3.27pt

Selected topics in the Analysis of Large Dimensional Data

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Outline

- Testing for Global Null and Multiple Testing

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- Model selection in multiple regression - Information Criteria

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- Regularization techniques (1)

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- Regularization techniques (2)

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Major problem - multiple comparisons, multiple testing (in PCA selection of nonzero singular values)

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We test $H_{0j} : \mu_{1j} = \mu_{2j}$ with a t-test $t_j = \frac{\bar{X}_{.j} - \bar{Y}_{.j}}{S(\bar{X}_{.j} - \bar{Y}_{.j})}$, where $S(\bar{X}_{.j} - \bar{Y}_{.j})$ is the estimate of the standard deviation of $\bar{X}_{.j} - \bar{Y}_{.j}$

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If n_1 and n_2 are large enough then $t_j \sim N(\mu_j, 1)$ with $\mu_j = \frac{\mu_{1j} - \mu_{2j}}{\sigma_{1j}/\sqrt{n_1} + \sigma_{2j}/\sqrt{n_2}}$ and $H_{0j} : \mu_j = 0$

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Thus to separate signal from noise we need $c = c(p) \to \infty$ as $p \to \infty$.

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Probability of type I error:

$$P_{H_0}\left(\bigcup_{j=1}^{p}\{|X_j| > c_{Bon}\}\right) \leq \sum_{j=1}^{p} P(\{|X_j| > c_{Bon}\}) = \alpha$$

Exact type I error of Bonferroni

Due to independence

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\begin{aligned}
P(\mathit{Type\ I\ Error}) &= 1 - P_{H_0}\left(\bigcap_{j=1}^{p}\{|X_j| < c_{Bon}\}\right) \\
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$\alpha = 0.05 \quad, n = 30000, P(\mathit{Type\ I\ Error}) \approx 0.0488$

Needle in haystack

Needle in haystack: for some $i \in \{1, , p\}$, $\mu_i = \mu^p$ and for all $j \neq i$, $\mu_j = 0$

How long needle can be found ?

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Is there a test procedure which can find a shorter needle ?

Neyman-Person test for the needle in haystack problem

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Neyman-Pearson optimal test (maximal power for a given type I error) has the form

Reject H_0 for large values of $L(x) = \frac{L_A(x)}{L_0(x)}$

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Bayesian model for "needle in haystack" problem

$$H_0 : \mu = (\mu_1, \ldots, \mu_p) = 0$$

$$H_A : \mu \sim \frac{1}{p} \sum_{i=1}^{p} \delta_{\mu_i}$$

$$\delta_{\mu_i} : P\left(\{\mu_i = \mu^p, \mu_j = 0 \text{ for } j \neq i\}\right) = 1$$

Interpretation: under H_A there is just one signal of known magnitude μ^p but we do not know where and assume a uniform distribution over $i \in \{1, \ldots, p\}$.

Neyman-Pearson test for the needle in haystack problem (2)

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Interpretation - Bonferroni correction has asymptotically optimal detection region under the needle in haystack problem

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$$\text{If } \ ||\mu||^2/p \rightarrow 0 \ \text{ then } \ \frac{||X||^2 - (p + ||\mu^2||)}{\sqrt{2p + 4||\mu||^2}} \rightarrow N(0,1)$$

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Power of the chi-square test converges to α if $\frac{||\mu||^2}{\sqrt{p}} \to 0$

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Power of Neyman-Pearson test converges to α when $\frac{||\rho||^2}{\sqrt{p}} \to 0$

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What happens in the middle ?

Sparse mixture model

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Neyman-Pearson test:

$$L = \Pi_{i=1}^{p}[(1-\epsilon) + \epsilon e^{\mu X_i - \mu^2/2}]$$

Ingster (1999) detection boundary

$$\epsilon^p = p^{-\beta}, \quad \frac{1}{2} < \beta < 1$$

when $\beta = 1/2$ we have about $k = p^{-1/2}p = \sqrt{p}$ signals,
when $\beta = 1$ then the number of signals $k = p^{-1}p = 1$ is equal to 1 (needle in the haystack)

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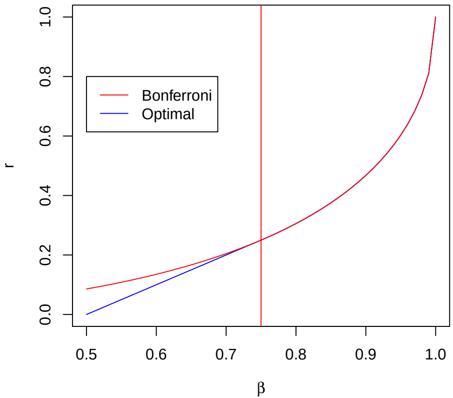
Donoho and Jin (2004): Bonferroni detection boundary coincides with the optimal detection boundary if $\beta \geq 3/4$

Ingster detection boundary



Interpretation: When the number of needles increases than they can be substantially shorter than $\sqrt{2\log p}$ to be detected by Bonferroni.

Lesson learned from theory of testing the global hypothesis

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The length of sparse needles to be detected depends on their number. If there are more of them, they can be shorter.

There is no a single optimal method for testing the global null hypothesis - the selection of the method should depend on the expectation on the signal sparsity.

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	W	R	p

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$$E(V) = \alpha p_0$$

Multiple testing

We now separately test each of hypotheses $H_{0i} : \mu_i = 0$

	H_0 accepted	H_0 rejected	
H_0 true	U	V	p_0
H_0 false	T	S	p_1
	W	R	p

$$FWER = P(V > 0), \qquad FDR = E\left(\frac{V}{R \vee 1}\right)$$

$$E(V) = \alpha p_0$$

$$\alpha = 0.05, p_0 = 5000 \to E(V) = 250$$

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Benjamini-Hochberg (1995) procedure:

(1) $|X|_{(1)} \geq |X|_{(2)} \geq \ldots \geq |X|_{(p)}$

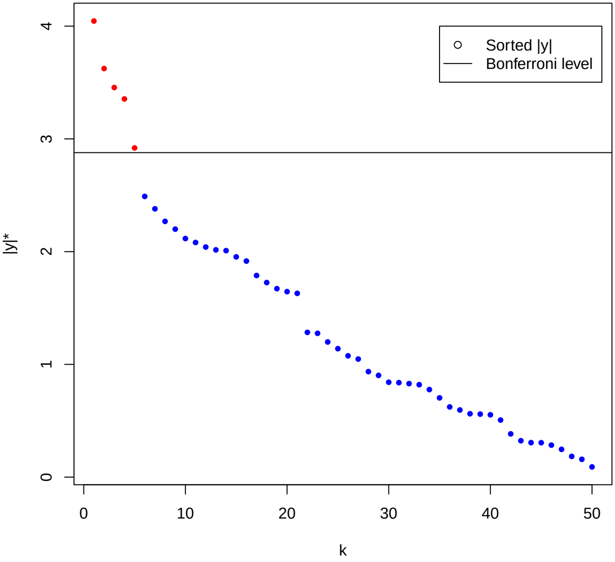
(2) Find the largest index i such that

$$|X|_{(i)} \geq \Phi^{-1}(1 - \alpha_i), \quad \alpha_i = \alpha\frac{i}{2p}, \tag{1}$$

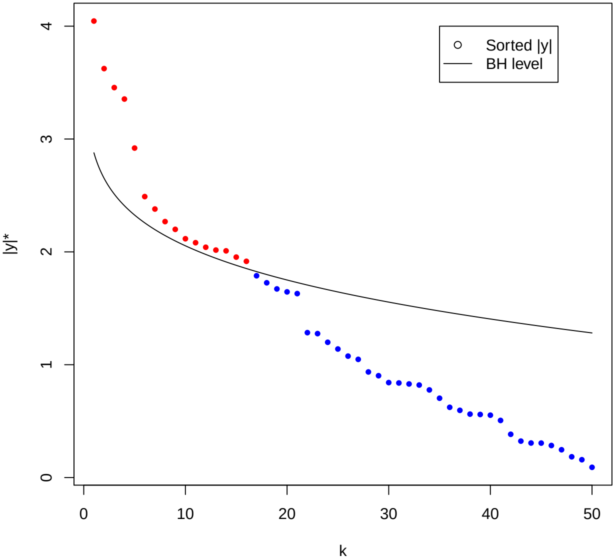
Call this index i_{SU}.

(3) Reject all $H_{(i)}$'s for which $i \leq i_{\mathrm{SU}}$

Bonferroni correction



Benjamini and Hochberg correction



FWER and FDR control

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(Benjamini, Yekutieli, 2001) When test statistics are "positively correlated"then BH controls FDR at or below the level $\alpha \frac{p_0}{p}$. Independently of the correlation structure FDR is controlled at or below the level $\alpha \frac{p_0}{p}$ if $|X|_{(j)}$ is compared to $\Phi^{-1}\left(1 - \frac{j\alpha}{2p \sum_{i=1}^{p} \frac{1}{i}}\right)$.

Estimating μ

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Can we do better ?

Bias-Variance Tradeoff

$$MSE(\hat{\mu}_i) = E(\hat{\mu}_i - \mu_i)^2 = B_i^2 + Var_i,$$

where $B_i = E\hat{\mu}_i - \mu_i$ is the bias of $\hat{\mu}_i$

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Can we improve MSE by introducing some bias and reducing the variance ?

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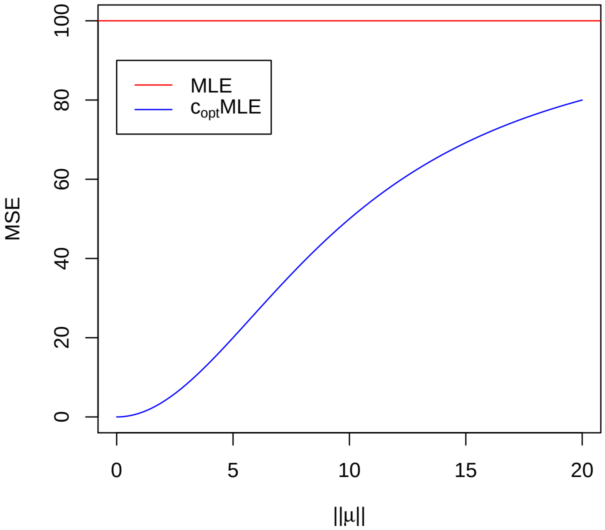
Using elementary calculus we can show that the optimal value of c is equal to

$$c_{opt} = argmin_{c\in R} MSE(c) = \frac{||\mu||^2}{||\mu||^2 + p\sigma^2} \in [0,1) \ .$$

Improvement in MSE, $p = 100, \sigma = 1$



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$$d = 1 \text{ if and only if } \mu_1 = \ldots = \mu_p$$

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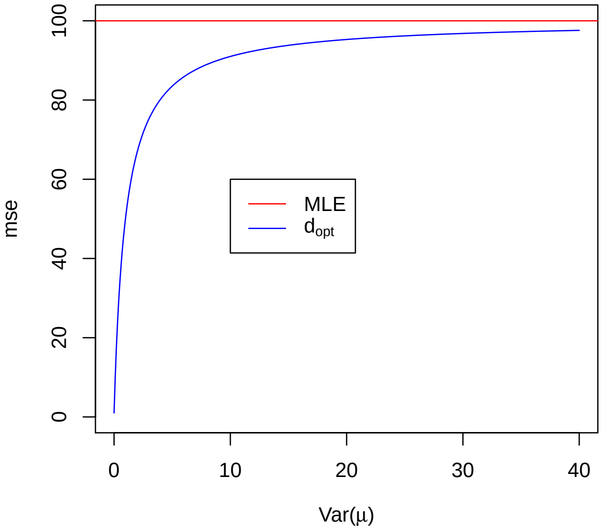

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If $p > 3$ then for both J-S estimators it holds
$E||\hat{\mu}_{JS} - \mu||^2 < E||\hat{\mu}_{MLE} - \mu||^2$

Hard thresholding

When signal is sparse even better results can be obtained by hard thresholding

$$\hat{\mu}_i = \begin{cases} X_i & \text{when} \quad H_{0i} \text{ is rejected} \\ 0 & \text{when} \quad H_{0i} \text{ is not rejected} \end{cases}, \tag{3}$$

where the decisions are made by Bonferroni or BH multiple testing procedures. Bonferroni is optimal for very sparse signals while BH "adapts" to the unknown sparsity (see Abramovich, Benjamini, Donoho and Johnstone, Ann.Statist. 2006)

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Hard thresholded estimator of μ using BH multiple testing rule adapts to the unknown sparsity and is asymptotically optimal in the sense discussed in (Abramovich, Benjamini, Donoho and Johnstone, Ann.Statist. 2006)