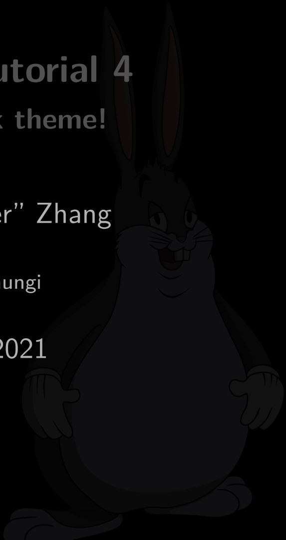
CSC363H5 Tutorial 4

this time in dark theme!



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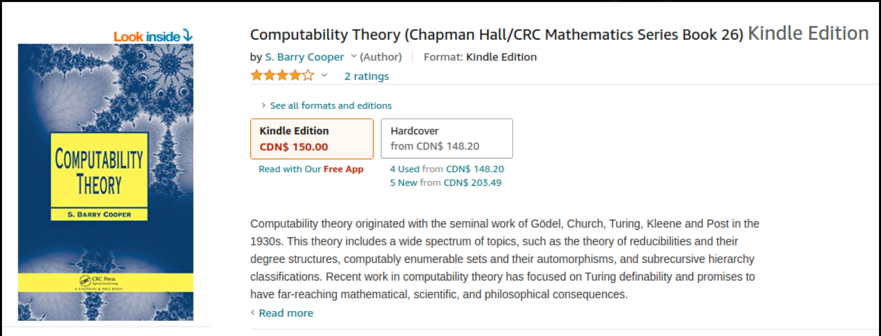
Learning objectives this tutorial

By the end of this tutorial, you should...

- Find a totally legal way to obtain the recommended textbook for this course.
- Read the recommended textbook sections (if you'd like, but I think it's really worth it!).
- Be able to state the *normal form theorem* and apply it to an example problem.
- Turn yourself into a moai. 🗿 🗿 🗿
- Have your internet revoked by your ISP due to piracy.
- Appreciate the time when I ran tutorials using light theme slides instead of dark theme slides like these.

Readings? What is this, English class?

Yea, I just discovered that a (recommended) textbook exists for this course! Available for the cheap price of only $150.



yea right.

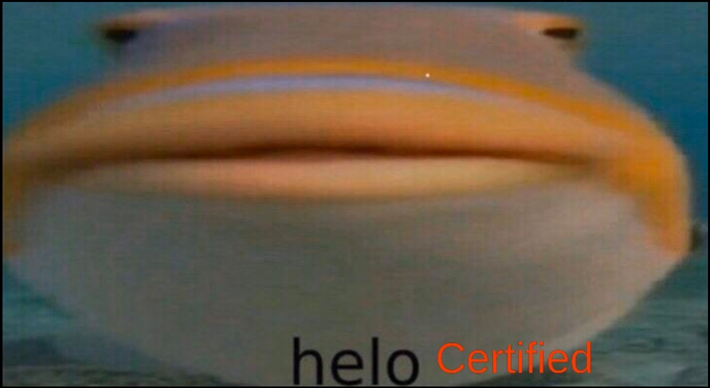
Trust me, it's worth a read! You will learn cool stuff like ~~how to solve homework problems~~ the proof of some theorems skipped in class.

Readings? What is this, English class?

Just some recommended readings[1] from me to reinforce lecture material:

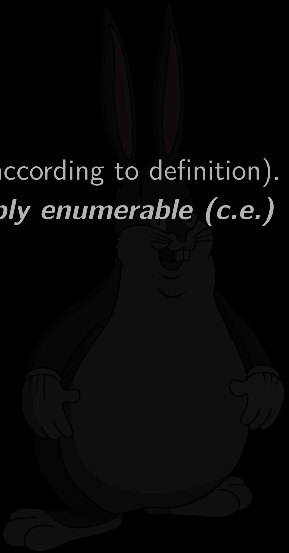
- Week 2: sections 2, 4.2, 4.3 (Note the book gives a different definition of Turing machine, but it is equivalent and worth a read!)
- Week 3: sections 5.1-5.3
- Week 4: sections 5.2, 10.1 (first page)

These recommended readings are certified by `helo_fish.jpg`.



[1] not official! just what i think would be useful.

Normal Form Theorem!



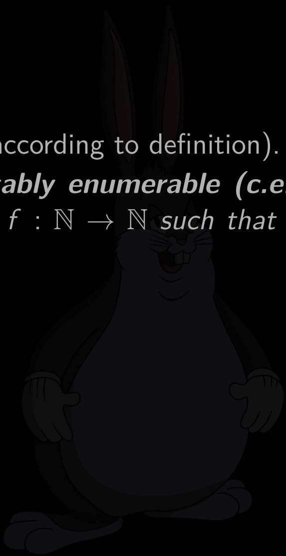
Task: Complete the following sentence (according to definition).

*We say a set $A \subseteq \mathbb{N}$ is **computably enumerable (c.e.)** if...*

Normal Form Theorem!

Task: Complete the following sentence (according to definition).

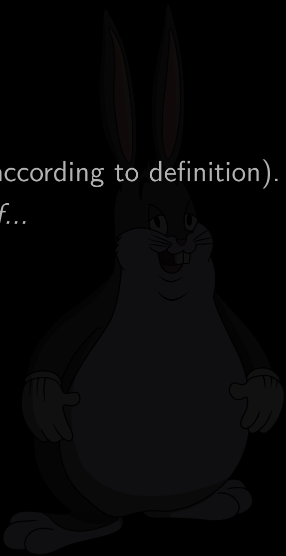
> *We say a set $A \subseteq \mathbb{N}$ is **computably enumerable (c.e.)** if $A = \emptyset$ or there exists a computable $f : \mathbb{N} \to \mathbb{N}$ such that $A = \mathit{range}(f)$.*



Normal Form Theorem!

Task: Complete the following sentence (according to definition).

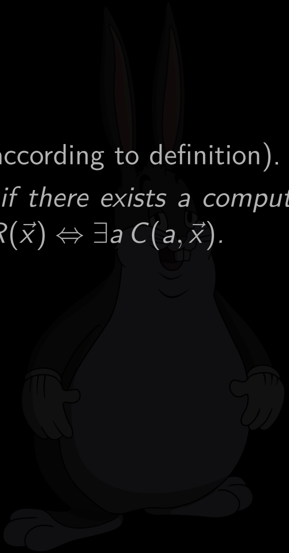
We say a relation $R(\vec{x})$ *is in* Σ^0_1 *if...*



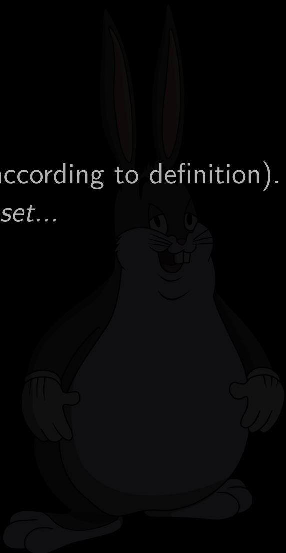
Normal Form Theorem!

Task: Complete the following sentence (according to definition).

We say a relation $R(\vec{x})$ *is in* Σ^0_1 *if there exists a computable relation* $C(a,\vec{x})$ *such that for all* $\vec{x}$*,* $R(\vec{x}) \Leftrightarrow \exists a\, C(a,\vec{x})$*.*



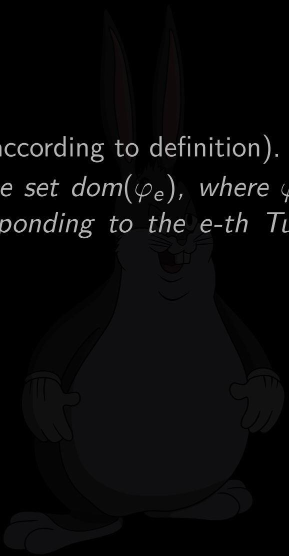
Normal Form Theorem!



Task: Complete the following sentence (according to definition).

For $e \in \mathbb{N}$, we denote by W_e the set...

Normal Form Theorem!



Task: Complete the following sentence (according to definition).

For $e \in \mathbb{N}$, we denote by W_e the set dom(φ_e), where φ_e is the partial recursive function corresponding to the e-th Turing machine.

Normal Form Theorem!

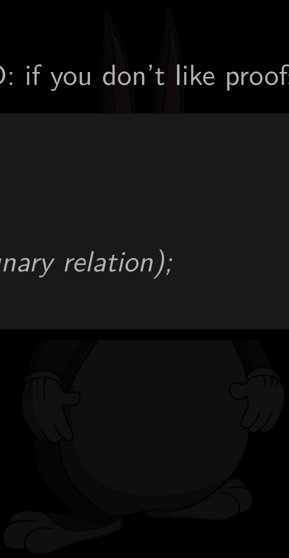
Finally, we get to the statement! :D (or D: if you don't like proofs)

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma^0_1$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).



Hope you like proofs.

(1) $\Rightarrow$ (2): A is c.e. $\Rightarrow A \in \Sigma^0_1$.

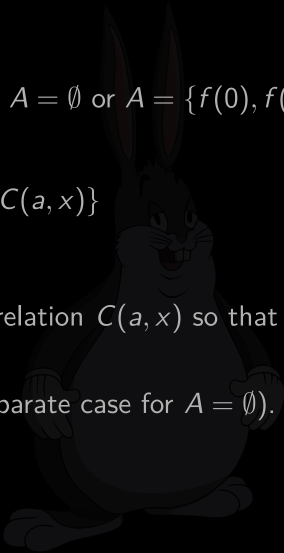
To prove this: suppose A is c.e., so either $A = \emptyset$ or $A = \{f(0), f(1), \ldots\}$. To show $A \in A \in \Sigma^0_1$, we want to write

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}$$

where $C(a, x)$ is a computable relation.

Task: Come up with such a computable relation $C(a, x)$ so that A satisfies the above equality.

(Hint: I think you will probably need a separate case for $A = \emptyset$).



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$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}$$

where $C(a, x)$ is a computable relation.

Answer: If $A = \emptyset$, let $C(a, x)$ be the empty relation (false for all a, x). Then for any $x \in \mathbb{N}$, there doesn't exist a such that $C(a, x)$ is true, so

$$\{x \in \mathbb{N} : \exists a\, C(a, x)\} = \emptyset = A.$$

Otherwise let $C(a, x) : f(a) = x$. Since the range of f is A, we have

$$\{x \in \mathbb{N} : \exists a\, C(a, x)\} = \{x \in \mathbb{N} : \exists a\, f(a) = x\} = A.$$

Hope you like proofs.

(2) $\Rightarrow$ (3): $A \in \Sigma^0_1 \Rightarrow A = W_e$ for some $e \in \mathbb{N}$.

To prove this: suppose $A \in \Sigma^0_1$, so there is a computable relation $C(a, x)$ so that

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$

To show $A = W_e$ for some $e \in \mathbb{N}$, we want to produce a Turing machine that only halts on A (since W_e is the set of inputs that make the eth Turing machine halt).

Task: Come up with such a Turing machine (informally describing what it does).

Hope you like proofs.

(2) $\Rightarrow$ (3): $A \in \Sigma^0_1 \Rightarrow A = W_e$ for some $e \in \mathbb{N}$.

To prove this: suppose $A \in \Sigma^0_1$, so there is a computable relation $C(a, x)$ so that

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To show $A = W_e$ for some $e \in \mathbb{N}$, we want to produce a Turing machine that only halts on A (since W_e is the set of inputs that make the eth Turing machine halt).

Answer: Let P_e be the Turing machine that does the following given an input $x \in \mathbb{N}$:

```
a = 0
while not C(a, x):
  a += 1
```

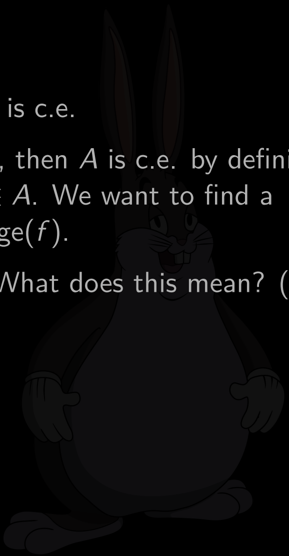
This program halts on input x if and only if $x \in A$.

Hope you like proofs.

(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

Task: Recall the definition of $\varphi_{e,s}(x)\downarrow$. What does this mean? (Week 4 slides)



Hope you like proofs.

(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

Task: Recall the definition of $\varphi_{e,s}(x)\downarrow$. What does this mean? (Week 4 slides)
Answer: $\varphi_{e,s}(x)\downarrow$ means that $x, e < s$ and the eth Turing machine takes s steps or less to halt on x.

So $x \in W_e$ if and only if $\varphi_{e,s}(x)\downarrow$ for some $s \in \mathbb{N}$. In other words, s is large enough so that $x < s$ and the Turing machine takes s steps or less to halt on x.

Hope you like proofs.

(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

Task: Show (or convince yourself) that the range of f is exactly A, where

$$f(\langle x, s\rangle) = \begin{cases} x & \varphi_{e,s}(x)\downarrow \\ p & \text{otherwise.} \end{cases}$$

This finishes the proof!

Task: Breathe in for 4 seconds, hold your breath for 7 seconds, then slowly release it for 8 seconds. Repeat for 3 minutes.

While doing this, imagine you are a 600-year-old giant 50-ton stone on Easter Island in the middle of the Pacific. You have access to a black box that can solve *all* problems in this universe, including non-computable problems such as deciding whether $x \in \mathbb{N}$ is in the halting set K, as it has access to any oracle in existence.
As you breath in and out, you feel your brain expanding. You are filled with *determination*. You can solve any problem. You acknowledge all the suffering and injustice you have received throughout your life, and have ascended beyond any worldly desire. A light breeze brushes up against your coarse stone surface. You feel *at peace*.
A *Gygis alba* lands on your head. You can hear it chirping amidst the light breeze. You are unbothered, because you have ascended beyond life.
You are a Moai. 🗿 🗿 🗿

Now that you have attained knowledge of this universe, you may prove the following:

> *Let $A \subseteq \mathbb{N}$. A is c.e. if and only if there exists a partial computable function f such that $A = range(f)$.*

Task: Prove the above using the normal form theorem (written below).

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma^0_1$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

Hint: For proving the $\Leftarrow$ direction, show A satisfies condition 2 in the above theorem.

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Now that you have attained knowledge of this universe, you may prove the following:

> *Let $A \subseteq \mathbb{N}$. A is c.e. if and only if there exists a partial computable function f such that $A = range(f)$.*

Proof: ($\Rightarrow$) Suppose A is c.e.. Then either $A = \emptyset$ or $A = \text{range}(f)$ for some *computable* function f. If $A = \emptyset$, then A is the range of the empty partial computable function (always undefined). Otherwise $A = \text{range}(f)$ (and f is partial computable since it is computable).

($\Leftarrow$) Let $A = \text{range}(f)$ for some partial computable f. Then $f = \varphi_e$ for some $e \in \mathbb{N}$ (since f can be emulated by a Turing machine). Define the relation

$$C(s, x, y) : \varphi_{e,s}(x) \downarrow= y.$$

Then

$$A = \{y \in \mathbb{N} : \exists (s, x)\, C(s, x, y)\}.$$

(Formally I should have used $C(\langle s, x \rangle, y)$ for the relation.)

Thanks for watching my video. For more information, please visit `sjorv.github.io` for a giveaway of two $GME shares.

If you'd like to cheat on the homework, please stay for office hours! :D

If not, then bye. ;-;
To help you on your homework, please try the following proof methods.

Common proof techniques

Proof by intimidation Trivial!

Proof by cumbersome notation The theorem follows immediately from the fact that $\left|\bigoplus_{k\in S}\left(\mathfrak{K}^{\mathbb{F}^\alpha(i)}\right)_{i\in\mathcal{U}_k}\right| \preccurlyeq \aleph_1$ when $[\mathfrak{H}]_{\mathcal{W}} \cap \mathbb{F}^\alpha(\mathbb{N}) \neq \emptyset$.

Proof by inaccessible literature The theorem is an easy corollary of a result proven in a hand-written note handed out during a lecture by the Yugoslavian Mathematical Society in 1973.

Proof by ghost reference The proof my be found on page 478 in a textbook which turns out to have 396 pages.

Circular argument Proposition 5.18 in [BL] is an easy corollary of Theorem 7.18 in [C], which is again based on Corollary 2.14 in [K]. This, on the other hand, is derived with reference to Proposition 5.18 in [BL].

Proof by authority My good colleague Andrew said he thought he might have come up with a proof of this a few years ago...

Internet reference For those interested, the result is shown on the web page of this book. Which unfortunately doesn't exist any more.

Proof by avoidance *Chapter 3:* The proof of this is delayed until Chapter 7 when we have developed the theory even further. *Chapter 7:* To make things easy, we only prove it for the case $z = 0$, but the general case in handled in Appendix C. *Appendix C:* The formal proof is beyond the scope of this book, but of course, our intuition knows this to be true.

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