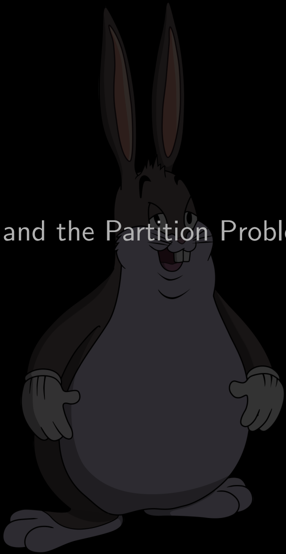
CSC363 Tutorial #10

Subset Sum, Partition problem

March 30, 2022

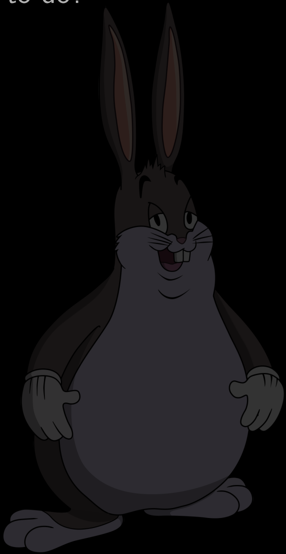
Learning objectives this tutorial

- Review the Subset Sum Problem.
- Introduce the Partition problem.
- Prove that the Subset Sum Problem and the Partition Problem p-reduce to each other.



Subset Sum Review

Task: Recall the (integer, multiset) Subset Sum (Decision) Problem. What are you given? What are you asked to do?



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$$\sum_{s' \in S'} s' = t.$$

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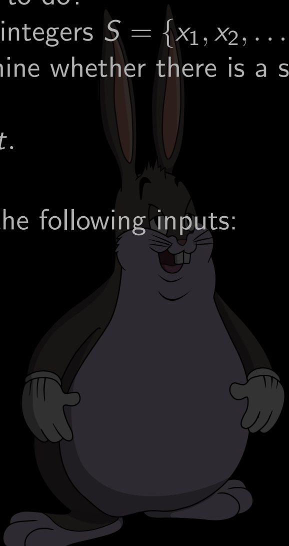
Task: Solve the subset sum problem for the following inputs:

- $S = \{18, 37, 20, 13, 33\}$, $t = 75$.
- $S = \{20, 21, 22, 36, 67\}$, $t = 90$.



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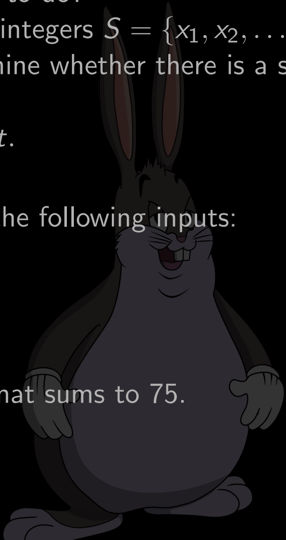
- There is no subset that sums to 90.



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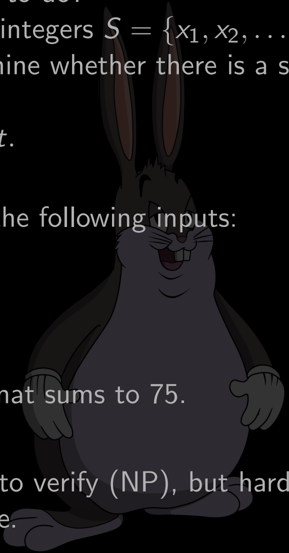
Remind yourself that Subset Sum is easy to verify (NP), but hard to solve (NP-hard), so Subset Sum is NP-complete.



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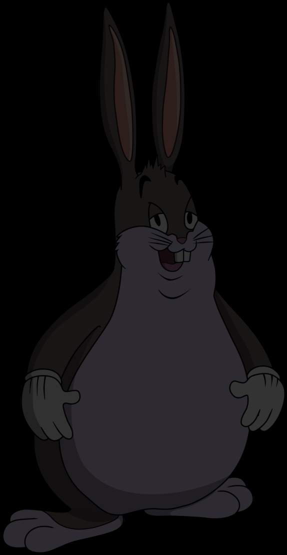

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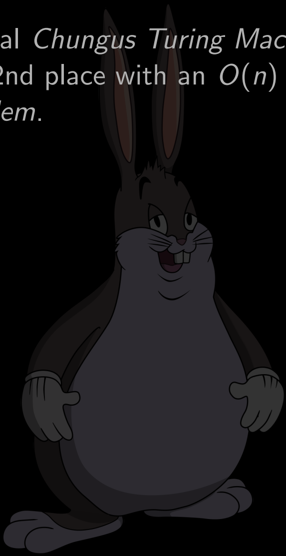
You team up with your friend in the annual *Chungus Turing Machine Contest*. Your team manages to achieve 2nd place with an $O(n)$ solution of the NP-complete *Chungus-Cover Problem*.



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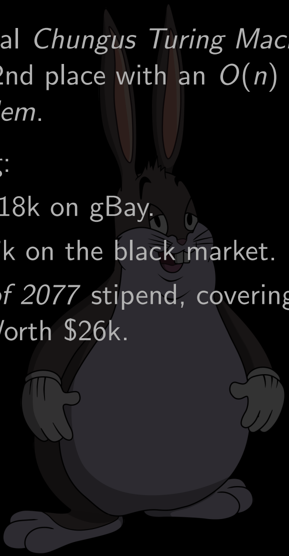

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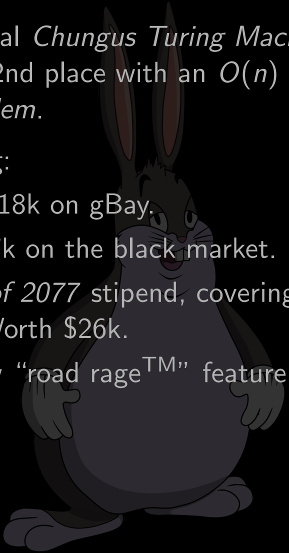

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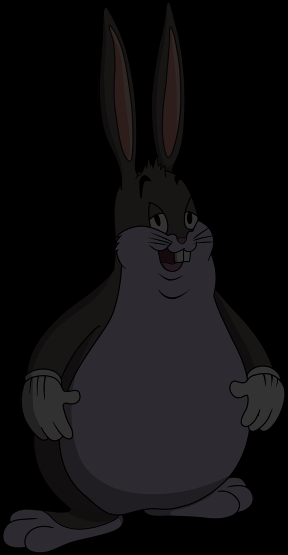

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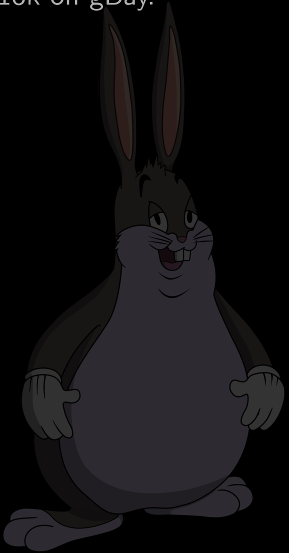

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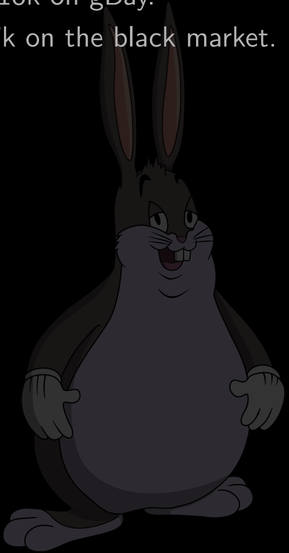

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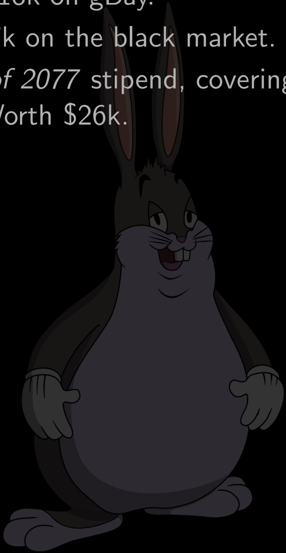

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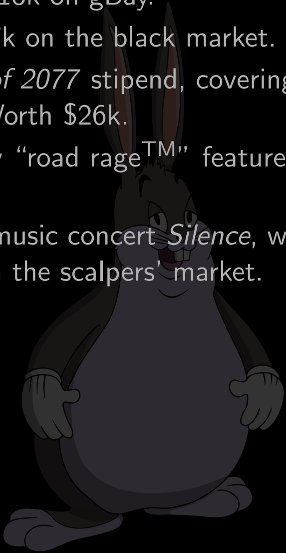

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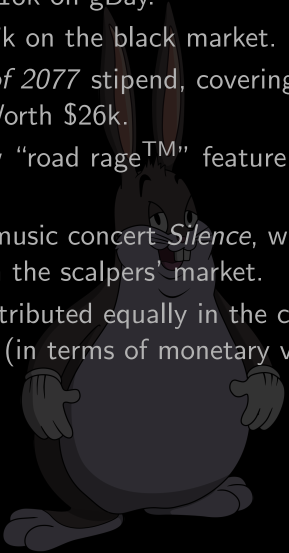

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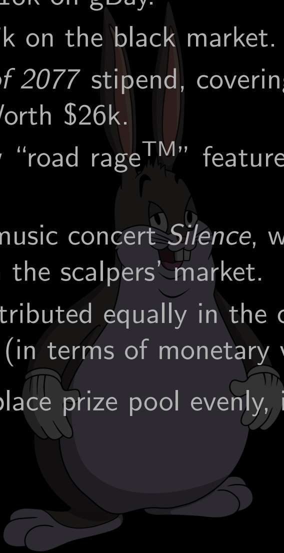

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Ans: Yes. You take the Chungus plushie (\$37k) and the stipend (\$26k), and your friend takes the rest. Your prize is $37k + 26k = 53k$, while your friend's prize is $18k + 15k + 15k + 15k = 53k$.

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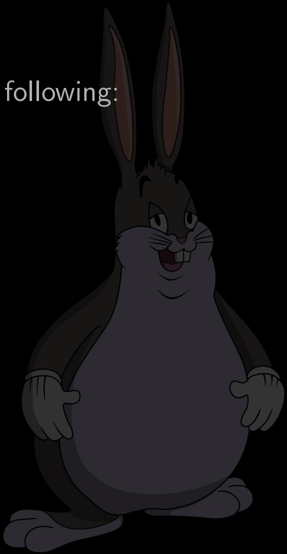
In general, the partition problem asks the following:



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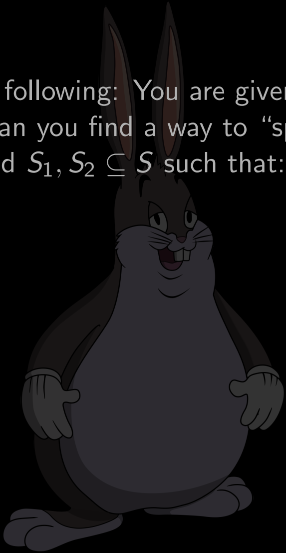

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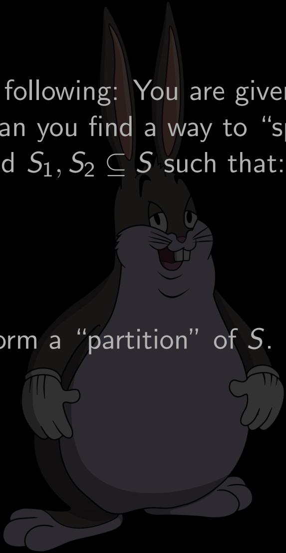
Note that 1. and 2. say that S_1 and S_2 form a "partition" of S.



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2. $S = \{18, 37, 20, 13, 33\}$.
3. $S = \{20, 21, 32, 36, 69\}$.
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Ans:

1. Yes. (This is the example I gave earlier)
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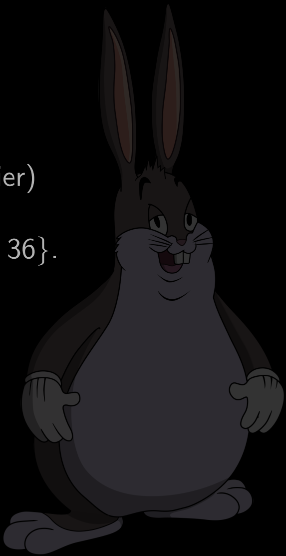

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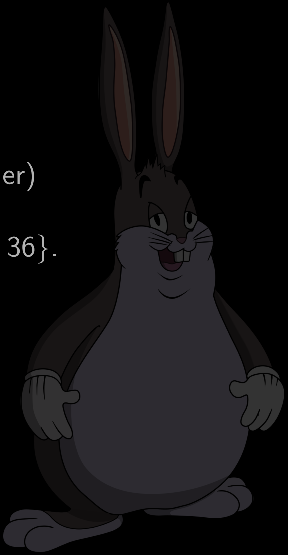

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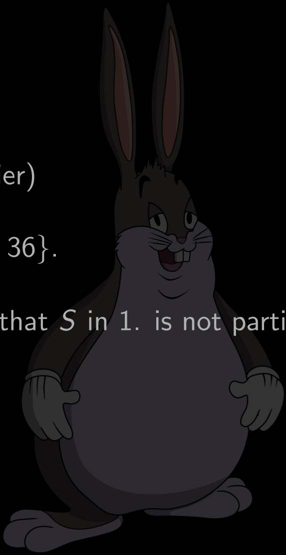

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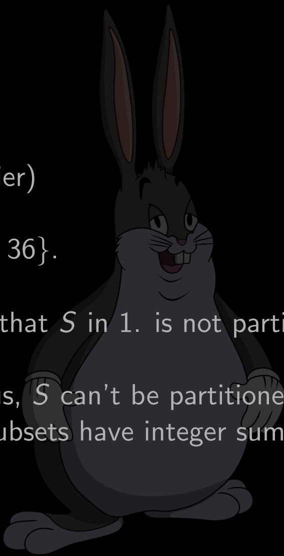

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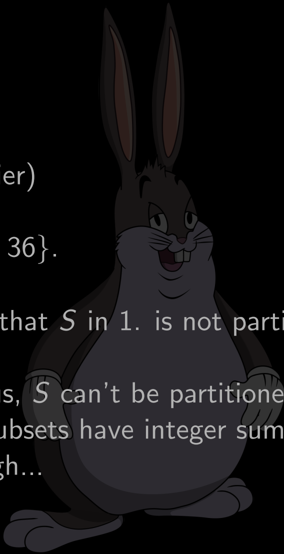

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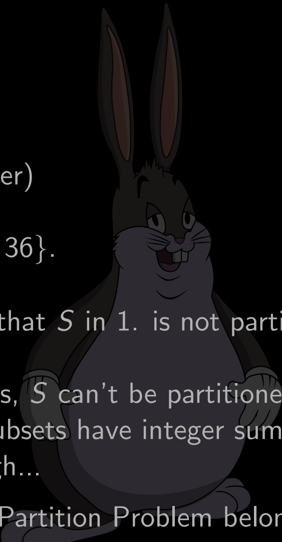

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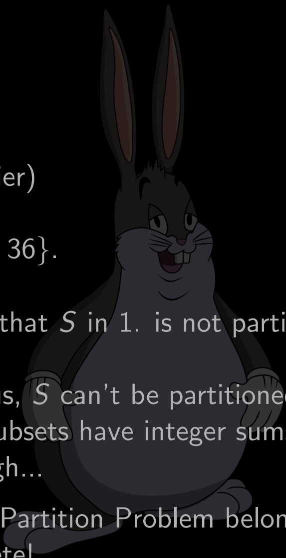
Ans: The partition problem is NP-complete!



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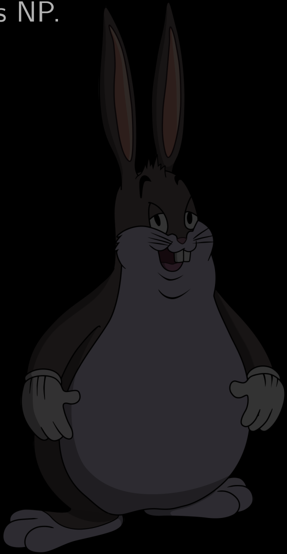
Task: Prove that the Partition Problem is NP.



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Ans: We can build a poly-time verifier V:

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V(S, S1):
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  S2 = S \ S1
  if sum(S1) = sum(S2):
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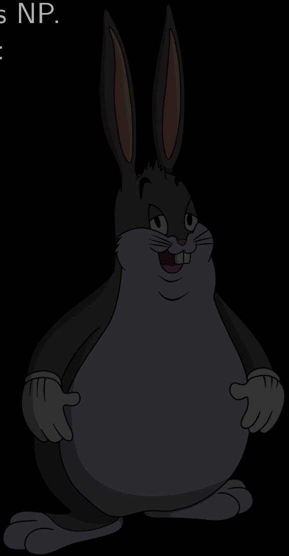

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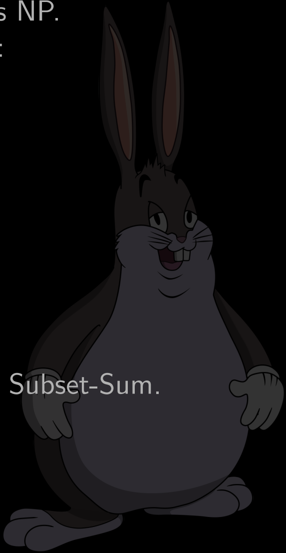
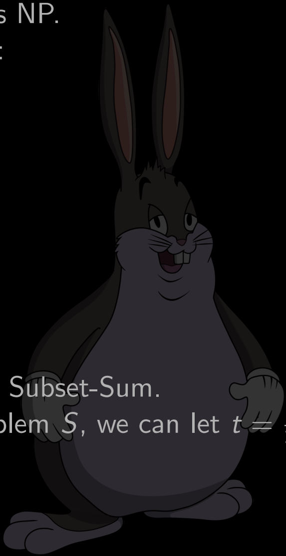
Unrelated Task: Show that Partition $\leq_p$ Subset-Sum.



Partition Problem is NPC

Task: Prove that the Partition Problem is NP.
Ans: We can build a poly-time verifier V:

```
V(S, S1):
  if S1 isn't a subset of S2:
    reject
  S2 = S \ S1
  if sum(S1) = sum(S2):
    accept
  reject
```



Unrelated Task: Show that Partition $\leq_p$ Subset-Sum.
Ans: For an instance of the partition problem S, we can let $t = \frac{1}{2}\sum S$. S is partitionable iff (S, t) is in subset sum.

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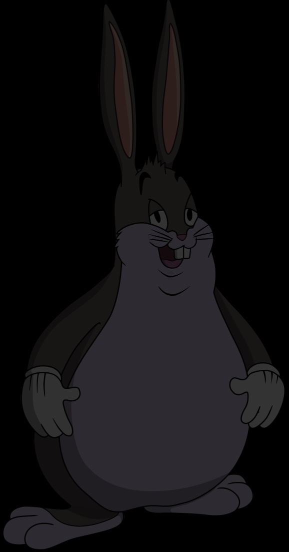
Now we show the Partition Problem is NP-hard, by showing Subset-Sum $\leq_p$ Partition. Let (S, t) be an instance of the Subset-Sum problem.



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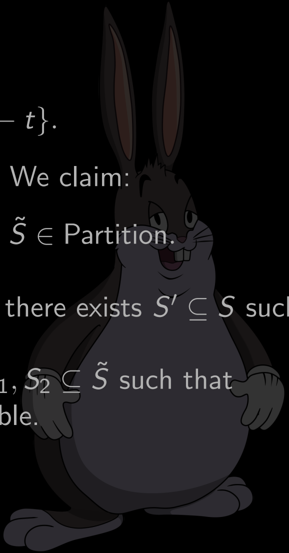
($\Rightarrow$) Suppose $(S, t) \in$ Subset-Sum. Then there exists $S' \subseteq S$ such that $\sum S' = t$.
Task: Using the above, find a partition $S_1, S_2 \subseteq \tilde{S}$ such that $\sum S_1 = \sum S_2$. This shows $\tilde{S}$ is partitionable.



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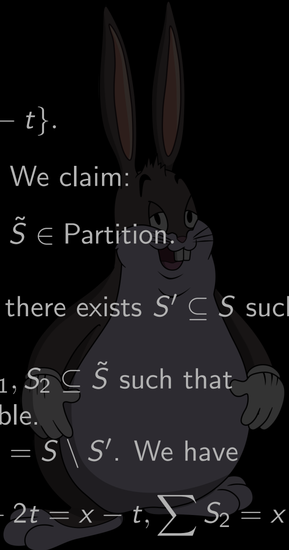
Ans: Defining $S_1 = S' \cup \{x - 2t\}$ and $S_2 = S \setminus S'$. We have

$$\sum S_1 = (\sum S') + x - 2t = t + x - 2t = x - t, \sum S_2 = x - t.$$



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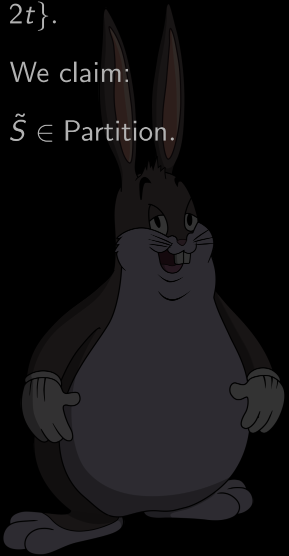

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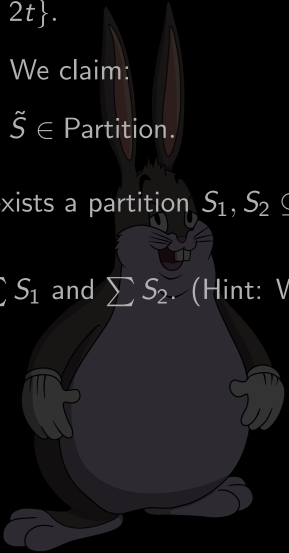

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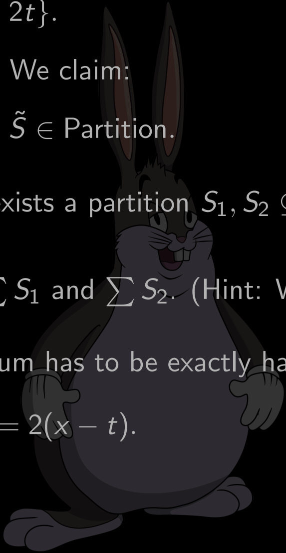
Ans: $\sum S_1 = \sum S_2 = x - t$, since each sum has to be exactly half of

$$\sum \tilde{S} = x + x - 2t = 2(x - t).$$



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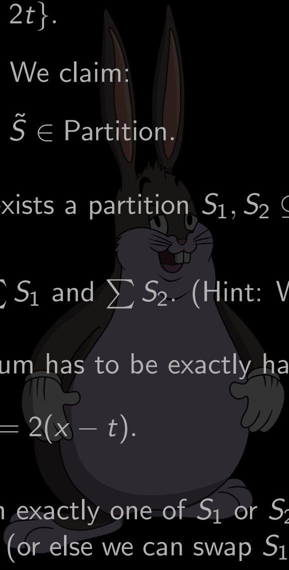

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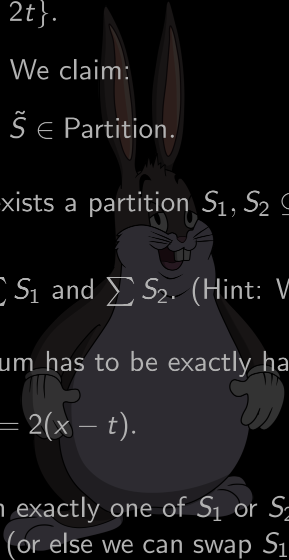

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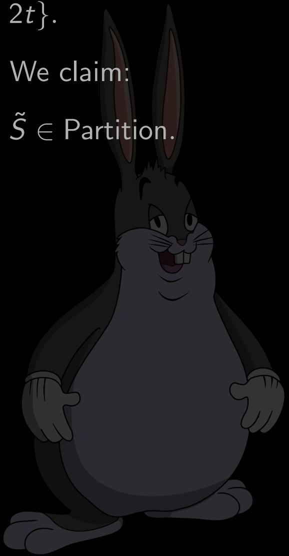

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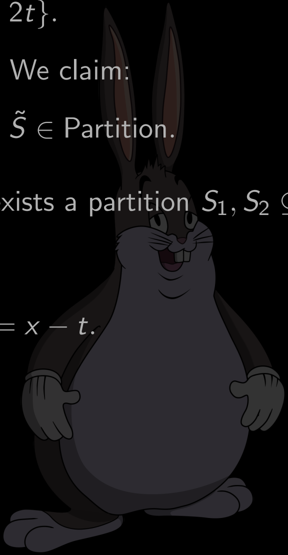

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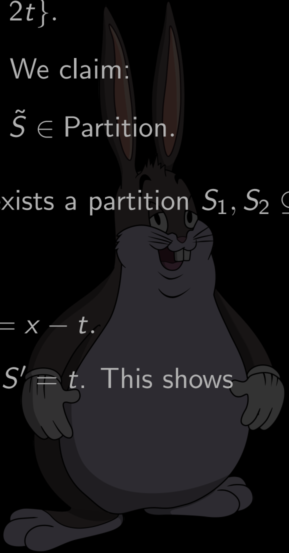

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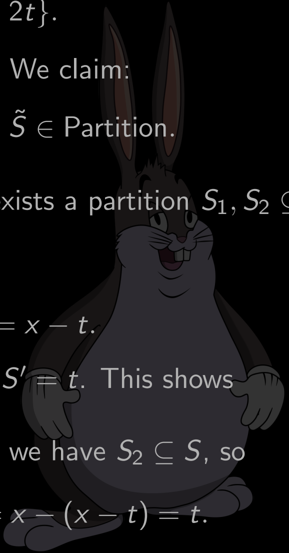
$$\sum S' = \sum S - \sum S_2 = x - (x - t) = t.$$



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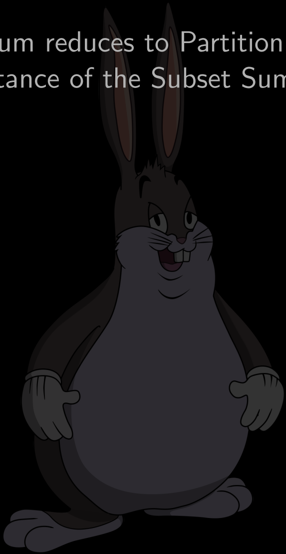
Let's try an example to see how Subset Sum reduces to Partition. Let $S = \{18, 37, 20, 13, 33\}$, $t = 75$ be an instance of the Subset Sum problem.



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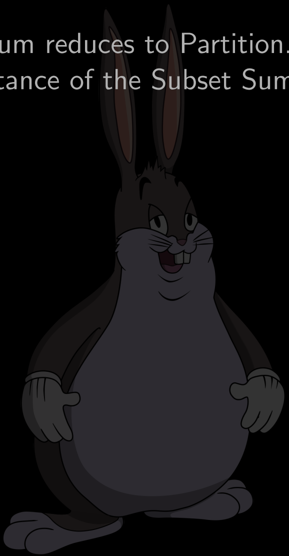

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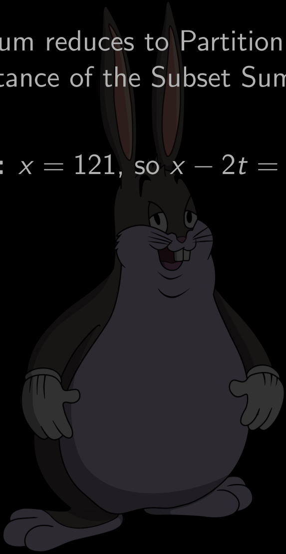

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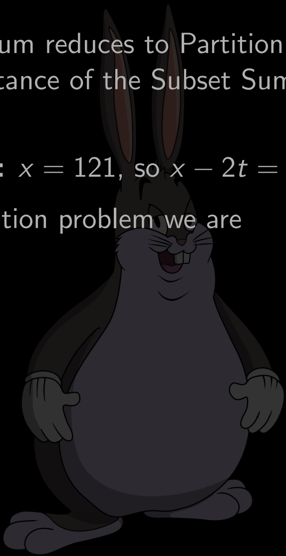

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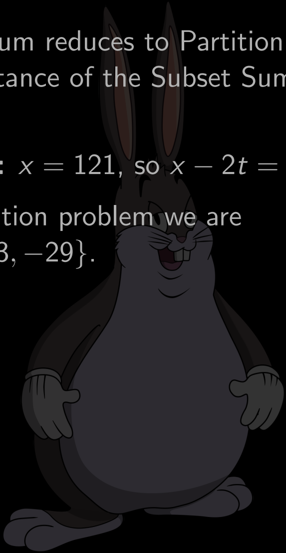

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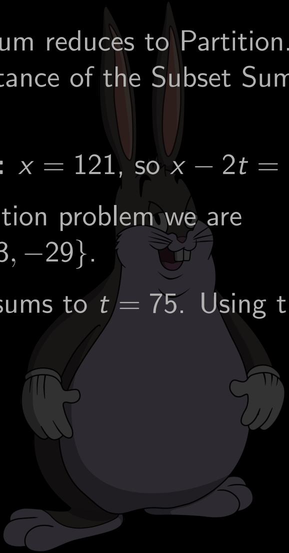

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Ans: Let $S_1 = S' \cup \{x - 2t\} = \{18, 37, 20, -29\}$, and $S_2 = \tilde{S} \setminus S_1 = \{20, 13, 33\}$. Both sum to 46.