Politecnico di Milano

Mathematical Engineering

Real and Functional Analysis

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Anno Accademico:

2025-2026

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February 23, 2026

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Measure Theory

σ-Algebra and Measurable Space

Let X be a set.

Definition. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be a σ**-algebra** if

1. $\emptyset \in \mathcal{A}$,
2. $E \in \mathcal{A} \Rightarrow E^C \in \mathcal{A}$,
3. $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a **measurable space** when $\mathcal{A}$ is a σ-algebra. The elements of $\mathcal{A}$ are called **measurable sets**.

Theorem. Let $S \subseteq \mathcal{P}(X)$; then there exists a σ-algebra $\sigma_0(S)$ such that

1. $S \subseteq \sigma_0(S)$,
2. For every σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ with $\mathcal{A} \supseteq S$, we have $\mathcal{A} \supseteq \sigma_0(S)$.

This $\sigma_0(S)$ is called the σ**-algebra generated by** S, and it is the minimal such σ-algebra.

Remark. A σ-algebra is a collection of subsets closed under complementation and countable unions. The generated σ-algebra $\sigma_0(S)$ is the smallest σ-algebra containing S. This is a fundamental concept because it allows us to define the smallest family of sets where a measure can be defined.

Borel Sets

Definition. Let (X, d) be a metric space. Let $\mathcal{G}$ (called **topology**) be a family of open sets in X. The σ-algebra $\sigma_0(\mathcal{G})$ is the **Borel** σ**-algebra**, denoted by $\mathcal{B}(X)$. The elements of $\sigma_0(\mathcal{G})$ are called **Borel sets**.

Remark. The following sets are Borel: open sets, closed sets, countable intersections of open sets, countable unions of closed sets.

Example. We know that $\mathcal{B}(\mathbb{R}) = \sigma_0(\mathcal{G})$, but we can use other families instead of open sets: $\mathcal{B}(\mathbb{R}) = \sigma_0(\mathcal{G}) = \sigma_0(I)$, where $I = \{(a,b) : a,b \in \mathbb{R}, a \leq b\}$ or $I = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ or $I = \{(a,b] : a,b \in \mathbb{R}, a \leq b\}$.
Also, $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K)$, where K is the family of N-dimensional closed or open rectangles.
Defining $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we have $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(I)$, where $I = \{(a, +\infty] : a \in \mathbb{R}\}$.

Remark. In metric spaces, the Borel σ-algebra is generated by open sets. It includes many familiar sets like open, closed, and sets constructed from them via countable operations. This is the natural domain for measures in topological spaces.

Measure

Let X be a set, and let $\mathcal{A}$ be a σ-algebra.

Definition. A function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a **measure** if

1. $\mu(\emptyset) = 0$,
2. For every disjoint family $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}$, we have $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$. This is called **$\sigma$-additivity**.

Definition. A measure μ is **finite** if $\mu(X) < \infty$. A measure μ is **σ-finite** if there exists $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}$ such that $X = \bigcup_{k=1}^{\infty} E_k$ and $\mu(E_k) < \infty$ for every $k \in \mathbb{N}$.

Definition. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a measure. $(X, \mathcal{A}, \mu)$ is called a **measure space**. If $\mu(X) = 1$, then $(X, \mathcal{A}, \mu)$ is a **probability space** and μ is a **probability measure**.

Example. The **counting measure**: $\mu : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ defined by

$$\mu(E) := \begin{cases} |E| & \text{if } E \subseteq X \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

where $|E|$ is the cardinality. It is finite if and only if X is finite, and σ-finite if and only if X is countable.
The **Dirac measure** δ_{x_0}, concentrated at x_0: Take $X \neq \emptyset$, $x_0 \in X$, define $\delta_{x_0} : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ by

$$\delta_{x_0}(E) := \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{otherwise} \end{cases}$$

Remark. Measures assign a ”size” to sets in a consistent way. Key examples include the counting measure (number of elements) and the Dirac measure (concentrated at a point). Finite and σ-finite measures are important for avoiding pathological behavior.

Properties of Measures

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

1. **Finite additivity**: For every finite disjoint family $\{E_1, \ldots, E_n\} \subseteq \mathcal{A}$, $\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k)$.
2. **Monotonicity**: For every $E, F \in \mathcal{A}$ with $E \subseteq F$, we have $\mu(E) \leq \mu(F)$.
3. **σ-subadditivity**: For every countable family $\{E_k\} \subset \mathcal{A}$ (not necessarily disjoint), $\mu\left(\bigcup_{k=1}^\infty E_k\right) \leq \sum_{k=1}^\infty \mu(E_k)$.
4. **Continuity from below**: For every increasing sequence $\{E_k\} \subseteq \mathcal{A}$ (i.e., $E_k \nearrow$), $\mu\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$.
5. **Continuity from above**: For every decreasing sequence $\{E_k\} \subseteq \mathcal{A}$ (i.e., $E_k \searrow$) with $\mu(E_1) < \infty$, $\mu\left(\bigcap_{k=1}^\infty E_k\right) = \mu\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$.

Remark. Measures are well-behaved: they are additive, monotone, and continuous with respect to increasing/decreasing sequences. These properties make them suitable for integration and limit processes.

Sets of Zero Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Definition. A set $N \subseteq X$ is said to be a **set of zero measure** if $N \in \mathcal{A}$ and $\mu(N) = 0$. A set $E \subseteq X$ is **negligible** if there exists $N \in \mathcal{A}$ such that $E \subseteq N$ and $\mu(N) = 0$.
Let $\mathcal{N}_\mu$ be the collection of sets of zero measure and $\mathcal{T}_\mu$ be the collection of negligible sets.

Example. $X = \{a, b, c\}$, $\mathcal{A} = \{\emptyset, \{a\}, \{b, c\}, X\}$ is a σ-algebra. Define $\mu(X) = \mu(\{a\}) := 1$, $\mu(\emptyset) = \mu(\{b, c\}) := 0$. This is a measure.
Then $N = \{b, c\} \in \mathcal{N}_\mu$, and $\{b\}, \{c\} \subseteq N$ so $\{b\}, \{c\} \in \mathcal{T}_\mu \setminus \mathcal{N}_\mu$.

Definition. A measure space $(X, \mathcal{A}, \mu)$ is said to be **complete** if $\mathcal{T}_\mu \subseteq \mathcal{A}$. In such a case, μ is a **complete measure** and $\mathcal{A}$ is a **complete σ-algebra**.

Remark. $\mathcal{T}_\mu = \mathcal{N}_\mu$ if and only if $(X, \mathcal{A}, \mu)$ is complete.

Remark. A complete measure space is one where all subsets of zero-measure sets are measurable. Completeness is desirable because it ensures that sets that are ”tiny” are also measurable.

Almost Everywhere

Consider a measure space $(X, \mathcal{A}, \mu)$.

Definition. A property P on X is said to be true **almost everywhere (a.e.)** if the set $\{x \in X : P(x) \text{ is false}\} \in \mathcal{N}_\mu$.

Example.

- $f, g : X \to \overline{\mathbb{R}}$ are equal a.e. if $\{x \in X : f(x) \neq g(x)\} \in \mathcal{N}_\mu$.
- $f : X \to \overline{\mathbb{R}}$ is finite a.e. if $\{x \in X : f(x) = \pm\infty\} \in \mathcal{N}_\mu$.
- $f : D \to \mathbb{R}$, with $D \in \mathcal{A}$, is said to be defined a.e. if $D^C \in \mathcal{N}_\mu$.

Remark. Equal a.e. is an equivalence relation in the set of functions $f : X \to \overline{\mathbb{R}}$.

Completion of a Measure Space

Let $(X, \mathcal{A}, \mu)$ be a measure space.
Define

$$\overline{\mathcal{A}} := \{E \subseteq X : \exists F, G \in \mathcal{A} \text{ such that } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

and $\overline{\mu} : \overline{\mathcal{A}} \to \overline{\mathbb{R}}_+$ by $\overline{\mu}(E) := \mu(F)$.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

1. $\overline{\mathcal{A}}$ is a σ-algebra with $\overline{\mathcal{A}} \supseteq \mathcal{A}$,
2. $\overline{\mu}$ is a complete measure extending μ, i.e., $\overline{\mu}|_{\mathcal{A}} = \mu$.

The space $(X, \overline{\mathcal{A}}, \overline{\mu})$ is the **completion** of $(X, \mathcal{A}, \mu)$, and it is the smallest complete measure space containing $(X, \mathcal{A}, \mu)$.

Remark. The completion of a measure space is the smallest complete extension. It ensures that all subsets of zero-measure sets are included in the σ-algebra.

Outer Measure

Let X be a set.

Definition. A function $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ is said to be an **outer measure** on X if

1. $\mu^*(\emptyset) = 0$,
2. $E_1 \subseteq E_2 \Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$,
3. σ**-subadditivity**: $\mu^*\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \mu^*(E_n)$ for every $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$.

Remark. If μ is a measure on $\mathcal{P}(X)$, then μ is also an outer measure.

Now let $K \subseteq \mathcal{P}(X)$ with $\emptyset \in K$. We want to say that K is the collection of elementary sets having a certain "measure", given by ν (called elementary measure). Let $\nu : K \to \overline{\mathbb{R}}_+$ be a function such that $\nu(\emptyset) = 0$.

Define the function $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ by

$$\mu^*(E) := \inf\left\{\sum_{n=1}^{\infty} \nu(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \subset K\right\}$$

if $E \subseteq X$ can be covered by a countable union of sets $I_n \in K$, and $\mu^*(E) := \infty$ otherwise.

Theorem. μ^* is an outer measure on X.

Proof.

1. Since $\emptyset \in K$, we have $\mu^*(\emptyset) \leq \nu(\emptyset) = 0 \Rightarrow \mu^*(\emptyset) = 0$.
2. **Monotonicity**: If $E_1 \subseteq E_2$, then any countable cover of E_2 is also a countable cover of E_1. From the definition of μ^*, it follows that $\mu^*(E_1) \leq \mu^*(E_2)$. If E_2 does not have a countable cover, then $\mu^*(E_1) \leq \mu^*(E_2) = \infty$.
3. **σ-subadditivity**: We need to show $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$. This is obvious if $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$. Suppose $\sum_{n=1}^{\infty} \mu^*(E_n) < \infty$. Then every $\mu^*(E_n) < \infty$. By the definition of μ^*, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\{I_{n,k}\} \subseteq K$ such that $E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k}$ and $\mu^*(E_n) + \varepsilon/2^n > \sum_{k=1}^{\infty} \nu(I_{n,k})$.

   Since $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}$ and $\{I_{n,k}\} \subseteq K$, it follows that

   $$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n,k=1}^{\infty} \nu(I_{n,k}) < \sum_{n=1}^{\infty}[\mu^*(E_n) + \varepsilon/2^n] = \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon.$$

   Since ε is arbitrary, the conclusion follows.

■

Remark. An outer measure is a function defined on all subsets, satisfying subadditivity. It is used to construct measures via the Carathéodory extension theorem.

Generation of a Measure

Let μ^* be an outer measure on X.

Definition. A set $E \subseteq X$ is said to be μ^***-measurable** if for every $Z \subseteq X$, we have

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \cap E^C).$$

This is called the **Carathéodory condition**.

Remark. The Carathéodory condition is the crucial definition for moving from an outer measure to a true measure. An outer measure is defined for all subsets of X but is only countably additive on certain "nice" sets. A set E is deemed "nice" or measurable if it cleanly splits every other set Z into two parts whose outer measures add up correctly. This condition ensures that the measure will be additive when we restrict it to the family of measurable sets.
If we take $X = Z$, we have $\mu^*(X) = \mu^*(E) + \mu^*(E^C)$, so $\mu^*(E) = \mu^*(X) - \mu^*(E^C)$. This can be seen as an inner measure.

Lemma. $E \subseteq X$ is μ^*-measurable if and only if $\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \cap E^C)$ for every $Z \subseteq X$.

Proof.
It is enough to show that for every $E \subseteq X$, $Z \subseteq X$ we have $\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^C)$, because then equality follows. Since $Z = Z \cap X = Z \cap (E \cup E^C) = (Z \cap E) \cup (Z \cap E^C)$, by subadditivity of μ^* we get

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^C).$$

■

Lemma. If $\mu^*(E) = 0$, then E is μ^*-measurable.

Proof.
For every $Z \subseteq X$, by monotonicity of μ^*:

$$\mu^*(Z \cap E) + \mu^*(Z \cap E^C) \leq \mu^*(E) + \mu^*(Z) = 0 + \mu^*(Z).$$

So $\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \cap E^C)$. By the preceding lemma, the Carathéodory condition is fulfilled; hence, E is μ^*-measurable. ■

Lebesgue Measure

Let $\mathcal{L} := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$.

Theorem. Let μ^* be an outer measure on X. Then

1. $\mathcal{L}$ is a σ-algebra,
2. The restriction of μ^* to $\mathcal{L}$, denoted $\mu = \mu^*|_{\mathcal{L}}$, is a complete measure on $\mathcal{L}$.

Lebesgue Measure in $\mathbb{R}$

Take $X = \mathbb{R}$. We choose (K, ν) with K the intervals and ν their length.
Let I be the family of open bounded intervals:

$$I := \{(a, b) : a, b \in \mathbb{R}, a \leq b\}.$$

So $\emptyset \in I$ when $a = b$.
The elementary measure $\lambda_0 : I \to \mathbb{R}_+$:

$$\lambda_0(\emptyset) := 0, \quad \lambda_0((a, b)) := b - a.$$

From (I, λ_0), we generate λ^* outer measure on $\mathbb{R}$:

$$\lambda^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \lambda_0(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \subset I \right\}$$

for every $E \subseteq \mathbb{R}$ that can be covered by a countable union of open bounded intervals, and $\lambda^*(E) := \infty$ otherwise.

Definition. λ^* generated by (I, λ_0) is called the **outer Lebesgue measure** on $\mathbb{R}$. λ^*-measurable sets are called **Lebesgue measurable sets**. The corresponding σ-algebra $\mathcal{L}(\mathbb{R})$ is called the **Lebesgue σ-algebra**. The measure $\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})}$ is called the **Lebesgue measure** on $\mathbb{R}$, and it is complete.

We want to study concretely what is the Lebesgue measure and σ-algebra.

Remark. Consider the interval (a, b). The outer measure is $b - a$. We can consider the covering of (a, b) made just by $I = (a, b)$: $\lambda^*((a, b)) \leq \lambda_0((a, b)) = b - a$.
Then we see $(a, b) \subseteq \bigcup_{n=1}^{\infty} I_n = \Omega$ an open set. Since it is open, $\Omega \supseteq [a, b] \Rightarrow \Omega \supseteq (a+\varepsilon, b-\varepsilon)$. The elementary measure of this covering: $b - a - 2\varepsilon = \lambda_0((a + \varepsilon, b - \varepsilon)) \leq \sum_{n=1}^{\infty} \lambda_0(I_n)$.
Passing to the infimum: $b - a \leq \lambda^*((a, b))$. So we have that $\lambda^*((a, b)) = b - a$.

Theorem. Any countable subset $E \subseteq \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

Proof.
First consider $\{a\}$ for $a \in \mathbb{R}$. $\{a\}$ can be covered by $\{a\} \subseteq (a-\varepsilon, a+\varepsilon)$. Then $\lambda^*(\{a\}) \leq \lambda_0((a-\varepsilon, a+\varepsilon)) = 2\varepsilon$.
Since ε is arbitrary, $\lambda^*(\{a\}) = 0$. So $\{a\}$ is measurable and $\lambda(\{a\}) = 0$.
Now consider $E = \bigcup_{n=1}^{\infty} \{a_n\}$ with $a_n \in \mathbb{R}$. Each set $\{a_n\}$ is measurable. This is a countable union of sets, so $E \in \mathcal{L}$. Also, λ^* is sub-additive; hence,

$$\lambda^*(E) = \lambda^* \left(\bigcup_{n=1}^{\infty} \{a_n\} \right) \leq \sum_{n=1}^{\infty} \lambda^*(\{a_n\}) = 0.$$

Then $E \in \mathcal{L}$ and $\lambda^*(E) = \lambda(E) = 0$. ■

Remark. The converse is not true.

Theorem. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. Every Borel set is Lebesgue measurable.

Proof.

1. Show that for every $a \in \mathbb{R}$, $(a, \infty) \in \mathcal{L}(\mathbb{R})$.

   Let $A \subseteq \mathbb{R}$ be any set. We assume that $a \notin A$; otherwise, we replace A by $A \setminus \{a\}$, which leaves the outer measure unchanged. We must show that $\lambda^*(A) \geq \lambda^*(A \cap (-\infty, a)) + \lambda^*(A \cap (a, +\infty))$.

   Since $\lambda^*(A)$ is defined as an infimum, to check this inequality it is necessary and sufficient to show that for any sequence $\{I_k\} \subset I$ that covers A, we have

   $$\sum_{k=1}^{\infty} \lambda_0(I_k) \geq \lambda^*(A_1) + \lambda^*(A_2)$$

   where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, +\infty)$.

   For each $k \in \mathbb{N}$, define $I'_k = I_k \cap (-\infty, a)$ and $I''_k = I_k \cap (a, +\infty)$. Then I'_k and I''_k are disjoint intervals and $\lambda_0(I_k) = \lambda_0(I'_k) + \lambda_0(I''_k)$.

   $\{I'_k\}$ is a countable cover of A_1 and $\{I''_k\}$ is a countable cover of A_2. Therefore,

   $$\lambda^*(A_1) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k), \quad \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I''_k)$$

   Hence,

   $$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) + \sum_{k=1}^{\infty} \lambda_0(I''_k) = \sum_{k=1}^{\infty} [\lambda_0(I'_k) + \lambda_0(I''_k)] = \sum_{k=1}^{\infty} \lambda_0(I_k).$$

2. Show that for any open set $\Omega \subseteq \mathbb{R}$, $\Omega \in \mathcal{L}(\mathbb{R})$.

   From Step 1, $(a, \infty) \in \mathcal{L}(\mathbb{R}) \Rightarrow (a, \infty)^C = (-\infty, a] \in \mathcal{L}(\mathbb{R})$. But also $(a, b) = (a, \infty) \cap (-\infty, b)$. These two belong to $\mathcal{L}$, so also (a, b) belongs to $\mathcal{L}$.

   Also, we know that $\Omega = \bigcup_{\substack{r,s \in \mathbb{Q} \\ r<s \\ (r,s) \in \Omega}} (r, s)$, so $\Omega \in \mathcal{L}(\mathbb{R})$.

3. Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

   $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra which contains the collection $\mathcal{G}$ of all open sets of $\mathbb{R}$. Therefore, if another σ-algebra $\mathcal{A}$ contains $\mathcal{G}$, it follows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{A}$. Since $\mathcal{L}(\mathbb{R})$ contains all open sets, we have $\mathcal{L}(\mathbb{R}) \supseteq \mathcal{B}(\mathbb{R})$.

■

Remark. From before we obtained $(a, b) \in \mathcal{L}(\mathbb{R})$ and $\lambda((a, b)) = b - a$.

Remark. $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$, so it is strictly contained.

Theorem. $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$.

Remark. Completion is a process of making a measure space complete. This theorem states that the Lebesgue σ-algebra is exactly what you get if you start with all Borel sets and then throw in every subset of every Borel set that has measure zero. This is often the most practical way to think about what a Lebesgue measurable set is.

What is a Lebesgue Measurable Set?

All Borel measurable sets are Lebesgue measurable, but we want to understand better how a Lebesgue measurable set is related to open and closed sets.

Lemma. [Excision property] If $A \in \mathcal{L}(\mathbb{R})$ with $\lambda^*(A) < \infty$ and $A \subseteq B$, then

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A).$$

Proof.
Since $A \in \mathcal{L}(\mathbb{R})$, the Carathéodory condition is satisfied, so (with $Z = B$):

$$\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^C) = \lambda^*(A) + \lambda^*(B \setminus A).$$

And the conclusion follows. ■

Theorem. [Regularity of Lebesgue measure] Let $E \subseteq \mathbb{R}$. The following statements are equivalent:

1. $E \in \mathcal{L}(\mathbb{R})$,
2. For every $\varepsilon > 0$, there exists $A \subseteq \mathbb{R}$ open such that $E \subseteq A$ and $\lambda^*(A \setminus E) < \varepsilon$,
3. There exists $G \subseteq \mathbb{R}$ of class G_δ (countable intersection of open sets) such that $E \subseteq G$ and $\lambda^*(G \setminus E) = 0$,
4. For every $\varepsilon > 0$, there exists $C \subseteq \mathbb{R}$ closed such that $C \subseteq E$ and $\lambda^*(E \setminus C) < \varepsilon$,
5. There exists $F \subseteq \mathbb{R}$ of class F_σ (countable union of closed sets) such that $F \subseteq E$ and $\lambda^*(E \setminus F) = 0$.

Proof.
Let's consider only the open sets: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

(1) $\Rightarrow$ (2) $E \in \mathcal{L}(\mathbb{R})$ and we assume that $\lambda(E) < \infty$. By the definition of outer measure, for every $\varepsilon > 0$, there exists $\{I_k\} \subset I$ of open bounded intervals which covers E and for which

$$\sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon.$$

Define the set $O := \bigcup_{k=1}^{\infty} I_k$. Then O is open and $E \subseteq O$. Therefore,

$$\lambda^*(O) \leq \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon.$$

This implies that $\lambda^*(O) - \lambda^*(E) < \varepsilon$. Given $E \in \mathcal{L}(\mathbb{R})$ and $\lambda^*(E) < \infty$, we can say $\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$.

(2) $\Rightarrow$ (3) For every $k \in \mathbb{N}$, choose $O_k \supseteq E$ open set such that $\lambda^*(O_k \setminus E) < 1/k$. Define $G := \bigcap_{k=1}^{\infty} O_k$. Then $G \in G_\delta$ and $G \supseteq E$. Moreover, for every $k \in \mathbb{N}$, $G \setminus E \subseteq O_k \setminus E$. By monotonicity,

$$\lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < 1/k.$$

For $k \to \infty$, we have $\lambda^*(G \setminus E) = 0$.

(3) $\Rightarrow$ (1) $G \setminus E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G \setminus E) = 0$. Now $G \in \mathcal{L}(\mathbb{R})$ since $G \in G_\delta \subset \mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$. Therefore, $E = G \cap (G \setminus E)^C \in \mathcal{L}(\mathbb{R})$.

■

Remark. This Regularity Theorem is incredibly powerful. It tells us that a set is measurable if and only if it can be approximated arbitrarily closely from the outside by open sets (which are too big by only a measure ε) and from the inside by closed sets (which are too small by only a measure ε). Furthermore, it states that every measurable set E is essentially a "nice" Borel set (a G_δ or F_σ) with a negligible set attached or removed. This bridges the abstract definition of measurability with a more intuitive geometric understanding.

Non-Measurable Sets

Theorem. [Vitali] Any Lebesgue measurable set $E \in \mathcal{L}(\mathbb{R})$ with $\lambda(E) > 0$ contains a subset that is not Lebesgue measurable.

Theorem. There exist disjoint sets $A, B \subset \mathbb{R}$ for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$.

Proof.
Assume by contradiction that $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for any $A, B \subset \mathbb{R}$ that are disjoint ($A \cap B = \emptyset$).
For every $E, Z \subseteq \mathbb{R}$:

$$\lambda^*(Z \cap E) + \lambda^*(Z \cap E^c) = \lambda^*((Z \cap E) \cup (Z \cap E^c)) = \lambda^*(Z).$$

If this is true for every $E \subseteq \mathbb{R}$, it fulfills the Carathéodory condition, so any $E \in \mathcal{L}(\mathbb{R})$. This is not possible because there exists the Vitali set. ■

Remark. This theorem highlights the need for the Carathéodory condition. The outer measure, by itself, is not a measure because it fails countable additivity on the entire power set of $\mathbb{R}$. We must restrict it to the smaller σ-algebra $\mathcal{L}(\mathbb{R})$ where additivity holds.

Lebesgue Measure in $\mathbb{R}^N$

Take $X = \mathbb{R}^N$. Take K as the family of N-dimensional open intervals (rectangles):

$$I^N := \left\{ \prod_{k=1}^{N} (a_k, b_k) : a_k, b_k \in \mathbb{R} \text{ and } a_k \leq b_k \right\}.$$

Take ν as the elementary volume function $\lambda_0^N : I^N \to [0, \infty)$:

$$\lambda_0^N(\emptyset) := 0, \quad \lambda_0^N\left(\prod_{k=1}^{N}(a_k, b_k)\right) := \prod_{k=1}^{N}(b_k - a_k).$$

$(K, \nu) = (I^N, \lambda_0^N)$

- This generates the N-dimensional Lebesgue outer measure $\lambda^{*,N}$,
- The $\lambda^{*,N}$-measurable sets form, with the Carathéodory condition, the N-dimensional Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^N)$,
- The restriction $\lambda^N := \lambda^{*,N}|_{\mathcal{L}(\mathbb{R}^N)}$ is the N-dimensional Lebesgue measure,
- The space $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \lambda^N)$ is a complete measure space.

Measurable Functions

Definition. Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be measurable spaces. A function $f : X \to X'$ is said to be **measurable** if $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{A}'$.

Remark. Continuity and measurability are similar and related. Given (X, d) and (X', d') metric spaces, let $\mathcal{G}$ be the collection of all open sets in X and $\mathcal{G}'$ the collection of all open sets in X'. Then $f : X \to X'$ is continuous if and only if $f^{-1}(E) \in \mathcal{G}$ for every $E \in \mathcal{G}'$.

Proposition. Given $(X, \mathcal{A})$, $(X', \mathcal{A}')$, and $(X'', \mathcal{A}'')$ measurable spaces, let $f : X \to X'$ and $g : X' \to X''$ be measurable. Then $g \circ f : X \to X''$ is measurable.

Theorem. [Characterization by Generating Sets] Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be measurable spaces. Let $C' \subseteq \mathcal{P}(X')$ be such that $\sigma_0(C') = \mathcal{A}'$. Then $f : X \to X'$ is measurable if and only if $f^{-1}(E) \in \mathcal{A}$ for every $E \in C'$.

Remark. The implication ($\Rightarrow$) is trivial since measurability is stronger than the characterizing condition, while ($\Leftarrow$) is useful. Indeed, to check if a function is measurable we should consider all sets in $\mathcal{A}'$, but if there is a set C' that generates the σ-algebra $\mathcal{A}'$, it is enough to consider all sets in C', which are usually fewer than all sets of $\mathcal{A}'$. Instead of considering the entire σ-algebra, we are just considering a small collection of sets.

Definition. Consider $(X, \mathcal{L})$ a measurable space. Consider (X', d') a metric space, on which we consider $(X', \mathcal{B}')$ measurable space, where $\mathcal{B}'$ is the Borel σ-algebra. Then $f : X \to X'$ is **Lebesgue measurable** if $f^{-1}(E) \in \mathcal{L}$ for every $E \in \mathcal{B}'$.

Definition. Let (X, d) and $(X, \mathcal{B})$, (X', d') and $(X', \mathcal{B}')$ be metric spaces with their Borel σ-algebras. Then $f : X \to X'$ is **Borel measurable** if $f^{-1}(E) \in \mathcal{B}$ for every $E \in \mathcal{B}'$.

Corollary. Consider $(X, \mathcal{L})$ a measurable space and (X', d') a metric space with $(X', \mathcal{B}')$ measurable space. Then $f : X \to X'$ is Lebesgue measurable if and only if $f^{-1}(E) \in \mathcal{L}$ for every open set $E \subseteq X'$.

Proof.
Let $C' = \{\text{open sets of } X'\}$, then $\sigma_0(C') = \mathcal{B}'$. By the preceding theorem, $f : X \to X'$ is $\mathcal{L}$-measurable if and only if $f^{-1}(E) \in \mathcal{L}$ for every $E \in \mathcal{B}'$. ■

Corollary. Given X, X' metric spaces, consider $f : X \to X'$ continuous. Then f is Borel measurable.

Proof.
f is continuous if and only if $f^{-1}(E) \in \mathcal{G}$ for every $E \in \mathcal{G}'$. By the preceding theorem, instead of taking $\mathcal{G}'$ we take the σ-algebra: $f^{-1}(E) \in \mathcal{G}$ for every $E \in \sigma_0(\mathcal{G}') = \mathcal{B}'$. But $\mathcal{G} \subset \mathcal{B}$. ■

Corollary. Given X, X' metric spaces, consider $f : X \to X'$ Borel measurable. Then f is Lebesgue measurable.

Proof.
We know $f^{-1}(E) \in \mathcal{B}$ for every $E \in \sigma_0(\mathcal{G}') = \mathcal{B}'$. But $\mathcal{B} \subseteq \mathcal{L}$. ■

Corollary. Given X, X' metric spaces, if $f : X \to X'$ is continuous, then f is Lebesgue measurable.

Theorem. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f$ is Lebesgue measurable.

Proof.
We have $(X \subseteq \mathbb{R}, \mathcal{L}(\mathbb{R}))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We know:

- f is $\mathcal{L}$-measurable if and only if $f^{-1}(E') \in \mathcal{L}(\mathbb{R})$ for every $E' \subseteq \mathbb{R}$ (Borel set).
- For every $E \subseteq \mathbb{R}$, $g^{-1}(E)$ is open if and only if g is continuous.

Then for every $E \subseteq \mathbb{R}$:
$$(g \circ f)^{-1}(E) = f^{-1}[g^{-1}(E)].$$
Since g is continuous, $g^{-1}(E) = E'$ is open. Thus, $(g \circ f)^{-1}(E) = f^{-1}(E') \in \mathcal{L}(\mathbb{R})$. ■

Remark. [1] If $g : \mathbb{R} \to \mathbb{R}$ is only Lebesgue measurable, then we arrive at $f^{-1}[g^{-1}(E)]$. But now $g^{-1}(E) \in \mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$, and we cannot conclude $f^{-1}[g^{-1}(E)] \in \mathcal{L}(\mathbb{R})$. So f, g Lebesgue measurable does not imply $g \circ f$ is Lebesgue measurable.

Remark. [2] Instead of continuity, we can also assume Borel measurability. If $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then we have $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. As before, we get $f^{-1}[g^{-1}(E)]$, and now $g^{-1}(E) \in \mathcal{B}(\mathbb{R})$, so we can conclude $f^{-1}[g^{-1}(E)] \in \mathcal{L}(\mathbb{R})$. Thus, $g \circ f$ is Lebesgue measurable.

Theorem. [Lusin] Let $f : E \subseteq \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Then for every $\varepsilon > 0$ there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set $F \subseteq E$ such that:

- $f = g$ in F,
- $\lambda(E \setminus F) < \varepsilon$.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Let $f, g : X \to \mathbb{R}$. If f is Lebesgue measurable and $f = g$ a.e., then g is Lebesgue measurable.

Real Valued Functions

Define:

$$\mathcal{M}(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}\}.$$

Notation: $(X, \mathcal{A}) = (X, \mathcal{A})$, $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})) = (X', \mathcal{A}')$.
Also define:

$$\mathcal{M}_+(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}, f \geq 0\}.$$

For every $\alpha \in \mathbb{R}$:

$$\begin{aligned}
\{f > \alpha\} &:= \{x \in X : f(x) > \alpha\} = f^{-1}((\alpha, +\infty]), \\
\{f \geq \alpha\} &:= \{x \in X : f(x) \geq \alpha\} = f^{-1}([\alpha, +\infty]), \\
\{f < \alpha\} &:= f^{-1}([-\infty, \alpha)), \\
\{f \leq \alpha\} &:= f^{-1}([-\infty, \alpha]).
\end{aligned}$$

Theorem. Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$. The following statements are equivalent:

1. f is measurable.
2. $\{f > \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.
3. $\{f \geq \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.
4. $\{f < \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.
5. $\{f \leq \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.

Let $f, g : X \to \overline{\mathbb{R}}$. Define:

$$\begin{aligned}
\{f < g\} &:= \{x \in X : f(x) < g(x)\}, \\
\{f \leq g\} &:= \{x \in X : f(x) \leq g(x)\}, \\
\{f = g\} &:= \{x \in X : f(x) = g(x)\}.
\end{aligned}$$

Theorem. Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Then:

1. $\{f < g\} \in \mathcal{A}$,
2. $\{f \leq g\} \in \mathcal{A}$,
3. $\{f = g\} \in \mathcal{A}$.

Theorem. Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$. Then:

$$\sup_{n \in \mathbb{N}} f_n, \quad \inf_{n \in \mathbb{N}} f_n \in \mathcal{M}(X, \mathcal{A}).$$

Proof.
For every $\alpha \in \mathbb{R}$:

$$\left\{\sup_{n\in\mathbb{N}} f_n > \alpha\right\} = \bigcup_{n=1}^{\infty} \{f_n > \alpha\} \in \mathcal{A}.$$

For the infimum, note that:

$$\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(-f_n).$$

■

Corollary. If $f, g \in \mathcal{M}(X, \mathcal{A})$, then:

$$\max\{f, g\}, \quad \min\{f, g\}, \quad f_\pm \in \mathcal{M}(X, \mathcal{A}).$$

Theorem. Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$. Then:

$$\liminf_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n \in \mathcal{M}(X, \mathcal{A}).$$

Proof.
Note that:

$$\limsup_{n\to\infty} f_n = \inf_{k\geq 1} \left(\sup_{n\geq k} f_n\right) \in \mathcal{M}(X, \mathcal{A}),$$

and

$$\liminf_{n\to\infty} f_n = -\limsup_{n\to\infty}(-f_n) \in \mathcal{M}(X, \mathcal{A}).$$

■

Theorem. Let $f, g : X \to \mathbb{R}$, with $f, g \in \mathcal{M}(X, \mathcal{A})$. Then:

$$f + g \in \mathcal{M}(X, \mathcal{A}), \quad fg \in \mathcal{M}(X, \mathcal{A}).$$

The Cantor Set

Construction

The Cantor set is constructed through an iterative process:

- **Step 0:** Start with $K_0 := [0, 1]$
- **Step 1:** Remove the open middle third $I_{0,1} := (1/3, 2/3)$, leaving:
$$K_1 = [0, 1/3] \cup [2/3, 1]$$
- **Step n:** After n steps, we have 2^n closed intervals $J_{n,k}$ of length $1/3^n$
- **Step n+1:** From each $J_{n,k}$, remove the open middle third $I_{n,k}$ of length $1/3^{n+1}$

Define the sets at each stage:

$$K_n := \bigcup_{k=1}^{2^n} J_{n,k}, \quad K := \bigcap_{n=0}^{\infty} K_n$$

The set K is the **Cantor set**.

Basic Properties

- **Closedness:** Each K_n is a finite union of closed intervals, hence closed. Since K is an intersection of closed sets, K is closed.
- **Measurability:** K is closed $\Rightarrow K \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$
- **Total length removed:** Define the removed set:

$$\Omega := \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}$$

  Since $I_{n,k}$ are open, Ω is open and measurable. The total length removed is:

$$\lambda(\Omega) = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot\frac{1}{1-2/3} = 1$$

  Hence, $\lambda(K) = \lambda([0,1]) - \lambda(\Omega) = 0$

Topological Properties

- **Uncountability:** K is uncountable (can be shown via ternary expansion)
- **Empty interior:** $\mathrm{int}(K) = \emptyset$. Any open interval contained in K would have positive measure, but $\lambda(K) = 0$
- **Density of complement:** $\overline{[0,1] \setminus K} = [0,1]$. For any $x_0 \in [0,1]$ and $r > 0$, the interval $(x_0 - r, x_0 + r)$ cannot be contained in K (since $\mathrm{int}(K) = \emptyset$), so it must intersect $[0,1] \setminus K$
- **Perfect set:** K equals its set of accumulation points. For any $\bar{x} \in K$ and $\varepsilon > 0$, choose n such that $1/3^n < \varepsilon$. Since $\bar{x} \in J_{n,k}$ for some k, the removed interval $I_{n,k} \subseteq (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ has endpoints in K different from $\bar{x}$

The Vitali-Lebesgue Function

Construction

Define a sequence of functions $L_n : [0,1] \to [0,1]$ recursively:

- **Base:** $L_0(x) = x$ (the identity function)
- **Inductive step:** On each interval $J_{n,k}$ of K_n, define L_{n+1} to be linear with slope $(3/2)^{n+1}$, preserving continuity
- On the removed intervals $[0,1] \setminus K_n$, keep L_{n+1} constant (equal to its values at the endpoints)

Explicitly:

$$L_n(x) := \begin{cases} \text{linear with slope } (3/2)^n & \text{on each } J_{n,k} \subset K_n \\ \text{constant} & \text{on } [0,1] \setminus K_n \end{cases}$$

Properties of the Sequence

- **Symmetry:** $L_n(x) + L_n(1-x) = 1$ for all $n \in \mathbb{N}$, $x \in [0,1]$
- **Monotonicity:** $L_{n+1}(x) \leq L_n(x)$ for all $n \in \mathbb{N}$, $x \in [0,1]$
- **Uniform convergence:** The maximum difference between successive functions is:

$$\sup_{x\in[0,1]} |L_{n+1}(x) - L_n(x)| = \frac{1}{2^{n+1}}$$

This follows from calculating the maximum difference on the first interval $J_{n,1} = [0, 1/3^n]$:

$$|L_{n+1}(x) - L_n(x)| = \left(\frac{3}{2}\right)^n \left(\frac{3}{2} - 1\right) x = \frac{1}{2^{n+1}} x$$

Hence, $\{L_n\}$ is uniformly Cauchy and converges uniformly to a function L

Properties of the Limit Function

The **Vitali-Lebesgue function** is defined as:

$$L(x) := \lim_{n\to\infty} L_n(x)$$

- **Continuity:** $L \in C([0,1])$ (uniform limit of continuous functions)
- **Monotonicity:** L is nondecreasing
- **Symmetry:** $L(x) + L(1-x) = 1$
- **Singular derivative:** $L'(x) = 0$ almost everywhere. On each $[0,1] \setminus K_n$, L is constant, so $L' = 0$. Since $\lambda(K) = 0$, $L' = 0$ a.e.
- **Non-constant:** Despite zero derivative a.e., $L(1) - L(0) = 1 - 0 = 1$

Remark. The Vitali-Lebesgue function is a classic example of a **singular function**: continuous, nondecreasing, with derivative zero almost everywhere, yet not constant. This demonstrates that the Fundamental Theorem of Calculus requires absolute continuity, not just continuity.

Applications

Application One

Consider a continuous function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. In general, given $E \subseteq \mathbb{R}$ with $E \in \mathcal{L}(\mathbb{R})$, can we say $f^{-1}(E) \in \mathcal{L}(\mathbb{R})$? The answer is no.

Proof.
Define:

$$h(x) := \frac{x + L(x)}{2}, \quad x \in [0,1].$$

Then h is continuous, strictly increasing, and surjective. Moreover, $h([0,1] \setminus K)$ is a union of open intervals with total length:

$$\lambda(h([0,1] \setminus K)) = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{2}.$$

Since $[0,1] = h([0,1]) = h(K) \cup h([0,1] \setminus K)$, we have $\lambda(h(K)) = \frac{1}{2}$.
There exists $\tilde{E} \subseteq h(K)$ such that $\tilde{E} \notin \mathcal{L}(\mathbb{R})$ (e.g., a Vitali set). On the other hand, $h^{-1}(\tilde{E}) \subseteq K$, so $h^{-1}(\tilde{E}) \in \mathcal{L}(\mathbb{R})$ since K has measure zero and the Lebesgue σ-algebra is complete.
Now consider $f := h^{-1}$ and $\tilde{F} := f(\tilde{E}) = h^{-1}(\tilde{E}) \in \mathcal{L}(\mathbb{R})$. Since f is continuous, but $\tilde{E} = f^{-1}(\tilde{F}) \notin \mathcal{L}(\mathbb{R})$, we obtain the counterexample. ■

Application Two

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \subsetneq (\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.

Proof.
From above, $\tilde{F} \in \mathcal{L}(\mathbb{R})$ and f is Lebesgue measurable (since continuous). If $\tilde{F} \in \mathcal{B}(\mathbb{R})$, then $f^{-1}(\tilde{F}) = \tilde{E} \in \mathcal{L}(\mathbb{R})$, which is false. Hence, $\tilde{F} \notin \mathcal{B}(\mathbb{R})$. ■

Application Three

If f is Borel measurable and g is Lebesgue measurable, then we can't say $g \circ f$ is Lebesgue measurable for sure.

Proof.
Let $f := h^{-1}$ as above. Take $\tilde{F} \in \mathcal{L}(\mathbb{R})$ and define $g := \chi_{\tilde{F}}$, the indicator function of $\tilde{F}$. Then g is Lebesgue measurable. Now:

$$(g \circ f)^{-1}((1/2, 3/2)) = f^{-1}(g^{-1}((1/2, 3/2))) = f^{-1}(\tilde{F}) = \tilde{E} \notin \mathcal{L}(\mathbb{R}).$$

Hence, $g \circ f$ is not Lebesgue measurable. ■

Application Four

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete.

Proof.
The Cantor set K is closed, hence Borel, and $\lambda(K) = 0$. There exists $\tilde{F} \subseteq K$ such that $\tilde{F} \notin \mathcal{B}(\mathbb{R})$ (as above). Hence, the Borel measure space is not complete. ■

Application Five

Define $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) := \begin{cases} 2 & x \in \tilde{F} \subset K, \\ 1 & x \in K \setminus \tilde{F}, \\ 0 & x \in \mathbb{R} \setminus K. \end{cases}$$

Then $f = g$ a.e. where $g \equiv 0$. g is Borel measurable (and Lebesgue measurable). But $\{f \geq 2\} = \tilde{F} \notin \mathcal{B}(\mathbb{R})$, so f is not Borel measurable. This shows the need for completeness to pass measurability a.e.

Measurable Functions: Further Properties

Corollary. Let $f : X \to \mathbb{R}$.

1. $f \in \mathcal{M}(X, \mathcal{A})$ if and only if $f_\pm \in \mathcal{M}(X, \mathcal{A})$.
2. $f \in \mathcal{M}(X, \mathcal{A})$ implies $|f| \in \mathcal{M}(X, \mathcal{A})$.

Proof.
(1) The forward implication is known. For the converse, note $f = f_+ - f_- \in \mathcal{M}(X, \mathcal{A})$.
(2) If $f \in \mathcal{M}(X, \mathcal{A})$, then $f_\pm \in \mathcal{M}(X, \mathcal{A})$ by (1), so $|f| = f_+ + f_- \in \mathcal{M}(X, \mathcal{A})$. ■

Lemma. Given $C \subseteq X$, the indicator function $\chi_C \in \mathcal{M}(X, \mathcal{A})$ if and only if $C \in \mathcal{A}$.

Proof.
We have:

$$\{\chi_C > \alpha\} = \begin{cases} X \in \mathcal{A} & \alpha < 0, \\ C \in \mathcal{A} & \alpha \in [0, 1), \\ \emptyset \in \mathcal{A} & \alpha \geq 1. \end{cases}$$

Hence, $\chi_C \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow C \in \mathcal{A}$. ■

Remark. In general, $|f| \in \mathcal{M}(X, \mathcal{A})$ does not imply $f \in \mathcal{M}(X, \mathcal{A})$. For example, let $E \subseteq X$ with $E \notin \mathcal{A}$. Define:

$$f(x) := \chi_E(x) - \chi_{E^c}(x) = \begin{cases} 1 & x \in E, \\ -1 & x \in E^c. \end{cases}$$

Then $\{f > 1/2\} = E \notin \mathcal{A}$, so $f \notin \mathcal{M}(X, \mathcal{A})$. But $|f| \equiv 1 \in \mathcal{M}(X, \mathcal{A})$.

Simple Functions

Definition. Let X be a set. A function $s : X \to \mathbb{R}$ is called a **simple function** if $s(X)$ is finite.

If $s(X) = \{c_1, c_2, \ldots, c_n\}$ with c_k distinct, define $E_k := \{x \in X : s(x) = c_k\}$. Then the **canonical form** of s is:

$$s = \sum_{k=1}^{n} c_k \chi_{E_k}.$$

The sets $\{E_k\}_{k=1}^n$ form a partition of X.

Remark. $s \in \mathcal{M}(X, \mathcal{A})$ if and only if $E_k \in \mathcal{A}$ for $k = 1, \ldots, n$.

Define:

$$\mathcal{S}(X, \mathcal{A}) := \{\text{measurable simple functions } f : X \to \mathbb{R}\},$$
$$\mathcal{S}_+(X, \mathcal{A}) := \{\text{measurable simple functions } f : X \to \mathbb{R}, f \geq 0\}.$$

Remark. If $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{L}(\mathbb{R}))$ and each E_k is an interval, then f is a step function.

Theorem. [Simple Approximation Theorem] Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence $\{s_n\}$ of simple functions such that $s_n \to f$ pointwise on X as $n \to \infty$. Furthermore:

1. If $f \in \mathcal{M}(X, \mathcal{A})$, then $\{s_n\} \subseteq \mathcal{S}(X, \mathcal{A})$.
2. If $f \geq 0$, then $\{s_n\}$ is increasing and $0 \leq s_n \leq f$.
3. If f is bounded, then $s_n \to f$ uniformly on X.

Proof.
For $f \geq 0$ bounded, say $0 \leq f \leq 1$, divide $[0, 1]$ into 2^n intervals of length 2^{-n}. Define:

$$E_k^{(n)} := \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}, \quad k = 0, 1, \ldots, 2^n - 1.$$

Then set:

$$s_n := \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_k^{(n)}}.$$

The sequence $\{s_n\}$ has the desired properties. ■

Essentially Bounded Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space. For any $N \in \mathcal{N}_\mu$ (null sets), define:

$$\alpha_N := \sup_{x \in N^c} f(x).$$

Note: if $N_2 \subseteq N_1$, then $\alpha_{N_2} \geq \alpha_{N_1}$.

Definition. The **essential supremum** of f is:

$$\operatorname*{ess\,sup}_X f := \inf \left\{ \sup_{x \in N^c} f(x) : N \in \mathcal{N}_\mu \right\}.$$

The **essential infimum** is:

$$\operatorname*{ess\,inf}_X f := \sup \left\{ \inf_{x \in N^c} f(x) : N \in \mathcal{N}_\mu \right\}.$$

Proposition. Let f be measurable. Then there exists $N \in \mathcal{N}_\mu$ such that:

$$\operatorname*{ess\,sup}_X f = \sup_{x \in N^c} f(x).$$

Moreover, $f(x) \leq \operatorname{ess\,sup}_X f$ almost everywhere.

Properties

1. If $f \in \mathcal{M}(X, \mathcal{A})$, then:

$$\operatorname*{ess\,sup}_X f = -\operatorname*{ess\,inf}_X(-f), \quad \operatorname*{ess\,sup}_X(kf) = k \operatorname*{ess\,sup}_X f \text{ for } k \geq 0.$$

2. If $f, g \in \mathcal{M}(X, \mathcal{A})$, then:

   (a) $f \leq g$ a.e. $\Rightarrow \operatorname{ess\,sup}_X f \leq \operatorname{ess\,sup}_X g$.

   (b) $\operatorname{ess\,sup}_X(f+g) \leq \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$.

   (c) $f = g$ a.e. $\Rightarrow \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$.

   (d) If $g \geq 0$ a.e., then $fg \leq (\operatorname{ess\,sup}_X f)g$ a.e.

Definition. A function $f \in \mathcal{M}(X, \mathcal{A})$ is **essentially bounded** if:

$$\operatorname*{ess\,sup}_X |f| < \infty.$$

Define:

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ such that } f \text{ is essentially bounded}\}.$$

Remark. Note: $L^\infty(X, \mathcal{A}, \mu) \neq \mathcal{L}^\infty(X, \mathcal{A}, \mu)$ (the latter is usually the space of equivalence classes).

Remark.

1. If $f \in \mathcal{L}^\infty$, then f is finite a.e., since $|f| \leq \operatorname{ess\,sup}_X f < \infty$ a.e.
2. f finite a.e. does not imply $f \in \mathcal{L}^\infty$.

Example. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by:

$$f(x) := \begin{cases} 1/|x| & x \neq 0, \\ +\infty & x = 0. \end{cases}$$

Then f is finite on $\mathbb{R} \setminus \{0\}$, so finite a.e., but $\operatorname{ess\,sup}_X f = +\infty$.

Example. Let:

$$f(x) := \begin{cases} \arctan(x) & x \in \mathbb{R} \setminus A, \\ +\infty & x \in A, \end{cases}$$

where $A = \{1/n : n \in \mathbb{N}\}$. Then $\sup_X f = +\infty$, but $\operatorname{ess\,sup}_X f = \pi/2$ since A is countable (measure zero). So $f \in \mathcal{L}^\infty$, but f is not bounded in the usual sense.

Lebesgue Integral

Integral of Non-Negative Measurable Simple Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $s \in S_+(X, \mathcal{A})$ be a non-negative measurable simple function. Then s can be written as:

$$s = \sum_{k=1}^{n} c_k \chi_{E_k},$$

where $c_1, \ldots, c_n \in \mathbb{R}_+$, $\{E_k\}$ is a partition of X, and $n \in \mathbb{N}$ is fixed.

Definition. The **integral of s over X** is defined as:

$$\int_X s \, d\mu := \sum_{k=1}^{n} c_k \mu(E_k).$$

If $E \in \mathcal{A}$, we set:

$$\int_E s \, d\mu := \int_X s\chi_E \, d\mu.$$

Note that $s\chi_E = \sum_{k=1}^{n} c_k \chi_{E_k \cap E}$, and therefore:

$$\int_E s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap E).$$

Remark. For any $E \in \mathcal{A}$, we have:

$$\int_X \chi_E \, d\mu = \mu(E).$$

Indeed, we can write $\chi_E(x) = \sum_{k=1}^{2} c_k \chi_{E_k}(x)$ with $E_1 = E$, $E_2 = E^c$, $c_1 = 1$, and $c_2 = 0$. Therefore:

$$\int_X \chi_E \, d\mu = c_1\mu(E_1) + c_2\mu(E_2) = \mu(E).$$

Remark. For any μ-null set $N \in \mathcal{N}_\mu$, we have:

$$\int_N s \, d\mu = 0.$$

Indeed:

$$\int_N s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap N) = 0,$$

because $(E_k \cap N) \subseteq N$ for all k.

Properties

Let $s, t \in S_+(X, \mathcal{A})$ be non-negative measurable simple functions.

1. If $c \geq 0$, then:
$$\int_X cs\,d\mu = c\int_X s\,d\mu.$$
2. The integral is additive:
$$\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu.$$
3. The integral is monotonic: if $s \leq t$, then:
$$\int_X s\,d\mu \leq \int_X t\,d\mu.$$
4. For $E, F \in \mathcal{A}$ with $E \subseteq F$:
$$\int_E s\,d\mu \leq \int_F s\,d\mu.$$

Proposition. Let $s \in S_+(X, \mathcal{A})$. Then the function $\varphi : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined by:
$$\varphi(E) := \int_E s\,d\mu, \quad \forall E \in \mathcal{A},$$
is a measure on $(X, \mathcal{A})$.

Proof.
We need to verify that φ satisfies the definition of a measure:

- $\varphi(\emptyset) = \int_\emptyset s\,d\mu = 0$, since $\mu(\emptyset) = 0$.
- σ-additivity: Let $\{E_k\}_{k=1}^\infty$ be a sequence of pairwise disjoint measurable sets, and let $E = \bigcup_{k=1}^\infty E_k$. Write $s = \sum_{l=1}^m d_l \chi_{F_l}$. Then:
$$\begin{aligned}\varphi(E) &= \int_E s\,d\mu = \sum_{l=1}^m d_l \mu(F_l \cap E)\\ &= \sum_{l=1}^m \sum_{k=1}^\infty d_l \mu(F_l \cap E_k) \quad \text{(by } \sigma\text{-additivity of } \mu)\\ &= \sum_{k=1}^\infty \sum_{l=1}^m d_l \mu(F_l \cap E_k) = \sum_{k=1}^\infty \int_{E_k} s\,d\mu = \sum_{k=1}^\infty \varphi(E_k).\end{aligned}$$

■

Integral of Non-Negative Measurable Functions

Let $f : X \to \overline{\mathbb{R}}_+$ be a non-negative measurable function, i.e., $f \in \mathcal{M}_+(X, \mathcal{A})$.

Definition. The **integral of f over X** is defined as:

$$\int_X f \, d\mu := \sup_{s \in S_f} \int_X s \, d\mu,$$

where $S_f := \{s \in S_+(X, \mathcal{A}) : s \leq f\}$.
If $E \in \mathcal{A}$, we set:

$$\int_E f \, d\mu := \int_X f \chi_E \, d\mu.$$

Remark. $S_f \neq \emptyset$ by the Simple Approximation Theorem. There exists an increasing sequence $\{s_n\} \subseteq S_f$ such that $s_n \leq s_{n+1}$ and $s_n \to f$ pointwise in X as $n \to \infty$.
It is also possible to define:

$$\int_X f \, d\mu := \lim_{n \to \infty} \int_X s_n \, d\mu.$$

This integral is independent of the choice of the approximating sequence $\{s_n\}$.

Properties

The integral of non-negative measurable functions inherits the same properties as the integral of simple functions.

Remark. For any $f \in \mathcal{M}_+(X, \mathcal{A})$ and any μ-null set $N \in \mathcal{N}_\mu$, we have:

$$\int_N f \, d\mu = 0.$$

This follows from the corresponding property for simple functions.

Theorem. [Chebyshev's Inequality] Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Then for any $c > 0$:

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu.$$

Proposition. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ such that $\int_X f\, d\mu < \infty$. Then f is finite almost everywhere in X.

Proof.
We need to show that $\mu(\{f = \infty\}) = 0$. Note that:

$$\{f = \infty\} = \bigcap_{n=1}^{\infty} \{f > n\}.$$

Let $E_n := \{f > n\}$. Then:

1. $\{E_n\}$ is a decreasing sequence,
2. By Chebyshev's inequality: $\mu(E_n) \leq \frac{1}{n} \int_X f\, d\mu$,
3. For $n = 1$, $\mu(E_1) < \infty$.

Therefore, by the continuity of the measure:

$$\mu(\{f = \infty\}) = \mu\left(\bigcap_{n=1}^{\infty} \{f > n\}\right) = \lim_{n \to \infty} \mu(E_n) \leq \lim_{n \to \infty} \frac{1}{n} \int_X f\, d\mu = 0.$$

■

Lemma. [Vanishing Lemma] Let $f \in \mathcal{M}_+(X, \mathcal{A})$ such that $\int_X f\, d\mu = 0$. Then $f = 0$ almost everywhere in X.

Proof.
We need to show that $\mu(\{f > 0\}) = 0$. Note that:

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \{f > 1/n\}.$$

Let $F_n := \{f > 1/n\}$. Then:

1. $\{F_n\}$ is an increasing sequence,
2. $\frac{1}{n}\chi_{F_n} \leq f\chi_{F_n}$.

By Chebyshev's inequality:

$$0 \leq \mu(F_n) \leq \frac{1}{1/n} \int_X f\, d\mu = 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore:

$$\mu(\{f > 0\}) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n \to \infty} \mu(F_n) = 0.$$

■

Theorem. [Monotone Convergence Theorem, Beppo Levi] Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ and $f : X \to \overline{\mathbb{R}}_+$ be such that:

1. $f_n \leq f_{n+1}$ in X for all n,
2. $f_n \to f$ pointwise in X as $n \to \infty$.

Then:

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \left(\lim_{n\to\infty} f_n \right) \, d\mu = \int_X f \, d\mu.$$

Proof.
Since $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ and $f_n \to f$ pointwise, we have $f \in \mathcal{M}_+(X, \mathcal{A})$.
As $\{f_n\}$ is increasing, by the monotonicity of the integral:

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu.$$

Thus $\{\int_X f_n \, d\mu\}$ is an increasing sequence of real numbers, and there exists:

$$\alpha := \lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

We now prove that $\alpha \geq \int_X f \, d\mu$.
For any $\varepsilon \in (0, 1)$ and $s \in S_f$, define:

$$E_n := \{(1 - \varepsilon)s \leq f_n\}, \quad \forall n \in \mathbb{N}.$$

We have:

1. $\{E_n\} \subseteq \mathcal{A}$,
2. $\{E_n\}$ is increasing (since $\{f_n\}$ is increasing),
3. $X = \bigcup_{n=1}^{\infty} E_n$.

To verify (3): let's note that clearly $\bigcup_{n=1}^{\infty} E_n \subseteq X$ and then let's check if $X \subseteq \bigcup_{n=1}^{\infty} E_n$. Let $x \in X$.

- If $f(x) = \lim_{n\to\infty} f_n(x) = +\infty$, then there exists $\bar{n} \in \mathbb{N}$ such that, for all $n > \bar{n}$, we have $(1 - \varepsilon)s(x) < f_n(x)$, so $x \in E_n$.
- If $f(x) < +\infty$, then there exists $\bar{n} \in \mathbb{N}$ such that, for all $n > \bar{n}$, we have $(1-\varepsilon)s(x) < (1-\varepsilon)f(x) < f_n(x)$, so again $x \in E_n$.

Now we have:

$$(1 - \varepsilon) \int_{E_n} s \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f_n \, d\mu.$$

Taking the limit as $n \to \infty$:

$$(1 - \varepsilon) \int_X s \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu = \alpha.$$

Since ε is arbitrary:

$$\int_X s \, d\mu \leq \alpha \Rightarrow \sup_{s\in S_f} \int_X s \, d\mu \leq \alpha \Rightarrow \int_X f \, d\mu \leq \alpha.$$

This completes the proof. ■

Remark. The step $\lim_{n\to\infty}\int_{E_n} s\,d\mu = \int_X s\,d\mu$ is justified because $\varphi(E) := \int_E s\,d\mu$ is a measure and $\{E_n\}$ is increasing, so by the continuity of the measure:

$$\lim_{n\to\infty}\varphi(E_n) = \varphi\left(\bigcup_{n=1}^{\infty} E_n\right) = \varphi(X) = \int_X s\,d\mu.$$

Lemma. [Fatou's Lemma] Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:

$$\liminf_{n\to\infty}\int_X f_n\,d\mu \geq \int_X \left(\liminf_{n\to\infty} f_n\right)\,d\mu.$$

Proof.
Note that $\liminf_{n\to\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$. Define:

$$\liminf_{n\to\infty} f_n := \sup_{k\geq 1}\inf_{n\geq k} f_n = \sup_{k\geq 1} g_k,$$

where $g_k := \inf_{n\geq k} f_n$. Then:

1. $\{g_k\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ and $\{g_k\}$ is increasing,
2. $g_k \leq f_k$ for all $k \in \mathbb{N}$,
3. $\liminf_{n\to\infty} f_n = \sup_{k\geq 1} g_k = \lim_{k\to\infty} g_k$.

From (2), we have:

$$\int_X g_k\,d\mu \leq \int_X f_k\,d\mu \quad \text{for all } k \in \mathbb{N}.$$

Taking $\liminf$ on both sides:

$$\liminf_{k\to\infty}\int_X g_k\,d\mu \leq \liminf_{k\to\infty}\int_X f_k\,d\mu.$$

From (1), $\{\int_X g_k\,d\mu\}$ is an increasing sequence, so:

$$\liminf_{k\to\infty}\int_X g_k\,d\mu = \lim_{k\to\infty}\int_X g_k\,d\mu.$$

By the Monotone Convergence Theorem and (1), (3):

$$\lim_{k\to\infty}\int_X g_k\,d\mu = \int_X \left(\lim_{k\to\infty} g_k\right)\,d\mu = \int_X \left(\liminf_{n\to\infty} f_n\right)\,d\mu.$$

Therefore:

$$\int_X \left(\liminf_{n\to\infty} f_n\right)\,d\mu \leq \liminf_{n\to\infty}\int_X f_n\,d\mu.$$

■

Remark. Consider $(X, \mathcal{A}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^{\#})$ (counting measure). Define:

$$f_n(x) = \chi_{\{n\}}(x) = \begin{cases} 1 & \text{if } x = n, \\ 0 & \text{if } x \neq n. \end{cases}$$

Then:

- $\lim_{n\to\infty} f_n = 0$, so $\liminf_{n\to\infty} f_n = 0$,
- $\int_{\mathbb{N}} (\liminf_{n\to\infty} f_n)\, d\mu^{\#} = 0$,
- For each $n \in \mathbb{N}$, $\int_{\mathbb{N}} f_n \, d\mu^{\#} = 1$,
- $\liminf_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^{\#} = 1$.

Thus, here we have a strict inequality:

$$\liminf_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^{\#} > \int_{\mathbb{N}} \left(\liminf_{n\to\infty} f_n \right) \, d\mu^{\#}.$$

Theorem. [Integration of Series] Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu \right).$$

Theorem. Let $f \in \mathcal{M}_+(X, \mathcal{A})$.

1. The function $\nu : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined by:

$$\nu(E) = \int_E f \, d\mu, \quad \forall E \in \mathcal{A},$$

   is a measure.

2. For any $g \in \mathcal{M}_+(X, \mathcal{A})$:

$$\int_X g \, d\nu = \int_X gf \, d\mu.$$

Proof.

1. We verify that ν is a measure:

   - $\nu(\emptyset) = \int_\emptyset f \, d\mu = 0$ since $\mu(\emptyset) = 0$,
   - σ-additivity: Let $\{E_k\} \subseteq \mathcal{A}$ be disjoint and $E = \bigcup_{k=1}^\infty E_k$. Then:

$$\begin{aligned}
\nu(E) = \int_X f\chi_E \, d\mu = \int_X f \sum_{k=1}^\infty \chi_{E_k} \, d\mu \\
= \sum_{k=1}^\infty \int_X f\chi_{E_k} \, d\mu \quad \text{(by MCT)} \\
= \sum_{k=1}^\infty \int_{E_k} f \, d\mu = \sum_{k=1}^\infty \nu(E_k).
\end{aligned}$$

2. First, let $g = s \in S_+(X, \mathcal{A})$ with $s = \sum_{k=1}^n c_k \chi_{F_k}$, where $\{F_k\}$ is a partition of X. Then:

$$\begin{aligned}
\int_X s \, d\nu &= \sum_{k=1}^n c_k \nu(F_k) = \sum_{k=1}^n c_k \int_{F_k} f \, d\mu \\
&= \int_X \left(\sum_{k=1}^n c_k f \chi_{F_k} \right) d\mu = \int_X f \left(\sum_{k=1}^n c_k \chi_{F_k} \right) d\mu = \int_X sf \, d\mu.
\end{aligned}$$

   For general $g \in \mathcal{M}_+(X, \mathcal{A})$, the result follows by approximation.

■

Remark.

1. For any $E \in \mathcal{A}$, if $\mu(E) = 0$ then $\nu(E) = 0$,
2. We say that $d\nu = f d\mu$ and $f = \frac{d\nu}{d\mu}$.

Null Sets and Integrals

Theorem. Let $f, g \in \mathcal{M}_+(X, \mathcal{A})$ such that $f = g$ almost everywhere in X. Then:

$$\int_X f \, d\mu = \int_X g \, d\mu.$$

Proof.
Let $N := \{f \neq g\}$, so $N \in \mathcal{A}$ and $\mu(N) = 0$. Then:

$$\int_N f \, d\mu = \int_N g \, d\mu = 0.$$

Therefore:

$$\int_X f \, d\mu = \int_N f \, d\mu + \int_{N^c} f \, d\mu = \int_{N^c} f \, d\mu = \int_{N^c} g \, d\mu = \int_N g \, d\mu + \int_{N^c} g \, d\mu = \int_X g \, d\mu.$$

■

Corollary. Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Then the following are equivalent:

1. $\int_X f \, d\mu = 0$,
2. $f = 0$ almost everywhere in X.

Proof.

$(1) \Rightarrow (2)$ This is the Vanishing Lemma.

$(2) \Rightarrow (1)$ If $f = 0$ a.e., then by the previous theorem with $g = 0$:

$$\int_X f \, d\mu = \int_X 0 \, d\mu = 0.$$

■

Integrable Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$.

Definition. A function $f : X \to \overline{\mathbb{R}}$ is said to be **integrable** on X if $f \in \mathcal{M}(X, \mathcal{A})$ and

$$\int_X f_+ d\mu < \infty, \quad \int_X f_- d\mu < \infty.$$

If $f \in \mathcal{M}(X, \mathcal{A})$, then $f_\pm \in \mathcal{M}(X, \mathcal{A})$, so the integrals are well-defined.
We define

$$\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ integrable on } X\}.$$

Definition. Let $f \in \mathcal{L}^1$. The **Lebesgue integral** of f on X is defined as

$$\int_X f d\mu := \int_X f_+ d\mu - \int_X f_- d\mu.$$

If $E \in \mathcal{A}$, we set

$$\int_E f d\mu := \int_X f \chi_E d\mu = \int_E f_+ d\mu - \int_E f_- d\mu = \int_X f_+ \chi_E d\mu - \int_X f_- \chi_E d\mu.$$

Proposition. Let $f : X \to \overline{\mathbb{R}}$. Then

1. $f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$,
2. $f \in \mathcal{L}^1 \iff f \in \mathcal{M},\ |f| \in \mathcal{L}^1$,
3. $f \in \mathcal{L}^1 \Rightarrow \left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.

Proof.

1. $f \in \mathcal{L}^1 \iff f \in \mathcal{M},\ \int_X f_\pm d\mu < \infty$.
$f_\pm \in \mathcal{L}^1 \iff f_\pm \in \mathcal{M},\ \int_X (f_+)_+ d\mu < \infty,\ \int_X (f_+)_- d\mu < \infty,\ \int_X (f_-)_+ d\mu < \infty,\ \int_X (f_-)_- d\mu < \infty.$
Since $f \in \mathcal{M} \iff f_\pm \in \mathcal{M}$ and
$$\int_X f_\pm d\mu < \infty \iff \int_X (f_+)_+ d\mu < \infty,\ \int_X (f_+)_- d\mu < \infty,\ \int_X (f_-)_+ d\mu < \infty,\ \int_X (f_-)_- d\mu < \infty,$$
the equivalence follows.
2. $f \in \mathcal{L}^1 \iff f \in \mathcal{M},\ \int_X f_\pm d\mu < \infty$.
Note that $f \in \mathcal{M} \Rightarrow |f| \in \mathcal{M}$.
Now, $|f| \in \mathcal{L}^1 \iff f \in \mathcal{L}^1,\ \int_X |f|_\pm d\mu < \infty$, but
$$\int_X |f|_\pm d\mu < \infty = \int_X |f| d\mu < \infty.$$
($\Rightarrow$) $f \in \mathcal{L}^1 \Rightarrow f \in \mathcal{M},\ |f| \in \mathcal{M}$, and
$$\int_X |f| d\mu = \int_X (f_+ + f_-) d\mu < \infty \Rightarrow |f| \in \mathcal{L}^1.$$
($\Leftarrow$) $|f| \in \mathcal{L}^1,\ f \in \mathcal{M} \Rightarrow f \in \mathcal{M}$, and
$$\int_X f_+ d\mu + \int_X f_- d\mu = \int_X (f_+ + f_-) d\mu = \int_X |f| d\mu < \infty.$$
The sum is finite and $\int_X f_+ d\mu$ and $\int_X f_- d\mu$ are both positive; hence, they are both finite. Therefore, $f \in \mathcal{L}^1$.
3. $$\left|\int_X f d\mu\right| = \left|\int_X f_+ d\mu - \int_X f_- d\mu\right| \leq \int_X (f_+ + f_-) d\mu = \int_X |f| d\mu.$$

■

Definition. By point (2), we have an alternative (and most common) definition of $\mathcal{L}^1$:

$$\mathcal{L}^1 = \left\{ f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f| d\mu < \infty \right\}.$$

For all $p \in [1, \infty)$, we define the **Lebesgue spaces**:

$$L^p = \left\{ f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|^p d\mu < \infty \right\}.$$

Remark. Note that $L^p \neq \mathcal{L}^p$.

Proposition. $\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space. That is, linear combinations of integrable functions remain integrable.

Proof.
Let $f, g \in \mathcal{L}^1$ and $\lambda \in \mathbb{R}$. Then $f_\pm, g_\pm$ are finite a.e. in X, so f, g are finite a.e. in X. Hence, $h := f + \lambda g$ is defined a.e. in X and $h \in \mathcal{M}(X, \mathcal{A})$. Moreover,

$$\int_X |h| d\mu \leq \int_X |f| d\mu + |\lambda| \int_X |g| d\mu < \infty,$$

so $h \in \mathcal{L}^1$. Therefore, L$\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space. ■

Remark. Let $f, g \in \mathcal{L}^1$ and $\lambda \in \mathbb{R}$. Then

1. $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$,
2. $\int_X \lambda f d\mu = \lambda \int_X f d\mu$.

Lemma. [Vanishing Lemma 2] Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ be such that

$$\int_E f d\mu = 0 \quad \forall E \in \mathcal{A}.$$

Then $f = 0$ a.e. in X.

Proof.
Define

$$E_+ := \{f \geq 0\} \in \mathcal{A}, \quad E_- := \{f \leq 0\} \in \mathcal{A}.$$

Then

$$\int_{E_+} f d\mu = 0 \Rightarrow f = 0 \text{ a.e. in } E_+,$$

$$\int_{E_-} f d\mu = 0 \Rightarrow f = 0 \text{ a.e. in } E_-.$$

Hence, $f = 0$ a.e. in $E_+ \cup E_- = X$. ■

Remark. The condition can also be written as

$$\int_X f\chi_E d\mu = 0 \quad \forall E \in \mathcal{A}.$$

Theorem. Let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$, and $f = g$ a.e. in X. Then $g \in \mathcal{L}^1$ and

$$\int_X f d\mu = \int_X g d\mu.$$

Proof.
Since $f_\pm = g_\pm$ a.e. in X, by previous results we have

$$\int_X f_\pm d\mu = \int_X g_\pm d\mu.$$

∎

Theorem. [Lebesgue Dominated Convergence Theorem] Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$, $f \in \mathcal{M}(X, \mathcal{A})$ be such that $f_n \to f$ a.e. in X as $n \to \infty$. Suppose there exists $g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ such that $|f_n| \leq g$ a.e. in X for all $n \in \mathbb{N}$.
Then $f_n, f \in \mathcal{L}^1$ and

$$\int_X |f_n - f| d\mu \xrightarrow[n\to\infty]{} 0.$$

In particular,

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu = \int_X f d\mu.$$

Proof.
For all $n \in \mathbb{N}$,

$$\int_X |f_n| d\mu \leq \int_X g d\mu < \infty.$$

Since $|f_n| \leq g$ a.e. in X, it follows that $|f| \leq g$ a.e. in X, so

$$\int_X f d\mu \leq \int_X g d\mu < \infty.$$

Hence, $f_n, f \in \mathcal{L}^1$, and f_n, f are finite a.e. in X.
Define for all $n \in \mathbb{N}$:

$$g_n := 2g - |f_n - f|.$$

Since $|f_n - f| \leq |f_n| + |f| \leq 2g$ a.e. in X, we have $g_n \geq 0$ a.e. in X, so $g_n \in \mathcal{M}_+$.
Now,

$$2\int_X g d\mu = \int_X \left(\lim_{n\to\infty} g_n\right) d\mu \leq \liminf_{n\to\infty} \int_X g_n d\mu = \liminf_{n\to\infty} \int_X [2g - |f_n - f|] d\mu$$

$$= 2\int_X g d\mu + \liminf_{n\to\infty} \int_X [-|f_n - f|] d\mu = 2\int_X g d\mu - \limsup_{n\to\infty} \int_X |f_n - f| d\mu.$$

Thus,

$$2\int_X g d\mu \leq 2\int_X g d\mu - \limsup_{n\to\infty} \int_X |f_n - f| d\mu,$$

so

$$\limsup_{n\to\infty} \int_X |f_n - f| d\mu \leq 0.$$

Hence,

$$0 \leq \liminf_{n\to\infty} \int_X |f_n - f| d\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| d\mu \leq 0,$$

and therefore

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0.$$

For the second part, observe that

$$0 \leq \left| \int_X f_n d\mu - \int_X f d\mu \right| = \left| \int_X (f_n - f) d\mu \right| \leq \int_X |f_n - f| d\mu \xrightarrow[n\to\infty]{} 0,$$

so

$$\int_X f_n d\mu \xrightarrow[n\to\infty]{} \int_X f d\mu.$$

■

Remark. If

1. $\mu(X) < \infty$,
2. $\exists M > 0$ such that $|f_n| \leq M$ a.e. in X,

then take $g := M$. Indeed, $g \in \mathcal{M}$ and

$$\int_X |g| d\mu = \int_X M d\mu = M\mu(X) < \infty \Rightarrow g \in \mathcal{L}^1.$$

Theorem. [Integration of Series] Let $\{f_n\} \subseteq \mathcal{L}^1$ be such that

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty.$$

Then

1. the series $\sum_{n=1}^{\infty} f_n$ converges a.e. in X,
2. $\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.

$\mathcal{L}^1$ and $\mathcal{L}^\infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space. Define the equivalence relation $fRg \iff f = g$ a.e. in X. Let $[f] := \{g \mid fRg\}$ and define

$$\mathcal{L}^1(X, \mathcal{A}, \mu) := L^1(X, \mathcal{A}, \mu)/R.$$

For simplicity, we often say $f \in \mathcal{L}^1$.

Lemma. $\mathcal{L}^1$ is a metric space with the distance

$$d(f, g) := \int_X |f - g| d\mu, \quad \forall f, g \in \mathcal{L}^1.$$

Proof.
The map $d : \mathcal{L}^1 \times \mathcal{L}^1 \to \mathbb{R}$ is well-defined since for all $f, g \in \mathcal{L}^1$,

$$\int_X |f - g| d\mu \leq \int_X |f| d\mu + \int_X |g| d\mu < \infty.$$

For all $f, g \in \mathcal{L}^1$, we have $d(f, g) \geq 0$ and $d(f, f) = 0$.
If $d(f, g) = 0$, then $\int_X |f - g| d\mu = 0$, so by the vanishing lemma, $|f - g| = 0$ a.e in X, hence $f = g$ a.e in X, i.e., $f = g$ in $\mathcal{L}^1$.
Symmetry: $d(f, g) = d(g, f)$.
Triangle inequality: for all $f, g, h \in \mathcal{L}^1$,

$$d(f, g) = \int_X |f - g| d\mu \leq \int_X |f - h| d\mu + \int_X |h - g| d\mu = d(f, h) + d(h, g).$$

■

Remark.

1. L^1 is not a metric space,
2. $\mathcal{L}^1$ is a metric space.

Define

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := L^\infty(X, \mathcal{A}, \mu)/R,$$

with the metric

$$d(f, g) := \operatorname*{ess\,sup}_X |f - g|.$$

Lemma. $\mathcal{L}^\infty$ is a metric space.

Comparisons

Peano-Jordan / Lebesgue Measure

Theorem. Let $E \subseteq \mathbb{R}^N$. If E is Peano-Jordan measurable, then $E \in \mathcal{L}(\mathbb{R}^N)$ and $m_{PJ}(E) = \lambda(E)$.

Example. The set $E = [0,1] \cap \mathbb{Q}$ is not Peano-Jordan measurable, but it is countable and so $E \in \mathcal{L}(\mathbb{R})$ with $\lambda(E) = 0$.

Riemann / Lebesgue Integral

Theorem. Consider $I = [a,b]$ and let $f \in R(I)$. Then $f \in L(I, \mathcal{L}(I), \lambda)$ and

$$\int_I f \, d\lambda = \int_a^b f(x)\, dx.$$

Example. Let $I = [0,1]$ and define $f := \chi_{I \cap \mathbb{Q}}$. Then:

1. $f \notin R(I)$
2. $f \in L^1$
3. $\int_I f \, d\lambda = \lambda(I \cap \mathbb{Q}) = 0$

Generalized Riemann Integral

Let $\alpha, \beta \in \overline{\mathbb{R}}$ and $I = [\alpha, \beta]$. Define

$$R^i(I) := \{f \colon I \to \mathbb{R} \mid f \text{ is integrable in the generalized sense}\}.$$

Theorem. If $f \in R^i(I)$, then $f \in \mathcal{M}(I, \mathcal{L}(I))$. Furthermore, if $f \in R^i(I)$ and $|f| \in R^i(I)$, then $f \in L^1$. Moreover,

$$\int_I f \, d\lambda = \int_\alpha^\beta f(x)\, dx.$$

Remark. [Dirichlet Integral] Define $f \colon [0, \infty) \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Clearly $f \in R^i(I)$ and

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

But notice that

$$\int_{\mathbb{R}_+} \left| \frac{\sin x}{x} \right| d\lambda = \int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty \quad \Rightarrow \quad f \notin L^1.$$

Types of Convergence

Convergences

Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$ with $f_n : X \to \mathbb{R}$ and $f : X \to \overline{\mathbb{R}}$.

Pointwise Convergence

$f_n \to f$ pointwise in X as $n \to \infty$ if and only if

$$\forall x \in X, \quad f_n(x) \to f(x) \quad \text{as } n \to \infty.$$

Uniform Convergence

$f_n \to f$ uniformly in X as $n \to \infty$ if and only if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Almost Everywhere Convergence

$f_n \to f$ a.e. in X as $n \to \infty$ if and only if

$$\{x \in X : f_n(x) \not\to f(x)\}^c \in \mathcal{N}_\mu.$$

Convergence in L^1

For $\{f_n\} \subseteq L^1$ and $f \in L^1$, we say $f_n \to f$ in L^1 as $n \to \infty$ if and only if

$$d(f_n, f) \to 0 \quad \text{as } n \to \infty,$$

which is equivalent to

$$\int_X |f_n - f| \, d\mu \to 0 \quad \text{as } n \to \infty.$$

Convergence in L^∞

For $\{f_n\} \subseteq L^\infty$ and $f \in L^\infty$, we say $f_n \to f$ in L^∞ as $n \to \infty$ if and only if

$$d(f_n, f) \to 0 \quad \text{as } n \to \infty,$$

which is equivalent to

$$\operatorname*{ess\,sup}_X |f_n - f| \to 0 \quad \text{as } n \to \infty.$$

Convergence in Measure

We say that $f_n \to f$ in measure as $n \to \infty$ if

$$\forall \varepsilon > 0, \quad \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0 \quad \text{as } n \to \infty.$$

Theorems

Theorem. Let $\mu(X) < \infty$ and let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X. If $f_n \to f$ a.e. in X, then $f_n \to f$ in measure.

Remark. When $\mu(X) = \infty$, convergence a.e. $\not\Rightarrow$ convergence in measure.

Example. Let $f_n : \mathbb{R} \to \mathbb{R}$ with $\lambda(\mathbb{R}) = \infty$ and define $f_n := \chi_{[n,+\infty)}$. Then $f_n \to 0$ pointwise in $\mathbb{R}$, but

$$\mu(\{f_n \geq 1/2\}) = \mu([n, +\infty)) = +\infty,$$

so $f_n \not\to 0$ in measure.

Remark. Convergence in measure $\not\Rightarrow$ convergence a.e. This can be shown using the counterexample of the Typewriter sequence.

Theorem. Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X. If $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. in X as $k \to \infty$.

Theorem. Let $f_n, f \in L^1(X, \mathcal{A}, \mu)$. If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof.
Suppose, by contradiction, that $f_n \not\to f$ is in measure. Then there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$\mu(\{|f_n - f| \geq \varepsilon\}) \geq \sigma \quad \text{for infinitely many } n \in \mathbb{N}.$$

Thus,

$$\int_X |f_n - f| \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon \, d\mu = \varepsilon \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon\sigma$$

for infinitely many $n \in \mathbb{N}$, which implies $f_n \not\to f$ in L^1. This is a contradiction. ■

Remark. If $f_n \to f$ in measure, this $\not\Rightarrow$ $f_n \to f$ in L^1.

Example. Let $f_n(x) = n\chi_{[0,1/n]}(x)$ for $x \in [0, 1]$ and $f = 0$. Then $f_n \to f$ a.e. in $[0, 1]$. Since $\lambda([0, 1]) = 1 < \infty$, we have $f_n \to 0$ in measure. However,

$$\int_{[0,1]} |f_n - 0| \, d\lambda = \int_0^1 f_n \, dx = \int_0^{1/n} n \, dx = n \cdot \frac{1}{n} = 1,$$

so $f_n \not\to f$ in $L^1([0, 1])$.

Corollary. If $f_n \to f$ in L^1, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. in X as $k \to \infty$.

Remark. To check if $\{f_n\}$ converges in L^1, first study the limit a.e. If $f := \lim_{n\to\infty} f_n$ exists a.e. in X, then f is the candidate limit for the L^1 convergence.

Typewriter Sequence (Rademacher Sequence)

Determine $k \in \mathbb{N}$ such that $2^k \leq n \leq 2^{k+1}$ (so $k = \lfloor \log_2 n \rfloor$). Define

$$f_n(x) := \chi_{\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right]}(x), \quad I_n := \left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right].$$

For example:

- $n = 1$, $k = 0$: $I_1 = [0, 1]$
- $n = 2$, $k = 1$: $I_2 = [0, 1/2]$
- $n = 3$, $k = 1$: $I_3 = [1/2, 1]$
- $n = 4$, $k = 2$: $I_4 = [0, 1/4]$
- $n = 5$, $k = 2$: $I_5 = [1/4, 1/2]$

We observe that for every $x \in [0, 1]$, there exists a countable set $J \subset \mathbb{N}$ such that $x \in I_n$ for all $n \in J$, and $x \notin I_n$ for all $n \in \mathbb{N} \setminus J$. It follows that

$$f_n(x) = 1 \text{ for infinitely many } n \in \mathbb{N}, \quad f_n(x) = 0 \text{ for infinitely many } n \in \mathbb{N}.$$

Thus, the limit $\lim_{n\to\infty} f_n(x)$ does not exist. Pointwise convergence does not occur, and this holds for every x, so we cannot have a.e. pointwise convergence.
However,

$$\int_{[0,1]} f_n \, d\lambda = \int_0^1 f_n \, dx = \int_{\frac{n-2^k}{2^k}}^{\frac{n-2^k+1}{2^k}} 1 \, dx = \frac{1}{2^k} \to 0 \quad \text{as } n \to \infty \ (k \to \infty).$$

Hence, $f_n \to 0$ in $L^1([0, 1])$ and $f_n \to 0$ in measure in $[0, 1]$.

Differentiation and Integration

Absolute Continuity

Remark. The following is not correct: Given $f : [a, b] \to \mathbb{R}$ continuous,

$$\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt \xrightarrow[h\to 0]{} f(x_0).$$

Let's consider L^1 now. In L^1 we cannot write the previous limit, since we don't know if $f(x_0)$ is defined. But we have the following:

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt = 0.$$

Definition. [Lebesgue Points] A point $x_0 \in [a, b]$ is a **Lebesgue point** for f if

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt = 0.$$

Theorem. If $f \in L^1((a, b))$, then almost every $x_0 \in [a, b]$ is a Lebesgue point for f. So almost every point is a regular and good point.

Theorem. [First Fundamental Theorem of Calculus in L^1] If $f \in L^1((a,b))$, then

$$F(x) := \int_{[a,x]} f \, d\lambda$$

is differentiable a.e. in (a,b) and $F'(x) = f(x)$ for a.e. $x \in (a,b)$.

Proof.
The key insight here is the connection between Lebesgue points and differentiability. Let's break down the proof step by step:

1. **Setting up the difference quotient**: For a Lebesgue point $x \in [a,b]$ and $h \neq 0$ with $x+h \in [a,b]$, we consider:
$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] \, dt.$$
This expresses the difference between the difference quotient and the function value at x.

2. **Applying the triangle inequality**: Taking absolute values and using the triangle inequality for integrals:
$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{|h|} \int_x^{x+h} |f(t) - f(x)| \, dt.$$

3. **Crucial observation**: The right-hand side is exactly the expression that defines a Lebesgue point! For x to be a Lebesgue point, we require:
$$\lim_{h \to 0} \frac{1}{|h|} \int_x^{x+h} |f(t) - f(x)| \, dt = 0.$$

4. **Conclusion**: Since almost every point is a Lebesgue point (by the previous theorem), and at Lebesgue points the limit above is zero, we conclude that:
$$\lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = 0,$$
which means $F'(x) = f(x)$ for almost every $x \in (a,b)$.

The beauty of this proof lies in how it connects the measure-theoretic concept of Lebesgue points with the differential calculus concept of derivative. ■

Absolutely Continuous Functions

Let $J = [a, b]$ and define

$$\mathcal{F}(J) := \{\text{finite collections of closed intervals } \subseteq J \text{ without interior points in common}\}.$$

Definition. A function $f : J \to \mathbb{R}$ is said to be **absolutely continuous** in J if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\{[a_k, b_k]\} \in \mathcal{F}(J)$ with $k = 1, \ldots, n$ for which

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

one has

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

We define $AC([a, b]) := \{f : [a, b] \to \mathbb{R} \text{ absolutely continuous}\}$.

Remark. Consider $n = 1$, $k = 1$:

$$\{[a_k, b_k]\} = \begin{cases} [x, y], & y \geq x \\ [y, x], & y < x \end{cases}.$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have $|f(y) - f(x)| < \varepsilon$. So f is uniformly continuous in $[a, b]$.

1. So an absolutely continuous function is uniformly continuous and therefore continuous:

$$f \in AC \Rightarrow f \in UC \Rightarrow f \in C^0.$$

2. To show that the implication doesn't go the other way, consider

$$f(x) = \begin{cases} x \sin(1/x), & x \in [-1, 1] \setminus \{0\} \\ 0, & x = 0 \end{cases}.$$

   Then $f \in UC([-1, 1])$ but $f \notin AC([-1, 1])$.

3. If f is Lipschitz in $[a, b]$, then $f \in AC([a, b])$.

   Proof.
   f is Lipschitz means there exists $L > 0$ such that $|f(y) - f(x)| \leq L|y - x|$ for all $x, y \in [a, b]$. For every $\varepsilon > 0$,

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} L|b_k - a_k| = L \sum_{k=1}^{n} |b_k - a_k| < L\delta < \varepsilon,$$

   provided that $\delta < \varepsilon/L$. ■

4. $f \in AC([a, b]) \not\Rightarrow f$ Lipschitz. Consider $f(x) = \sqrt{x}$ on $[0, 1]$. This function is not Lipschitz (it is Hölder continuous), but

$$f(x) = \sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}} \, dt \in AC$$

   because $1/(2\sqrt{t}) \in L^1$.

Theorem. [Absolute Continuity of the Integral] Let $f \in M_+(X, \mathcal{A})$ be such that $\int_X f\,d\mu < \infty$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $E \in \mathcal{A}$ with $\mu(E) < \delta$, we have

$$\int_E f\,d\mu < \varepsilon.$$

Proof.
This proof demonstrates a fundamental property of integrable functions. Let's examine the strategy:

1. **Constructing approximating sets**: We define $F_n = \{f < n\}$, which are measurable sets where f is bounded. The sequence $\{F_n\}$ is increasing and covers almost all of X since f is finite almost everywhere (because $\int_X f\,d\mu < \infty$).

2. **Using continuity of the integral**: By the Monotone Convergence Theorem (or continuity from below):

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_{F_n} f\,d\mu.$$

   This means that for large n, most of the integral's mass is concentrated on the sets F_n where f is bounded.

3. **Controlling the tail**: For any $\varepsilon > 0$, there exists $\bar{n}$ such that:

$$\int_{F_n^C} f\,d\mu < \frac{\varepsilon}{2} \quad \text{for all } n > \bar{n}.$$

   This bounds the integral over the "bad" set where f is large.

4. **Splitting the integral**: For any measurable set E with $\mu(E) < \delta$, we split:

$$\int_E f\,d\mu = \int_{E\cap F_n} f\,d\mu + \int_{E\cap F_n^C} f\,d\mu.$$

5. **Estimating both parts**:

   - On $E \cap F_n$, f is bounded by n, so:

$$\int_{E\cap F_n} f\,d\mu \leq n \cdot \mu(E) < n\delta.$$

   - On $E \cap F_n^C$, we use the tail estimate:

$$\int_{E\cap F_n^C} f\,d\mu \leq \int_{F_n^C} f\,d\mu < \frac{\varepsilon}{2}.$$

6. **Choosing** δ: Taking $\delta = \frac{\varepsilon}{2n}$, we get:

$$\int_E f\,d\mu < n \cdot \frac{\varepsilon}{2n} + \frac{\varepsilon}{2} = \varepsilon.$$

This proof shows that for integrable functions, the contribution from small sets can be made arbitrarily small, which is a crucial property in measure theory. ■

Corollary. Let $I = [a, b]$ and $f \in L^1(I)$. Then $F(x) := \int_{[a,x]} f \, d\lambda$ is absolutely continuous in I.

Proof.
This corollary shows that indefinite integrals of L^1 functions are absolutely continuous:

1. **Setup**: Let $\{[a_k, b_k]\}$ be a finite collection of non-overlapping intervals in $[a, b]$ with:
$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

2. **Estimating the variation**: For $F(x) = \int_{[a,x]} f \, d\lambda$, we have:
$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| = \sum_{k=1}^{n} \left| \int_{[a_k, b_k]} f \, d\lambda \right|.$$

3. **Using triangle inequality**:
$$\sum_{k=1}^{n} \left| \int_{[a_k, b_k]} f \, d\lambda \right| \leq \sum_{k=1}^{n} \int_{[a_k, b_k]} |f| \, d\lambda.$$

4. **Combining integrals**: Since the intervals are disjoint:
$$\sum_{k=1}^{n} \int_{[a_k, b_k]} |f| \, d\lambda = \int_E |f| \, d\lambda,$$
where $E = \bigcup_{k=1}^{n} [a_k, b_k]$ and $\lambda(E) < \delta$.

5. **Applying absolute continuity**: By the previous theorem, since $|f| \in L^1$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda(E) < \delta$ implies $\int_E |f| \, d\lambda < \varepsilon$.

■

Theorem. [Second Fundamental Theorem of Calculus] Let $\varphi : [a,b] \to \mathbb{R}$. The following are equivalent:

1. $\varphi \in AC([a,b])$
2. φ is differentiable a.e. in $[a,b]$, $\varphi' \in L^1([a,b])$, and

$$\varphi(x) - \varphi(a) = \int_{[a,x]} \varphi' \, d\lambda, \quad \forall x \in [a,b].$$

Remark. Take $\varphi : [a,b] \to \mathbb{R}$ with $\varphi \in C^1((a,b]) \cap C^0([a,b])$ and $\varphi' \in L^1([a,b])$. Then condition (2) is fulfilled.

Proof.
For every $\xi \in (a,b)$, we have $\varphi \in C^1([\xi,b])$, so we can use the fundamental formula of calculus:

$$\varphi(x) - \varphi(\xi) = \int_{[\xi,x]} \varphi'(t) \, dt.$$

Now notice that $\varphi(x) - \varphi(\xi) \xrightarrow[\xi \to a^+]{} \varphi(x) - \varphi(a)$. For the right-hand side:

$$\int_{[\xi,x]} \varphi'(t) \, dt = \int_{[a,x]} \varphi'(t) \chi_{[\xi,x]}(t) \, dt.$$

We have $\varphi'(t)\chi_{[\xi,x]}(t) \in M(X,\mathcal{A})$ and $\varphi'(t)\chi_{[\xi,x]}(t) \xrightarrow[\xi \to a^+]{} \varphi'(t)$ in $[a,x]$, with $|\varphi'(t)\chi_{[\xi,x]}(t)| \leq \varphi'(t) \in L^1([a,b])$ for all $t \in [a,x], \xi$. By the Dominated Convergence Theorem, we conclude that

$$\varphi(x) - \varphi(a) = \int_{[a,x]} \varphi'(t) \, dt.$$

■

Sobolev Spaces

Proposition. The following are equivalent:

- $u \in AC([a,b])$
- $u \in C^0([a,b])$, u is differentiable a.e., $u' \in L^1([a,b])$, and

$$\int_a^b u'\varphi \, dx = -\int_a^b u\varphi' \, dx, \quad \forall \varphi \in C_c^\infty((a,b)).$$

We can also consider $\forall \varphi \in \mathrm{Lip}([a,b])$ with $\varphi(b) = \varphi(a) = 0$.

Definition. [Weak Derivative] Let $u \in L^p((a,b))$, $p \in [1,\infty)$. We say that $w \in L^1_{\text{loc}}((a,b))$ is the **weak derivative** of u if

$$\int_a^b u\varphi' \, dx = -\int_a^b w\varphi \, dx, \quad \forall \varphi \in C_c^\infty((a,b)).$$

We write $u' \equiv w$.

Definition. We define the **Sobolev spaces**:

$$W^{1,p}(I) := \{u : I \to \mathbb{R} : u \in L^p(I),\, w \in L^p(I)\}.$$

When $p = 1$, we have $W^{1,1}(I)$; when $p = 2$, we have $W^{1,2}(I) \equiv H^1(I)$.

Remark. Both u and w are equivalence classes.

Lemma. [Vanishing Lemma, 3rd Version] Let $u \in L^1([a,b])$. Suppose that

$$\int_a^b u\varphi\, dx = 0, \quad \forall \varphi \in C_c^\infty((a,b)).$$

Then $u = 0$ a.e. in $[a,b]$.

Proposition. If the weak derivative w exists, it is unique.

Proof.
The uniqueness of weak derivatives is fundamental to the theory of Sobolev spaces:

1. **Assumption of non-uniqueness**: Suppose w_1 and w_2 are both weak derivatives of u. Then for any test function $\varphi \in C_c^\infty((a,b))$:
$$\int_a^b w_1\varphi\, dx = -\int_a^b u\varphi'\, dx = \int_a^b w_2\varphi\, dx.$$
2. **Difference vanishes**: Subtracting gives:
$$\int_a^b (w_1 - w_2)\varphi\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty((a,b)).$$
3. **Applying the Vanishing Lemma**: The Vanishing Lemma (3rd version) states that if a function in L^1 integrates to zero against all test functions, then it must be zero almost everywhere.
4. **Conclusion**: Therefore, $w_1 = w_2$ almost everywhere, meaning the weak derivative is unique as an element of L^1 (i.e., unique up to sets of measure zero).

This proof highlights the power of test functions in distribution theory - they can ”detect” when two functions are essentially the same. ■

Remark. In principle, weak derivative and a.e. derivative are different. However, by the previous proposition, they coincide if $u \in AC$. Therefore, $AC(I) \subseteq W^{1,1}(I)$.

Theorem. $AC(I) = W^{1,1}(I)$.

Proof.
This is a fundamental characterization of absolutely continuous functions:
Forward direction ($AC(I) \subseteq W^{1,1}(I)$):

1. If $u \in AC(I)$, then by the Second Fundamental Theorem of Calculus, u is differentiable almost everywhere, $u' \in L^1(I)$, and:
$$u(x) - u(a) = \int_a^x u'(t)\, dt.$$
2. **Verifying weak derivative condition**: For any $\varphi \in C_c^\infty((a,b))$, we can integrate by parts (justified by absolute continuity):
$$\int_a^b u\varphi'\, dx = [u\varphi]_a^b - \int_a^b u'\varphi\, dx = -\int_a^b u'\varphi\, dx,$$
since φ has compact support in (a,b), so $\varphi(a) = \varphi(b) = 0$.
3. Therefore, u' is indeed the weak derivative of u, so $u \in W^{1,1}(I)$.

Reverse direction ($W^{1,1}(I) \subseteq AC(I)$):

1. **Constructing a candidate**: Given $u \in W^{1,1}(I)$ with weak derivative w, define:
$$z(x) = \int_a^x w(t)\, dt.$$
2. **Properties of z**: Since $w \in L^1(I)$, the integral function z is absolutely continuous (by the previous corollary).
3. **Comparing u and z**: For any test function $\varphi \in C_c^\infty((a,b))$:
$$\int_a^b z\varphi'\, dx = -\int_a^b z'\varphi\, dx = -\int_a^b w\varphi\, dx = \int_a^b u\varphi'\, dx,$$
where the last equality uses that w is the weak derivative of u.
4. **Difference has zero weak derivative**: Therefore:
$$\int_a^b (z-u)\varphi'\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty((a,b)).$$
5. **Constant difference**: By the corollary to the Vanishing Lemma, this implies $z - u$ is constant almost everywhere, say $z - u = c$ a.e.
6. **Conclusion**: Since z is absolutely continuous and differs from u by only a constant, u is also absolutely continuous.

This proof beautifully connects the classical notion of absolute continuity with the modern distributional approach to derivatives. ∎

To explain the previous "Therefore", we used a corollary of the Vanishing Lemma:

Corollary. Let $u \in L^1([a, b])$. Suppose that

$$\int_a^b u\varphi' \, dx = 0, \quad \forall \varphi \in C_c^\infty((a, b)).$$

Then u is constant a.e. in $[a, b]$. Because this means that the weak derivative of u is 0.

Derivative of Measures

Let $(X, \mathcal{A})$ be a measurable space, with μ, ν measures.

Definition. A function $\phi \in M_+(X, \mathcal{A})$ is said to be the **Radon-Nikodym derivative** of ν with respect to μ if

$$\nu(E) = \int_E \phi d\mu \quad \forall E \in \mathcal{A}.$$

We write $\phi = \frac{d\nu}{d\mu}$.

Definition. We say that ν is **absolutely continuous** with respect to μ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0.$$

We write $\nu \ll \mu$.

Theorem. [Radon-Nikodym] Let $(X, \mathcal{A})$ be a measurable space, with μ, ν measures. Suppose that $\nu \ll \mu$ and that μ is σ-finite. Then $\frac{d\nu}{d\mu}$ exists and is unique μ-almost everywhere.

Remark. The Radon-Nikodym theorem establishes that under the conditions $\nu \ll \mu$ and μ σ-finite, the measure ν can be "reconstructed" from μ by integrating the derivative $\frac{d\nu}{d\mu}$. This is the measure-theoretic analogue of the Fundamental Theorem of Calculus.

Metric Spaces

Let (X, d) be a metric space.

Definition. For any $r > 0$ and $x_0 \in X$, the **open ball** of radius r centered at x_0 is defined as:

$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

Let $\{x_n\} \subset X$ be a sequence.

Definition. A sequence $\{x_n\}$ is said to be **bounded** if there exist $x_0 \in X$ and $k > 0$ such that:

$$d(x_n, x_0) \leq k, \quad \forall n \in \mathbb{N}.$$

Definition. A sequence $\{x_n\}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $\nu_\varepsilon \in \mathbb{N}$ such that:

$$d(x_n, x_m) < \varepsilon, \quad \forall n, m > \nu_\varepsilon.$$

Definition. A metric space (X, d) is **complete** if every Cauchy sequence $\{x_n\} \subset X$ converges in X.

Example.

1. $(\mathbb{R}^n, |\cdot|_2)$ is complete.
2. $(C^0([a, b]), d_\infty)$ is complete, where
$$d_\infty(f, g) := \sup_{x \in [a,b]} |f(x) - g(x)|.$$
3. $(C^0([a, b]), d)$ with the metric
$$d(f, g) := \int_a^b |f(x) - g(x)|\, dx$$
is **not** complete. Its completion is the space $L^1([a, b])$.

Density and Separability

Definition. A subset $A \subset X$ is **dense** in X if $\overline{A} = X$, where the closure $\overline{A} = A \cup \partial A$ consists of all points $x \in X$ for which there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$ as $n \to \infty$.

Definition. A metric space X is **separable** if it contains a countable dense subset $A \subset X$.

Example. $\mathbb{R}$ is separable, since $\mathbb{Q}$ is a countable dense subset.

Theorem. The space $C^0([a, b])$ is separable.

Compactness in Metric Spaces

Definition. A subset $E \subseteq X$ is **compact** if every open cover of E has a finite subcover.

Definition. A subset $E \subseteq X$ is **sequentially compact** if every sequence $\{x_n\} \subseteq E$ has a convergent subsequence with limit in E.

Theorem. In a metric space X, a subset $E \subseteq X$ is compact if and only if it is sequentially compact.

Proposition. If $E \subseteq X$ is compact, then E is closed and bounded. The converse is not true in general.

Remark. In $\mathbb{R}^n$, a subset $E \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Compactness in $C^0(X)$

Let X be a compact metric space, and define:

$$C^0(X) := \{f : X \to \mathbb{R} \text{ continuous}\},$$

equipped with the metric:

$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

This is a complete metric space.

Definition. A family $\mathcal{A} \subset C^0(X)$ is **equicontinuous** if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $f \in \mathcal{A}$ and for all $x, y \in X$:

$$d(x, y) < \delta_\varepsilon \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Definition. A subset $E \subseteq X$ is **relatively compact** if its closure $\overline{E}$ is compact.

Theorem. [Ascoli-Arzelà] Let X be a compact metric space and $\mathcal{F} \subset C^0(X)$. Then:

$$\mathcal{F} \text{ is bounded and equicontinuous} \quad \Longleftrightarrow \quad \mathcal{F} \text{ is relatively compact.}$$

In particular, if $\mathcal{F}$ is closed, then:

$$\mathcal{F} \text{ is compact} \quad \Longleftrightarrow \quad \mathcal{F} \text{ is bounded and equicontinuous.}$$

Application to Sequences of Functions

Consider a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C^0(X)$, where $X = [a, b] \subset \mathbb{R}$ or more generally $X = K \subseteq \mathbb{R}^n$ with K compact.

- **Equicontinuity of** $\{f_n\}$: The choice of $\delta = \delta(\varepsilon)$ must be independent of n.
- **Boundedness of** $\{f_n\}$: There exists $M > 0$, independent of n, such that:

$$\sup_{x\in X} |f_n(x)| \leq M, \quad \forall n \in \mathbb{N}.$$

The family $\{f_n\}$ is relatively compact if and only if its closure $\overline{\{f_n\}}$ is compact. This is equivalent to the following: for every subsequence $\{f_{n_i}\} \subseteq \overline{\{f_n\}}$, there exists a further subsequence $\{f_{n_{i_k}}\} \subset \overline{\{f_n\}}$ and a function $f \in C^0(X)$ such that:

$$f_{n_{i_k}} \to f \quad \text{uniformly in } X \text{ as } k \to \infty.$$

In particular, if we consider the original sequence $\{f_n\}$ itself, this implies the existence of a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and a function $f \in C^0(X)$ such that $f_{n_k} \to f$ uniformly.

Corollary. If $\{f_n\} \subset C^0(X)$ is bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and a function $f \in C^0(X)$ such that $f_{n_k} \to f$ uniformly in $C^0(X)$.

Corollary. Let $\{f_n\} \subset C^0(X)$ with X a compact metric space. Assume $\{f_n\}$ is bounded and equicontinuous. Then there exists $\{f_{n_k}\} \subset \{f_n\}$ and $f \in C^0(X)$ such that $f_{n_k} \to f$ for $k \to \infty$ in $C^0(X)$.

Corollary. Let $\{f_n\} \subset C^1([a,b])$ and suppose there exists $c > 0$ such that:

1. $\sup_{[a,b]} |f_n| \leq c, \quad \forall n \in \mathbb{N}$
2. $\sup_{[a,b]} |f_n'| \leq c, \quad \forall n \in \mathbb{N}$

Then there exists $\{f_{n_k}\} \subset \{f_n\}$ and $f \in C^0([a,b])$ such that $f_{n_k} \to f$ for $k \to \infty$ in $C^0([a,b])$.

Proof.
The proof of this corollary illustrates the power of the Arzelà-Ascoli theorem:

1. **Boundedness in** C^0: Condition (1) directly gives that $\{f_n\}$ is uniformly bounded:
$$\sup_{[a,b]} |f_n| \leq c \quad \text{for all } n.$$
2. **Equicontinuity from derivatives**: Condition (2) implies the sequence is equi-Lipschitz. By the Mean Value Theorem, for any $x, y \in [a,b]$:
$$|f_n(y) - f_n(x)| = |f_n'(\xi_n)| \cdot |y - x| \leq c|y - x|,$$
for some ξ_n between x and y.
3. **Uniform equicontinuity**: This Lipschitz condition gives uniform equicontinuity. For any $\varepsilon > 0$, take $\delta = \varepsilon/c$. Then $|x - y| < \delta$ implies:
$$|f_n(y) - f_n(x)| \leq c|x - y| < c \cdot \frac{\varepsilon}{c} = \varepsilon \quad \text{for all } n.$$
4. **Applying Arzelà-Ascoli**: The Arzelà-Ascoli theorem states that a uniformly bounded and equicontinuous sequence of functions on a compact metric space has a uniformly convergent subsequence.
5. **Conclusion**: Therefore, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly to some $f \in C^0([a,b])$.

This result is particularly important in the calculus of variations and PDE theory, where it's often used to extract convergent subsequences from minimizing sequences. ■

Remark. Consider $C^1([a,b])$ with the metric:
$$d(f,g) := \sup_{[a,b]} |f - g| + \sup_{[a,b]} |f' - g'|,$$
which is a complete metric space. Then the two conditions in the corollary mean that $\{f_n\} \subset C^1$ is bounded.

Normed and Banach Spaces

Let X be a vector space.
A **norm** is a function $\|\cdot\| : X \to [0,\infty)$ such that:

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$, $x \in X$
- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

The pair $(X, \|\cdot\|)$ is called a **normed space**.
Define the metric $d(x,y) := \|x-y\|$. Then (X,d) is a metric space.

Examples

Example. $\mathbb{R}^n$, $\dim \mathbb{R}^n = n < \infty$. For $p \in [1,\infty)$:

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad \|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$$

Example. $C^0([a,b])$, $\|f\|_{C^0} := \max_{[a,b]} |f|$.

$$C^k([a,b]), \quad \|f\|_{C^k} := \sum_{i=0}^k \max_{[a,b]} |f^{(i)}|$$

where $f^{(0)} = f$.

Example. For $p \in [1,\infty)$, define:

$$L^p(X,\mathcal{A},\mu) := \left\{ f \in M(X,\mathcal{A}) : \int_X |f|^p d\mu < \infty \right\}, \quad \|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}$$

For $p = \infty$:

$$L^\infty(X,\mathcal{A},\mu), \quad \|f\|_\infty := \operatorname*{ess\,sup}_X |f|$$

In L^p, we identify functions that are equal almost everywhere.

Example. $AC([a,b])$, the space of absolutely continuous functions, with norm:

$$\|f\|_{AC} = |f(a)| + \|f'\|_1 \quad \text{or} \quad \|f\|_{AC} = \|f\|_1 + \|f'\|_1$$

Example. $W^{1,p}([a,b])$, the Sobolev space, with norm:

$$\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f'\|_{L^p}$$

Example. For $p \in [1, \infty)$, the space of sequences:

$$\ell^p := \left\{ x = (x^{(1)}, x^{(2)}, \dots) : \|x\|_p := \left(\sum_{k=1}^{\infty} |x^{(k)}|^p \right)^{1/p} < \infty \right\}$$

For $p = \infty$:

$$\ell^\infty := \left\{ x : \|x\|_\infty := \sup_{k \in \mathbb{N}} |x^{(k)}| < \infty \right\}$$

Remark. If $p < q$, then $\ell^p \subset \ell^q$.

Remark. $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#) = \ell^p$, where $\mu^\#$ is the counting measure.

Sequences and Series

Let $(X, \|\cdot\|)$ be a normed space, $\{x_n\} \subset X$, $x \in X$.

Definition. We say $x_n \to x$ as $n \to \infty$ if $\|x_n - x\| \to 0$ as $n \to \infty$, i.e., $d(x_n, x) \to 0$.

Remark. If $x_n \to x$, then $\|x_n\| \to \|x\|$. The converse is not true.

Proof.
By the reverse triangle inequality: $|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0$. ∎

Definition. A sequence $\{x_n\} \subset X$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > \bar{n}$.

Remark. Every convergent sequence is Cauchy. The converse is not true.

Definition. A sequence $\{x_n\}$ is **bounded** if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Remark. Every Cauchy sequence is bounded.

Definition. For a sequence $\{x_n\} \subset X$, define the partial sums $s_n := \sum_{k=1}^n x_k$. The series $\sum_{k=1}^\infty x_k$ is **convergent** if the sequence $\{s_n\}$ converges to some $x \in X$, i.e., $\|s_n - x\| \to 0$. We write $x = \sum_{k=1}^\infty x_k$.

Remark. If $\sum_{k=1}^\infty \|x_k\|$ converges, it does not necessarily imply that $\sum_{k=1}^\infty x_k$ converges.

Completeness

Definition. A normed space $(X, \|\cdot\|)$ is **complete** if the metric space (X, d) is complete, i.e., every Cauchy sequence in X converges in X. A complete normed space is called a **Banach space**.

Example. All the examples above are Banach spaces.

Theorem. [Criterion for completeness]

1. Let X be a Banach space and $\{x_n\} \subset X$. If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.
2. Let X be a normed space. If for every $\{x_n\} \subset X$ such that $\sum_{n=1}^{\infty} \|x_n\|$ converges implies that $\sum_{n=1}^{\infty} x_n$ converges, then X is Banach.

Finite and Infinite Dimension

Definition. A vector space X has **infinite dimension** if for every $n \in \mathbb{N}$, there exists a set of n linearly independent vectors in X.

Remark. If Y is a finite-dimensional vector subspace of a normed space X, then Y is closed. If Y has infinite dimension, it is not necessarily closed.

Definition. Let $(X, \|\cdot\|)$ and $(X, \|\cdot\|_\#)$ be normed spaces. Two norms $\|\cdot\|$ and $\|\cdot\|_\#$ are **equivalent** if there exist constants $m, M > 0$ such that:

$$m\|x\| \leq \|x\|_\# \leq M\|x\| \quad \text{for all } x \in X.$$

Theorem. If $\dim X < \infty$, then all norms on X are equivalent.

Remark. $(C^0([a,b]), \|\cdot\|_\infty)$ is a Banach space, but $(C^0([a,b]), \|\cdot\|_1)$ is not complete. Hence, these norms are not equivalent. In fact, $\dim C^0 = \infty$.

Balls and Closure

Let X be a normed space, $x_0 \in X$, $r > 0$.

- Open ball: $B_r(x_0) := \{x \in X : \|x - x_0\| < r\}$
- Closed ball: $\overline{B}_r(x_0) := \{x \in X : \|x - x_0\| \le r\}$

The closure of $B_r(x_0)$ is denoted $\overline{B_r(x_0)}$. In general, $\overline{B_r(x_0)} = \overline{B}_r(x_0)$, but in a metric space, it is possible that $\overline{B_r(x_0)} \subsetneq \overline{B}_r(x_0)$.

Compactness

Example. Let X be a normed space with $\dim X < \infty$, and let $E \subsetneq X$ be a vector subspace. Then there exists $x \in X$ with $\|x\| = 1$ and $\operatorname{dist}(x, E) \ge 1$.
Indeed, since $E \subsetneq X$, there exists $y \in X \setminus E$. Let $\eta := \operatorname{Proj}_E y$ and $x := (y - \eta)/\|y - \eta\|$. Then $\|x\| = 1$ and $\operatorname{dist}(x, E) \ge 1$.

Lemma. [Riesz] Let X be a normed space and $E \subsetneq X$ a closed vector subspace. Then for every $\varepsilon > 0$, there exists $x \in X$ with $\|x\| = 1$ and $\operatorname{dist}(x, E) \ge 1 - \varepsilon$, where $\operatorname{dist}(x, E) := \inf_{\xi \in E} \|x - \xi\|$.

Proof.
Let $y \in X \setminus E$. Then $d := \operatorname{dist}(y, E) > 0$ since E is closed. For $\varepsilon \in (0, 1)$, choose $\eta \in E$ such that:

$$d \le \|y - \eta\| \le \frac{d}{1 - \varepsilon} > d$$

Define $x := (y - \eta)/\|y - \eta\|$. Then $\|x\| = 1$. For any $\xi \in E$:

$$\|x - \xi\| = \left\| \frac{y - \eta}{\|y - \eta\|} - \xi \right\| = \frac{1}{\|y - \eta\|} \|y - (\eta + \xi\|y - \eta\|)\| \ge \frac{d}{\|y - \eta\|} \ge 1 - \varepsilon$$

since $\eta + \xi\|y - \eta\| \in E$. Hence, $\operatorname{dist}(x, E) \ge 1 - \varepsilon$. ∎

Theorem. [Riesz] Let X be a normed space. If $\dim X = \infty$, then the closed unit ball $\overline{B}_1(0)$ is not compact.

Proof.
We construct a sequence with no convergent subsequence.
Pick $x_1 \in \overline{B}_1(0)$ and let $Y_1 := \operatorname{span}\{x_1\}$. Since $\dim Y_1 = 1 < \infty$, Y_1 is closed.

- If $X = Y_1$, then $\dim X < \infty$, and we are done.
- Otherwise, by Riesz's lemma with $\varepsilon = 1/2$, there exists $x_2 \in \overline{B}_1(0)$ such that $\|x_2 - x_1\| \geq 1/2$.

Let $Y_2 := \operatorname{span}\{x_1, x_2\}$, which is closed.

- If $X = Y_2$, then $\dim X < \infty$.
- Otherwise, continue this process inductively:
  - Given $x_1, \ldots, x_n$ with $\|x_i - x_j\| \geq 1/2$ for $i \neq j$
  - Let $Y_n := \operatorname{span}\{x_1, \ldots, x_n\}$, which is closed
  - If $X = Y_n$, then $\dim X < \infty$
  - Otherwise, by Riesz's lemma, there exists $x_{n+1} \in \overline{B}_1(0)$ with $\|x_{n+1} - x_i\| \geq 1/2$ for all $i = 1, \ldots, n$

If $\dim X = \infty$, this process continues indefinitely, producing a sequence $\{x_n\} \subset \overline{B}_1(0)$ with $\|x_i - x_j\| \geq 1/2$ for all $i \neq j$.
This sequence has no convergent subsequence, so $\overline{B}_1(0)$ is not sequentially compact, hence not compact. ■

Corollary. Let X be a normed space. Then:

$$\exists K \subseteq X \text{ closed and bounded} \iff \dim X < \infty$$

Lebesgue Spaces

L^p Spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in (1, \infty)$. Define:

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \left\{ f \in M(X, \mathcal{A}) : \int_X |f|^p d\mu < \infty \right\}$$

Define an equivalence relation $fRg \iff f = g$ almost everywhere. Then:

$$L^p(X, \mathcal{A}, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu)/R$$

We typically write $f \in L^p$.

Lemma. Let $p \in [1, \infty)$, $a, b \geq 0$. Then:

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof.
The function $x \mapsto x^p$ is convex on $[0, \infty)$, so:

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$$

Multiplying by 2^p gives the result. ■

Lemma. L^p is a vector space.

Proof.
Let $f, g \in L^p$, $\lambda \in \mathbb{R}$. Then $f + \lambda g$ is measurable, and:

$$\int_X |f + \lambda g|^p d\mu \leq 2^{p-1} \left(\int_X |f|^p d\mu + |\lambda|^p \int_X |g|^p d\mu \right) < \infty$$

Hence, $f + \lambda g \in L^p$. ■

Lemma. [Young's inequality] Let $p \in (1, \infty)$, $a, b > 0$. Then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof in the next page ↓.

Proof.
Define the function $\varphi(x) := e^x$, which is convex on $\mathbb{R}$. Then for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have:

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$$

Now consider:

$$ab = e^{\log a} \cdot e^{\log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q}$$

Apply the convexity inequality with:

- $t = \frac{1}{p}, 1 - t = \frac{1}{q}$
- $x = \log a^p$
- $y = \log b^q$

This gives:

$$ab = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \le \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q}$$

■

Definition. Two exponents $p, q \in [1, \infty]$ are **conjugate** if:

- $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, or
- $p = 1, q = \infty$, or
- $p = \infty, q = 1$

Theorem. [Hölder's inequality] Let $f, g \in M(X, \mathcal{A})$, and let $p, q \in [1, \infty]$ be conjugate. Then:

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Proof.
Case 1: $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\|f\|_p\|g\|_q = \infty$, the inequality is trivial. If $\|f\|_p\|g\|_q = 0$, then $f = 0$ or $g = 0$ a.e., so $fg = 0$ a.e., and the inequality holds. Assume $0 < \|f\|_p, \|g\|_q < \infty$. For $x \in X$, let:

$$a = \frac{|f(x)|^p}{\|f\|_p^p}, \quad b = \frac{|g(x)|^q}{\|g\|_q^q}$$

By Young's inequality:

$$\frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_q} \le \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}$$

Integrating both sides:

$$\frac{1}{\|f\|_p\|g\|_q}\int_X |f(x)g(x)| d\mu \le \frac{1}{p} + \frac{1}{q} = 1$$

So $\|fg\|_1 \le \|f\|_p\|g\|_q$.
Case 2: $p = 1$, $q = \infty$. Then $|g| \le \|g\|_\infty$ a.e., so:

$$\|fg\|_1 = \int_X |fg| d\mu \le \|g\|_\infty \int_X |f| d\mu = \|f\|_1\|g\|_\infty$$

■

Theorem. [Minkowski's inequality] Let $f, g \in M(X, \mathcal{A})$, $p \in [1, \infty]$. Then:

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

Proof.
Case 1: $p \in (1, \infty)$.

$$\|f+g\|_p^p = \int_X |f+g|^p d\mu = \int_X |f+g||f+g|^{p-1} d\mu \le \int_X |f||f+g|^{p-1} d\mu + \int_X |g||f+g|^{p-1} d\mu$$

By Hölder's inequality:

$$\int_X |f||f+g|^{p-1} d\mu \le \|f\|_p \||f+g|^{p-1}\|_q, \quad \int_X |g||f+g|^{p-1} d\mu \le \|g\|_p \||f+g|^{p-1}\|_q$$

Note that $\||f+g|^{p-1}\|_q = \left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q} = \|f+g\|_p^{p/q}$. So:

$$\|f+g\|_p^p \le (\|f\|_p + \|g\|_p) \|f+g\|_p^{p/q}$$

Dividing by $\|f+g\|_p^{p/q}$ (if zero, trivial) gives:

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Case 2: $p = 1$.

$$\|f+g\|_1 = \int_X |f+g| d\mu \le \int_X (|f| + |g|) d\mu = \|f\|_1 + \|g\|_1$$

Case 3: $p = \infty$.

$$\|f+g\|_\infty = \operatorname*{ess\,sup}_X |f+g| \le \operatorname*{ess\,sup}_X (|f| + |g|) \le \operatorname*{ess\,sup}_X |f| + \operatorname*{ess\,sup}_X |g| = \|f\|_\infty + \|g\|_\infty$$

■

Corollary. $L^p(X, \mathcal{A}, \mu)$ is a normed space with the norm $\|f\|_p$.

Proof.
Clearly $\|f\|_p = 0 \iff f = 0$ a.e., and $\|\lambda f\|_p = |\lambda| \|f\|_p$. Minkowski's inequality gives the triangle inequality.
■

Theorem. [Inclusion of L^p spaces] Suppose $\mu(X) < \infty$. Then for $1 \le p \le q \le \infty$, we have:

$$L^q(X, \mathcal{A}, \mu) \subseteq L^p(X, \mathcal{A}, \mu)$$

Moreover, there exists $C > 0$ such that $\|f\|_p \le C\|f\|_q$ for all $f \in L^q$.

Proof.
Case 1: $q = \infty$.

$$\|f\|_p^p = \int_X |f|^p d\mu \le \|f\|_\infty^p \mu(X) \Rightarrow \|f\|_p \le \mu(X)^{1/p}\|f\|_\infty$$

Case 2: $q < \infty$.
By Hölder's inequality with $r = \frac{q}{q-p}$, $s = \frac{q}{p}$:

$$\|f\|_p^p = \int_X |f|^p d\mu = \int_X 1 \cdot |f|^p d\mu \le \left(\int_X 1^r d\mu\right)^{1/r} \left(\int_X (|f|^p)^s d\mu\right)^{1/s} = \mu(X)^{\frac{q-p}{q}} \|f\|_q^p$$

So $\|f\|_p \le \mu(X)^{\frac{q-p}{pq}} \|f\|_q$. ■

Remark. Note that it is necessary that $\mu(X) < \infty$.
If $\mu(X) = \infty$, in general the preceding inclusion is false.
Consider $f(x) = \frac{1}{x}$ for $x \in (1, \infty)$ with $\lambda((1, \infty)) = \infty$.
Then $f \in L^2((1, \infty))$, but $f \notin L^1((1, \infty))$.
Hence $L^2((1, \infty)) \not\subseteq L^1((1, \infty))$.

Remark. For ℓ^p spaces with counting measure, $\mu^\#(\mathbb{N}) = \infty$, so the above theorem does not apply. In fact, $\ell^p \subset \ell^q$ for $p \le q$.

Theorem. [Interpolation inequality] Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \le p \le q \le \infty$. If $f \in L^p \cap L^q$, then $f \in L^r$ for all $r \in (p, q)$, and:

$$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}$$

where $\alpha \in (0, 1)$ satisfies $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

Proof.
Write $|f|^r = |f|^{\alpha r}|f|^{(1-\alpha)r}$. Apply Hölder's inequality with exponents $P = \frac{p}{\alpha r}$, $Q = \frac{q}{(1-\alpha)r}$. Then:

$$\|f\|_r^r = \int_X |f|^r d\mu \le \left(\int_X |f|^p d\mu\right)^{\alpha r/p} \left(\int_X |f|^q d\mu\right)^{(1-\alpha)r/q} = \|f\|_p^{\alpha r}\|f\|_q^{(1-\alpha)r}$$

Taking the r-roots gives the result. ■

Completeness of L^p Spaces

Theorem. $L^p(X, \mathcal{A}, \mu)$ is a Banach space for all $p \in [1, \infty]$.

Proof.
We prove the case $p \in [1, \infty)$. It suffices to show that if $\{f_n\} \subset L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p$ converges, then $\sum_{n=1}^{\infty} f_n$ converges in L^p.
Let $g_k := \sum_{n=1}^{k} |f_n|$. By Minkowski's inequality:

$$\|g_k\|_p \leq \sum_{n=1}^{k} \|f_n\|_p \leq M := \sum_{n=1}^{\infty} \|f_n\|_p < \infty$$

Let $g(x) := \sum_{n=1}^{\infty} |f_n(x)|$. Then $\{g_k\}$ is an increasing sequence of non-negative measurable functions. By the Beppo-Levi (Monotone Convergence) Theorem:

$$\lim_{k \to \infty} \int_X g_k^p d\mu = \int_X g^p d\mu \leq M^p$$

So $g \in L^p$, hence $g < \infty$ a.e., and $\sum_{n=1}^{\infty} f_n$ converges absolutely a.e. Let $s(x) := \sum_{n=1}^{\infty} f_n(x)$, $s_k(x) := \sum_{n=1}^{k} f_n(x)$. Then $s_k \to s$ a.e., and $|s_k - s|^p \to 0$ a.e. Moreover:

$$|s_k - s|^p \leq \left(\sum_{n=k+1}^{\infty} |f_n| \right)^p \leq g^p \in L^1$$

By the Dominated Convergence Theorem:

$$\lim_{k \to \infty} \int_X |s_k - s|^p d\mu = 0$$

So $\|s_k - s\|_p \to 0$, hence $\sum_{n=1}^{\infty} f_n$ converges in L^p. ■

Remark. [Dominated Convergence Theorem in L^p] Let $\{f_n\} \subset M(X, \mathcal{A})$, $f \in M(X, \mathcal{A})$, and $f_n \to f$ a.e.

1. If there exists $g \in L^1(X)$ such that $|f_n - f|^p \leq g$ a.e. for all n, then $f_n \to f$ in L^p.
2. If there exists $g \in L^p(X)$ such that $|f_n|^p \leq g$ a.e. for all n, then $f_n \to f$ in L^p.

Separability

Let $\Omega \subseteq \mathbb{R}^N$ be open and Lebesgue measurable.

1. $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, \infty)$.
2. $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, \infty)$.

Theorem. $L^p(\Omega)$ is separable for all $p \in [1, \infty)$.

Lemma. Let X be a metric space. Suppose there exists an uncountable family $\{A_i\}_{i\in I}$ of open sets such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Then X is not separable.

Proof.
Suppose X is separable, so there exists a countable dense set $\{c_n\}$. For each $i \in I$, $A_i \cap \{c_n\} \neq \emptyset$, so there exists $n(i)$ such that $c_{n(i)} \in A_i$. The map $i \mapsto n(i)$ is injective because $A_i \cap A_j = \emptyset$ for $i \neq j$. But I is uncountable and $\mathbb{N}$ is countable, contradiction. ∎

Theorem. $L^\infty(\mathbb{R}, \mathcal{L}, \lambda)$ is not separable.

Proof.
Consider the family $\{\chi_{[-\alpha,\alpha]} : \alpha > 0\} \subset L^\infty(\mathbb{R})$. For $\alpha \neq \alpha'$, $\|\chi_{[-\alpha,\alpha]} - \chi_{[-\alpha',\alpha']}\|_\infty = 1$. Let $A_\alpha := B_{1/2}(\chi_{[-\alpha,\alpha]})$. Then $\{A_\alpha\}$ is an uncountable family of disjoint open sets. By the lemma, L^∞ is not separable. ∎

ℓ^p Spaces

- ℓ^p is a Banach space for all $p \in [1, \infty]$.
- ℓ^p is separable for all $p \in [1, \infty)$.
- ℓ^∞ is not separable.

Linear Operators

Definition. Let X, Y be vector spaces. An operator $T : X \to Y$ is called a **linear operator** if

$$T(\alpha v_1 + \beta v_2) = T(\alpha v_1) + T(\beta v_2)$$

for all $v_1, v_2 \in X$ and all $\alpha, \beta \in \mathbb{R}$. If $Y = \mathbb{R}$, then T is called a **functional**.

Remark. For any linear operator T, we have $T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$.

Definition. Let X, Y be normed spaces. We say that $T : X \to Y$ is **bounded** if there exists $M > 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Definition. Let X, Y be normed spaces. An operator $T : X \to Y$ is **continuous at** $x_0 \in X$ if and only if for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$ as $n \to \infty$, we have

$$T(x_n) \to T(x_0) \quad \text{as } n \to \infty.$$

We say T is **continuous** if it is continuous at every $x_0 \in X$.

Definition. Let X, Y be normed spaces. An operator $T : X \to Y$ is **Lipschitz** if and only if there exists $L > 0$ such that

$$\|T(x) - T(y)\|_Y \leq L\|x - y\|_X \quad \text{for all } x, y \in X.$$

Theorem. Let $T : X \to Y$ be a linear operator between two normed spaces. Then the following statements are equivalent:

1. T is bounded.
2. T is Lipschitz.
3. T is continuous at $x_0 = 0$.
4. T is continuous.

Proof.

(1) $\Rightarrow$ (2) If T is bounded, then $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$. Then for all $x, y \in X$:

$$\|T(x) - T(y)\|_Y = \|T(x-y)\|_Y \leq M\|x-y\|_X,$$

so T is Lipschitz with constant M.

(2) $\Rightarrow$ (3) Let $\{x_n\} \subset X$ with $x_n \to 0$. Then $\|x_n\|_X \to 0$. We have:

$$0 \leq \|T(x_n) - T(0)\|_Y = \|T(x_n)\|_Y \leq L\|x_n - 0\|_X = L\|x_n\|_X \to 0,$$

so $\|T(x_n) - T(0)\|_Y \to 0$, hence $T(x_n) \to T(0)$.

(3) $\Rightarrow$ (1) Suppose by contradiction that T is not bounded.
Then there exists a sequence $\{x_n\} \subset X$, $x_n \neq 0$, such that

$$\|T(x_n)\|_Y \geq n\|x_n\|_X.$$

Define $S_n = \frac{x_n}{n\|x_n\|_X}$. Then

$$\|S_n\|_X = \frac{1}{n} \to 0,$$

so $S_n \to 0$. However,

$$T(S_n) = \frac{1}{n\|x_n\|_X}T(x_n),$$

and

$$\|T(S_n)\|_Y = \frac{1}{n\|x_n\|_X}\|T(x_n)\|_Y \geq \frac{1}{n\|x_n\|_X} \cdot n\|x_n\|_X = 1.$$

Thus, $T(S_n) \not\to T(0) = 0$, contradicting continuity at 0.

(4) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let $x_0 \in X$ and $\{x_n\} \subset X$ with $x_n \to x_0$.
Then $\|T(x_n) - T(x_0)\|_Y = \|T(x_n - x_0)\|_Y \leq L\|x_n - x_0\|_X \to 0$, so $T(x_n) \to T(x_0)$.

■

Remark. Let X, Y be normed spaces, and $T : X \to Y$ be linear with $\dim(X) < \infty$. Then T is continuous.

Example. Define $T : \ell^2 \to \ell^2$ by

$$T(x) = \left(\frac{x^{(1)}}{1}, \frac{x^{(2)}}{2}, \dots, \frac{x^{(k)}}{k}, \dots \right)$$

for $x = \{x^{(k)}\} \in \ell^2$. Then T is linear and continuous (hence bounded), since

$$\|T(x)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} \left(\frac{x^{(k)}}{k} \right)^2 \leq \sum_{k=1}^{\infty} (x^{(k)})^2 = \|x\|_{\ell^2}^2.$$

Definition. Let X, Y be normed spaces. We denote by $\mathcal{L}(X, Y)$ (or $\mathcal{B}(X, Y)$) the set of all linear continuous operators from X to Y. When $X = Y$, we write $\mathcal{L}(X) \equiv \mathcal{L}(X, X)$.

Remark. $\mathcal{L}(X, Y)$ is a vector space. If $T \in \mathcal{L}(X, Y)$, then there exists $M > 0$ such that

$$\|T(x)\|_Y \leq M \quad \text{for all } x \in X \text{ with } \|x\|_X \leq 1.$$

Hence,

$$\sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y \in \mathbb{R}_+.$$

We can show that $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}})$ is a normed space, where the **operator norm** is defined by

$$\|T\|_{\mathcal{L}} := \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y.$$

Proposition. The operator norm can be equivalently defined as:

$$\|T\|_{\mathcal{L}} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y,$$

$$\|T\|_{\mathcal{L}} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|T(x)\|_Y,$$

$$\|T\|_{\mathcal{L}} = \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

Proof.
The equivalence between the first and second follows from homogeneity: If $\|x\|_X \leq 1$, $x \neq 0$, then

$$\|T(x)\|_Y = \|x\|_X \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y \leq \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y,$$

so the supremum over $\|x\|_X \leq 1$ is attained on the unit sphere. The equivalence with the third follows from

$$\frac{\|T(x)\|_Y}{\|x\|_X} = \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y.$$

∎

Theorem. Let X be a normed space and Y be a Banach space. Then $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}})$ is a Banach space.

Uniform Boundedness Principle (Banach–Steinhaus Theorem)

Theorem. [Baire's Theorem] Let X be a complete metric space. If $\{A_n\}_{n\in\mathbb{N}} \subset X$ is a sequence of open dense sets, then

$$\overline{\bigcap_{n=1}^{\infty} A_n} = X.$$

Equivalently, if $\{C_n\}_{n\in\mathbb{N}} \subset X$ is a sequence of closed sets with $\bigcup_{n=1}^{\infty} C_n = X$, then there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Int}(C_{n_0}) \neq \emptyset$.

Definition. Let X, Y be Banach spaces and $\mathcal{F} \subset \mathcal{L}(X,Y)$. We say that $\mathcal{F}$ is **pointwise bounded** if for every $x \in X$, there exists $M_x > 0$ such that

$$\sup_{T\in\mathcal{F}} \|T(x)\|_Y \leq M_x.$$

We say that $\mathcal{F}$ is **uniformly bounded** if there exists $M > 0$ such that

$$\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}} \leq M.$$

Theorem. [Uniform Boundedness Principle] Let X, Y be Banach spaces and $\mathcal{F} \subset \mathcal{L}(X,Y)$. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is uniformly bounded.

Proof.
For each $n \in \mathbb{N}$, define

$$C_n := \{x \in X : \|T(x)\|_Y \leq n \text{ for all } T \in \mathcal{F}\}.$$

1. Each C_n is closed: Let $\{x_k\} \subset C_n$ with $x_k \to x_0 \in X$. Then $T(x_k) \to T(x_0)$ for all $T \in \mathcal{F}$, and since $\|T(x_k)\|_Y \leq n$, we have $\|T(x_0)\|_Y \leq n$, so $x_0 \in C_n$.
2. $\bigcup_{n=1}^{\infty} C_n = X$: For each $x \in X$, pointwise boundedness implies $\sup_{T\in\mathcal{F}} \|T(x)\|_Y < \infty$, so $x \in C_n$ for some n.

By Baire's Theorem, there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Int}(C_{n_0}) \neq \emptyset$. Hence, there exists a closed ball $\overline{B}_\varepsilon(x_0) \subset C_{n_0}$. For any $z \in X$ with $\|z\|_X \leq \varepsilon$, we have $z + x_0 \in \overline{B}_\varepsilon(x_0) \subset C_{n_0}$, so

$$\|T(z)\|_Y = \|T(z+x_0) - T(x_0)\|_Y \leq \|T(z+x_0)\|_Y + \|T(x_0)\|_Y \leq n_0 + n_0 = 2n_0.$$

For any $x \in X \setminus \{0\}$ and $T \in \mathcal{F}$,

$$\|T(x)\|_Y = \frac{\|x\|_X}{\varepsilon} \left\| T\left(\varepsilon \frac{x}{\|x\|_X}\right) \right\|_Y \leq \frac{2n_0}{\varepsilon} \|x\|_X.$$

Thus, $\|T\|_{\mathcal{L}} \leq \frac{2n_0}{\varepsilon} := M$ for all $T \in \mathcal{F}$, so $\mathcal{F}$ is uniformly bounded. ■

Corollary. Let X, Y be Banach spaces and $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$. Suppose that for every $x \in X$, the limit $\lim_{n\to\infty} T_n(x)$ exists. Define $T : X \to Y$ by $T(x) := \lim_{n\to\infty} T_n(x)$. Then T is linear and bounded, hence $T \in \mathcal{L}(X,Y)$.

Proof.
Linearity is obvious. For each $x \in X$, the sequence $\{T_n(x)\}$ is convergent and hence bounded. Thus, $\{T_n\}$ is pointwise bounded. By the Uniform Boundedness Principle, there exists $M > 0$ such that

$$\|T_n(x)\|_Y \leq M\|x\|_X \quad \text{for all } n \in \mathbb{N},\ x \in X.$$

Taking the limit as $n \to \infty$, we get

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X,$$

so T is bounded. ∎

Open Mapping Theorem

Definition. Let X, Y be metric spaces and $T : X \to Y$. We say T is **open** if $T(A)$ is open in Y for every open set $A \subset X$.

Theorem. [Open Mapping Theorem] Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. If T is surjective, then T is an open mapping.

Corollary. [Inverse Bounded/Continuous Mapping] Let $T \in \mathcal{L}(X,Y)$ be bijective. Then $T^{-1} \in \mathcal{L}(Y,X)$.

Proof.
Since T is bijective, $T^{-1} : Y \to X$ is linear. We show T^{-1} is continuous. Let $E \subset X$ be open. Then $(T^{-1})^{-1}(E) = T(E)$ is open by the Open Mapping Theorem, so T^{-1} is continuous. ∎

Closed Graph Theorem

Definition. Let X, Y be normed spaces and $T : X \to Y$ a linear operator. We say T is **closed** if for every sequence $\{x_n\} \subset X$ with $x_n \to x$ in X and $T(x_n) \to y$ in Y, we have $T(x) = y$.

Remark. If $T \in \mathcal{L}(X,Y)$, then T is closed. The converse is not true in general.

Definition. The **graph** of $T : X \to Y$ is the set

$$\mathrm{graph}(T) := \{(x, T(x)) : x \in X\} \subseteq X \times Y.$$

We say $\mathrm{graph}(T)$ is closed if whenever $(x_n, T(x_n)) \to (x, y)$ in $X \times Y$, then $(x, y) \in \mathrm{graph}(T)$.

T is closed if and only if $\mathrm{graph}(T)$ is closed.

Corollary. Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces. Suppose there exists $M > 0$ such that

$$\|x\|_2 \leq M\|x\|_1 \quad \text{for all } x \in X.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, i.e., there exists $m > 0$ such that

$$\|x\|_1 \leq m\|x\|_2 \quad \text{for all } x \in X.$$

Proof.
Consider the identity map $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ defined by $I(x) = x$. Then I is linear, bijective, and bounded (hence continuous) by assumption. By the Inverse Bounded Mapping Theorem, $I^{-1} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is also bounded. Thus, there exists $m' > 0$ such that

$$\|I^{-1}(x)\|_1 \leq m'\|x\|_2 \quad \text{for all } x \in X.$$

Since $I^{-1}(x) = x$, we get $\|x\|_1 \leq m'\|x\|_2$. Taking $m = m'$ gives the result. ∎

Theorem. [Closed Graph Theorem] Let X, Y be Banach spaces and $T : X \to Y$ a linear closed operator. Then $T \in \mathcal{L}(X, Y)$.

Proof.
Define the **graph norm** on X by

$$\|x\|_2 := \|x\|_X + \|T(x)\|_Y.$$

Then $(X, \|\cdot\|_2)$ is a Banach space (since T is closed). Clearly, $\|x\|_X \leq \|x\|_2$. By the previous corollary, there exists $M \geq 1$ such that

$$\|x\|_2 \leq M\|x\|_X \quad \text{for all } x \in X.$$

Hence,

$$\|T(x)\|_Y \leq \|x\|_2 \leq M\|x\|_X,$$

so T is bounded. ∎

Dual Spaces

Let X be a normed space. The dual of X, denoted X^* or X', is defined as:

$$X^* := \mathcal{L}(X, \mathbb{R})$$

which is a Banach space with the norm:

$$\|L\|_* := \sup_{\substack{\|x\|=1 \\ x \in X}} |L(x)|.$$

Example. [Dual of L^p]
Let $X = L^p(X, \mathcal{A}, \mu)$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
For $g \in L^q$, define:

$$L_g : L^p \to \mathbb{R}, \quad L_g(f) := \int_X f g \, d\mu.$$

L_g is linear due to the linearity of the integral. Indeed, for all $\alpha \in \mathbb{R}$ and $f_1, f_2 \in L^p$:

$$L_g(\alpha_1 f_1 + \alpha_2 f_2) = \int_X (\alpha_1 f_1 + \alpha_2 f_2) g \, d\mu = \alpha_1 \int_X f_1 g \, d\mu + \alpha_2 \int_X f_2 g \, d\mu = \alpha_1 L_g(f_1) + \alpha_2 L_g(f_2).$$

L_g is bounded. Indeed, for all $f \in L^p$:

$$|L_g(f)| = \left| \int_X f g \, d\mu \right| \leq \int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q = \|f\|_p M,$$

where the last inequality follows from Hölder's inequality. Hence, $L_g \in (L^p)^*$.
Now we compute $\|L_g\|_*$. From the previous inequality, we know $\|L_g\|_* \leq \|g\|_q$. Consider the special function:

$$\varphi := \frac{|g|^{q-2} g}{\|g\|_q^{q-1}}.$$

Then:

$$L_g(\varphi) = \int_X \varphi g \, d\mu = \int_X \frac{|g|^{q-2} g}{\|g\|_q^{q-1}} g \, d\mu = \frac{1}{\|g\|_q^{q-1}} \int_X |g|^q \, d\mu = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q.$$

Therefore, $\|L_g\|_* = \|g\|_{L^q}$.

Example. [Finite-Dimensional Case]
Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space with $\dim V = n$.
Let $L : V \to \mathbb{R}$ be linear (and thus continuous in finite dimensions), so $L \in V^*$.
We can prove that there exists a unique $y \in V$ such that:

$$L(x) = \langle x, y \rangle \quad \forall x \in V.$$

Proof.
Existence. Consider an orthonormal basis of V: $B = \{\vec{v}_1, \dots, \vec{v}_n\}$. Then for all $x \in V$:

$$x = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n, \quad \text{where } \alpha_i = \langle x, \vec{v}_i \rangle.$$

So:

$$\begin{aligned} L(x) &= \alpha_1 L(\vec{v}_1) + \cdots + \alpha_n L(\vec{v}_n) \\ &= \langle x, \vec{v}_1 \rangle L(\vec{v}_1) + \cdots + \langle x, \vec{v}_n \rangle L(\vec{v}_n) \\ &= \langle x, \vec{v}_1 L(\vec{v}_1) + \cdots + \vec{v}_n L(\vec{v}_n) \rangle \\ &= \langle x, y \rangle. \end{aligned}$$

Uniqueness. Suppose there exists $y' \in V$ such that $L(x) = \langle x, y' \rangle$ for all $x \in V$. Then:

$$0 = L(x) - L(x) = \langle x, y \rangle - \langle x, y' \rangle = \langle x, y - y' \rangle \quad \forall x \in V,$$

which implies $y - y' = 0$, so $y = y'$. ■

Hence, $V \simeq V^*$.
Now we compute the norm $\|L\|_*$. We have:

$$|L(x)| = |\langle x, y \rangle| \leq \|x\| \|y\| = \|x\| M \quad \forall x \in V,$$

so $\|L\|_* \leq \|y\|$. Moreover:

$$\left| L\left(\frac{y}{\|y\|} \right) \right| = \left| \frac{1}{\|y\|} \langle y, y \rangle \right| = \frac{\|y\|^2}{\|y\|} = \|y\|.$$

Therefore, $\|L\|_* = \|y\|$.

Example. [Extension in $\mathbb{R}^2$]
Let $X = \mathbb{R}^2$ and Y be a vector subspace of X. Consider $\phi \in Y^*$ (i.e., $\phi : Y \to \mathbb{R}$ is linear and hence continuous).
The problem is to find $\psi \in X^*$ such that:

- $\psi = \phi$ in $Y \subset X$
- $\|\psi\|_* = \|\phi\|_*$

Since $\phi \in Y^*$, there exists a unique $\eta \in Y \subset X$ such that $\phi(x) = \langle \eta, x \rangle$ for all $x \in Y$.
Define $\psi : X \to \mathbb{R}$ by $\psi(x) = \langle \eta, x \rangle$ for all $x \in X$. From the preceding example, ψ is linear and bounded, and $\|\psi\|_* = \|\eta\| = \|\phi\|_*$.

Hahn-Banach Theorem

Theorem. [Continuous Extension] Let X be a normed space, Y a vector subspace of X, and $f \in Y^*$. Then there exists $F \in X^*$ such that:

- $F(y) = f(y)$ for all $y \in Y$
- $\|F\|_{X^*} = \|f\|_{Y^*}$

Separation Form

Let $X = \mathbb{R}^2$ and A, B be disjoint convex subsets of X. We want to find a line that separates the two sets: $H = \{f(x) = \alpha\}$.

Definition. Let X be a normed space, $\alpha \in \mathbb{R}$, and $f \in X^*$. We define the **closed hyperplane** as $H = \{x \in X : f(x) = \alpha\}$.

Definition. We say that H **separates** $A \subseteq X$ and $B \subseteq X$ if:

$$f(a) \leq \alpha \leq f(b) \quad \forall a \in A, b \in B.$$

We say that H **strictly separates** $A \subseteq X$ and $B \subseteq X$ if there exists $\varepsilon > 0$ such that:

$$f(a) \leq \alpha - \varepsilon \quad \text{and} \quad f(b) \geq \alpha + \varepsilon \quad \forall a \in A, b \in B.$$

Theorem. [Separation Form] Let X be a normed space. If $\emptyset \neq A \subseteq X$ and $\emptyset \neq B \subseteq X$ are disjoint convex sets and A is open, then there exists a closed hyperplane H which separates A and B.

Remark. If A, B are disjoint convex sets with A closed and B compact, then there exists H which strictly separates A and B.

Consequences of the Hahn-Banach Theorem

Corollary. Let X be a normed space and $x_0 \in X \setminus \{0\}$. Then there exists $L_{x_0} \in X^*$ such that:

$$\|L_{x_0}\|_{X^*} = 1 \quad \text{and} \quad L_{x_0}(x_0) = \|x_0\|.$$

Proof.
Let $Y := \operatorname{Span}\{x_0\} = \{\lambda x_0 : \lambda \in \mathbb{R}\}$, a vector subspace of X. Define $L_0 : Y \to \mathbb{R}$ by $L_0(\lambda x_0) := \lambda \|x_0\|$.
By the Hahn-Banach theorem (continuous extension), there exists $\tilde{L}_0 : X \to \mathbb{R}$ with $\tilde{L}_0 \in X^*$, $\|\tilde{L}_0\| = \|L_0\|$, and:

$$\|L_0\| = \sup_{\substack{\|\lambda x_0\|=1 \\ \lambda x_0 \in Y}} |L_0(\lambda x_0)| = 1.$$

Moreover, $\tilde{L}_0(x_0) = L_0(x_0) = 1 \cdot \|x_0\| = \|x_0\|$. Set $L_{x_0} := \tilde{L}_0$. ∎

Corollary. Let $y, z \in X$. Assume $L(y) = L(z)$ for all $L \in X^*$. Then $y = z$.

Proof.
Suppose, by contradiction, that there exist $y \neq z$ such that $L(y) = L(z)$ for all $L \in X^*$. Define $x := y - z \neq 0$. Then:

$$L(x) = L(y - z) = L(y) - L(z) = 0 \quad \forall L \in X^*.$$

By the preceding corollary, there exists $L_x \in X^*$ such that $L_x(x) = \|x\| \neq 0$. But $L(x) = 0$ and $L_x(x) \neq 0$, a contradiction. ■

Corollary. Let $Y \subseteq X$ be a vector subspace with $\overline{Y} \neq X$ and $x_0 \in X \setminus \overline{Y}$. Then there exists $L \in X^*$ such that $L(x_0) \neq 0$ and $L|_Y = 0$.

Proof.
Let $Z := \{\lambda x_0 + y : y \in Y, \lambda \in \mathbb{R}\} \subset X$, a vector subspace. Define $L_0 : Z \to \mathbb{R}$ by $L_0(\lambda x_0 + y) := \lambda$. Then:

$$L_0(x_0) = L_0(1 \cdot x_0 + 0) = 1 \neq 0, \quad \ker(L_0) = \{\lambda x_0 + y \in Z : L_0(\lambda x_0 + y) = 0\} = Y.$$

So $L_0|_Y = 0$. By the Hahn-Banach theorem, there exists $\tilde{L}_0 \in X^*$ such that $\tilde{L}_0 = L_0$ in $Z \supseteq Y$. Set $L := \tilde{L}_0$. Then $L|_Y = 0$ and $L(x_0) = L_0(x_0) = 1 \neq 0$. ■

Reflexive Spaces

Definition. Let X be a normed space and X^* its dual.
The dual of X^*, i.e., $(X^*)^* \equiv X^{**}$, is called the **bidual** or **second dual**.
For each $x \in X$, define $\Lambda_x : X^* \to \mathbb{R}$ by $\Lambda_x(L) := L(x)$ for all $L \in X^*$.
Λ_x is linear, and:

$$|\Lambda_x(L)| = |L(x)| \leq \|L\|_* \|x\|_X = \|L\|_* M \quad \forall L \in X^*,$$

so Λ_x is bounded. Therefore, $\Lambda_x \in X^{**}$ and $\|\Lambda_x\|_{X^{**}} \leq \|x\|_X$.

Definition. The map $\tau : X \to X^{**}$ defined by $\tau(x) := \Lambda_x$ for all $x \in X$ is called the **canonical map** (or evaluation map).

Theorem. We have

1. τ is linear and $\|\tau(x)\|_{X^{**}} = \|x\|_X$ for all $x \in X$.
2. τ is injective.

Proof.
(1) Linearity is obvious. We already proved that $\|x\|_X \geq \|\tau(x)\|_{X^{**}}$. It remains to show the opposite inequality. By a corollary of the Hahn-Banach theorem, for each $x \in X \setminus \{0\}$, there exists $L \in X^*$ such that $\|L\|_{X^*} = 1$ and $L(x) = \|x\|_X$. Therefore:

$$\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \sup_{\|L\|_{X^*}=1} |\Lambda_x(L)| = \sup_{\|L\|_{X^*}=1} |L(x)| \geq \|x\|_X.$$

So $\|\tau(x)\|_{X^{**}} = \|x\|_X$ for all $x \in X$.
(2) For all $x_1, x_2 \in X$:

$$\tau(x_1) = \tau(x_2) \Leftrightarrow \Lambda_{x_1} = \Lambda_{x_2} \Leftrightarrow L(x_1) = L(x_2) \quad \forall L \in X^* \Rightarrow x_1 = x_2,$$

by a corollary of the Hahn-Banach theorem. So τ is injective. ■

Remark. $\tau(X)$ is closed in X^{**}. Indeed, X is complete $\Rightarrow \tau(X)$ is complete (because τ is an isometry). A complete metric space in X^{**} implies closedness.

Definition. If $\tau(X) = X^{**}$ (i.e., τ is surjective), then X is said to be **reflexive**.

Remark.

1. If X is reflexive, then $\tau : X \to X^{**}$ is bijective.
2. X is reflexive $\Leftrightarrow$ for all $\phi \in X^{**}$ and $L \in X^*$, we have $\phi(L) = L(x)$ where $x := \tau^{-1}(\phi)$.

Definition. A normed space X is **uniformly convex** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| > \varepsilon$, we have:

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Theorem. [Milman-Pettis] If X is a Banach space that is uniformly convex, then X is reflexive.

Theorem. For all $p \in (1, \infty)$, $L^p(\Omega)$ is uniformly convex.

Corollary. By the previous theorem, for all $p \in (1, \infty)$, $L^p(\Omega)$ is reflexive.

Remark. $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.

Clarkson Inequalities

- **Case $p \geq 2$:**
$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right) \quad \forall f, g \in L^p(\Omega).$$
- **Case $1 < p < 2$:**
$$\left\| \frac{f+g}{2} \right\|_p^q + \left\| \frac{f-g}{2} \right\|_p^q \leq \left(\frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right) \right)^{q/p},$$
where q is the conjugate exponent of p.

Proposition. L^p is uniformly convex for all $p \in (1, \infty)$.

Proof.
Take any $\varepsilon > 0$, and $f, g \in L^p$ with $\|f\|_p \leq 1$, $\|g\|_p \leq 1$, and $\|f - g\|_p > \varepsilon$.
Case $p \geq 2$: We have $\|f - g\|_p > \varepsilon$, so $\|f - g\|_p^p > \varepsilon^p$. By Clarkson's inequality:
$$\left\| \frac{f+g}{2} \right\|_p^p < 1 - \left(\frac{\varepsilon}{2} \right)^p \Leftrightarrow \left\| \frac{f+g}{2} \right\|_p < 1 - \delta,$$
where $\delta = 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^p \right]^{1/p} > 0$.
Case $1 < p < 2$: The proof is similar. ■

Dual of L^p

Let $L^p(X, \mathcal{A}, \mu)$ with $1 < p < \infty$, and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$.
Define $\Lambda : L^p \to \mathbb{R}$ by $\Lambda(f) := \int_X fg \, d\mu$ for all $f \in L^p$.
Then $\Lambda \in (L^p)^*$ and $\|\Lambda\|_{(L^p)^*} = \|g\|_{L^q}$.

Theorem. [Riesz Representation Theorem] Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in (1, \infty)$. For any $\Lambda \in (L^p(X, \mathcal{A}, \mu))^*$, there exists a unique $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that:
$$\Lambda(f) = \int_X fg \, d\mu \quad \forall f \in L^p.$$
Furthermore, $\|\Lambda\|_{(L^p)^*} = \|g\|_{L^q}$.

Remark. The same holds when $p = 1$, $q = \infty$, provided that μ is σ-finite.

Dual of L^∞

Consider $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \lambda)$ with $\Omega \in \mathcal{L}(\mathbb{R}^N)$. Let $g \in L^1$. Define $L_g : L^\infty \to \mathbb{R}$ by:
$$L_g(f) := \int_\Omega fg \, d\lambda \quad \forall f \in L^\infty.$$

Then:

- L_g is linear.
- $|L_g(f)| \leq \|f\|_\infty \|g\|_1 \Rightarrow \|L_g\|_{(L^\infty)^*} \leq \|g\|_1$.

So L_g is bounded, and hence $L^1 \subseteq (L^\infty)^*$.
Take $f := \operatorname{sgn}(g)$. Then:

$$|L_g(f)| = \int_\Omega |g|\, d\mu = \|g\|_1 \Rightarrow \|L_g\|_{(L^\infty)^*} = \|g\|_1.$$

But $(L^\infty)^* \supsetneq L^1$. There exists $L \in (L^\infty)^*$ that is not of the form L_g with $g \in L^1$.
Indeed, consider $L_0 \in [C_c^0(\mathbb{R}^N)]^*$ with $(C_c^0(\mathbb{R}^N), \|\cdot\|_\infty)$ and $C_c^0(\mathbb{R}^N)$ a vector subspace of L^∞. Define $L_0(f) := f(0)$ for all $f \in C_c^0(\mathbb{R}^N)$. Then:

- L_0 is linear.
- $|L_0(f)| = |f(0)| \leq \|f\|_\infty$ for all $f \in C_c^0(\mathbb{R}^N)$.

So L_0 is bounded. By the Hahn-Banach theorem, there exists $L \in (L^\infty(\mathbb{R}^N))^*$ which is a continuous extension of L_0.

Remark. Claim: There does not exist $g \in L^1(\mathbb{R}^N)$ such that $L(f) = \int_{\mathbb{R}^N} fg\, d\lambda$ for all $f \in L^\infty(\mathbb{R}^N)$.

Proof.
Suppose, by contradiction, that such a g exists. Then for all $f_1 \in C_c^0(\mathbb{R}^N)$ with $f_1(0) = 0$:

$$L(f_1) = L_0(f_1) = f_1(0) = 0, \quad \text{but also} \quad L(f_1) = \int_{\mathbb{R}^N} f_1 g\, d\lambda.$$

This implies $g = 0$ a.e. in $\mathbb{R}^N$. Then $L(f) = \int_{\mathbb{R}^N} f \cdot 0\, d\lambda = 0$ for all $f \in L^\infty(\mathbb{R}^N)$.
But take $f_2 \in C_c^0(\mathbb{R}^N)$ with $f_2(0) \neq 0$. Then:

$$0 = L(f_2) = L_0(f_2) = f_2(0) \neq 0,$$

a contradiction. ∎

Remark. For a normed space X:

- If X^* is separable, then X is separable.
- If X is not separable, then X^* is not separable.

Take $X = L^\infty$. Suppose, by contradiction, that $(L^\infty)^* = L^1$. Then L^∞ not separable $\Rightarrow$ L^1 not separable, which is false since L^1 is separable.

Summary of L^p Spaces

Space	Completeness	Separability	Reflexivity	Dual
L^p $(1 < p < \infty)$	Yes	Yes	Yes	L^q $(\frac{1}{p} + \frac{1}{q} = 1)$
L^1	Yes	Yes	No	L^∞ (if μ is σ-finite)
L^∞	Yes	No	No	$\supsetneq L^1$

Weak Convergence

Definition. [Weak Convergence] Let X be a normed space, $\{x_n\} \subset X$, $x \in X$.
We say that $x_n \rightharpoonup x$ as $n \to \infty$ if
$$L(x_n) \to L(x) \quad \forall L \in X^*$$

Remark. $x_n \to x \Rightarrow x_n \rightharpoonup x$, but not vice versa.
Indeed, for all $L \in X^*$:
$$|L(x_n) - L(x)| = |L(x_n - x)| \leq \|L\|_{X^*} \|x_n - x\|_X$$
and since $\|x_n - x\|_X \to 0$, we have $L(x_n) \to L(x)$, which by definition means $x_n \rightharpoonup x$.

Remark. If $X = \mathbb{R}^N$, then $x_n \to x \iff x_n \rightharpoonup x$.

Weak Convergence in L^p

Theorem. [Weak Convergence in L^p] Let $\Omega \subset \mathbb{R}^N$ be measurable. For $p \in [1, \infty)$, the following are equivalent:

1. $f_n \rightharpoonup f$ in $L^p(\Omega)$
2. $T(f_n) \to T(f)$ for all $T \in (L^p)^*$
3. $\int_\Omega f_n g \, d\lambda \to \int_\Omega f g \, d\lambda$ for all $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$
4. $\int_\Omega f_n \varphi \, d\lambda \to \int_\Omega f \varphi \, d\lambda$ for all $\varphi \in C_c^1(\Omega)$

The equivalence between (2) and (3) follows from the Riesz Representation Theorem.

Weak Convergence in ℓ^p

Theorem. [Weak Convergence in ℓ^p] For $1 \leq p < \infty$, let $\Lambda \in (\ell^p)^*$.
Then there exists a unique $y \equiv (y^{(k)}) \in \ell^q$ such that
$$\Lambda(x) = \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \qquad \forall x \equiv (x^{(k)}) \in \ell^p$$
Then, $x_n \rightharpoonup x$ in ℓ^p if and only if
$$\sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \to \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \qquad \forall y \equiv (y^{(k)}) \in \ell^q$$

Remark. In general, weak convergence does not imply strong convergence.
Consider $X = \ell^2$ and $\{l_n\} \subset \ell^2$ where $l_n \equiv (l_n^{(k)}) = \delta_{kn}$ (the sequence with 1 in the n-th position and 0 elsewhere).
Then $l_n \rightharpoonup 0$ since for all $y \in \ell^2$:

$$\sum_{k=1}^{\infty} l_n^{(k)} y^{(k)} = y^{(n)} \to 0 \quad \text{as } n \to \infty$$

However, $\|l_n\|_{\ell^2} = 1$ for all $n \in \mathbb{N}$, so $l_n \not\to 0$ in ℓ^2.

Properties of Weak Convergence

Theorem. [Uniqueness of Weak Limit] If $\{x_n\}$ weakly converges, then the weak limit is unique.

Proof.
Suppose, by contradiction, that $x_n \rightharpoonup x_1$ and $x_n \rightharpoonup x_2$ with $x_1 \neq x_2$.
Then for all $L \in X^*$:

$$|L(x_n) - L(x_1)| \to 0 \quad \text{and} \quad |L(x_n) - L(x_2)| \to 0$$

Hence $L(x_1) = L(x_2)$ for all $L \in X^*$, which implies $x_1 = x_2$ by a corollary of the Hahn-Banach theorem. Contradiction! ■

Proposition. [Boundedness of Weakly Convergent Sequences] If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded.

Theorem. [Lower Semicontinuity of Norm] If $x_n \rightharpoonup x$, then

$$\liminf_{n\to\infty} \|x_n\|_X \geq \|x\|_X.$$

$x \mapsto \|x_n\|$ is lower semicontinuous with respect to weak convergence.

Proof.
Let $x \in X \setminus \{0\}$. By a corollary of the Hahn-Banach theorem, there exists $L \in X^*$ with $\|L\|_{X^*} = 1$ such that $L(x) = \|x\|$. Then

$$0 < \|x\| = L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} |L(x_n)|$$

On the other hand,

$$|L(x_n)| \leq \|L\|_{X^*} \|x_n\|_X = \|x_n\|_X$$

Hence

$$\|x\| = \liminf_{n\to\infty} |L(x_n)| \leq \liminf_{n\to\infty} \|x_n\|_X$$

■

Theorem. [Joint Convergence] If $x_n \rightharpoonup x$ in X and $L_n \to L$ in X^*, then $L_n(x_n) \to L(x)$ in $\mathbb{R}$.

Proof.
We have:

$$L_n(x_n) - L(x) = L_n(x_n) - L(x_n) + L(x_n) - L(x)$$

So

$$|L_n(x_n) - L(x)| \le |L_n(x_n) - L(x_n)| + |L(x_n) - L(x)|$$

The first term satisfies:

$$|L_n(x_n) - L(x_n)| = |(L_n - L)(x_n)| \le \|L_n - L\|_{X^*} \|x_n\|_X$$

Since $\{x_n\}$ is bounded (by the previous proposition) and $\|L_n - L\|_{X^*} \to 0$, this term tends to 0.
The second term $|L(x_n) - L(x)| \to 0$ by weak convergence of x_n to x. ■

Theorem. [Weak Continuity of Bounded Linear Operators] Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$.
If $x_n \rightharpoonup x$ in X, then $T(x_n) \rightharpoonup T(x)$ in Y.

Proof.
Let $L \in Y^*$ and define $\Lambda : X \to \mathbb{R}$ by $\Lambda(x) = L[T(x)]$.
Since T is linear and continuous, and L is linear and continuous, $\Lambda \in X^*$.
By weak convergence of x_n to x, we have:

$$\Lambda(x_n) \to \Lambda(x) \quad \text{i.e.,} \quad L[T(x_n)] \to L[T(x)]$$

Since this holds for all $L \in Y^*$, we conclude that $T(x_n) \rightharpoonup T(x)$ in Y. ■

Weak* Convergence

Definition. We say that $\{L_n\} \subset X^*$ **weakly* converges** to $L \in X^*$ whenever

$$L_n(x) \xrightarrow[n\to\infty]{} L(x), \quad \forall x \in X.$$

We write $L_n \overset{*}{\rightharpoonup} L$ for $n \to \infty$.

Example: L^∞

$(\mathbb{R}^N, \mathcal{L}_{\mathbb{R}^N}), \lambda)$, $\Omega \in \mathcal{L}(\mathbb{R}^N)$, $\{f_n\} \subset L^\infty(\Omega) = X^*$, $f \in L^\infty(\Omega)$.
Define:

$$L_n(g) := \int_\Omega f_n g \, d\lambda, \quad \forall g \in L^1(\Omega) = X$$

$$L(g) := \int_\Omega f g \, d\lambda, \quad \forall g \in L^1(\Omega)$$

Then $L_n \in [L^1(\Omega)]^* = L^\infty(\Omega)$ by the Riesz Theorem.

$$L_n \overset{*}{\rightharpoonup} L \text{ for some } L \in (L^1)^*$$

$$\Updownarrow$$

$$\int_\Omega f_n g\, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f g\, d\lambda, \quad \forall g \in L^1(\Omega)$$
$$\Updownarrow$$
$$f_n \overset{*}{\rightharpoonup} f \text{ in } L^\infty(\Omega)$$

Remark. The same holds in ℓ^∞:
$$x_n \overset{*}{\rightharpoonup} x \text{ in } \ell^\infty$$
$$\Updownarrow$$
$$\sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty} x^{(k)} y^{(k)}, \quad \forall y \equiv y^{(k)} \in \ell^1$$

Theorem. $L_n \rightharpoonup L$ in $X^* \Rightarrow L_n \overset{*}{\rightharpoonup} L$.
And viceversa, if X is reflexive.

Proof.

$$L_n \rightharpoonup L \text{ in } X^*$$
$$\Updownarrow \text{ def}$$
$$\Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L), \quad \forall \Lambda \in X^{**}$$
$$\Downarrow$$
$$\Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L), \quad \forall \Lambda \in \tau(X) \subseteq X^{**}$$
$$\Updownarrow$$
$$L_n(x) \to L(x), \quad \forall x \in X$$
$$\Updownarrow$$
$$L_n \overset{*}{\rightharpoonup} L$$

And viceversa: X is reflexive $\iff \tau(X) = X^{**}$. ■

Theorem. Let X be a Banach space.

- $\{L_n\} \subset X^*$ weakly* converges $\Rightarrow$ the weak* limit is unique;
- $L_n \overset{*}{\rightharpoonup} L \Rightarrow \{L_n\}$ is bounded in X^*;
- $L_n \overset{*}{\rightharpoonup} L \Rightarrow \liminf_{n\to\infty} \|L_n\|_{X^*} \geq \|L\|_{X^*}$;
- $\begin{cases} x_n \to x \\ L_n \overset{*}{\rightharpoonup} L \end{cases} \Rightarrow L_n(x_n) \xrightarrow[n\to\infty]{} L(x)$

Banach-Alaoglu Theorem

Theorem. [Banach-Alaoglu] Let X be a separable Banach space. Then any bounded sequence $\{L_n\} \subset X^*$ admits a subsequence that weakly* converges to some $L \in X^*$.

Remark. $\{f_n\} \subset L^\infty$ bounded.
L^∞ is the dual of L^1. Every function in L^∞ identifies an element of $(L^1)^*$.
Define:

$$L_n(g) := \int_\Omega f_n g \, d\lambda, \quad \forall g \in L^1$$

$$|L_n(g)| \leq \|f_n\|_\infty \|g\|_1 \leq c\|g\|_1, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \|L_n\|_{(L^1)^*} \leq c, \quad \forall n \in \mathbb{N}$$

So the sequence $\{L_n\}$ is bounded in $(L^1)^*$.
Now we can apply the Banach-Alaoglu theorem:

$$\exists \{L_{n_h}\} \subset \{L_n\} : L_{n_h} \overset{*}{\rightharpoonup} L \text{ for some } L \in (L^1)^*.$$

This means that:

$$\exists \{f_{n_h}\} \subset \{f_n\} : L_{n_h}(g) \overset{*}{\rightharpoonup} L(g), \quad \forall g \in L^1$$

We can also say:

$$\exists! f \in L^\infty : L(g) := \int_\Omega f g \, d\lambda, \quad \forall g \in L^1$$

So, by the two of them:

$$\Rightarrow \int_\Omega f_{n_h} g \, d\lambda \xrightarrow[h \to \infty]{} \int_\Omega f g \, d\lambda$$

$$\iff f_{n_h} \overset{*}{\rightharpoonup} f, \quad \forall g \in L^1$$

Therefore, any bounded sequence $\{f_n\} \subset L^\infty$ admits a subsequence $\{f_{n_h}\}$ which weakly* converges to some $f \in L^\infty$.

Remark. The same holds also in ℓ^∞.

Corollary. Let X be a separable and reflexive Banach space. Then any bounded sequence $\{x_n\} \subset X$ admits a subsequence which weakly converges.

Proof.
X separable and reflexive $\Rightarrow X^*$ is separable too.

- $X^* = Y$ separable
- $\{\tau(x_n)\} \subset X^{**} = Y^*$ bounded, where $\tau(x_n)$ is the canonical evaluation map

So Y is separable and we have a sequence in Y^*.
We apply the Banach-Alaoglu theorem for $Y = X^*$, therefore obtaining:

$$\exists\{\tau(x_{n_h})\} \subset \{\tau(x_n)\} : \tau(x_{n_h}) \overset{*}{\rightharpoonup} \Lambda, \quad \Lambda \in Y^* = X^{**}$$

$$\Updownarrow$$

$$[\tau(x_{n_h})](f) \to \Lambda(f), \quad \forall f \in Y = X^*$$

where $[\tau(x_{n_h})](f) = f(x_{n_h})$ and $\Lambda(f) = f(x)$ with $x := \tau^{-1}(\Lambda)$.

$$\Updownarrow$$

$$x_{n_h} \xrightharpoonup[h\to\infty]{} x$$

■

Compact Operators

Let X, Y be Banach spaces.

Definition. $K : X \to Y$ linear is said to be **compact** if

$$\forall E \subseteq X \text{ bounded}, \quad \overline{K(E)} \subseteq Y \text{ is compact.}$$

Remark. [Equivalent definition] $\forall\{x_n\} \subset X$ bounded, $K(x_n)$ has a subsequence which converges in Y strongly.

Theorem. $K : X \to Y$ linear, compact $\Rightarrow K \in \mathcal{L}(X, Y)$.

Proof.
Let $B \subset X$ be the closed unit ball.
$\Rightarrow K(B)$ is relatively compact $\Rightarrow K(B)$ is bounded.
Therefore $\exists M > 0$ such that $\forall x \in X$, $\|x\|_X \leq 1$ we have $\|K(x)\|_Y \leq M$.
$\Rightarrow K$ is bounded.
Since K is also linear $\Rightarrow K \in \mathcal{L}(X, Y)$. ■

Definition. $T \in \mathcal{L}(X, Y)$ is a **finite rank operator** if $\dim \operatorname{Im}(T) < \infty$.

Remark. $T \in \mathcal{L}(X, Y)$, $\operatorname{rank}(T)$ finite $\Rightarrow T$ is compact. Not viceversa.

Theorem. $\{K_n\} \subset \mathcal{L}(X, Y)$, $\operatorname{rank}(K_n) < \infty, \forall n \in \mathbb{N}$.
Assume $K_n \xrightarrow[n\to\infty]{} K$ in $\mathcal{L}(X, Y)$.
Then K is compact.

Remark. The converse is not true.

Theorem. X, Y Banach spaces.
$K \in \mathcal{L}(X, Y)$, $\dim(Y) = \infty$
K is compact $\Rightarrow$ K is not surjective.

Proof.
Suppose that, by contradiction, K is surjective.
Now consider $B_1(0) \subset X$ open.
$K(B_1(0)) \subset Y$, $K(0) = 0 \in K(B_1(0))$.
Given K surjective, we can apply the open mapping theorem.
So Y is open (open set maps into an open set).

$$\Rightarrow \exists \delta > 0 : B_\delta(0) \subseteq K(B_1(0))$$
$$B_\delta(0) \subseteq Y, \quad \overline{B_\delta(0)} \subseteq \overline{K(B_1(0))}$$

By assumption, K is compact, so $\overline{K(B_1(0))}$ is compact in Y.
Now we have a closed ball compact in Y: $\overline{B_\delta(0)} \subseteq Y$, $\dim(Y) = \infty$.
This contradicts the Riesz theorem. ■

Definition. $\mathcal{K}(X, Y) := \{K \in \mathcal{L}(X, Y) : K \text{ compact}\}$
If $X = Y$, we write $\mathcal{K}(X, X) = \mathcal{K}(X)$.

Theorem. ***Part 1***. If $T \in \mathcal{K}(X,Y)$, then

$$x_n \xrightharpoonup[n\to\infty]{} x \Rightarrow T(x_n) \xrightarrow[n\to\infty]{} T(x).$$

This is called **weak-strong continuity (WSC)**.

Part 2. If X is separable and reflexive, and $T \in \mathcal{L}(X,Y)$ satisfies the WSC property, then $T \in \mathcal{K}(X,Y)$.

Remark. In general WSC property is equivalent to compactness.

Proof.
Part 1.
Consider $\{x_n\}$ weakly convergent to x for some $x \in X$.
We apply a corollary of the Banach-Steinhaus theorem $\Rightarrow$ $\{x_n\}$ is bounded.
Since T is compact by assumption:

$$\exists\{x_{n_k}\}, y \in Y : T(x_{n_k}) \xrightarrow[k\to\infty]{} y$$

Take any $L \in Y^*$. We consider:

$$(L \circ T)(x_{n_k}) = L[T(x_{n_k})]$$

Since $L \circ T \in X^*$, by definition of weak convergence:

$$(L \circ T)(x_{n_k}) \xrightarrow[k\to\infty]{} (L \circ T)(x)$$

Strong convergence implies weak convergence:

$$L[T(x_{n_k})] \xrightarrow[k\to\infty]{} L(y)$$

Thus, $y = T(x)$ and $T(x_{n_k}) \xrightarrow[k\to\infty]{} T(x)$.
Now we want to show that $T(x_n) \xrightarrow[n\to\infty]{} T(x)$.
If this is not true, $\exists\varepsilon > 0$, $\{x_{n_h}\}$ such that:

$$\|T(x_{n_h}) - T(x)\|_Y > \varepsilon$$

However, $x_{n_h} \rightharpoonup x$ by hypothesis, since $\{x_{n_h}\} \subset \{x_n\}$.
By the previous argument:

$$\exists\{x_{n_{h_i}}\} \subset \{x_{n_h}\} : T(x_{n_{h_i}}) \xrightarrow[i\to\infty]{} T(x)$$

And we are done.
Part 2.
Let $\{x_n\} \subset X$ be bounded, X reflexive.
Banach-Alaoglu theorem $\Rightarrow \exists\{x_{n_k}\}, x \in X : x_{n_k} \rightharpoonup x$
By hypothesis, T fulfills WSC, thus $T(x_{n_k}) \xrightarrow[k\to\infty]{} T(x)$.
So for any bounded $\{x_n\} \subset X$, there exists a subsequence such that we have strong convergence.
This is the definition of compact operator.
Done. ■

Hilbert Spaces

Pre-Hilbert and Hilbert Spaces

Definition. A vector space H with a scalar/inner product is called a **pre-Hilbert space** or **inner product space**.
Define $\|x\| := \sqrt{\langle x, x\rangle}$ for all $x \in H$. This is a norm.
If $(H, \langle\cdot,\cdot\rangle)$ is pre-Hilbert, then $(H, \|\cdot\|)$ is a normed space. So all theorems from normed spaces hold.
Thus (H, d) is a metric space with $d(x, y) = \|y - x\|$.
A pre-Hilbert space which is complete is called a **Hilbert space**.

Examples

Example. $C^0([a, b], \langle\cdot,\cdot\rangle)$ with

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx$$

is only pre-Hilbert.

Example. $L^2(X, \mathcal{A}, \mu)$ with

$$\langle f, g\rangle := \int_X f(x)g(x)\,d\mu$$

is Hilbert, because L^2 is complete.

Example. ℓ^2 with

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x^{(n)} y^{(n)}$$

is Hilbert.

Example. $W^{1,2}((a, b)) \equiv H^1((a, b)) := \{f \in L^2((a, b)) : f' \in L^2((a, b))\}$ with

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx + \int_a^b f'(x)g'(x)\,dx$$

is Hilbert, where f' is the weak derivative.

Remark. L^p is Hilbert if and only if $p = 2$.

Parallelogram Identity

Theorem. [Parallelogram Identity] Let H be a pre-Hilbert space. Then

$$\left\|\frac{a+b}{2}\right\|^2 + \left\|\frac{a-b}{2}\right\|^2 = \frac{1}{2}\left(\|a\|^2 + \|b\|^2\right), \quad \forall a, b \in H.$$

Remark. If $\|\cdot\|$ satisfies the parallelogram identity, then $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Remark. Consider $H = L^p$ with $1 < p < \infty$. For $p \geq 2$,

$$\left\|\frac{a+b}{2}\right\|_p^p + \left\|\frac{a-b}{2}\right\|_p^p \leq \frac{1}{2}\left(\|a\|_p^p + \|b\|_p^p\right), \quad \forall a, b \in H.$$

This is the first Clarkson inequality.

Convex Sets and Projection Theorem

Definition. Let E be a normed space, $K \subseteq E$. K is **convex** if for all $x, y \in K$ the segment joining x to y belongs to K.

Theorem. [Projection Theorem] Let H be a Hilbert space and $K \subset H$ a closed convex subset.
Part 1. For every $f \in H$, there exists a unique $u \in K$ such that

$$\|f - u\| = \min_{v \in K} \|f - v\| := \operatorname{dist}(f, K).$$

Part 2. Moreover, u satisfies the above property if and only if

$$u \in K, \quad \langle f - u, v - u\rangle \leq 0, \quad \forall v \in K.$$

Proof in the next page ↓.

Proof.
Part 1. Existence. Let $\{v_n\} \subset K$ be a minimizing sequence, i.e.,

$$\{v_n\} \subset K, \quad d_n := \|f - v_n\| \to \inf_{v \in K} \|f - v\| =: d.$$

We claim $\{v_n\}$ is Cauchy. By the parallelogram law with $a = f - v_n$ and $b = f - v_m$:

$$\left\| f - \frac{v_n + v_m}{2} \right\|^2 + \left\| \frac{v_n - v_m}{2} \right\|^2 = \frac{1}{2}(d_n^2 + d_m^2).$$

Since K is convex, $(v_n + v_m)/2 \in K$, so

$$\left\| f - \frac{v_n + v_m}{2} \right\|^2 \geq d.$$

Thus,

$$\left\| \frac{v_n - v_m}{2} \right\|^2 = \frac{1}{2}(d_n^2 + d_m^2) - \left\| f - \frac{v_n + v_m}{2} \right\|^2 \leq \frac{1}{2}(d_n^2 + d_m^2) - d^2.$$

Taking limit as $n, m \to \infty$, we get $\|(v_n - v_m)/2\|^2 \to 0$, so $\{v_n\}$ is Cauchy.
Since H is complete, $\exists u \in H$ such that $v_n \to u$. As K is closed, $u \in K$. Then

$$d \leq \|f - u\| \leq \|f - v_n\| + \|v_n - u\| = d_n + \|v_n - u\| \to d + 0,$$

so $\|f - u\| = d$.
Part 1. Uniqueness. Suppose by contradiction that $u, \tilde{u} \in K$ with $u \neq \tilde{u}$ and

$$d = \|f - u\| = \|f - \tilde{u}\|.$$

Apply the parallelogram identity with $a = f - u$, $b = f - \tilde{u}$:

$$\left\| f - \frac{u + \tilde{u}}{2} \right\|^2 + \left\| \frac{u - \tilde{u}}{2} \right\|^2 = \frac{1}{2}(d^2 + d^2) = d^2.$$

Let $\left\| \frac{u - \tilde{u}}{2} \right\|^2 =: \sigma > 0$. Then

$$d^2 \leq \left\| f - \frac{u + \tilde{u}}{2} \right\|^2 = d^2 - \sigma,$$

a contradiction.
Part 2. Not included. ■

Theorem. [Projection in L^p] Let $1 < p < \infty$, $K \subset L^p(X, \mathcal{A}, \mu)$ a closed convex subset. Then for every $f \in L^p$, there exists a unique $u \in K$ such that

$$\|f - u\|_p = \min_{v \in K} \|f - v\|_p = \operatorname{dist}(f, K).$$

Proof.
Same argument as above, with the parallelogram law replaced by the first Clarkson inequality. ■

Definition. $u := \operatorname{Proj}_K f$ is the element of K of minimal distance from f.

Projection Theorem: Special Case

Corollary. Let H be a Hilbert space and $M \subset H$ a closed vector subspace. Let $f \in H$. Then

$$u = \operatorname{Proj}_M f \iff u \in M, \quad \langle f - u, v \rangle = 0, \quad \forall v \in M.$$

This is the **orthogonal projection**.

Proof.
($\Rightarrow$) Condition (2) in the general theorem gives:

$$u \in M, \quad \langle f - u, v - u \rangle \leq 0, \quad \forall v \in M.$$

Since M is a vector space, $tv \in M$ for all $t \in \mathbb{R}$, so

$$\langle f - u, tv - u \rangle \leq 0, \quad \forall v \in M, \forall t \in \mathbb{R}.$$

By homogeneity and linearity: $t\langle f - u, v \rangle \leq \langle f - u, u \rangle$ for all $v \in M,\ t \in \mathbb{R}$.
If $\langle f - u, v \rangle > 0$ for some v, choosing

$$t > \frac{\langle f - u, u \rangle}{\langle f - u, v \rangle}$$

gives a contradiction.
Hence

$$\langle f - u, v \rangle = 0 \quad \text{for all } v \in M.$$

($\Leftarrow$) If u satisfies condition (3), then for $\xi \in M$, set $v = \xi - u \in M$:

$$\langle f - u, \xi - u \rangle = 0, \quad \forall \xi \in M,$$

which is condition (2), so $u = \operatorname{Proj}_M f$. ■

Remark. In L^p, for $f \in L^p$ and $M \subset L^p$ a closed vector subspace, $u = \operatorname{Proj}_M f$ satisfies

$$\int_X |f - u|^{p-2}(f - u)v \, d\mu = 0, \quad \forall v \in M \subset L^p.$$

This replaces the orthogonality condition. For $p = 2$, this is exactly the scalar product in L^2.

Dual of Hilbert Spaces

Let $(V, \langle\cdot,\cdot\rangle)$ with $\dim V = n \Rightarrow V^* = V$. For $\dim V = \infty$, what happens?
Let H be a Hilbert space, $f \in H$. Define $\varphi : H \to \mathbb{R}$ by $\varphi(u) := \langle f, u\rangle$ for all $u \in H$. Then $\varphi \in H^*$ and $\|\varphi\|_{\mathcal{L}} = \|f\|_H$, so $H \subseteq H^*$. Is $H^* \subseteq H$?

Theorem. [Riesz Representation Theorem] Let H be a Hilbert space. For any $\varphi \in H^*$, there exists a unique $f \in H$ such that

$$\varphi(u) = \langle f, u\rangle, \quad \forall u \in H,$$

and $\|\varphi\|_{\mathcal{L}} = \|f\|_H$.

Remark. In the proof, we find $f = \varphi(g) \cdot g$.

Proof.
Existence. Let $M := \varphi^{-1}(0)$, the kernel of φ, a closed vector subspace. If $M = H$, take $f = 0$. Otherwise, $M \subsetneq H$. We claim there exists $g \in H$ with $\|g\| = 1$ and $g \in M^\perp$.
Indeed, let $g_0 \in H \setminus M$, $g_1 := \mathrm{Proj}_M\, g_0$, and define

$$g := \frac{g_0 - g_1}{\|g_0 - g_1\|}.$$

Then $\|g\| = 1$ and $g \in M^\perp$.
For any $u \in H$, set $v := u - \lambda g$ with $\lambda = \varphi(u)/\varphi(g)$. Note that

$$\varphi(g) = \frac{\varphi(g_0) - \varphi(g_1)}{\|g_0 - g_1\|} = \frac{\varphi(g_0)}{\|g_0 - g_1\|}.$$

Then $\varphi(v) = \varphi(u) - \lambda\varphi(g) = 0$, so $v \in M$. Hence

$$\langle g, v\rangle = 0 \Rightarrow \langle g, u - \lambda g\rangle = 0 \Rightarrow \langle g, u\rangle = \lambda\|g\|^2 = \lambda,$$

so $\langle g, u\rangle = \varphi(u)/\varphi(g) \Rightarrow \varphi(u) = \varphi(g)\langle g, u\rangle = \langle f, u\rangle$ with $f = \varphi(g) \cdot g$.
Uniqueness. Suppose $f_1, f_2 \in H$ both represent φ. Then

$$0 = \langle f_1 - f_2, u\rangle, \quad \forall u \in H.$$

Choosing $u = f_1 - f_2$ gives $\|f_1 - f_2\|^2 = 0$, so $f_1 = f_2$.
The norm equality $\|\varphi\|_{\mathcal{L}} = \|f\|_H$ follows as in exercises. ∎

Remark. By similar arguments, we get the Riesz Representation Theorem for L^p spaces ($1 < p < \infty$).

Lax-Milgram Theorem

Let H be a pre-Hilbert space, $B : H \times H \to \mathbb{R}$ a bilinear form.

Definition. We say B is **bounded/continuous** if there exists $\alpha > 0$ such that

$$|B(u,v)| \leq \alpha\|u\|\|v\|, \quad \forall u,v \in H. \tag{1}$$

We say B is **coercive** if there exists $\beta > 0$ such that

$$B(u,u) \geq \beta\|u\|^2, \quad \forall u \in H. \tag{2}$$

Let $f \in H^*$. Consider the problem: find $u \in H$ such that

$$B(u,v) = f(v), \quad \forall v \in H. \tag{3}$$

Theorem. [Lax-Milgram] Let H be a Hilbert space, $B : H \times H \to \mathbb{R}$ a bilinear form that is continuous (1) and coercive (2). Let $f \in H^*$. Then there exists a unique $u \in H$ satisfying (3).

Proof.
1. For fixed $u \in H$, the map $v \mapsto B(u,v)$ is linear and continuous. By Riesz, there exists a unique $w \in H$ such that

$$B(u,v) = \langle w,v\rangle, \quad \forall v \in H. \tag{*}$$

2. Define $A : H \to H$ by $A(u) := w$. Then A is linear and continuous since

$$\|A(u)\|^2 = \langle A(u), A(u)\rangle = B(u, A(u)) \leq \alpha\|u\|\|A(u)\|,$$

so $\|A(u)\| \leq \alpha\|u\|$ for all $u \in H$.
3. **(i)** A **is injective.** By (2) and (*):

$$\beta\|u\|^2 \leq B(u,u) = \langle A(u), u\rangle \leq \|u\|\|A(u)\| \Rightarrow \|A(u)\| \geq \beta\|u\|.$$

If $u \neq 0$, then $\|A(u)\| > 0$, so $\mathrm{Ker}(A) = \{0\}$.
(ii) $\mathrm{Im}(A)$ **is closed.** Let $\{v_n\} \subseteq \mathrm{Im}(A)$ with $v_n \to v$ in H. Write $v_n = A(u_n)$. Then

$$\beta\|u_n - u_m\| \leq \|A(u_n) - A(u_m)\| = \|v_n - v_m\|,$$

so $\{u_n\}$ is Cauchy. Since H is complete, $u_n \to u$ for some $u \in H$. By continuity of A, $v_n = A(u_n) \to A(u)$, so $v = A(u) \in \mathrm{Im}(A)$.
4. A is surjective. If not, $\mathrm{Im}(A) \subsetneq H$. Since $\mathrm{Im}(A)$ is closed, there exists $\eta \in H$ with $\eta \in \mathrm{Im}(A)^\perp$. Then

$$0 < \beta\|\eta\|^2 \leq B(\eta,\eta) = \langle A(\eta), \eta\rangle = 0,$$

a contradiction.
5. By Riesz, there exists a unique $y \in H$ such that $f(v) = \langle y, v\rangle$ for all $v \in H$. Since A is bijective, there exists a unique $u \in H$ with $A(u) = y$. Then

$$B(u,v) = \langle A(u), v\rangle = \langle y, v\rangle = f(v), \quad \forall v \in H.$$

6. **Uniqueness.** Suppose $u, \tilde{u} \in H$ both satisfy (3). Then

$$B(u - \tilde{u}, v) = 0, \quad \forall v \in H.$$

Taking $v = u - \tilde{u}$ gives

$$0 < \beta\|u - \tilde{u}\|^2 \leq B(u - \tilde{u}, u - \tilde{u}) = 0,$$

a contradiction. ■

Remark. If B is symmetric, then B defines a new inner product on H. The norms $\|\cdot\|_* = \sqrt{B(u,u)}$ and $\|\cdot\|$ are equivalent by (1) and (2). Then $(H, B(\cdot,\cdot))$ is also Hilbert, and Riesz gives $f(v) = B(g,v)$ for some $g \in H$.

Remark. If B is symmetric, then u solves (3) if and only if

$$u \in H, \quad \frac{1}{2}B(u,u) - f(u) = \min_{v \in H} \left\{ \frac{1}{2}B(v,v) - f(v) \right\}.$$

This is a minimization problem.

Orthonormal Bases

Definition. Let H be a Hilbert space. A sequence $\{e_n\} \subset H$ is an **orthonormal basis** (or Hilbert basis) if:

1. $\langle e_n, e_m \rangle = 0$ for all $n \neq m \in \mathbb{N}$, and $\|e_n\| = 1$ for all $n \in \mathbb{N}$.
2. $\mathrm{Span}(\{e_n\})$ is dense in H.

Remark. Condition (2) is equivalent to: for every $u \in H$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\left\| u - \sum_{i=1}^{n} \alpha_i e_i \right\| < \varepsilon.$$

Theorem. Let H be a Hilbert space with orthonormal basis $\{e_n\}$. Then for every $u \in H$:

$$u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k, \quad \|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2.$$

Conversely, for any sequence $\{\alpha_k\} \in \ell^2$, the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to some $u \in H$ with

$$\langle u, e_k \rangle = \alpha_k, \quad \|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2.$$

Remark. This is the **abstract Fourier series**. The convergence is in the Hilbert space norm:

$$\left\| \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k - u \right\| \to 0 \quad (n \to \infty).$$

The $\langle u, e_k \rangle$ are the **Fourier coefficients**, and $\langle u, e_k \rangle e_k$ is the projection of u onto $\mathrm{Span}(e_k)$. The identity $\|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$ is **Parseval's identity**.

Theorem. Every separable Hilbert space has an orthonormal basis.

Example. 1. L^2: trigonometric functions $\sin$, $\cos$, or polynomials (from ODEs).
2. ℓ^2: standard basis $e_n^{(k)} = \delta_{nk}$.

Theorem. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in a Hilbert space H. Then $e_n \rightharpoonup 0$ weakly, but $e_n \not\to 0$ strongly.

Proof.
For any $f \in H$, Parseval's identity gives $\|f\|^2 = \sum_{k=1}^{\infty} \langle f, e_n \rangle^2 < \infty$, so $\langle f, e_n \rangle \to 0$ as $n \to \infty$. This means $F(e_n) \to 0$ for all $F \in H^*$, i.e., $e_n \rightharpoonup 0$ weakly. But $\|e_n\| = 1$ for all n, so $e_n \not\to 0$ strongly. ■

Theorem. Any Hilbert space is reflexive.

Proof.
As in L^p, H is uniformly convex (using the parallelogram law instead of Clarkson's inequality). The conclusion follows from the Milman-Pettis theorem. ■

Linear and Continuous Operators on Hilbert Spaces

Symmetric Operators

Definition. Let H be a Hilbert space. An operator $T \in \mathcal{L}(H)$ is called **symmetric** if

$$\langle T(x), y\rangle = \langle x, T(y)\rangle, \quad \forall x, y \in H.$$

Remark. An alternative formula for the operator norm is:

$$\|T\|_{\mathcal{L}} = \sup_{\|x\|=1} |\langle T(x), x\rangle|.$$

Fredholm Alternative

Let $f \in H, T \in \mathcal{L}(H)$. Consider the equation:

$$u - T(u) = f \tag{E}$$

Theorem. [Fredholm Alternative] Let H be a Hilbert space, $T \in \mathcal{K}(H)$ a compact operator, and T symmetric. Then:

1. $n := \dim \operatorname{Ker}(I - T) < \infty$
2. $\operatorname{Im}(I - T)$ is closed, and $\operatorname{Im}(I - T) = (\operatorname{Ker}(I - T))^{\perp}$
3. $\operatorname{Ker}(I - T) = \{0\} \iff \operatorname{Im}(I - T) = H$
   (i.e., injective $\iff$ surjective)

Remark. Property (3) holds automatically if $\dim H < \infty$. However, in infinite-dimensional spaces, injectivity $\not\Rightarrow$ surjectivity and vice versa.

Solvability of Equation (E)

There are two cases:
Case 1: For all $f \in H$, $u - T(u) = f$ has a unique solution $u \in H$.
Case 2: $u - T(u) = 0$ admits n linearly independent solutions, and equation (E) is solvable if and only if $f \in [\operatorname{Ker}(I - T)]^{\perp}$ (orthogonality condition).

Remark. In Case 1, property (3) gives both existence (surjectivity) and uniqueness (injectivity).
In Case 2, $\dim \operatorname{Ker}(I-T) = n$ by property (1), and $[\operatorname{Ker}(I-T)]^{\perp} = \operatorname{Im}(I-T)$ by property (2). If $f \in \operatorname{Im}(I-T)$, solutions exist but are not unique.

The Spectrum

Let E be a Banach space, $T \in \mathcal{L}(E)$.

Definition. The **resolvent set** is

$$\rho(T) := \{\lambda \in \mathbb{R} : (T - \lambda I) : E \to E \text{ is bijective}\}.$$

The **spectrum** is

$$\sigma(T) := \mathbb{R} \setminus \rho(T).$$

Definition. A real number λ is an **eigenvalue** if $\mathrm{Ker}(T - \lambda I) \neq \{0\}$, i.e., there exists $v \in E \setminus \{0\}$ such that $T(v) = \lambda v$.
$\mathrm{Ker}(T - \lambda I)$ is called the **eigenspace**, and its elements are **eigenvectors**.
The **point spectrum** is

$$EV(T) \equiv \sigma_p(T) := \{\text{all eigenvalues of } T\}.$$

Remark. $EV(T) \subseteq \sigma(T)$. If $\dim E < \infty$, then $EV(T) = \sigma(T)$.

Example. Consider the right-shift operator $T : \ell^2 \to \ell^2$ defined by

$$T(x) = (0, x^{(1)}, x^{(2)}, \dots) \quad \text{for } x = \{x^{(k)}\}_{k \in \mathbb{N}} \in \ell^2.$$

For $\lambda = 0$: T is injective $\Rightarrow 0 \notin EV(T)$, but T is not surjective $\Rightarrow 0 \in \sigma(T)$. Thus in general $EV(T) \subset \sigma(T)$.

Remark. The spectrum is a compact subset of $\mathbb{R}$ and

$$\sigma(T) \subseteq [-\|T\|_{\mathcal{L}}, \|T\|_{\mathcal{L}}].$$

Theorem. [Structure of the Spectrum] Let E be a Banach space, $T \in \mathcal{K}(E)$ compact, with $\dim E = \infty$. Then:

1. $0 \in \sigma(T)$
2. $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$
3. One of the following holds:
   (a) $\sigma(T) = \{0\}$
   (b) $\sigma(T) \setminus \{0\}$ is a finite set
   (c) $\sigma(T) \setminus \{0\}$ is a sequence of real numbers converging to 0

Theorem. [Spectral Theorem] Let H be a separable Hilbert space and $T \in \mathcal{K}(H)$ symmetric. Then there exists an orthonormal basis of H consisting of eigenvectors of T.