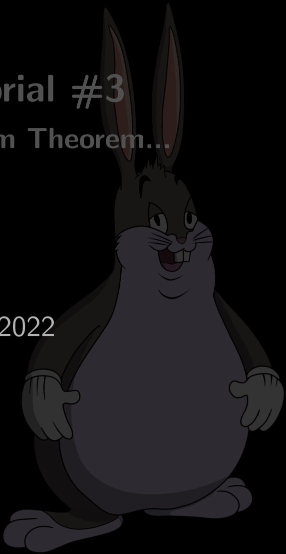
CSC363 Tutorial #3

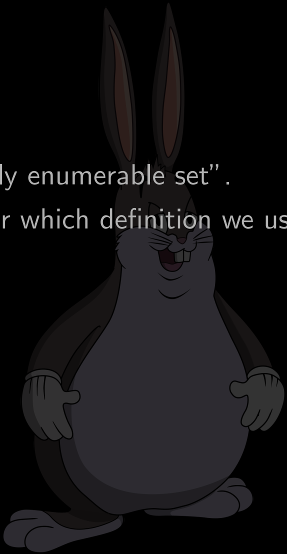
CE sets, Normal Form Theorem...

February 02, 2022



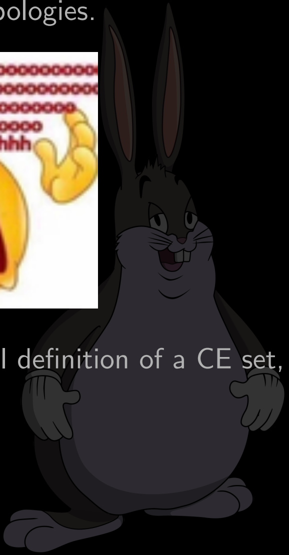
Learning objectives this tutorial

- Talk about the definition “computably enumerable set”.
- Conclude that it doesn’t really matter which definition we use!



Computably Enumerable Sets

Assignment 1 recall time! My sincerest apologies.



Question: What was our original informal definition of a CE set, from the first assignment?

Computably Enumerable Sets

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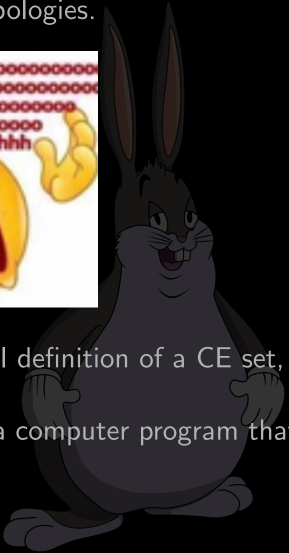
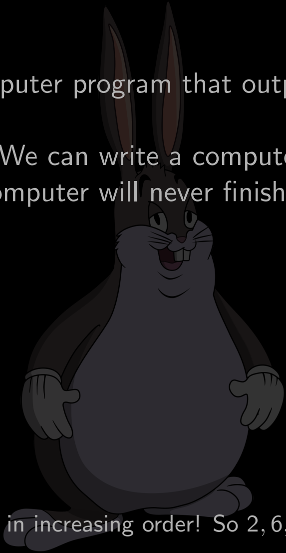


Question: What was our original informal definition of a CE set, from the first assignment?
Ans: A set $M \subseteq \mathbb{N}$ is CE if we can write a computer program that outputs the elements of M in a list.

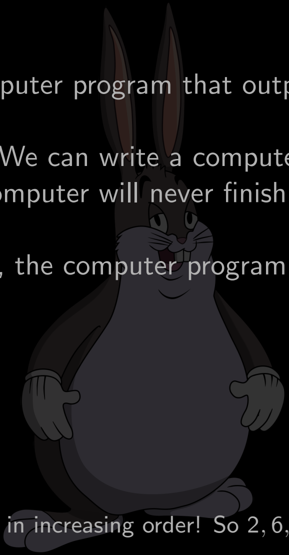
Computably Enumerable Sets



A set $M \subseteq \mathbb{N}$ is CE if we can write a computer program that outputs the elements of M in a list.
But how do we "output" an infinite set? We can write a computer program that prints $2, 4, 6, 8, \ldots$, but a computer will never finish outputting all the even numbers!

[1] It is not necessary that we print the numbers in increasing order! So $2, 6, 4, 8, \ldots$ is also a valid way to enumerate the evens.

Computably Enumerable Sets



A set $M \subseteq \mathbb{N}$ is CE if we can write a computer program that outputs the elements of M in a list.

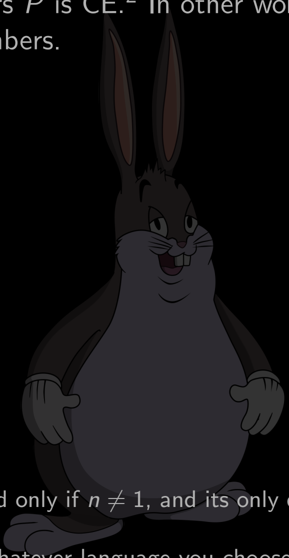
But how do we "output" an infinite set? We can write a computer program that prints $2, 4, 6, 8, \ldots$, but a computer will never finish outputting all the even numbers!

What we mean here is: given any $m \in M$, the computer program will eventually print out m.[1]

[1] It is not necessary that we print the numbers in increasing order! So $2, 6, 4, 8, \ldots$ is also a valid way to enumerate the evens.

Computably Enumerable Sets

Task: Show that the set of prime numbers P is CE.[2] In other words, write a program[3] that prints out the prime numbers.



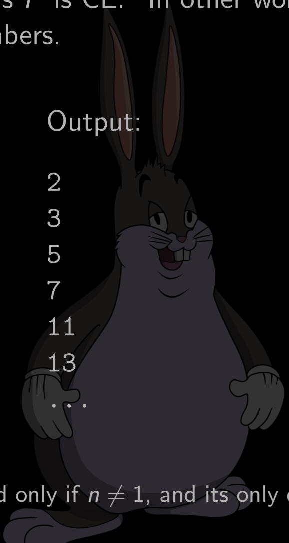
[2]Recall that a natural number n is prime if and only if $n \neq 1$, and its only divisors are 1 and n

[3]In Python, C, Minecraft, ChungusCode, or whatever language you choose!

Computably Enumerable Sets

Task: Show that the set of prime numbers P is CE.[2] In other words, write a program[3] that prints out the prime numbers.
Ans:

```
i = 2
while True:
  is_prime = True
  for j in range(i):
    if i % j == 0:
      is_prime = False
  if is_prime:
    print(i)
  i += 1
```



Output:

2
3
5
7
11
13
...

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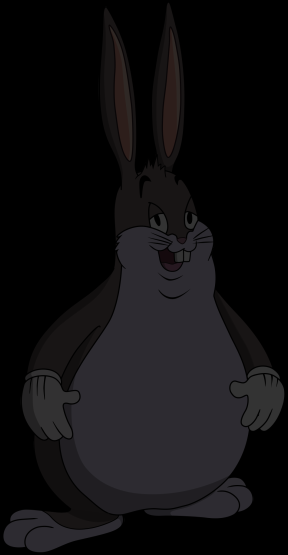
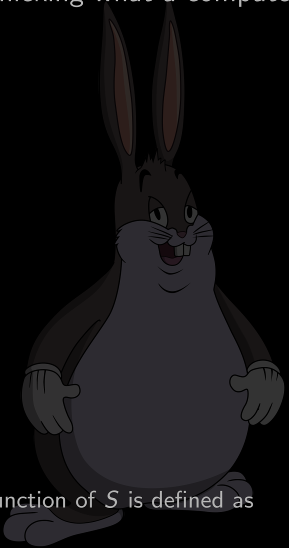
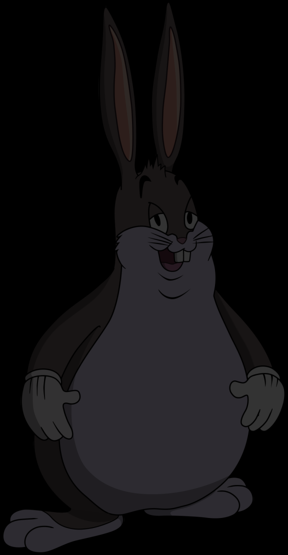


Formal definition of CE set

Recall in Lecture 3 that we built up a set of functions called the "partial recursive" functions, in an attempt to mimicking what a computer can do.



[4]Recall: If $S \subseteq \mathbb{N}$ is a set, the characteristic function of S is defined as
$$\chi_S(n) = \begin{cases} 1 & n \in \mathbb{N} \\ 0 & n \notin \mathbb{N}. \end{cases}$$



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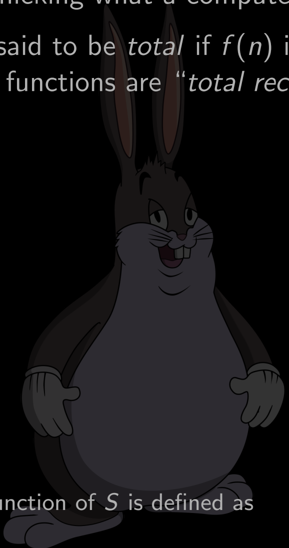
A partial recursive function $f : \mathbb{N} \to \mathbb{N}$ is said to be *total* if $f(n)$ is defined for all $n \in \mathbb{N}$. Some synonyms for "total" functions are "*total recursive*" and "**computable**".



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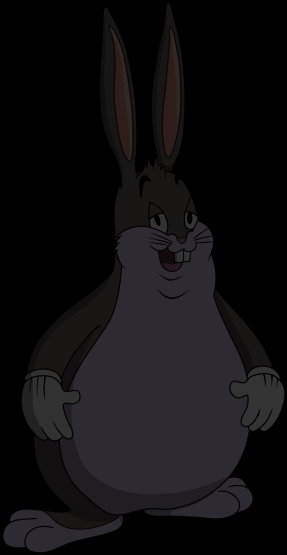

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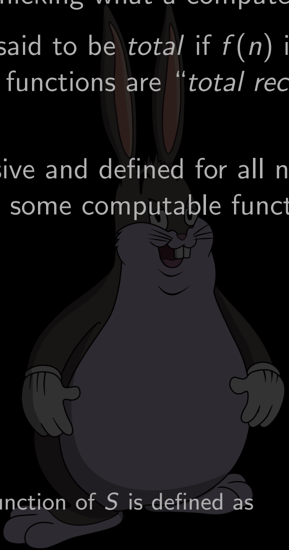
All primitive recursive functions are recursive and defined for all natural numbers, so they are all computable! But some computable functions are not primitive recursive.



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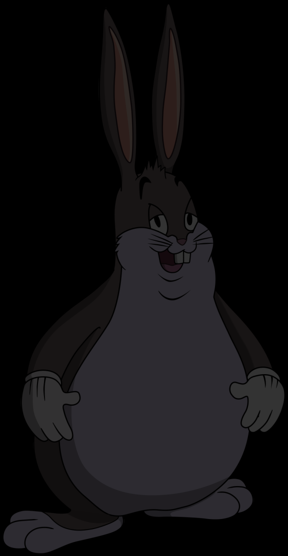

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Correction to last week's tutorial: Again, we lied to you!

- Last week's definition: A computable set is a set whose characterstic function[4] is primitive recursive.
- This week's definition: A computable set is a set whose characterstic function is computable (as we have just defined).

[4]Recall: If $S \subseteq \mathbb{N}$ is a set, the characteristic function of S is defined as
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Formal definition of CE set

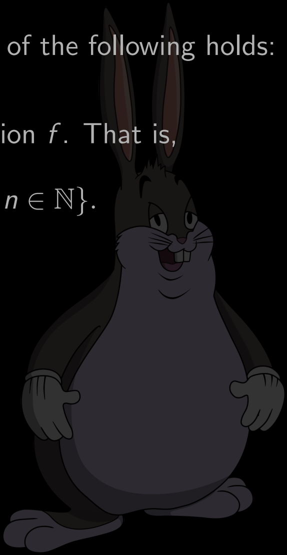
Now we will present the formal definition of a CE set (from Lecture 3 also).

Definition: A set $S \subseteq \mathbb{N}$ is **CE** when one of the following holds:

- $S = \emptyset$;
- S is the range of a computable function f. That is,

$$S = \{f(n) : n \in \mathbb{N}\}.$$

Write this down!!



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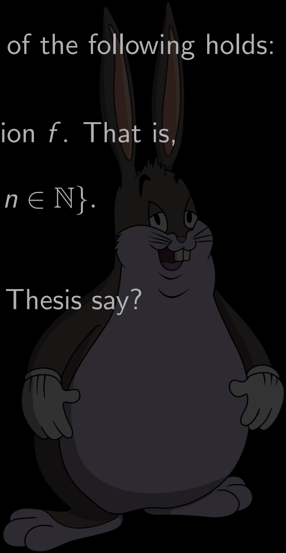
Question: What does the Church-Turing Thesis say?



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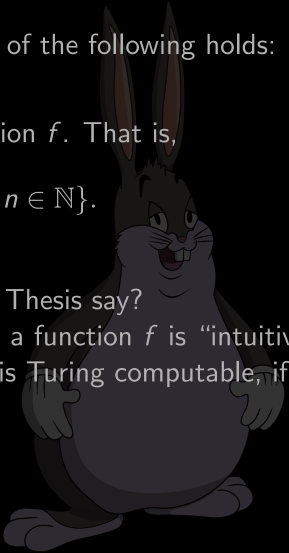
Question: What does the Church-Turing Thesis say?

Ans: The Church-Turing Thesis says that a function f is "intuitively computable" iff it is total recursive (iff it is Turing computable, iff it is URM computable, etc).



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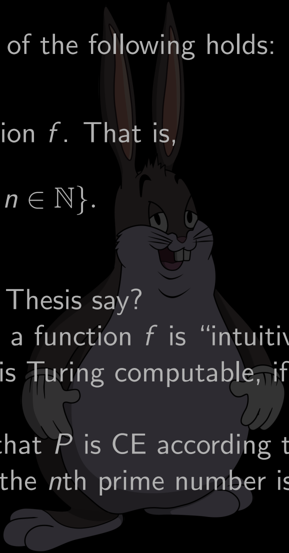
Task: Let P be the set of primes. Show that P is CE according to the above definition, by showing that $f(n) =$ the nth prime number is computable using the CT Thesis.



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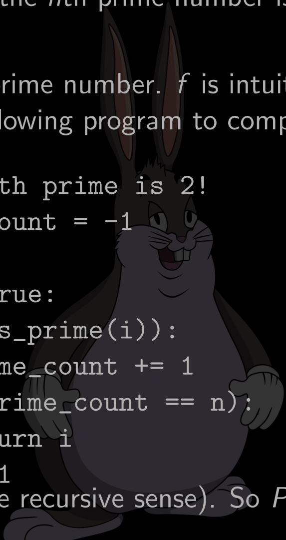
Task: Let P be the set of primes. Show that P is CE according to the above definition, by showing that $f(n) =$ the nth prime number is computable using the CT Thesis.

Ans: Define $f : \mathbb{N} \to \mathbb{N}$, $f(n) =$ the nth prime number. f is intuitively computable, because we can write the following program to compute f:

```
def is_prime(i):
  for j in range(i):
    if i % j == 0:
      return False
  return True
```

```
def f(n):
  # the 0th prime is 2!
  prime_count = -1
  i = 2
  while True:
    if (is_prime(i)):
      prime_count += 1
    if (prime_count == n):
      return i
    i += 1
```



By the CT Thesis, f is computable (in the recursive sense). So P, which is the range of f, is a CE set.

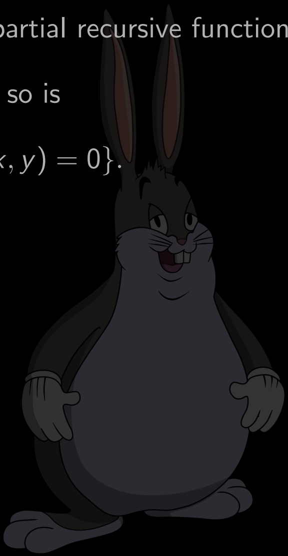
Equivalent definition 2

We will now prove the following:

$$S \text{ is CE} \Leftrightarrow S \text{ is the domain of a partial recursive function.}$$

Recall: if $g(x, y)$ is partial recursive, then so is

$$f(x) = \min\{y : g(x, y) = 0\}.$$



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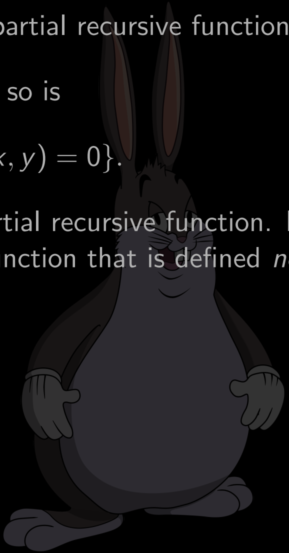

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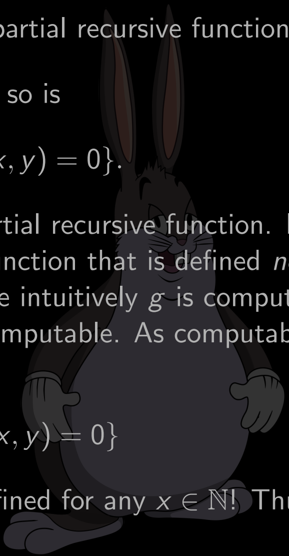
Task: Show that $\emptyset$ is the domain of a partial recursive function. In other words, come up with a partial recursive function that is defined *nowhere*!
Ans: Define $g(x, y) = 1$ for all x, y. Since intuitively g is computable (just return 1 regardless of input), g is computable. As computable functions are (partial) recursive,

$$f(x) = \min\{y : g(x, y) = 0\}$$

is also partial recursive. But $f(x)$ is undefined for any $x \in \mathbb{N}$! Thus domain$(f) = \emptyset$.



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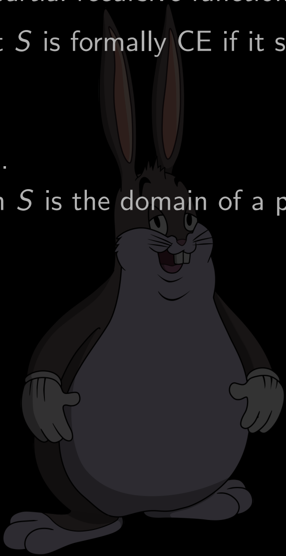
$$S \text{ is CE} \Rightarrow S \text{ is the domain of a partial recursive function.}$$

Let's prove the theorem! Recall that a set S is formally CE if it satisfied one of the following:

- $S = \emptyset$.
- $S = \text{range}(f)$ for some computable f.

Task: Show that if S is formally CE, then S is the domain of a partial recursive function.



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- $S = \emptyset$.
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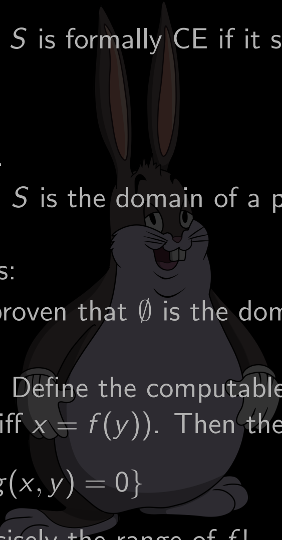
Task: Show that if S is formally CE, then S is the domain of a partial recursive function.

Ans: Suppose S is CE. We have two cases:

- $S = \emptyset$: On the previous slide, we've proven that $\emptyset$ is the domain of a partial recursive function.
- $S = \text{range}(f)$ where f is computable. Define the computable function $g(x, y) = |x - f(y)|$ (so $g(x, y) = 0$ iff $x = f(y)$). Then the function

  $$h(x) = \min\{x : g(x, y) = 0\}$$

  is partial recursive. h's domain is precisely the range of f!



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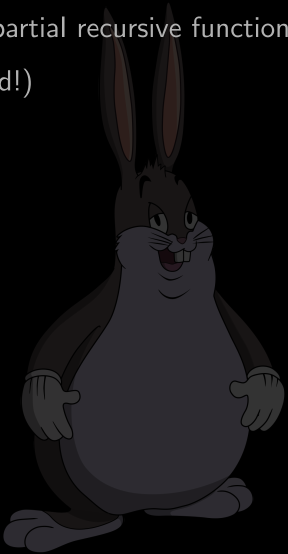
What about the other direction? (It's hard!)



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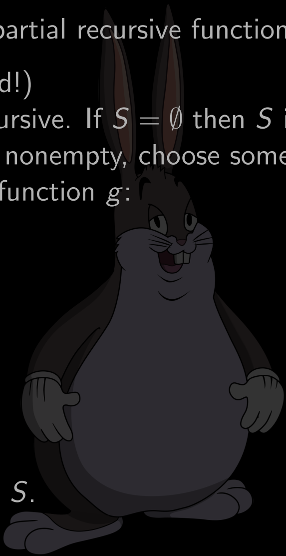
What about the other direction? (It's hard!)
Let $S = \text{domain}(f)$, where f is partial recursive. If $S = \emptyset$ then S is CE and we're done, so suppose $S \neq \emptyset$. Since S is nonempty, choose some $p \in S$. We may define the following computable function g:

```
def g(x, s):
  try to compute f(x) for s steps
  if f(x) returns within s steps:
    return x
  else:
    return p
```

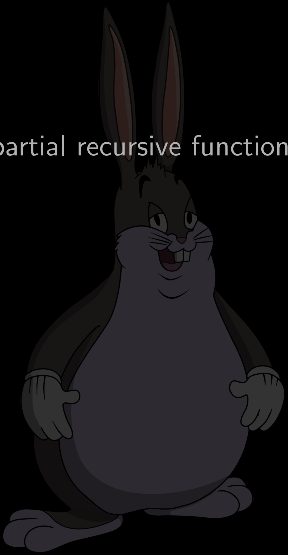


Task: Show that the range of g is indeed S.

Equivalent definition 2

So we've proven the following!

$$S \text{ is CE} \Leftrightarrow S \text{ is the domain of a partial recursive function.}$$



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It also turns out that

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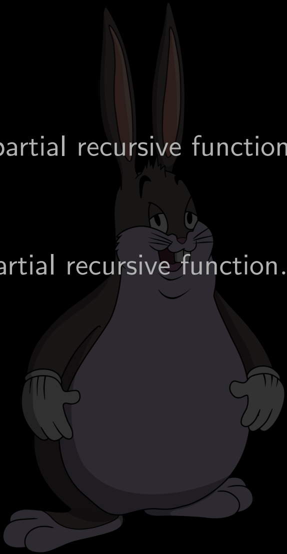
But we don't have time to prove this! :(



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This equivalence of definitions is called the **Normal Form Theorem**.