CSC363 Tutorial 11

almost done!

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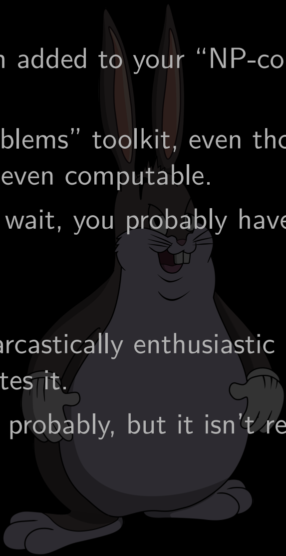
Learning objectives this tutorial

By the end of this tutorial, you should...

- Have one more NP-complete problem added to your “NP-complete problems” toolkit.
- Add a problem to your “NP-hard problems” toolkit, even though it’s kinda useless since the problem isn’t even computable.
- Probably work on assignment 5? Oh wait, you probably have other courses with higher priority... :(
- Be scared for the final exam! D:
- Feel like sushi_enjoyer is just being sarcastically enthusiastic about CSC363 material when he himself hates it.

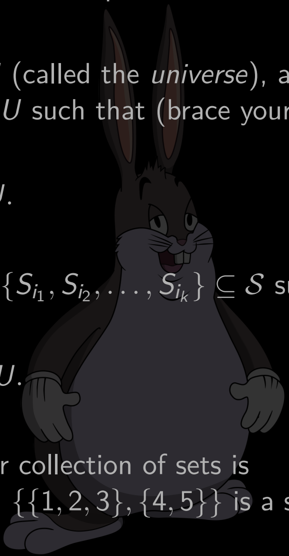
Big Chungus certified readings: chapter 8 probably, but it isn’t really necessary tbh.



do you like proving NP-completeness? D:

well too bad! you'll have to do it for the upcoming problem set.

Set cover



We will now describe the **set cover problem** and prove it is NP-complete, because why not.

Suppose we are given a set of elements U (called the *universe*), and a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of U such that (brace yourself, $\bigcup$)

$$\bigcup_{i=1}^{n} S_i = U.$$

A **set cover** of U is a subcollection $\mathcal{S}' = \{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\} \subseteq \mathcal{S}$ such that

$$\bigcup_{m=1}^{k} S_{i_k} = U.$$

For example, if $U = \{1, 2, 3, 4, 5\}$, and our collection of sets is $\mathcal{S} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$, then $\{\{1, 2, 3\}, \{4, 5\}\}$ is a set cover.

Set cover

Task: Let $U = \{$🍣, 🧃, 🥕, 🥐, 🥖, 🍙$\}$, and

$\mathcal{S} = \{\{$🍣, 🥖$\}, \{$🧃, 🥖, 🍙$\}, \{$🧃, 🥕, 🥖$\}, \{$🥐, 🍙$\}, \{$🥐$\}, \{$🍣, 🥕, 🥐$\}\}$.

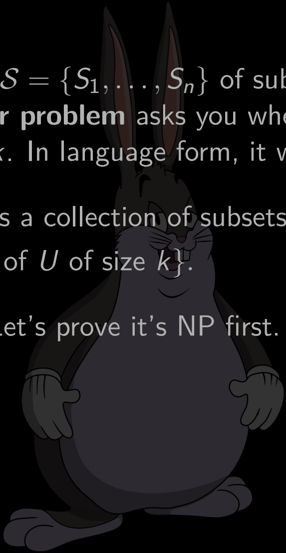
Find the smallest set cover of U.
Answer: The smallest set cover is $\{\{$🧃, 🥖, 🍙$\}, \{$🍣, 🥕, 🥐$\}\}$, which is of size 2.

Set cover is NP-complete!

Now given a universal set U, a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of U, and a natural number k, the **set cover problem** asks you whether it is possible to find a set cover for U of size k. In language form, it would be

$$\text{Set-Cover} = \{(U, \mathcal{S}, k) : U \text{ is a set, } \mathcal{S} \text{ is a collection of subsets of } U, \\ \text{and there is a set cover of } U \text{ of size } k\}.$$

Turns out this problem is NP-complete! Let's prove it's NP first.
Task: Prove that Set-Cover is NP.

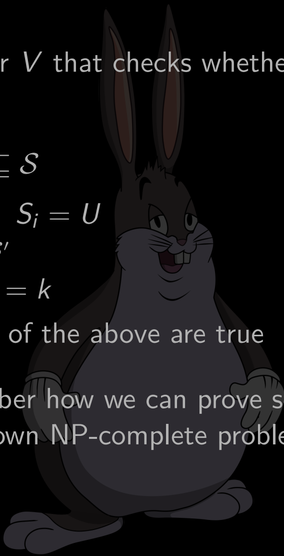


Set cover is NP-complete!

Task: Prove that Set-Cover is NP.

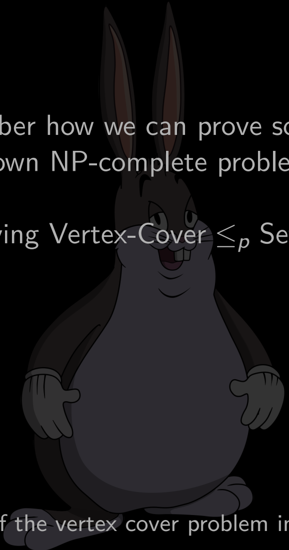
Answer: We can build a poly-time verifier V that checks whether a given subcollection $\mathcal{S}'$ of $\mathcal{S}$ is a set cover for U.

$$
\begin{aligned}
V(U, \mathcal{S}, k, \mathcal{S}') : \quad & \text{Check if } \mathcal{S}' \subseteq \mathcal{S} \\
& \text{Check if } \bigcup_{S_i \in \mathcal{S}'} S_i = U \\
& \text{Check if } |\mathcal{S}'| = k \\
& \text{Accept iff all of the above are true}
\end{aligned}
$$



Now we prove it is NP-complete. Remember how we can prove something is NP-complete by showing that some known NP-complete problem reduces to it?

Set cover is NP-complete!



Now we prove it is NP-complete. Remember how we can prove something is NP-complete by showing that some known NP-complete problem reduces to it?

Task: Show that Set-Cover $\in$ NP by proving Vertex-Cover $\leq_p$ Set-Cover.[1]

[1] Recall: this involves converting an instance of the vertex cover problem into an instance of set cover problem in poly-time.

Set cover is NP-complete!

Answer: Suppose we are given an instance (G, k) of the vertex cover problem. We may transform it into a set cover problem $(U_G, \mathcal{S}_G, k_G)$ with the property that

$$(G, k) \in \text{Vertex-Cover} \Leftrightarrow (U_G, \mathcal{S}_G, k_G) \in \text{Set-Cover}.$$

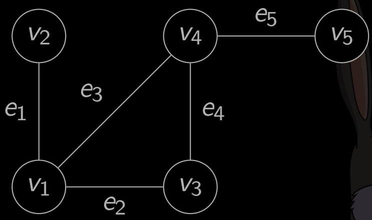
Let $v_1, \ldots, v_n$ be the vertices of G, and $e_1, \ldots, e_m$ the edges of G. Define $U_G, \mathcal{S}_G, k_G$ as follows:

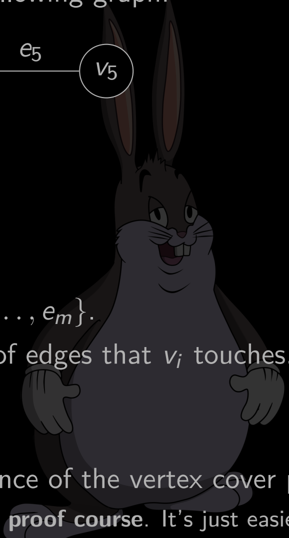
- U_G will consist of all the edges $\{e_1, \ldots, e_m\}$.
- For each vertex v_i, let S_i be the set of edges that v_i touches. Then let $\mathcal{S}_G = \{S_1, \ldots, S_n\}$.
- $k_G = k$.

This transformation takes poly-time with respecc to the size of (G, k). We claim $(G, k) \in \text{Vertex-Cover} \Leftrightarrow (U_G, \mathcal{S}_G, k_G) \in \text{Set-Cover}$.

Set cover is NP-complete!

We'll "prove" $(G, k) \in$ Vertex-Cover $\Leftrightarrow (U_G, \mathcal{S}_G, k_G) \in$ Set-Cover via example.[2] Suppose $k = 2$ and G is the following graph:



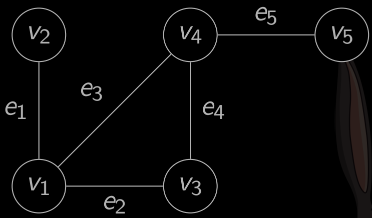


- U_G will consist of all the edges $\{e_1, \ldots, e_m\}$.
- For each vertex v_i, let S_i be the set of edges that v_i touches. Then let $\mathcal{S}_G = \{S_1, \ldots, S_n\}$.
- $k_G = k$.

Task: Find U_G, $\mathcal{S}_G$, and k_G for this instance of the vertex cover problem.

[2] Please, please, please, **do not do this in any proof course**. It's just easier for illustrate with an example.

Set cover is NP-complete!



$U_G = \{e_1, e_2, e_3, e_4, e_5\}$.

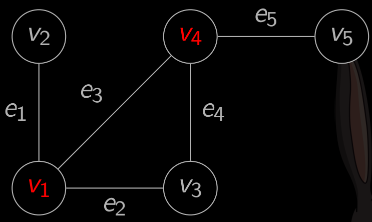
We have $S_1 = \{e_1, e_2\}$, $S_2 = \{e_1\}$, $S_3 = \{e_2, e_4\}$, $S_4 = \{e_3, e_4, e_5\}$, $S_5 = \{e_5\}$. So

$$\mathcal{S}_G = \{S_1, \dots, S_5\} = \{\{e_1, e_2\}, \{e_1\}, \{e_2, e_4\}, \{e_3, e_4, e_5\}, \{e_5\}\}.$$

$k_G = 2$ since $k = 2$.

Task: Find a vertex cover for G. What would the corresponding set cover be?

Set cover is NP-complete!



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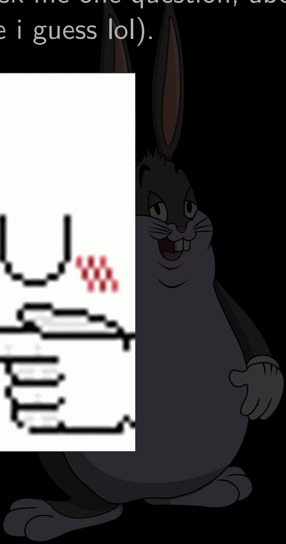
$$\mathcal{S}_G = \{S_1, \ldots, S_5\} = \{\{e_1, e_2\}, \{e_1\}, \{e_2, e_4\}, \{e_3, e_4, e_5\}, \{e_5\}\}.$$

$k_G = 2$ since $k = 2$.

v_1, v_4 form a vertex cover of G. $\mathcal{S}' = \{S_1, S_4\}$ forms a set cover of U.

Break time!

No sushi juice this time. But you get to ask me one question, about pretty much anything (as long as it's appropriate i guess lol).



no more brake time with uwu

Alright so we now have one more problem that we know is NP-complete. I'm so excited! Anyone? ;-;

Let's add an NP-hard problem to the back of our memory! This one is actually a bit tricky to prove though...

Task: What does HP stand for?

no more brake time with uwu

Task: What does HP stand for?
Answer: **H**elo **P**hish.

$$\mathrm{HP} = \{(M, w) : M \text{ is a Turing machine that halts on input } w\}.$$

We will prove HP is NP-Hard by showing 3SAT $\leq_p$ HP.[3]

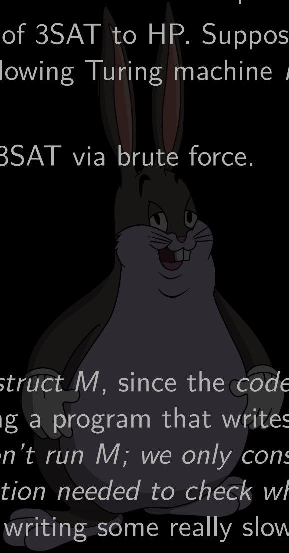
[3] In fact, any computable language A satisfies $A \leq_p$ HP! You can just adapt the proof I'm about to show.

HP is NP-complete

$$\mathsf{HP} = \{(M, w) : M \text{ is a Turing machine that halts on input } w\}.$$

We will construct the following reduction of 3SAT to HP. Suppose φ is a given instance of 3SAT. Construct the following Turing machine M:

$$\begin{aligned} M(\varphi):\quad & \text{Check whether } \varphi \in \text{3SAT via brute force.} \\ & \text{If } \varphi \in \text{3SAT:} \\ & \quad \text{Accept} \\ & \text{Else:} \\ & \quad \text{Loop} \end{aligned}$$



Notice that *it takes constant time to construct* M, since the *code* of M doesn't depend on φ at all. It's like writing a program that writes a fixed Python script into a text file. Also, *we don't run M; we only construct it, and bypass the exponential time computation needed to check whether $\varphi \in$ 3SAT via brute force.* Again, it's like writing some really slow code to a text file but not running it.

HP is NP-complete

$$\mathrm{HP} = \{(M, w) : M \text{ is a Turing machine that halts on input } w\}.$$

We will construct the following reduction of 3SAT to HP. Suppose φ is a given instance of 3SAT. Construct the following Turing machine M:

$M(\varphi)$: Check whether $\varphi \in$ 3SAT via brute force.
If $\varphi \in$ 3SAT:
  Accept
Else:
  Loop

Task: Show $\varphi \in \text{3SAT} \Leftrightarrow (M, \varphi) \in \mathrm{HP}$, where M is as above. Then convince yourself that we can replace 3SAT with any computable language, and the same proof would work.

buy

`helo_fish.jpg` is sad to see you go ;-;
only one more week left! D: