \rho\eta \\ &= \frac{\rho^2}{4} > 0\end{aligned} \tag{4.2.16}$$

therefore we get the desired asymptotic.

For the last interval $x \in [0, M]$, we use the fact that for any $\epsilon > 0$, we can fix t_0 such that for each $t > t_0$, $M/\sqrt{t} < \epsilon$. And for such t, we have

$$\begin{aligned}J_3(t) &= \int_0^M e^{-\frac{x^2}{2t}} e^{\alpha x} \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh\left(\frac{xy}{t}\right) dy d\mu(x) \\ &\leq \sqrt{t}e^{\frac{\alpha^2 t}{2}} \int_0^M \mu(x) L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right) d\mu(x)\end{aligned} \tag{4.2.17}$$

And since L is decreasing,

$$(4.2.17) \leq \sqrt{t}e^{\frac{\alpha^2 t}{2}} \int_0^M L\left(\sqrt{t}\alpha - \epsilon\right) d\mu(x) \tag{4.2.18}$$

Finally using (4.2.14) and that μ is a probability measure,

$$\begin{aligned}(4.2.18) &\leq \sqrt{t} \int_0^M e^{\alpha\epsilon\sqrt{t}} e^{-\frac{\epsilon^2}{2}} d\mu(x) \\ &\leq \sqrt{t}e^{\alpha\epsilon\sqrt{t} - \frac{\epsilon^2}{2}} \\ &= o\left(\sqrt{t}e^{\frac{(\alpha-\rho)^2 t}{2}}\right) = o(J_1(t))\end{aligned} \tag{4.2.19}$$

$\square$

We now turn to computing the limiting density.

$$
\begin{aligned}
P_\mu(X_t \in dy, \tau > t) &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty \underbrace{e^{-\frac{x^2}{2t}} e^{\alpha x} e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh\left(\frac{xy}{t}\right)}_{=g(x,y,t)} d\mu(x) \right) \\
&= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\underbrace{\int_0^M g(x,y,t) d\mu(x)}_{=K_3(t,y)} + \underbrace{\int_M^{st} g(x,y,t) d\mu(x)}_{=K_1(t,y)} + \underbrace{\int_{st}^\infty g(x,y,t) d\mu(x)}_{=K_2(t,y)} \right)
\end{aligned}
\tag{4.2.20}
$$

Where M, s are the same as (4.2.1).

Proposition 4.2.2. *Under assumption* (4.0.2),

$$
\begin{aligned}
P_\mu(X_t \in dy, \tau > t) &\sim \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} K_1(t,y) \\
&\sim c e^{-\frac{(2\alpha\rho - \rho^2)t}{2}} e^{-\alpha y} \sinh((\alpha - \rho) y)
\end{aligned}
\tag{4.2.21}
$$

where c is the constant in (4.2.2) *which only depends on μ.*

Proof. Using similar estimation method and sequence of measures $(\nu_t^+, \nu_t^-, t \geq 0)$ as before, we can see that for each $y \in \mathbb{R}_+$

$$
\begin{aligned}
\limsup_{t\to\infty} K_1(t,y) &= \limsup_{t\to\infty} c\sqrt{2\pi t} \int_{M/t}^{s} e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) d\nu_t^+(x) \\
&= c\sqrt{2\pi t} e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}} e^{-\alpha y} \sinh((\alpha - \rho + \epsilon) y)
\end{aligned}
\tag{4.2.22}
$$

$$\begin{aligned}\liminf_{t\to\infty} K_1(t,y) &= \liminf_{t\to\infty} c\sqrt{2\pi t}\int_{M/t}^{s} e^{-\frac{y^2}{2t}}e^{-\alpha y}\sinh(xy)d\nu_t^-(x)\\ &= c\sqrt{2\pi t}e^{\frac{(\alpha-\rho-\epsilon)^2 t}{2}}e^{-\alpha y}\sinh((\alpha-\rho-\epsilon)y)\end{aligned} \tag{4.2.23}$$

and therefore

$$K_1(t,y) \sim c\sqrt{2\pi t}e^{\frac{(\alpha-\rho)^2 t}{2}}e^{-\alpha y}\sinh((\alpha-\rho)y) \tag{4.2.24}$$

For $K_2(t,y)$ we use the upper bound in (4.2.2) to get the following estimate.

$$\begin{aligned}K_2(t,y) &\le e^{-\frac{y^2}{2t}}e^{-\alpha y}\int_{st}^{\infty} e^{-\frac{x^2}{2t}}e^{(\alpha-\rho+\epsilon)x}e^{\frac{xy}{t}}dx\\ &= \sqrt{t}e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}}e^{(-\rho+\epsilon)y}L\left(\sqrt{t}s-\sqrt{t}(\alpha-\rho+\epsilon)-\frac{y}{\sqrt{t}}\right)\end{aligned} \tag{4.2.25}$$

Since $s-(\alpha-\rho+\epsilon)>0$ for small enough ϵ, the argument for L above is strictly positive and increasing. Therefore by (4.2.14),

$$(4.2.25) \le \sqrt{t}e^{\frac{(\alpha-\rho)^2 t}{2}}\exp\left(-\frac{(s-(\alpha-\rho))^2 t}{2}+(2(\alpha-\rho)-s)\epsilon t\right)e^{(s-\alpha+2\epsilon)y} \tag{4.2.26}$$

Again, $s-(\alpha-\rho)>0$ and ϵ is arbitrarily small so the middle term above is exponentially decaying. We conclude that

$$(4.2.26) = o\left(\sqrt{t}e^{\frac{(\alpha-\rho)^2 t}{2}}\right) = o(K_1(t,y)) \tag{4.2.27}$$

Finally for $K_3(t,y)$ we can directly apply the dominated convergence theorem.

$$K_3(t,y) \sim \int_0^M e^{\alpha x} e^{-\alpha y} \sinh(0) d\mu(x)$$
$$= o(1) = o(K_1(t,y)) \tag{4.2.28}$$

□

We can now prove Theorem 4.0.2.

Proof of Theorem 4.0.2. The fact that ϵ is arbitrarily small in (4.2.6) proves the first part of the theorem. Also proposition 4.2.1 and proposition 4.2.2 combined satisfies the hypothesis of lemma 3.2.1, therefore the QLD exists and concludes the second part of the theorem. □

Chapter 5

Infinite Exponential Moments

In this chapter, we consider cases in which μ has no finite exponential moment. In Section 5.1 we will prove Lemma 2.2.1 to see that adequate scaling is necessary to obtain a non-trivial quasi-limiting behavior. In Section 5.2, to determine the right scaling, we will estimate the tail distribution of the surviving process in Proposition 5.2.2. In Section 5.3 we will use it prove the following theorem, which is the backbone of Principle 2.2.3.

Theorem 5.0.1. *Suppose* $\mu([x,\infty)) = \exp(-F(x))$ *where* $F(x)$ *is strictly increasing smoothly varying function with index* $\beta < 0.5$. *Then*

$$\lim_{t\to\infty} P_\mu\left(X_t > \frac{c}{F'(\alpha t)} \;\middle|\; \tau > t\right) = e^{-c} \tag{5.0.1}$$

This theorem will then be subdivided into specific cases, to present concrete results and examples.

5.1 Non-existence of QLD for heavy-tailed distributions

We first show the following proposition, which extends Theorem 1.1.4 from QSD to its domain of attraction and that if μ is in the domain of attraction of a QSD then the survival rate $P_\mu(\tau > t)$ must be exponential.

Proposition 5.1.1. *Suppose μ is in the domain of attraction of a QSD π. Then $P_\mu(\tau > t) = O(c^t)$ for some $0 < c < 1$.*

Proof. By the Markov property,

$$\begin{aligned} P_\mu(\tau > s + t) &= P_\mu(\tau > t, P_{X_t}(\tau > s)) \\ &= P_\mu(P_{X_t}(\tau > s) \mid \tau > t) P_\mu(\tau > t) \end{aligned} \tag{5.1.1}$$

Write $f(x) = P_x(\tau > s)$. Since π is the QLD of μ, for arbitrary $\epsilon > 0$ there is some t_0 such that for each $t > t_0$,

$$\left| P_\mu(P_{X_t}(\tau > s) \mid \tau > t) - \mathbb{E}_\pi(f) \right| < \epsilon \tag{5.1.2}$$

π is a QSD so $\mathbb{E}_\pi(f) = P_\pi(\tau > s) < 1$, and therefore $P_\mu(P_{X_t}(\tau > s) \mid \tau > t) \leq c(s)$ for some constant $0 < c(s) < 1$.

Let $c = c(1)$. Inductively we have the following.

$$\begin{aligned} &P_\mu(\tau > t_0 + 1) \leq c P_\mu(\tau > t_0) \\ &P_\mu(\tau > t_0 + 2) \leq c P_\mu(\tau > t_0 + 1) \leq c^2 P_\mu(\tau > t_0) \\ &\qquad\qquad \vdots \\ &P_\mu(\tau > t_0 + n) \leq c^n P_\mu(\tau > t_0) \end{aligned} \tag{5.1.3}$$

And we have the desired asymptotic survival rate. $\square$

It is important to remark that the above proposition is true for any general QSD; Assumption 1.2.1 is not necessary. Also, while it is not necessary for our context, we suspect that $c = e^{-\lambda}$, where $\mathcal{L}^*\pi = -\lambda\pi$ as in Section 1.2.

We now prove Lemma 2.2.1 and that a scaling is necessary in order to obtain a non-trivial limit result.

Proof of Lemma 2.2.1. Pick $b > 0$ such that $\sinh(\alpha b) > \frac{1}{4}e^{\alpha b}$ then by Proposition 2.1.1 we have the following.

$$
\begin{aligned}
P_\mu(\tau > t) &\geq P_\mu(X_0 > \alpha t, X_t > b, \tau > t) \\
&= \frac{e^{-\frac{\alpha^2 t}{2}}t}{\sqrt{2\pi t}} \int_\alpha^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_b^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y}(e^{xy} - e^{-xy}) dy dx \\
&\geq \frac{e^{-\frac{\alpha^2 t}{2}}t}{4\sqrt{2\pi t}} \int_\alpha^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_b^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} e^{xy} dy dx \\
&= \frac{t}{4\sqrt{2\pi}} \int_\alpha^\infty \mu(tx) L\left(\frac{b}{\sqrt{t}} + \sqrt{t}(\alpha - x)\right) dx \\
&\geq \frac{1}{8}\mu([t\alpha, \infty))
\end{aligned}
$$

This implies that $P_\mu(\tau > t)$ is at least as heavy as the tail distribution of μ. By Proposition 5.1.1, any initial distribution μ that has heavier-than-exponential tail distribution cannot converge to a QSD. $\square$

5.2 Distribution of the surviving processes

The method we develop here works for a large class of distributions μ, yet both scaling and limit distributions may depend on the choice of μ.

Recall that we work under the Assumption 2.2.2. We can write the density of μ as follows.

$$\text{If } \beta > 0 \text{ then } \mu(x) = F'(x)\exp(-F(x)) = F'(x)\mu([x,\infty)) \tag{5.2.1}$$

Note that by [3, Proposition 1.8.1] , $F'(x)$ is smooth varying with index $\beta - 1$.

We turn to the tail distribution. By the above assumption, μ has a continuous density, which we also denote by μ.

$$\begin{aligned}
P_\mu(X_t > a_t, \tau > t) &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty e^{-\frac{x^2}{2t}} e^{\alpha x} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}} \right) dy d\mu(x) \right) \\
&= \frac{e^{-\frac{\alpha^2 t}{2}} t}{\sqrt{2\pi t}} \int_0^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha t x} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} (e^{xy} - e^{-xy}) dy dx \\
&= \frac{t}{\sqrt{2\pi}} \left(\underbrace{\int_0^\infty \mu(tx) L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x \right) dx}_{=J_1(t)} \right. \\
&\qquad \left. - \underbrace{\int_0^\infty \mu(tx) e^{2\alpha t x} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x \right) dx}_{=J_2(t)} \right)
\end{aligned} \tag{5.2.2}$$

We first notice that from the second term J_2,

$$\begin{aligned} e^{2\alpha tx} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x\right) dx &\leq e^{2\alpha tx} \frac{e^{-\frac{a_t^2/t + t\alpha^2 + tx^2 + 2a_t\alpha + 2a_t x + 2\alpha tx}{2}}}{a_t/\sqrt{t} + \sqrt{t}\alpha + \sqrt{t}x} \\ &= \frac{e^{-\frac{t(\alpha-x)^2}{2}} e^{-a_t^2/(2t)} e^{-a_t(\alpha+x)}}{a_t/\sqrt{t} + \sqrt{t}\alpha + \sqrt{t}x} \end{aligned} \tag{5.2.3}$$

If $a_t \gg \epsilon\sqrt{t}$ then the term $e^{-a_t^2/(2t)}$ will let J_2 decay faster (in exponential sense) than $\mu(tx)$. In fact, unless $a_t = o(\sqrt{t})$ and $x \in (\alpha - t^{-1/2+\epsilon}, \alpha + t^{-1/2+\epsilon})$, J_2 decays exponentially faster than $\mu(tx)$.

Furthermore, when we define $J_{1,A}(t), J_{2,A}(t)$ to be integrated over some sub-interval A of $\mathbb{R}_+$ instead of the entire $\mathbb{R}_+$ as follows:

$$\begin{aligned} J_{1,A}(t) &= \int_A \mu(tx) L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx \\ J_{2,A}(t) &= \int_A \mu(tx) e^{2\alpha tx} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x\right) dx \end{aligned} \tag{5.2.4}$$

since $P_\mu(X_0 \in \cdot, X_t \in \cdot, \tau > t) \geq 0$ always, we can claim that $J_{2,A} = O(J_{1,A})$ on the same sub-interval $A \in \mathbb{R}_+$.

For the first term J_1, we split the integration.

$$
\begin{aligned}
J_1(t) = & \underbrace{\int_0^{\alpha+a_t/t-\eta_t} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)dx}_{=J_{1,1}(t)} \\
& + \underbrace{\int_{\alpha+a_t/t-\eta_t}^{\alpha+a_t/t+\epsilon_t} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)dx}_{=J_{1,2}(t)} \\
& + \underbrace{\int_{\alpha+a_t/t+\epsilon_t}^{\infty} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)dx}_{=J_{1,3}(t)}
\end{aligned} \tag{5.2.5}
$$

where η_t, ϵ_t is to be picked depending on μ.

The goal now is to get an accurate asymptotic on the survival rate.

Proposition 5.2.1. *Suppose μ satisfies 2.2.2. Then for any $\eta_t \gg t^{\beta-1}$,*

$$
\log J_{1,1}(t) \ll \log \mu([t\alpha + a_t, \infty)) \tag{5.2.6}
$$

Proof. Suppose $\eta_t \gg t^{\frac{\beta-1}{2}}$. Then we have the following estimate.

$$
\begin{aligned}
J_{1,1}(t) &\leq L(\sqrt{t}\eta_t) \int_0^{\alpha+a_t/t-\eta_t} \mu(tx)dx \\
&\leq L(\sqrt{t}\eta_t) \\
&\leq \exp\left(-\frac{t(\eta_t)^2}{2}\right) \\
&\ll \exp\left(\frac{t^{-\beta}}{2}\right) \sim \mu([t\alpha + a_t, \infty))
\end{aligned} \tag{5.2.7}
$$

Now suppose $t^{\frac{\beta-1}{2}} \gg \eta_t \gg t^{\beta-1}$. Pick $t^{(\beta-1)/2} \ll \eta_t^1 = t^{r_1}$ such that by (5.2.7),

$$\underbrace{\int_0^{\alpha+a_t/t-\eta_t^1} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,1,1}(t)} \ll \mu([t\alpha + a_t, \infty))$$

Now we want to pick $t^{r_2} = \eta_t^2 \ll \eta_t^1$ such that

$$\underbrace{\int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,1,2}(t)} \ll \mu([t\alpha + a_t, \infty))$$

Using integration by parts,

$$\begin{aligned} J_{1,1,2}(t) &= \int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx \\ &= -\frac{1}{t}\mu([tx,\infty)) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)\Bigg|_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \\ &\quad + \frac{1}{\sqrt{t}} \int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \exp\left(\frac{-t(x-(\alpha+a_t/t))^2}{2}\right) \mu([tx,\infty)) dx \\ &\le \frac{1}{t}\left(-\mu(t\alpha + a_t - t\eta_t^2, \infty) L(\sqrt{t}\eta_t^2) + \mu(t\alpha + a_t - t\eta_t^1, \infty) L(\sqrt{t}\eta_t^1)\right) \\ &\quad + \frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty) L(\sqrt{t}\eta_t^2) \end{aligned} \tag{5.2.8}$$

Since both $\mu([x,\infty))$ and $L(x)$ are decreasing function, the driving term of (5.2.8) is the last one. And since $\mu(x,\infty) = \exp(-F(x))$ where F is an increasing regularly varying function with index β,

$$
\begin{aligned}
\frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^2) &\sim \frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^2) \\
&\leq \frac{1}{t}\exp\left(-F((t\alpha + a_t) - t^{1+r_1})\right)\exp\left(-\frac{t^{1+2r_2}}{2}\right) \\
&\sim \frac{1}{t}\exp\left(-F(t\alpha + a_t)\right)\exp\left(t^{\beta-1+1+r_1}\right)\exp\left(-\frac{t^{1+2r_2}}{2}\right) \\
&\sim \frac{1}{t}\mu(t\alpha + a_t, \infty)\exp\left(t^{\beta+r_1} - \frac{t^{1+2r_2}}{2}\right)
\end{aligned} \tag{5.2.9}
$$

If $\beta + r_1 < 1 + 2r_2$ we get the desired asymptotic. That is, we need $r_2 > \dfrac{(\beta-1)+r_1}{2}$, and combining with $t^{(\beta-1)/2} \ll \eta_t^1$ we can pick

$$
\eta_t^2 \gg t^{\frac{(\beta-1)+r_1}{2}} \sim t^{\frac{3(\beta-1)}{4}}
$$

to get

$$
\begin{aligned}
\int_0^{\alpha+a_t/t-\eta_t^2} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)dx &= J_{1,1,1}(t) + J_{1,1,2}(t) \\
&\ll \mu([t\alpha + a_t, \infty))
\end{aligned} \tag{5.2.10}
$$

Recursively, we can pick $\eta_t^n \gg t^{(\beta-1)(1-(1/2)^n)}$ such that

$$
J_{1,1,n}(t) = \int_{\alpha+a_t/t-\eta_t^{n-1}}^{\alpha+a_t/t-\eta_t^n} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right)dx \ll \mu([t\alpha + a_t, \infty))
$$

So for sufficiently large n we have

$$
\eta_t = \eta_t^n \gg t^{(\beta-1)(1-(1/2)^n)} \gg t^{\beta-1}
$$

$$J_{1,1}(t) = \sum_{i=1}^{n} J_{1,1,i}(t) \ll \mu([t\alpha + a_t, \infty))$$

which completes the proof. □

Proposition 5.2.2. *Suppose μ satisfies 2.2.2 and $\beta > 0$. If $a_t \gg \sqrt{t}$,*

$$P_\mu(X_t > a_t, \tau > t) \sim \frac{t}{\sqrt{2\pi}} J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty)) \tag{5.2.11}$$

Proof. Pick η_t and ϵ_t as follows.

$$t^{\beta-1} \ll \eta_t \ll 1, \qquad \epsilon_t = t^{-b}, \qquad \beta < b < 0.5 \tag{5.2.12}$$

This choice yields the following asymptotic.

$$\eta_t \to 0, \quad \eta_t \ll a_t/t, \quad \epsilon_t \ll a_t/t, \quad \sqrt{t}\epsilon_t \to \infty, \quad F'(t\alpha + a_t)\epsilon_t \ll 1/t \tag{5.2.13}$$

For $J_{1,2}(t)$, we first observe that the interval $(\alpha + a_t/t - \eta_t, \alpha + a_t/t + \epsilon_t)$ close in to $\alpha + a_t/t$. Moreover, while L does vary between 0 and $\sqrt{\pi/2}$ within the interval, μ does not vary much from $\mu(t(\alpha + a_t/t))$ inside the interval, and therefore we can use the intermediate value theorem. Also, we split the integration to get the following bound for $J_{1,2}(t)$.

$$
\begin{aligned}
J_{1,2}(t) &\sim \mu(t(\alpha + a_t/t)) \\
&\qquad \times \left(\int_{\alpha+a_t/t-\eta_t}^{\alpha+a_t/t} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx + \int_{\alpha+a_t/t}^{\alpha+a_t/t+\epsilon_t} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx \right) \\
&\le \mu(t\alpha + a_t) \left(\frac{1}{\sqrt{t}} \int_0^{\sqrt{t}\eta_t} L(y)dy + \int_0^{\epsilon_t} \sqrt{2\pi} dx \right) \\
&\le \mu(t\alpha + a_t) \left(\frac{1}{\sqrt{t}} \int_0^{\infty} L(y)dy + \sqrt{2\pi}\epsilon_t \right) \\
&\sim \sqrt{2\pi}\mu(t\alpha + a_t)\epsilon_t
\end{aligned}
$$

Note that the first integration is essentially the expected value of a half-normal distribution, and second integration is estimated using the fact that L is bounded above.

To estimate $J_{1,3}(t)$, since $\sqrt{t}\epsilon_t \to \infty$, it follows that $L(\sqrt{t}\epsilon_t) \to \sqrt{2\pi}$ and we can use IVT to get the sharp estimate.

$$
\begin{aligned}
J_{1,3}(t) &\sim \sqrt{2\pi} \int_{\alpha+a_t/t+\epsilon_t}^{\infty} \mu(tx)dx \\
&\sim \sqrt{2\pi}\frac{1}{t}\mu([t\alpha + a_t, \infty))
\end{aligned}
\tag{5.2.14}
$$

Proposition 5.2.1 shows that $J_{1,1}(t) = o(J_{1,3}(t))$.

For $J_{1,2}(t)$, we combine (5.2.1) and (5.2.13) to get the following asymptotic comparison.

$$\begin{aligned} J_{1,2}(t) &\leq \sqrt{2\pi}\mu(t\alpha + a_t)\epsilon_t \\ &\sim \sqrt{2\pi}F'(t\alpha + a_t)\epsilon_t\mu([t\alpha + a_t, \infty)) \\ &\ll \frac{\sqrt{2\pi}}{t}\mu([t\alpha + a_t, \infty)) \\ &\sim J_{1,3}(t) \end{aligned} \tag{5.2.15}$$

Finally from the choice of ϵ_t we have $b < 0.5$, and therefore

$$\begin{aligned} J_{2,3}(t) &\leq \int_{\alpha + a_t/t + \epsilon_t}^{\infty} \mu(tx)e^{-\frac{t(\alpha - x)^2}{2}} e^{-a_t^2/(2t)} e^{-a_t(\alpha + x)} dx \\ &\leq e^{-\frac{t\epsilon_t^2}{2}}\mu([t\alpha + a_t, \infty)) = o(J_{1,3}(t)) \end{aligned} \tag{5.2.16}$$

We conclude that

$$P_\mu(X_t > a_t, \tau > t) \sim (1 + o(1))\frac{t}{\sqrt{2\pi}}J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty)) \tag{5.2.17}$$

$\square$

We can extend this proposition to the cases where F is slowly varying. In such cases, we expect the tail distribution $\mu(x, \infty)$ itself to be smoothly varying.

Corollary 5.2.3. *Suppose $\mu([x, \infty)) = G(x)$, where G is smoothly varying function with index $-\kappa < 0$. Then $P_\mu(X_t > a_t, \tau > t) \sim \frac{t}{\sqrt{2\pi}}J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty))$.*

Proof. It suffices to show that $J_{1,2}(t) = o(J_{1,3}(t))$. The smooth varying condition yields the following relation [3, 1.8.1']

$$t\mu(t\alpha + a_t) \sim \mu([t\alpha + a_t, \infty)) \tag{5.2.18}$$

Since we have $\epsilon_t \ll 1$,

$$J_{1,2}(t) \leq \mu(t\alpha + a_t)\epsilon_t = o\left(\frac{1}{t}\mu([t\alpha + a_t, \infty))\right) = o(J_{1,3}(t)) \tag{5.2.19}$$

so we have the desired asymptotic. □

5.3 Quasi-limiting behavior of heavy-tailed distributions

Proposition 5.2.2 and Corollary 5.2.3 show why the second part of Assumption 2.2.2 is necessary. We need the right a_t that will yield nontrivial result on the limit

$$\lim_{t\to\infty} P_\mu(X_T > a_t \mid \tau > t) = \lim_{t\to\infty} \frac{P_\mu(X_t > a_t, \tau > t)}{P_\mu(\tau > t)} \tag{5.3.1}$$

Due to Proposition 5.2.2 this boils down to comparing $\mu(t\alpha, \infty)$ and $\mu(t\alpha + a_t, \infty)$.

Proof of Theorem 5.0.1. If μ satisfies assumption 2.2.2, setting $a_t = R(t, c)$ gives the following.

$$\begin{aligned} \mu([t\alpha + a_t, \infty)) &= \exp(-F(t\alpha + R(t, c)) \\ &\sim \exp(-(F(t\alpha) + c)) \\ &= e^c \mu(t\alpha, \infty) \end{aligned} \tag{5.3.2}$$

We make few comments on the observation (2.2.2). If smooth enough, F has the Taylor expansion

$$F(t\alpha + R(t, c)) = F(t\alpha) + F'(t\alpha)R(t, c) + o(F'(t))$$

therefore by choosing $R(t,c) = \dfrac{c}{F'(t\alpha)}$, we get $F(t\alpha + R(t,c)) - F(t\alpha) = c + o(F'(t))$. Since F has index $\beta < 1$, $F'(t) = o(1)$ so condition (2.2.1) is satisfied.

We further observe that with the choice $R(t,c) = \dfrac{c}{F'(t\alpha)}$,

$$\begin{aligned} F'(t\alpha + R(t,c)) &= F'(t\alpha) + F''(t\alpha)R(t,c) + o(F''(t)) \\ &= F'(t\alpha) + \frac{cF''(t\alpha)}{F'(t\alpha)} + o(F''(t)) \\ &= F'(t\alpha) + o(1) \end{aligned} \tag{5.3.3}$$

Therefore we get $F'(t\alpha + R(t,c)) \sim F'(t\alpha)$, and consequently,

$$\begin{aligned} \mu(t\alpha + R(t,c)) &= F'(t\alpha + R(t,c)) \exp(-F(t\alpha + R(t,c)) \\ &\sim F'(t\alpha) \exp(-(F(t\alpha) + c)) \\ &= e^{-c}\mu(t\alpha) \end{aligned} \tag{5.3.4}$$

Putting together Proposition 5.2.2, Corollary 5.2.3, (5.3.2), and (5.3.4) completes the proof. □

We present some concrete results here.

Corollary 5.3.1. *Suppose $\mu([x,\infty)) = e^{-x^\beta}$ with $\beta \in (0, 0.5)$. Then*

$$\lim_{t\to\infty} P_\mu \left(\frac{X_t}{t^{1-\beta}} > c \,\middle|\, \tau > t \right) = \exp(-\beta\alpha^{\beta-1}c) \tag{5.3.5}$$

that is, the limiting distribution is exponential with parameter $\beta\alpha^{\beta-1}$.

Proof. From proposition 5.2.2 we get

$$P_\mu(X_t > a_t, \tau > t) \sim \mu([t\alpha + a_t, \infty)) \tag{5.3.6}$$

Pick $a_t = c \cdot t^{1-\beta}$. Then by the generalized binomial theorem,

$$(t\alpha + a_t)^\beta = (t\alpha)^\beta + c\beta\alpha^{\beta-1} + o(1)$$

Note that $F'(t\alpha) = \beta(t\alpha)^{\beta-1}$. By substituting $\overline{c} = ct^{1-\beta}((\alpha t)^\beta)' = c\beta\alpha^{\beta-1}$, Theorem 5.0.1 gives us the desired result. □

Example 5.3.2. If μ is a Weibull distribution with scale parameter $\lambda > 0$ and shape parameter $0 < \beta < 0.5$, the limiting distribution of $P_\mu\left(\frac{X_t}{t^{1-\beta}} > c \,\middle|\, \tau > t\right)$ is exponential distribution with rate $\beta\left(\frac{\alpha}{\lambda}\right)^{\beta-1}$.

Corollary 5.3.3. *Suppose $\mu([x,\infty)) = G(x)$, where G is smoothly varying function with index $-\kappa < 0$. Then*

$$\lim_{t\to\infty} P_\mu\left(\frac{X_t}{t} > c \,\middle|\, \tau > t\right) = \left(\frac{\alpha + c}{\alpha}\right)^{-\kappa} \tag{5.3.7}$$

that is, the limiting distribution is Lomax (shifted Pareto) distribution with shape parameter κ and scale parameter α.

Proof. Since $G(x) = \exp(\log(G(x)))$ and $\log(G(x))$ is a slowly varying function ($\beta = 0$), the natural choice for $R(t,c)$ would be $a_t = R(t,c) = tc$. Indeed, by the uniform convergence theorem of regular varying function, [3, Theorem 1.5.2]

$$\lim_{t\to\infty} \frac{G(t\alpha + tc)}{G(t)} = (\alpha + c)^{-\kappa} \tag{5.3.8}$$

Therefore we have

$$\frac{P_\mu(X_t > tc, \tau > t)}{P_\mu(\tau > t)} \sim \frac{(\alpha + c)^{-\kappa} G(t)}{\alpha^{-\kappa} G(t)} \tag{5.3.9}$$

which gives us the desired result. $\square$

Note that when $\beta = 0$, μ is a distribution with regular or slowly varying tail. In such cases it is often more convenient to work with the asymptotic result $P_\mu(X_t > \overline{R}(t,c), \tau > t) \sim \mu([t\alpha + \overline{R}(t,c), \infty))$ directly to find the right scaling factor $\overline{R}$. We conclude this section with showing the quasi-limiting behavior of μ which itself has slowly varying tail.

Example 5.3.4. If μ is a Half-Cauchy distribution (Cauchy distribution supported on $\mathbb{R}^+$), the limiting distribution of $P_\mu\left(\frac{X_t}{t} > c \;\middle|\; \tau > t\right)$ is Lomax distribution with shape parameter 1 and scale parameter α.

Corollary 5.3.5. *Suppose* $\mu([x,\infty)) \sim \frac{1}{\ln x}$ *as* $x \to \infty$. *Then*

$$\lim_{t\to\infty} P_\mu\left(\frac{\ln X_t}{\ln t} > c \mid \tau > t\right) = \begin{cases} 1 & c \leq 1; \\ \frac{1}{c} & c > 1. \end{cases}$$

that is, the limiting distribution is Pareto distribution with shape parameter 1 *and scale parameter* 1.

Proof. $\mu([x,\infty)) \sim \exp(-\ln\ln x)$ so we can apply Corollary 5.2.3. Since we have $\overline{R}(t,c) = t^c$,

$$\begin{aligned} \frac{P_\mu(X_t > t^c, \tau > t)}{P_\mu(\tau > t)} &\sim \frac{\ln(t\alpha)}{\ln(t\alpha + t^c)} \\ &\sim \begin{cases} \frac{\ln t + \ln\alpha}{\ln t + \ln\alpha} \to 1 & c < 1 \\ \frac{\ln t + \ln\alpha}{\ln t + \ln(\alpha+1)} \to 1 & c = 1 \\ \frac{\ln t + \ln\alpha}{c\ln t} \to \frac{1}{c} & c > 1 \end{cases} \end{aligned} \tag{5.3.10}$$

which gives us the desired result. □

Notice that in our last example with super-heavy tail initial distribution, the scaled limiting distribution does not depend on the drift parameter α of the BM.

Chapter 6

Additional Topics

6.1 Disjoint Combination of Sub-critical Exponential Tails : Alternating Behavior

In this section we present a simple construction of measure μ that satisfies [8, Theorem 1.4].

The general setting here is that we have a strictly increasing sequence of real numbers (a_k) with $a_0 = 0$. From this sequence we get sequence of intervals $(A_k : A_k = [a_k, a_{k+1}))$. We define the density of μ, which we denote by $\mu(x)$ as usual, as follows.

$$\mu(x) = C \sum_k 1_{A_k} \mu_k(x) \tag{6.1.1}$$

C is normalizing constant. $\mu_k(x)$ are nonnegative integrable functions. We will assume the following on each μ_k.

Assumption 6.1.1. Each of μ_k is defined as follows.

$$\mu_k(x) = \begin{cases} 1_{A_k}\mu_{\rho_1}(x) & k \text{ odd} \\ 1_{A_k}\mu_{\rho_2}(x) & k \text{ even} \end{cases} \tag{6.1.2}$$

where $\mu_{\rho_1}, \mu_{\rho_2}$ both satisfies Assumption 4.0.2 with respective ρ_1, ρ_2.

Roughly speaking, the density function of μ is alternating between two different exponential tail with parameters ρ_1 and ρ_2, both less than α. Under this assumption we will rewrite (6.1.1) as follows.

$$(6.1.1) = C\sum_k 1_{A_k} e^{-\rho_k x} \tag{6.1.3}$$

Note that $\rho_k = \rho_1$ if k odd, $\rho_k = \rho_2$ if k even.

We use (6.1.3) to analyze (3.1.4).

$$\begin{aligned} (3.1.4) &= C\frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}}\sum_k\left(\int_{a_k}^{a_{k+1}} e^{-\rho_k x - \frac{x^2}{2t} + \alpha x}\int_0^\infty e^{-\frac{y^2}{2t} - \alpha y}\left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}}\right) dydx\right) \\ &= C\frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} t\sum_k\left(\int_{a_k/t}^{a_{k+1}/t} e^{-\frac{tx(x-2\gamma_k)}{2}}\int_0^\infty e^{-\frac{y^2}{2t} - \alpha y}\sinh(xy)dydx\right) \end{aligned} \tag{6.1.4}$$

where $\gamma_k = \alpha - \rho_k$.

We observe few things here.

1. Like the case we saw in section 4.2, the sequence of measures ν_{t,ρ_1} that correspond to μ_{ρ_1} and ν_{t,ρ_2} that correspond to μ_{ρ_2} both weakly converge to $\delta_{\gamma_1}, \delta_{\gamma_2}$ respectively. However, as $t \to \infty$ each interval $[a_k/t, a_{k+1}/t)$ also shrinks and approaches 0.

2. The idea behind assumption 6.1.1 is that we only need to consider the base distribution μ_{ρ_k} that appear infinitely many times in the construction of μ, since those are the only part of μ that will survive in the long term. So without loss of generality, we may assume $\{\rho_k\}$ is cyclic, and even further assume that it is an alternating sequence of ρ_1, ρ_2 where $\rho_1 > \rho_2$.

Define constants A, B, θ as follows.

$$0 < A < \gamma_1 < \gamma_2 < B, \qquad \theta = \frac{B}{A} > 1 \tag{6.1.5}$$

From the first observation (convergence of measure) above, there is some $t_K = \theta^K$ such that for arbitrary $\epsilon > 0$, $\gamma_k \in \{\gamma_1, \gamma_2\}$ and each $t \geq t_K$,

$$1 - \int_A^B \sqrt{\frac{t}{2\pi}} e^{-\frac{t(x-\gamma_k)^2}{2}} dx < \epsilon \tag{6.1.6}$$

Construct the sequences (a_k), (t_n) as follows.

$$a_0 = 0, \quad a_1 = A\theta^{-N+1}, \quad a_{k+1} = \theta a_k, \quad t_n = \theta^n \tag{6.1.7}$$

Proposition 6.1.2. *Under assumption 6.1.1, constructions* (6.1.5) *and* (6.1.7),

$$
\begin{aligned}
P_\mu(\tau > t_{n_1}) &\sim Ce^{-\frac{(\alpha^2-\gamma_1^2)t_{n_1}}{2}} \left(\frac{1}{\rho_1} - \frac{1}{2\alpha-\rho_1}\right) \\
P_\mu(\tau > t_{n_2}) &\sim Ce^{-\frac{(\alpha^2-\gamma_2^2)t_{n_2}}{2}} \left(\frac{1}{\rho_2} - \frac{1}{2\alpha-\rho_2}\right)
\end{aligned} \tag{6.1.8}
$$

where (n_1) *is the sequence of odd integer and* (n_2) *is the sequence of even integer.*

Proof. We first mention that the technical details of this proof follows the proof of proposition 4.2.1.

First assume that $\alpha - \rho_2/2 < B$. In this case, we can directly apply proposition 4.2.1 since the interval (A, B) fully contains the critical interval of $J_1(t)$.

Now assume $\gamma_2 < B < \alpha - \rho_2/2$. In this case, we can reset $J_1(t)$ to be the integration (with respect to x) over (M, Bt) and add a new term $J_{1.5}(t)$ which represents the interval $(Bt, (\alpha - \rho_2/4)t)$, that is,

$$
\begin{aligned}
J_{1.5}(t) &= \int_{Bt}^{(\alpha-\rho_2/4)t} \mu(x)e^{-\frac{x^2}{2t}}e^{\alpha x}h\left(t, \frac{x}{t}\right)dx \\
&= t\int_{B}^{\alpha-\rho_2/4} \mu(tx)e^{-\frac{tx^2}{2}}e^{\alpha tx}h(t,x)dx
\end{aligned} \tag{6.1.9}
$$

Since both γ_1, γ_2 are both in (A, B), we still have same asymptotic for $J_1(t)$.

On the other hand, since $h(t, x) < c$ for some constant c in $(B, (\alpha - \rho_2/4))$, we have

$$\begin{aligned}
J_{1.5}(t) &\leq c\int_B^{\alpha-\rho_k/4} d\nu_{t,\rho_k} \\
&\leq c\sqrt{t}e^{\frac{(\alpha-\rho_k)^2 t}{2}}\int_{(B-\gamma_k)\sqrt{t}}^{\infty} e^{-\frac{z^2}{2}}dz \\
&\leq c\sqrt{t}e^{\frac{(\alpha-\rho_k)^2 t}{2}}e^{-\frac{(B-\gamma_2)^2 t}{2}} \\
&= o\left(\sqrt{t}e^{\frac{(\alpha-\rho_k)^2 t}{2}}\right) = o(J_1(t))
\end{aligned} \tag{6.1.10}$$

So in both cases, we observe that $P_\mu(\tau > t_{n_k})$ has same asymptotic as the uniform tail case of μ_{ρ_k}. And since the choice of ρ_k on each sub-intervals $[a_k, a_{k+1})$ are arbitrary, different subsequence converge to different $P_\mu(\tau > t_{n_k})$. $\square$

Now we study what condition is required on (a_k) to find a corresponding sequence (t_n) that will make proposition 6.1.2 work. We need two conditions here:

$$\begin{aligned}
\lim_{k\to\infty}\frac{a_k}{t_k} &= A > 0 \\
\lim_{k\to\infty}\frac{a_{k+1}}{t_k} &= B > A
\end{aligned} \tag{6.1.11}$$

For some monotone increasing t_k. One can easily see it must be the case that $a_k = \Theta(c^k)$ for some $c > 1$ to satisfy (6.1.11). It is also worth noting that if $a_k = \Theta(k^n)$ for some n then (6.1.11) cannot hold.

We conclude this section with the following theorem.

Theorem 6.1.3. *Suppose μ satisfies assumption 6.1.1, and the following two conditions.*

1. $a_k = \Theta(c^k), \quad c > 1$

2. $c\min(\gamma_l) > \max(\gamma_l), \quad l \in \{1,2\}$

Then there is some sequence t_k that on the different subsequences t_{k_l}, the conditional distribution converge to π_{γ_l} respectively.

Proof. The additional conditions are equivalent to the existence of constants A, B such that (6.1.5) is satisfied with $\theta = c$.

Construct (t_k) as follows.

$$t_1 = \frac{a_K}{A}, \quad t_{k+1} = ct_k \tag{6.1.12}$$

where (6.1.6) would hold for each $t > a_K$.

The satisfies the hypothesis of proposition 6.1.2, and therefore theorem 4.0.2 can be applied to different subsequence t_{k_l} that will converge to different π_{γ_l}. □

6.2 Completing Infinite Exponential Moments

By Lemma 2.2.1, if the initial distribution μ of X_0 does not have any exponential moment, it is not in the domain of attraction of any QSD.

So it is natural to try to extend the first part of Assumption 2.2.2 to $\beta < 1$. That would yield a complete description between the initial distribution described in terms of regular varying functions, and the quasi-limiting behavior of those, as the case $\beta \geq 1$ corresponds to the results of the domain of attraction of QSDs.

From Principle 2.2.3, we see a clear relation between the index φ of the scaling factor $R(x,c)$ and the index β of the initial distribution exponent $F(x)$, which is that $\varphi = 1-\beta$ when $\beta \neq 0$. So the intuitive expectation is that the relation $\varphi = 1-\beta$ should extend to any $\beta < 1$. The fact that we get QSDs when $\beta = 1$ (which means that the scaling factor is a constant, so $\varphi = 0$) also supports this expectation.

Conjecture 6.2.1. Suppose μ satisfies Assumption 2.2.2 but $0.5 < \beta < 1$. Then from (5.2.5), we get the following estimate.

$$\begin{aligned}
&J_{1,1}(t) \ll J_{1,2}(t), J_{1,3}(t) \\
&J_{1,2}(t) \sim \frac{\mu(t\alpha + a_t)}{\sqrt{t}} \\
&J_{1,3}(t) \leq \sqrt{2\pi}\frac{\mu([t\alpha + a_t, \infty))}{t}
\end{aligned}$$

As a result, $P_\mu(X_t > a_t, \tau > t) \sim \sqrt{\dfrac{t}{2\pi}}\mu(t\alpha + a_t)$.

The difficulty is that when we attempt to estimate $J_1(t)$ in (5.2.5), we cannot pick η_t to both satisfy $J_{1,1}(t) \ll J_{1,2}(t)$ and $\eta_t \ll a_t/t$, keeping us from using IVT to estimate $J_{1,2}(t)$. We believe that $J_{1,2}(t) \sim t^{-1/2}\mu(t\alpha + a_t)$, and $\beta > 1/2$ would result in $J_{1,2}(t) \gg J_{1,3}(t)$. If the estimation holds true then $\beta = 1/2$ would be an interesting case, as it would serve as the critical point which $J_{1,2}(t) \sim J_{1,3}(t)$.

6.3 General Disjoint Combination of Tails

In section 6.1 we showed a specific construction of initial distribution μ that is compound (that is, μ is the form of (6.1.1)). The natural question to ask would be the limiting behavior for general situations. Mainly, we have three cases to study:

1. What happens if the sequence (a_k) does not grow exponentially?

2. What happens if μ is disjoint combination of μ_1, μ_2 where μ_1 or μ_2 does not necessarily satisfy assumption 4.0.2? That is, what if μ is a disjoint combination of "lighter" or "heavier" tails?

3. Theorem 6.1.3 provides a discrete sequence of (t_k) such that the QLD does not converge. What is the continuous time behavior?

We provide some crude observation on the latter two questions.

For the second question, suppose μ is defined on $\mathbb{R}_+$ such that μ is exponential (with parameter $\rho < \alpha$) on even A_k and subexponential/regular varying (with density f) on odd A_k. That is,

$$\mu(x) = C\left(\sum_{\text{even}} 1_{A_k} e^{-\rho x} + \sum_{\text{odd}} 1_{A_k} f(x)\right)$$

Then we can write (3.1.4) as follows.

$$\begin{aligned}(3.1.4) = &\underbrace{C\sum_{\text{even}} \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}}\left(\int_{A_k} e^{-\rho x} e^{\alpha x}\int_0^\infty e^{-\alpha y}\left(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}\right)dydx\right)}_{J_1(t)} \\ &+ \underbrace{C\sum_{\text{odd}} \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}}\left(\int_{A_k} f(x) e^{\alpha x}\int_0^\infty e^{-\alpha y}\left(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}\right)dydx\right)}_{J_2(t)}\end{aligned} \tag{6.3.1}$$

From the previous computation, we have

$$J_1(t) = O\left(e^{\frac{(-\alpha^2 + (\alpha-\gamma)^2)t}{2}}\right)$$

On the other hand for $J_2(t)$,

$$J_2(t) \sim \frac{1}{t} \sum_{a_k > \alpha t, k \text{ odd}} \mu(A_k)$$

Therefore $J_1(t) = o(J_2(t))$. This observation tells that when we have a distribution where the tail is a compound combination of exponential tail and any tail heavier than the exponential tail, the exponential tail is eventually negligible and the heavier portion will always dominate. So the QSD would not exist, and we expect the quasi-limiting behavior of the dominating (heavy) tail.

For the third question, recall the settings we used in proposition 6.1.2. The general idea is that we have two different sequence of measures $\nu_{t,\rho_1} \rightharpoonup \delta_{\gamma_1}$ and $\nu_{t,\rho_2} \rightharpoonup \delta_{\gamma_2}$. So we can pick a discrete time sequence (t_k) such that at each time frame, only one of γ_1, γ_2 will hit the critical interval (A, B). However if we focus on the continuous spectrum of time, there will be times when both or neither γ_1, γ_2 hit the critical interval simultaneously.

Our guess is that when both hit the critical interval, we expect the following asymptotic. (we assume that $\rho_1 > \rho_2$)

$$\begin{aligned} P_\mu(\tau > t) &\sim Ce^{-\frac{\alpha^2 t}{2}} \left(e^{\frac{(\alpha-\rho_1)^2 t}{2}} \left(\frac{1}{\rho_1} - \frac{1}{2\alpha - \rho_1} \right) + e^{\frac{(\alpha-\rho_2)^2 t}{2}} \left(\frac{1}{\rho_2} - \frac{1}{2\alpha - \rho_2} \right) \right) \\ &\sim Ce^{-\frac{\alpha^2 t}{2}} e^{\frac{(\alpha-\rho_2)^2 t}{2}} \left(\frac{1}{\rho_2} - \frac{1}{2\alpha - \rho_2} \right) \end{aligned} \tag{6.3.2}$$

This is strictly following the procedure of proposition 4.2.1 and comparing $J_{1,\rho_1}(t), J_{1,\rho_2}(t)$ which corresponds to both γ_1, γ_2 hitting the critical interval.

Now suppose at some time t_0 both γ_1, γ_2 do NOT hit the critical interval. Recall that ν_{t,ρ_k} are essentially $C_t N(\gamma_k, 1/t)$, that is, the Normal distribution with fixed mean and shrinking variance with normalizing factor. So the overall picture at t_0 is a disjoint combination of Normal curve with both peaks missing, that is we only have the right and left tails of the Normal distribution. However, since the overall measure must be scaled up to be a probability measure, we need to determine which of the partial tails is the most significant one. What we believe is that at a fixed time, the tail defined closest to its peak will dominate all others.

Lemma 6.3.1. *Suppose we have the same setting as proposition 6.1.2, and moreover we have a sequence* (t_n) *where we have* ν_t *to be a combination of* $N\left(\gamma_1, \frac{1}{t}\right)$ *and* $N\left(\gamma_2, \frac{1}{t}\right)$ *where both* γ_1, γ_2 *misses the critical interval* $(A, B) = \left(\frac{a_k}{t_n}, \frac{a_{k+1}}{t_n}\right)$*. Then* ν_t *converges to* δ_γ *where* $\gamma = \frac{a_m}{t_n}$ *for some* m *such that either* $|\gamma - \gamma_1|$ *or* $|\gamma - \gamma_2|$ *is minimized.*

Proof. Let a be the minimal distance between the desired γ and its closest peak. Then for any $b > a$,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\sqrt{t/2\pi}\int_a^\infty e^{-tx^2/2}dx}{\sqrt{t/2\pi}\int_b^\infty e^{-tx^2/2}dx} &= \lim_{t\to\infty} \frac{\int_{a\sqrt{t}}^\infty e^{-x^2/2}dx}{\int_{b\sqrt{t}}^\infty e^{-x^2/2}dx} \\
&= \lim_{t\to\infty} \frac{-e^{-a^2t/2}}{-e^{-b^2t/2}} \\
&= \lim_{t\to\infty} e^{-(a^2-b^2)t/2} = \infty
\end{aligned} \tag{6.3.3}$$

∎

What we have shown here is that ν_t may converge to a spectrum of δ_γ depending on the choice of sequence of (t_n). Still, it remains to be shown that if such partial convergence is subject to the Principle 2.1.2. We also note that the above lemma could yield convergence to δ_γ where $\gamma > \alpha$, which we believe would require separate justification that we have additional condition $\gamma < \alpha$.

6.4 Rate of Convergence to QSD

When X_0 follows the initial distribution μ that is in the domain of attraction of a QSD π, we want to know what is the convergence rate. As previously mentioned, the rate of convergence to a QSD in continuous state space models is largely unknown and is open to future research. Since the convergence to a QSD is convergence in distribution, it makes sense to compare the total variation distance between an initial distribution and its QSD.

Definition 6.4.1. The total variation distance between two probability measures μ and π on a sigma-algebra $\mathcal{F}$ of the sample space Ω is

$$d(\mu, \pi) = \sup_{A \in \mathcal{F}} |\mu(A) - \pi(A)|$$

Since any QSD π has a smooth density and we may assume μ also have a smooth density (because we are only interested in convergence rate and $(X_t \mid \tau > t)$ have smooth density for any $t > 0$) we can rewrite the above definition as follows.

$$d(\mu, \pi) = \frac{1}{2} \int_{\mathbb{R}_+} |\mu - \pi| dx \tag{6.4.1}$$

Both μ and π in right hand side above are its density function.

While our convergence is not in probability, we will also present here the common definition of convergence rate in probability.

Definition 6.4.2. Let $(X_t)_{t\in\mathbb{R}_+}$ be a sequence of $\mathbb{R}_+$-valued random variables that converges to X in probability. Let $(r_t)_{t\in\mathbb{R}_+} \subset (0,\infty)$ a sequence with $r_t \to 0$. We say the sequence $(X_t)_{t\in\mathbb{R}_+}$ converges in probability with rate $(r_t)_{t\in\mathbb{R}_+}$, often denoted as $|X_t - X| \in O_p(r_t)$, if and only if for all $\epsilon > 0$ there is $K_\epsilon > 0$ such that $P(r_t^{-1}|X_t - X| > K_\epsilon) < \epsilon$ for all t.

Since we have explicit solution to QSDs one might try marginalizing X to find r_t. Computing the other part of the formula would be more difficult as it involves Proposition 2.1.1, but the asymptotic methods we used throughout Chapter 4 could play out in this subject as well.

6.5 QSDs of Ornstein-Uhlenbeck processes

Lladser and San Martin[7] showed the family of QSDs and their domain of attraction for regular Ornstein-Uhlenbeck(O-U) process X_t defined by the following.

$$dX_t = dB_t - \alpha X_t dt, \quad \alpha > 0 \tag{6.5.1}$$

We may ask two questions in this model:

- Similar to our results in Chapter 5, is there an initial distribution μ which the tail distribution is too heavy to be in any domain of attraction? If there is, what is the necessary scaling for such distribution?

- Unlike our model that is BM with constant drift, the fact that 0 is the absorbing state is relevant in O-U process. What happens if the absorbing state is $N \neq 0$ instead?

We present here the transition density $P_x(X_t \in dy, \tau > t)$ for both cases, which we expect to serve similar purpose as Proposition 2.1.1.

Proposition 6.5.1. *Suppose X_t is a O-U process defined as* (6.5.1) *with* 0 *being the absorbing state. Then we have the following.*

$$\begin{aligned} &P_x(X_t \in dy, \tau > t) \\ &= \sqrt{\frac{4\alpha}{\pi(1-e^{-2\alpha t})}} \exp\left(-\frac{\alpha(e^{-\alpha t}x)^2}{1-e^{-2\alpha t}} - \frac{\alpha y^2}{1-e^{-2\alpha t}}\right) \sinh\left(\frac{2\alpha e^{-\alpha t}xy}{1-e^{-2\alpha t}}\right) \end{aligned} \tag{6.5.2}$$

Next we propose the case when the absorbing state is $-N$ for some $N > 0$. We comment that the case for absorbing state $+N$ yield similar result.

Proposition 6.5.2. *Suppose X_t is a O-U process defined as* (6.5.1) *with $-N$ ($N > 0$) being the absorbing state. Then for each $y > -N$, we have the following.*

$$\begin{aligned} &P_x(X_t \in dy, \tau > t) \\ &= \sqrt{\frac{4\alpha}{\pi(1-e^{-2\alpha t})}} \exp\left(-\frac{\alpha(e^{-\alpha t}x)^2}{1-e^{-2\alpha t}} - \frac{\alpha y^2}{1-e^{-2\alpha t}}\right) \\ &\times \left(\exp\left(\frac{2\alpha e^{-\alpha t}xy}{1-e^{-2\alpha t}}\right) - \exp\left(\frac{-2\alpha e^{-\alpha t}xy - 4(Ne^{-\alpha t}x + Ny + N^2)}{1-e^{-2\alpha t}}\right)\right) \end{aligned} \tag{6.5.3}$$

From the above proposition we can define ν_t as follows and study the limiting behavior.

$$d\nu_t(x) = \exp\left(-\frac{\alpha(e^{-\alpha t}x)^2}{1-e^{-2\alpha t}}\right) d\mu(x)$$

Notice that from the above definition alone $\nu_t \to \mu$ as $t \to \infty$.

Bibliography

[1] M. S. Bartlett, *On theoretical models for competitive and predatory biological systems*, Biometrika **44** (1957), 27–42. MR 86727

[2] ———, *Stochastic population models in ecology and epidemiology*, Methuen's Monographs on Applied Probability and Statistics, Methuen& Co., Ltd., London; John Wiley& Sons, Inc., New York, 1960. MR 0118550

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Cambridge Univ. Press, 2001.

[4] Mioara Buiculescu, *On quasi-stationary distributions for multi-type Galton-Watson processes*, J. Appl. Probability **12** (1975), 60–68. MR 365734

[5] Pierre Collet, Servet Martínez, and Jaime San Martín, *Quasi-stationary distributions: Markov chains, diffusions and dynamical systems*, Springer Science & Business Media, 2012.

[6] Pablo A. Ferrari and Nevena Marić, *Quasi stationary distributions and Fleming-Viot processes in countable spaces*, Electron. J. Probab. **12** (2007), no. 24, 684–702. MR 2318407

[7] Manuel Lladser and Jaime San Martín, *Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process*, J. Appl. Probab. **37** (2000), no. 2, 511–520. MR 1781008

[8] Servet Martinez, Pierre Picco, and Jaime San Martin, *Domain of attraction of quasi-stationary distributions for the Brownian motion with drift*, Adv. in Appl. Probab. **30** (1998), no. 2, 385–408. MR 1642845

[9] Servet Martínez and Jaime San Martín, *Quasi-stationary distributions for a Brownian motion with drift and associated limit laws*, J. Appl. Probab. **31** (1994), no. 4, 911–920. MR 1303922

[10] Servet Martínez, Jaime San Martín, and Denis Villemonais, *Existence and uniqueness of a quasistationary distribution for Markov processes with fast return from infinity*, J. Appl. Probab. **51** (2014), no. 3, 756–768. MR 3256225

[11] Henry Scheffé, *A useful convergence theorem for probability distributions*, Ann. Math. Statistics **18** (1947), 434–438. MR 21585

[12] Erik A. van Doorn, *Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes*, Adv. in Appl. Probab. **23** (1991), no. 4, 683–700. MR 1133722

[13] Denis Villemonais, *Distributions quasi-stationnaires et méthodes particulaires pour l'approximation de processus conditionnés*, Ph.D. thesis, Ecole Polytechnique X, 2011.

[14] Sewall Wright, *Evolution in Mendelian populations*, Genetics **16** (1931), no. 2, 97.

[15] A. M. Yaglom, *Certain limit theorems of the theory of branching random processes*, Doklady Akad. Nauk SSSR (N.S.) **56** (1947), 795–798. MR 0022045

[16] Jun Ye, *Quasi-stationary distributions for the radial Ornstein-Uhlenbeck processes*, Acta Math. Sci. Ser. B (Engl. Ed.) **28** (2008), no. 3, 513–522. MR 2433799