Assignment 3

Due on Mar 5

Q1 (the join operator)

Consider the following operator $\oplus$ defined on $P(\mathbb{N})$ as follows:

$$A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$$

(a) [20 marks] Show that:
   1. $A \leq_m A \oplus B$ and $B \leq_m A \oplus B$
   2. For any arbitrary set C, if $A \leq_m C$ and $B \leq_m C$, then $A \oplus B \leq_m C$

In other words, $deg_m(A \oplus B)$ is the least upper bound to $deg_m(A)$ and $deg_m(B)$.

Recall that $deg_m(A) = \{S : A \equiv_m S\}$ which is the equivalence class of A based on the equivalence relation $\equiv_m$ ($A \equiv_m S$ iff $A \leq_m S$ *and* $A \geq_m S$).

(b) [10 marks] Show that the same holds if $\leq_m$ is replaced with $\leq_T$ (Turing reducibility)

Q2 (Cylinder) [20 marks]

A set A is a *cylinder* if there is a set B such that $A \equiv_1 B \times \mathbb{N}$.

Show that, if S is a cylinder, then for any set C: $C \leq_m S \Longrightarrow C \leq_1 S$.

Remark: In some situations, this can be used to explain why saying 1-complete is the same as saying m-complete even though the first is generally stronger.

Q3 (ω-jump) [20 marks]

In lecture, we defined $A^{(n)}$ the *n*-th jump of a set A for any natural number *n.*

We have also mentioned that $A^{(n)} \lneqq_T A^{(n+1)}$ for every *n.*

Here, we will define an infinite iteration of the jump, $A^{(\omega)}$, as follows:

$$A^{(\omega)} = \{\langle m, n\rangle : m \in A^{(n)}\}$$

Show that $A^{(n)} \lneqq_T A^{(\omega)}$ for every $n \in \mathbb{N}$.

Q4 [10 marks]

Let $\{A_n : n \in \mathbb{N}\}$ be a collection of c.e. sets. Is $\bigcup_n A_n$ c. e.? Under which condition can this be true? Briefly write your thoughts.

Q5 [20 marks]

Prove that for all A and B, if B is in the class P, and $A \leq_p B$ then A is in the class P