CSC363 Tutorial #8

Nondeterministic Turing machines

March 15, 2023

Things covered in this tutorial

- ⋆ What's a nondeterministic Turing machine?
- ⋆ How do nondeterministic Turing machines accept/reject inputs?
- ⋆ How is nondeterminism useful to the concepts we learn in this course?

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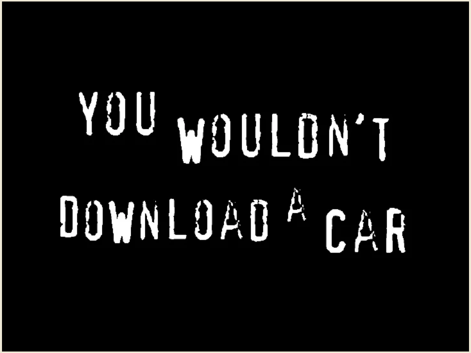

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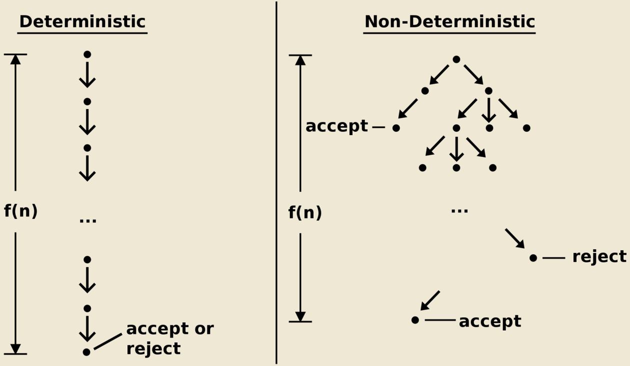
Nondeterminism

(Recall the difference between a DFA and an NFA, if you remember!)

Turing machines are *deterministic*: given an input, they have only one possible sequence of execution.

Nondeterministic Turing Machines (NTMs) have multiple possible sequences of execution.



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(Deterministic) Turing Machines

Task: Fill in . . . .

A Turing Machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ where:

- $\star$ Q is . . .
- $\star$ Σ is . . .
- $\star$ Γ is . . .
- $\star$ $\delta : \ldots \to \ldots$ is the transition function.
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- ⋆ Q is the set of states.
- ⋆ Σ is the *input alphabet*.
- ⋆ Γ is the *tape alphabet* (and satisfies $\Gamma \subseteq \Sigma$).
- ⋆ $\delta : (Q \times \Gamma) \to (Q \times \Gamma \times \{L, R\})$ is the transition function.
- ⋆ q_0 is the starting state.
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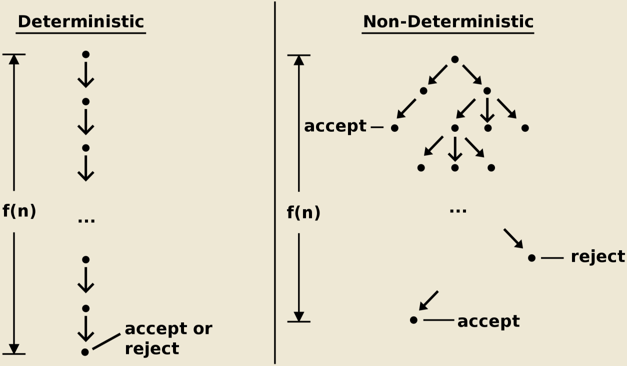
Multiple possible transitions!



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(Nondeterministic) Turing Machines

A NTM M accepts input x when M has *some* execution path that ends in q_{accept}. Otherwise:

- If all execution paths end in q_{reject}, M rejects input x.
- Otherwise, it loops.

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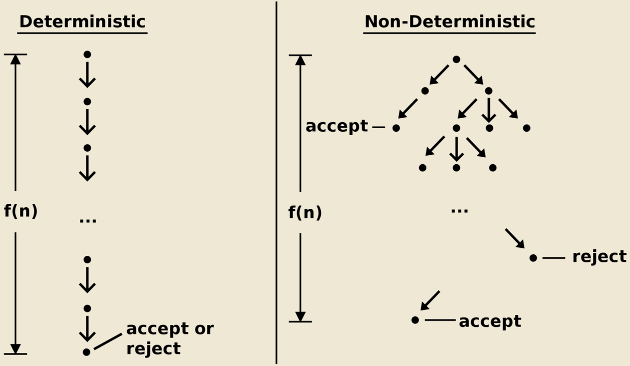
In some sense, think of NTMs like job applications. If you have one job offer, you're good! Otherwise :(



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This is a specific case of the subset sum problem. Given a finite set of natural numbers S and a *target* $t \in \mathbb{N}$, can we find a $S' \subseteq S$ such that S' sums to t?

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subset_sum(S, t):
  for every subset S' of S:
    if S' sums to t:
      return True
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What is the runtime of `subset_sum(S, t)`, in terms of $|S|$ (the size of S)? $O(2^{|S|})$.

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Subset Sum: Nondeterministic Solution

Wait! What about the following code? Does this solve subset sum?

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def subset_sum_2(S, t):
  choose a random subset S' of S
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No :(

Remember that Turing machines are *deterministic*: TMs can't generate a random number.

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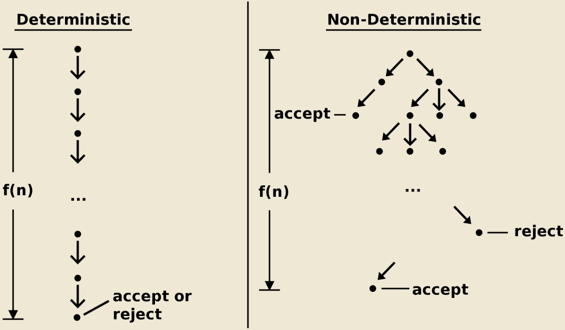
But this pseudocode is valid for a *nondeterministic* Turing machine.



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Subset Sum: Nondeterministic Solution

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def subset_sum_2(S, t):
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Note that this NTM accepts (S, t) iff there is a subset of S that sums to t.

A NTM M **accepts** input x when M has *some* execution path that ends in q_{accept}. Otherwise:

- ⋆ If all execution paths end in q_{reject}, M **rejects** input x.
- ⋆ Otherwise, it loops.

What is the *runtime* of this NTM?

Runtime of a NTM

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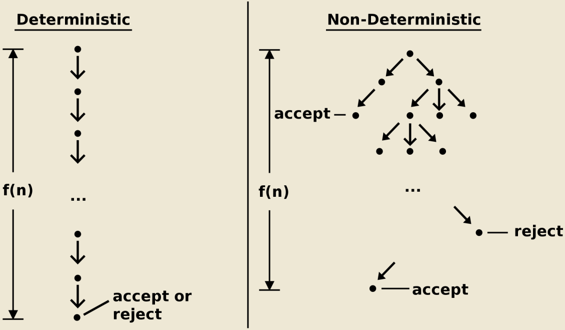

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In other words, it's $f(n)$ in the above diagram.

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Answer: Yes!

Conclusion:

- ⋆ The subset sum problem can be solved in polynomial time by a NTM.
- ⋆ We don't know if subset sum can be solved in polynomial time by a TM.

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$\mathrm{P} \subseteq \mathrm{NP}$ because every Turing machine is a nondeterministic Turing machine.

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P versus NP problem: Is $\mathrm{P} = \mathrm{NP}$?

Examples of problems in NP

The following languages are in NP, but we don't know if any of them are in P.

- ⋆ Subset Sum.
- ⋆ Boolean Satisfiability Problem.
- ⋆ Graph Isomorphism Problem.
- ⋆ Vertex Cover Problem.
- ⋆ Knapsack Problem.
- ⋆ Hamiltonian Path Problem.
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Showing that any one of those problems is not in P will net you $1 million USD.

Last question of the day!

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Answer: Yes.

```
def simulate_NTM(ntm, input):
  while True:
    execute ntm(input) one step.
    if there are multiple possible transitions,
    spawn a thread here to simulate each possible transition
```