Firstly, some more practice on continuous functions.

Problem 1

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$. Construct a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all $x \in [a, b]$.
2. Give an example to show that this doesn't need to be true if we only assume f is continuous on (a, b).

Problem 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Furthermore, suppose f is continuous at 0. Show that f is continuous everywhere. *Hint: Show that $f(0) = 0$ first.*

A function $f : D \to \mathbb{R}$ is **uniformly continuous** when

$$(\forall \epsilon > 0)(\exists \delta)(\forall x, y \in D)[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon].$$

Problem 3

1. Describe the (subtle but important) difference between "uniformly continuous" and "continuous".
2. Briefly explain why uniform continuity implies continuity. In other words, show if $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous everywhere in D.
3. Give an example of a function $f : D \to \mathbb{R}$ that is continuous but not uniformly continuous.

Problem 4

Decide whether each of the following functions is not continuous, continuous, or uniformly continuous. You do not have to give a formal proof.

1. $f : [0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$.
2. $f : (0, \infty) \to \mathbb{R}, f(x) = \dfrac{1}{x}$.
3. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin(x)$.
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin(x) + \cos(x) + 99x$.

Problem 5

Give an example of a bounded and continuous function that is not uniformly continuous. You may choose the domain of this function to your liking.