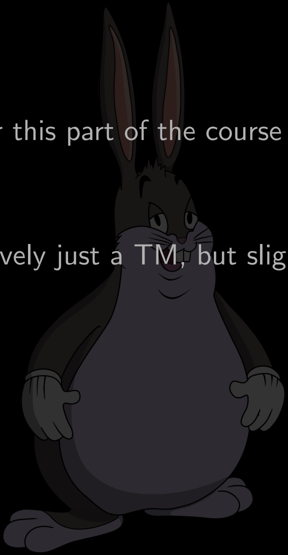
CSC363 Tutorial #7

Alternative TMs

March 9, 2022

Learning objectives this tutorial

- Define some aliases we'll be using for this part of the course (complexity).
- Describe a "multi-tape" TM.
- Show that a multi-tape TM is effectively just a TM, but slightly better.



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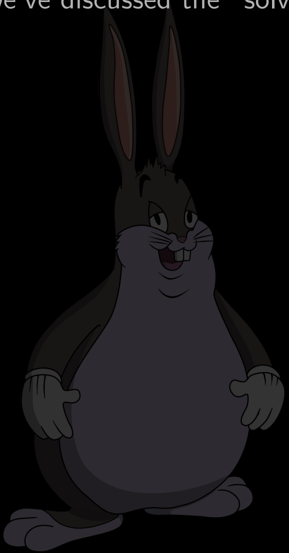

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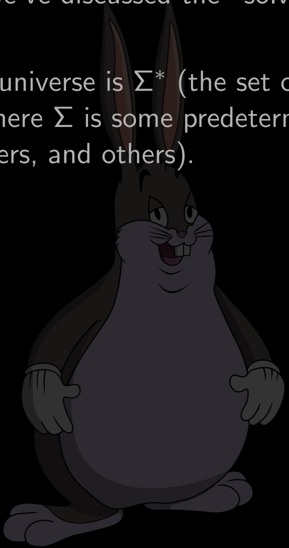

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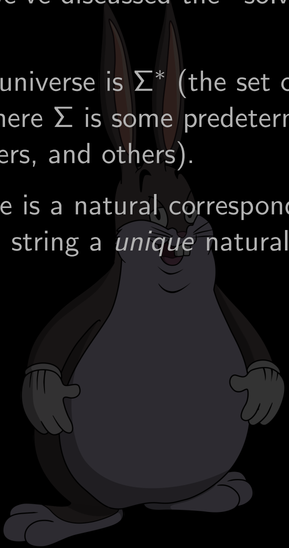

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Ans: Gödel Numbers!

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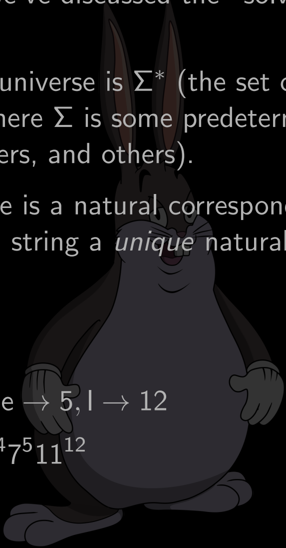
$$\mathsf{godel} \to 2^7 3^{15} 5^4 7^5 11^{12}$$



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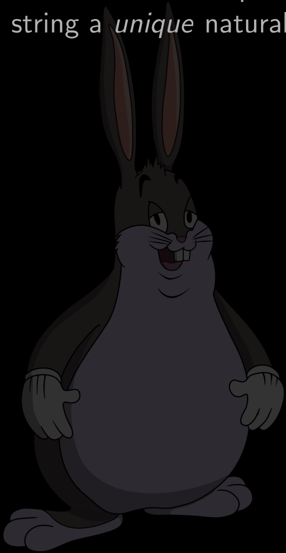

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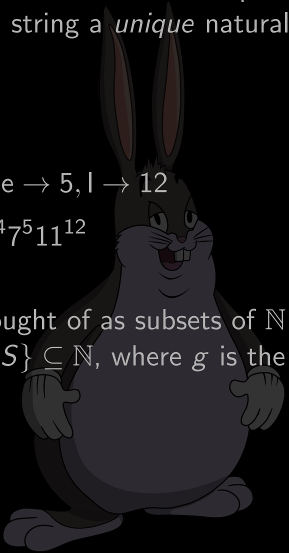

So in this sense, subsets of Σ^* can be thought of as subsets of $\mathbb{N}$ by mapping $S \subseteq \Sigma^*$ to $g(S) = \{g(w) : w \in S\} \subseteq \mathbb{N}$, where g is the Gödel mapping function.

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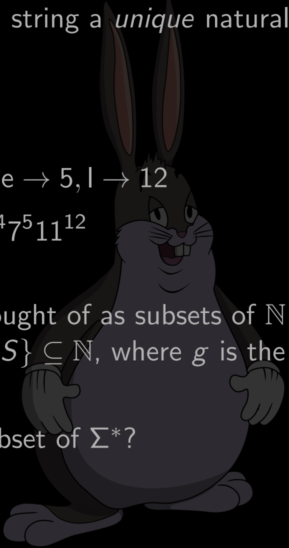

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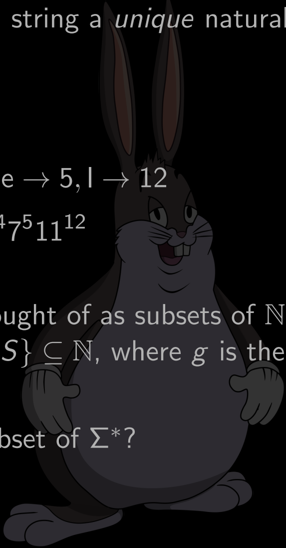

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Ans: *Language*.

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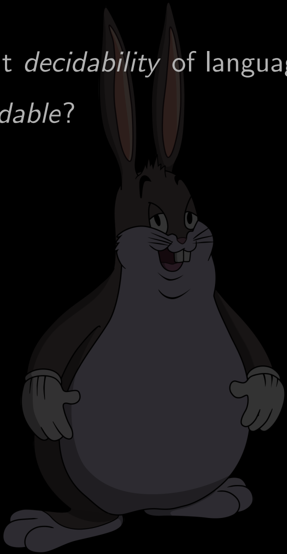

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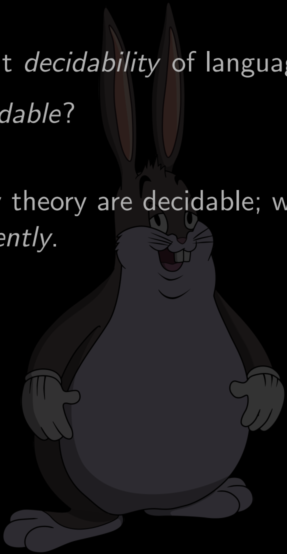

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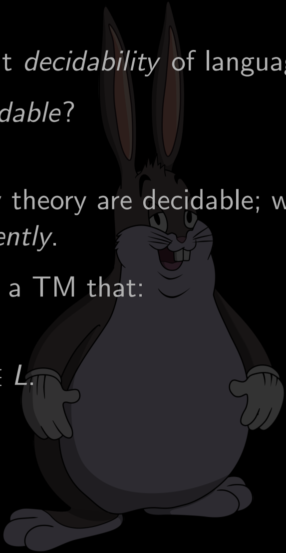

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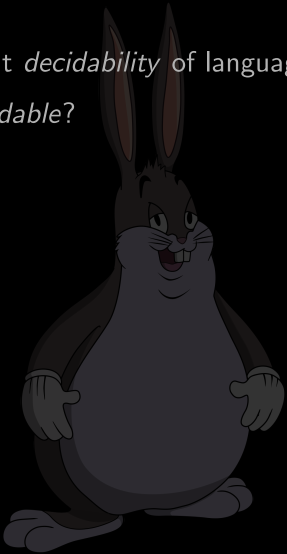

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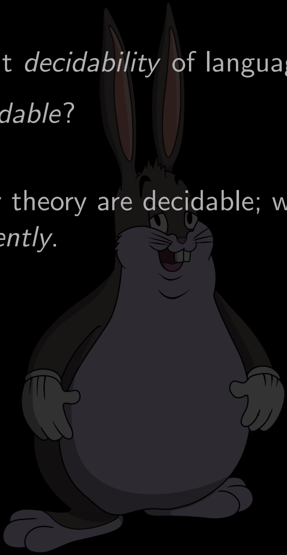

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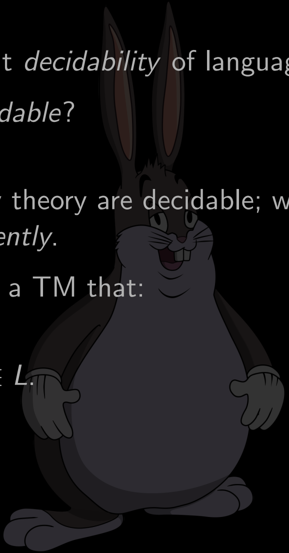
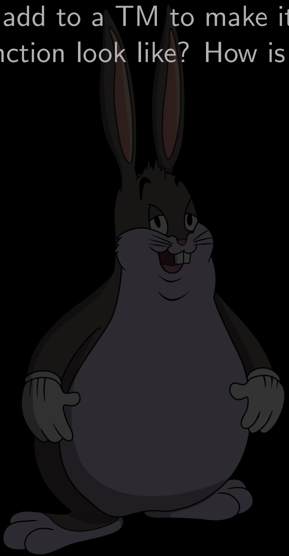


Multi-tape TM

Task: Describe some attributes we could add to a TM to make it "multi-tape". What will the transition function look like? How is input handled?



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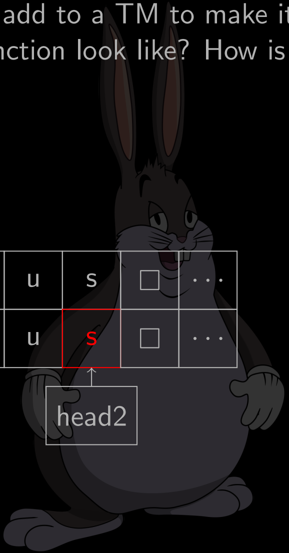
Here’s what I have in mind:

head1

$\cdots$	$\square$	r	e	d	s	u	s	$\square$	$\cdots$
$\cdots$	$\square$	a	m	o	g	u	s	$\square$	$\cdots$

head2

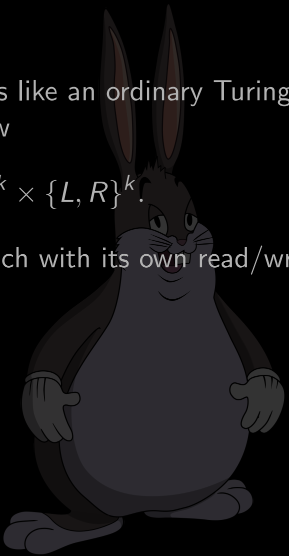


Multi-tape TM

Definition: A k-**tape Turing Machine** is like an ordinary Turing Machine, but its transition function is now

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k.$$

In effect, this gives us k distinct tapes, each with its own read/write head. We read and write k symbols at once.



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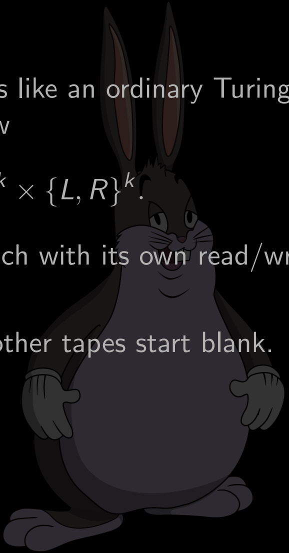
The input is placed on the first tape; all other tapes start blank.



Multi-tape TM Example

Let's construct a multi-tape TM over the language $\{0, 1\}$ that accepts palindromes. Here's what we will do:

head1

$\cdots$	$\square$	1	0	0	0	0	1	$\square$	$\cdots$
$\cdots$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\cdots$

head2

1. Copy the string on tape 1 to tape 2.
2. Move head1 to the beginning of the first tape.
3. Compare characters from head1 and head2, scanning right and left respectively.

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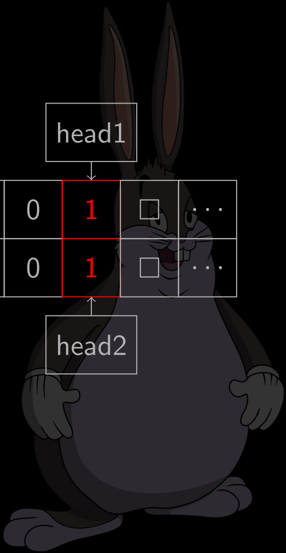

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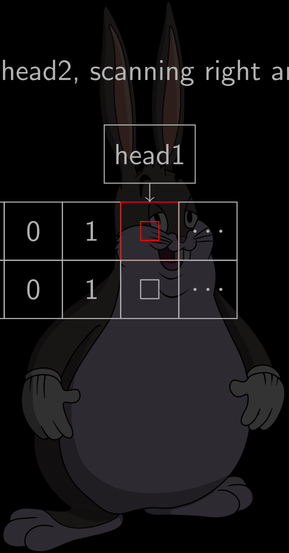
3. Compare characters from head1 and head2, scanning right and left respectively.



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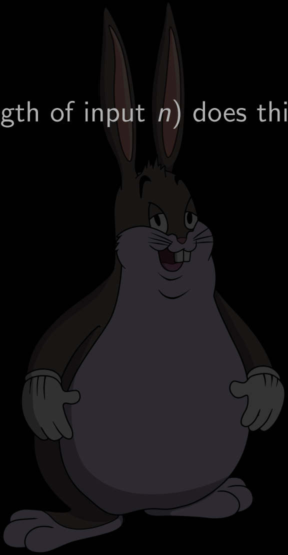
Question: What runtime (in terms of length of input n) does this 2-tape TM have?



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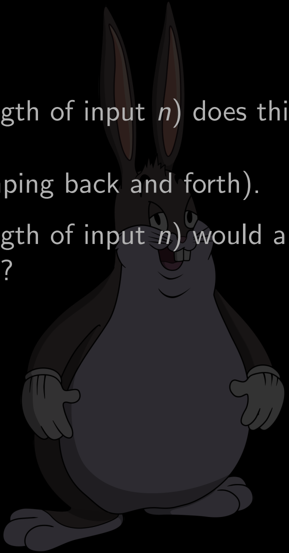

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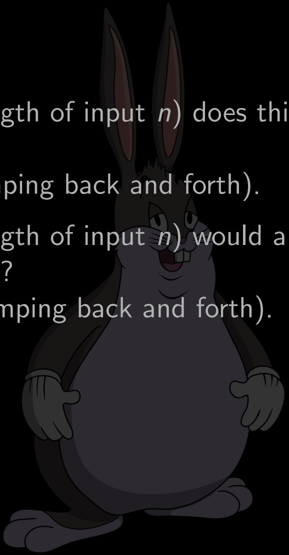
Ans: $O(n^2)$ (because we have to keep jumping back and forth).



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Task: Construct a $O(n)$ 2-tape TM that decides the language $\{0^n 1^n : k \in \mathbb{N}\}$. You may use a high-level description if you want.

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Ans: Here's the procedure I have in mind.

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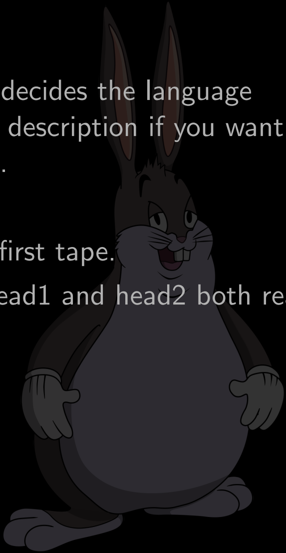
3. Compare character by character; if head1 and head2 both read 0 or both read 1, then reject.



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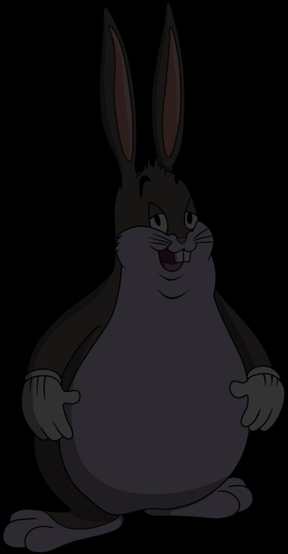
Question: How fast can we decide $\{0^n 1^n : k \in \mathbb{N}\}$ same language with a TM?



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Ans: Naively, $O(n^2)$. The procedure is as follows:

1. Cross out a 0; move to the right end of the string.
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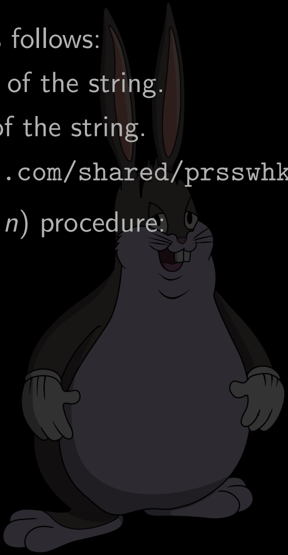

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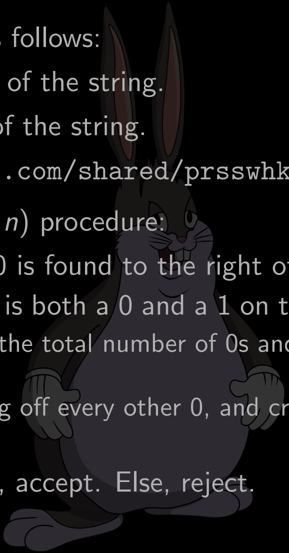
3. If the tape doesn't have any 0s or 1s, accept. Else, reject.



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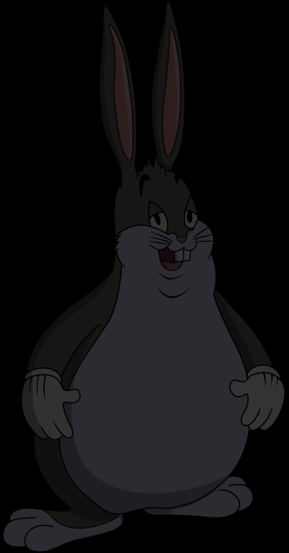

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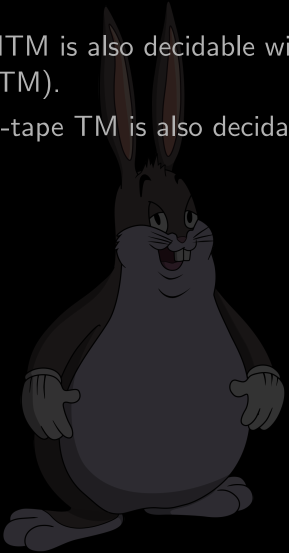
- Everything that is decidable with a k-tape TM is also decidable with a TM.



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In fact, there is an even stronger theorem.

Theorem: Everything that is decidable with a k-tape TM in $O(f(n))$ time is decidable with a TM in $O((f(n))^2)$ time. (See Sipser page 137)

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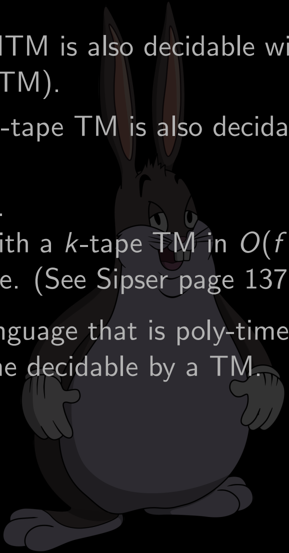
Task: Using the above, show that any language that is poly-time decidable by a k-tape TM is also poly-time decidable by a TM.



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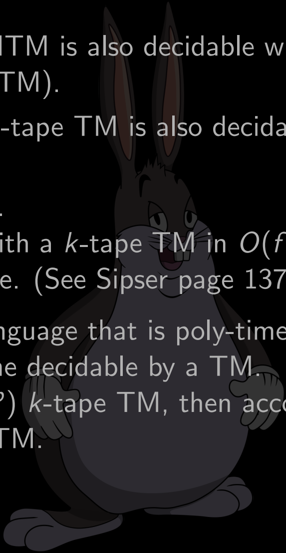

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In effect, this shows that multi-tape TMs are “better”, but don't fundamentally change the set of poly-time decidable languages.