

Recall the formal definition of a limit of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $\lim_{x \rightarrow c} f(x) = L$  if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \implies |f(x) - L| < \epsilon]$$

**Problem 1**

Let  $f(x) = 3x - 4$ . Let  $\epsilon = \frac{1}{100}$ . Find a  $\delta > 0$  so that for all  $x \in \mathbb{R}$ ,

$$0 < |x - 2| < \delta \implies |f(x) - 2| < \frac{1}{100}$$

**Problem 2**

Show that  $\lim_{x \rightarrow 0} \frac{1}{x} \neq 100$  by proving the negation:

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}) \left[ 0 < |x| < \delta \wedge \left| \frac{1}{x} - 100 \right| \geq \epsilon \right]$$

Hint: since we are allowed to choose  $\epsilon$  and  $x$ , it suffices to let  $\epsilon = 1$  and only consider positive values of  $x$ .

**Problem 3**

For each of the following problems you may draw a graph to support your reasoning instead of giving a full proof.<sup>a</sup>

1. Define  $h : \mathbb{R} \rightarrow \mathbb{R}$ ,  $h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ . Does  $\lim_{x \rightarrow 0} h(x)$  exist?
2. Define  $h : \mathbb{R} \rightarrow \mathbb{R}$ ,  $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ . Does  $\lim_{x \rightarrow 0} h(x)$  exist?
3. Define  $h : (-1, 1) \setminus \{0\} \rightarrow \mathbb{R}$ ,  $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ . Does  $\lim_{x \rightarrow 0} h(x)$  exist?

<sup>a</sup>Don't do this on your assignment.

**Problem 4**

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$\lim_{x \rightarrow \infty} f(x) = L.$$

1. Write down the definition of

$$\lim_{x \rightarrow \infty} f(x) = L.$$

2. Write down the definition of

$$\lim_{y \rightarrow 0^+} f\left(\frac{1}{y}\right) = L.$$

3. Show that

$$\lim_{y \rightarrow 0^+} f\left(\frac{1}{y}\right) = L.$$

**Problem 5**

Recall that if  $f, g$  are defined in some interval around  $c \in \mathbb{R}$ , and

$$\lim_{x \rightarrow c} f(x) = M \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = N,$$

then

$$\lim_{x \rightarrow c} [f(x) + g(x)] = M + N \quad \text{and} \quad \lim_{x \rightarrow c} [f(x)g(x)] = MN.$$

This problem shows why it is necessary for  $\lim_{x \rightarrow c} f(x)$  and  $\lim_{x \rightarrow c} g(x)$  to exist in the above.

1. Give an example of an  $f, g$  defined in an interval around  $c \in \mathbb{R}$  such that  $\lim_{x \rightarrow c} [f(x) + g(x)]$  exists but  $\lim_{x \rightarrow c} f(x)$  or  $\lim_{x \rightarrow c} g(x)$  don't exist.
2. Give an example of an  $f, g$  defined in an interval around  $c \in \mathbb{R}$  such that  $\lim_{x \rightarrow c} [f(x)g(x)]$  exists but  $\lim_{x \rightarrow c} f(x)$  or  $\lim_{x \rightarrow c} g(x)$  don't exist.

**Problem 6**

Use squeeze theorem in the following questions.

1. Write down the statement of squeeze theorem.
2. Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Show that  $\lim_{x \rightarrow 0} f(x) = 0$ .

3. Show that for any function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\lim_{x \rightarrow c} |f(x)| = 0$$

if and only if

$$\lim_{x \rightarrow c} f(x) = 0.$$