# MAT102H5 - Introduction to Mathematical Proofs - Summer 2021 -UTM

Problem Set 1 - TO BE SUBMITTED SAT. MAY 15, 11:59pm

**Submit the following 8 problems in Crowdmarks. Late submission will be given 0, no excuse will be accepted. You are supposed to submit earlier than the deadline so in case of any issue you will have time to deal with it.**

**Only 3 problems will be graded. The problem set is 8 points. 6 points for problems, and 2 points for clean and neat writing. To get those 2 points, you must:**
**submit each question on its place in Crowdmark,**
**write the answer of each question in a separate sheet,**
**you will not get the point if your handwriting is hard to read, DO NOT write with script handwriting.**

**Question 1.** Let $P(x) = ax^2 + bx + c$ with $a \neq 0$. The polynomial $P(x)$ is factorable in real numbers if we can find real numbers $\alpha, \beta, \gamma$ and $\lambda$ so that $P(x) = (\alpha x + \beta)(\gamma x + \lambda)$.

a) Show that $P(x)$ is factorable iff (if and only if) $b^2 - 4ac \geq 0$.

b) Show that $3x^2 - 5x + 10$ is not factorable in real numbers.

**Question 2.** If $ax^2 + bx + c = a(x - \alpha)^2$ with $a \neq 0$, find $\alpha$ in terms of $a, b$ and $c$.

**Question 3.** Prove that for any numbers $x$ and $y$ with $x \neq 0$ we have $2y \leq \frac{y^2}{x^2} + x^2$.

**Question 4.** Prove that for any $x, y \geq 0$, we have $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$. (Hint. Consider the cases $x \leq y$ and $y \leq x$ separately. )

**Question 5.** If $1 < x < 2$ find a bound for $|\frac{x^3+x^2-1}{x-6}|$. (Hint. If $1 < x < 2$ find $M > 0$ such that $|\frac{x^3+x^2-1}{x-6}| < M$.

**Question 6.** The goal of this exercise is to make sure you understand $\sqrt{x^2} = |x|$ and in general $\sqrt{x^2} \neq x$.
a) Check your textbook (course notes) to find the fact that for any real number a such that $a \geq 0$, there is a unique number $\sqrt{a} \geq 0$. Which fact is that?

b) Evaluate the following (write them without square root): $\sqrt{(-5)^2} = - - - - - -$ , $\sqrt{5^2} = - - - - - -$ ,

If $a > 0$ : $\sqrt{a^2} = - - - - - -$ , $\sqrt{(-a)^2} = - - - - - - -$.

c) If $x$ is any real number, prove that $\sqrt{x^2} = |x|$.

d) Bob is trying to prove an inequality. After few steps he got $(a^2b^2+c^2d^2)^2 \geq 4a^2b^2c^2d^2$. The only assumption on $a, b, c$ and $d$ is that they are real numbers. For the next step Bob took square root from both sides of the inequality and he got:

$$(a^2b^2 + c^2d^2) \geq 2abcd.$$

What is wrong?

**Question 7.** For each statement, decide whether it is true or false. Justify your answer briefly.

a) $\{(x, y) : x, y \in \mathbb{R}, x - 1 = 0\} \subseteq \{(x, y) : x, y \in \mathbb{R}, x^2 - x = 0\}$.

b) $\mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{Z}$.

**Question 8.** Give an example of a set A, for which $A \cap [1,4] = A \cap \mathbb{N}$ and $A \setminus \mathbb{Z} \neq \emptyset$.

The following problems from your course notes are only for extra practice. DO NOT submit them.

1.5.1 , 1.5.3 , 1.5.5 , 1.5.6 , 1.5.7 , 1.5.8 , 1.5.9 , 1.5.12 , 1.5.14 , 1.5.15 , 1.5.16 , 1.5.17 , 1.5.18 , 1.5.19 , 1.5.21 , 1.5.22

2.5.1 , 2.5.2 , 2.5.3 , 2.5.5 - 2.5.8 , 2.5.10 - 2.5.13 , 2.5.15 - 2.5.18 , 2.5.21 - 2.5.24 .