CSC363 Tutorial #6

More reductions, Arithmetic Hierarchy

March 1, 2023

Things covered in this tutorial

- ⋆ What is a *m*-reduction?
- ⋆ What is the *arithmetic hierarchy*?
- ⋆ Why are we learning all of this?
- ⋆ Why did I enroll in this course?
- ⋆ Can I get a hint for A4?

You know why you enrolled in this course.

Reductions, once again...

Recall: Given two languages A and B, we say A **Turing reduces to** B ($A \leq_T B$) if given an oracle for B, you can build a decider for A.

The HP-oracle.

Reductions, once again...

Task: Let

$$\mathrm{HP} = \{x : M_x(x) \text{ halts}\}.$$

$$\overline{\mathrm{HP}} = \{x : M_x(x) \text{ loops}\}.$$

Prove that $\overline{\mathrm{HP}} \leq_T \mathrm{HP}$.

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Answer: Assuming we have a decider `in_HP` for HP, we can build the following decider for $\overline{\mathrm{HP}}$:

```
in_HPbar(x):
  if in_HP(x):
    reject
  accept
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But $\overline{\mathrm{HP}}$ is not c.e., yet HP is c.e.. Why is $\overline{\mathrm{HP}} \leq_T \mathrm{HP}$?

m-reductions address this issue with Turing reductions.

m-reductions

Also known as *many-one reductions*. We say A m-**reduces to** B ($A \leq_m B$) if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

$$x \in A \Leftrightarrow f(x) \in B.$$

The *m*-reduction is a *stronger* version of the Turing reduction.

m-reductions

Scenario: The aliens come back, and see that we have been abusing their HP-oracle to illegally decide unrecognizable languages like $\overline{\mathrm{HP}}$.

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They confiscate the HP-oracle from humans.

m-reductions

However, the aliens are still willing to solve the halting problem for you.

m-reductions

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Now say you wanted to decide whether something is in A_{TM}. Recall

$$A_{\mathrm{TM}} = \{(x, w) : M_x(e) \text{ accepts}\}.$$

You just have to do the following:

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You just have to do the following:

⋆ Given an instance (x, w) of A_{TM}, construct the following Turing machine T:

```
T(z):
  ignore z
  run M_x(w) # might loop!
  if M_x(w) rejects: loop
  else: halt
```

This machine T has a number; call this machine's number $f(x, w)$.

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  This machine T has a number; call this machine's number $f(x, w)$.
- ⋆ Ask the aliens whether $f(x, w) \in \mathrm{HP}$, with a bribe of [REDACTED].

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Ok... how is this different from Turing reductions?

Answer: To show $A \leq_T B$, you assume that you have a B-oracle, and build a decider for A.

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Answer: To show $A \leq_T B$, you assume that you have a B-oracle, and build a decider for A.

To show $A \leq_m B$, you assume that you have a B-oracle, and build a decider for A, *with the following restrictions*:

- ⋆ You must call the B-oracle exactly once – no more, no less.
- ⋆ **You must return whatever the B-oracle returns**; negating the return value of the B-oracle (or performing any modification to the return value) is illegal.
  In other words, the last line of your decider for A must be `return in_B(...)`.

m-reductions

Task: We've shown $\overline{\mathrm{HP}} \leq_T \mathrm{HP}$ using the following decider for $\overline{HP}$:

```
in_HPbar(x):
  if in_HP(x):
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  reject
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Why isn't the above proof acceptable for showing that $\overline{\mathrm{HP}} \leq_m \mathrm{HP}$?

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Why isn't the above proof acceptable for showing that $\overline{\mathrm{HP}} \leq_m \mathrm{HP}$?

Answer: Remember; in a *m*-reduction proof of $A \leq_m B$, you must return whatever the B-oracle returns. You can't make any modifications (such as negation) to what the B-oracle returns.
The last line of your decider for A must be `return in_B(...)`.

m-reductions

Example: show that $\{\text{even numbers}\} \leq_m \{\text{odd numbers}\}$.

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Proof. I want to use the following procedure, using an oracle for the odd numbers:

```
is_even(x):
  if is_odd(x):
    reject
  accept
```

Unfortunately, this is not allowed...

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Example: show that $\{\text{even numbers}\} \leq_m \{\text{odd numbers}\}$.

Proof.

```
is_even(x):
  t = x + 1
  return is_odd(t)
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Proof.

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is_even(x):
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This is acceptable!

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We say A m-**reduces to** B ($A \leq_m B$) if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

$$x \in A \Leftrightarrow f(x) \in B.$$

The above function f is called the *reduction function*.[1]

[1] This function does not have to be injective.

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Task: Show that:

- $A \leq_m \{0, 1\}$, where A is a computable set.
- $A \not\leq_m \emptyset$, where A is any nonempty set.
- $A \leq_m \mathrm{HP} = \{x : M_x(x) \text{ halts}\}$, where A is a c.e. set.

[1] This function does not have to be injective.

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Note that if A is c.e., then $A \leq_m \mathrm{HP}$.

Is the converse true? If A is any set with $A \leq_m \mathrm{HP}$, does it follow that A is c.e.?

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Yes. In fact, if $A \leq_m B$ and B is c.e., then so is A. (Think about how you would recognize membership in A!)

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Yes. In fact, if $A \leq_m B$ and B is c.e., then so is A. (Think about how you would recognize membership in A!)

Consequently, $\overline{\mathrm{HP}} \not\leq_m \mathrm{HP}$.

Arithmetic Hierarchy

(I included this because the assignment's explanation might not be clear enough)

In assignment 3 questions 1 and 4, you prove that any set A is c.e. if and only if there is a computable binary relation R such that

$$A = \{x : \exists y \; R(x, y)\}.$$

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For example:

- ⋆ The set of even numbers E is c.e., since

$$E = \{x : \exists y \ R(x, y)\}$$

  where $R(x, y)$ is true iff

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- ⋆ The set of even numbers E is c.e., since

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- ⋆ The halting set is c.e., since

  $$\mathrm{HP} = \{x : \exists s\ \phi(x, s)\}$$

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Answer: $R(x, y, s)$ is true iff $\phi_x(y)$ halts in s steps or less.

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- ⋆ Tot can be written in the form $\{x : \forall y \exists z\ R(x, y, z)\}$.
- ⋆ ??? can be written in the form $\{x : \exists y_1 \forall y_2 \exists y_3\ R(x, y_1, y_2, y_3)\}$.

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- ⋆ Tot can be written in the form $\{x : \forall y \exists z\ R(x, y, z)\}$.
- ⋆ Cof can be written in the form $\{x : \exists y_1 \forall y_2 \exists y_3\ R(x, y_1, y_2, y_3)\}$.

$\mathrm{Cof} = \{x :$ There are finitely many inputs y for which $M_x(y)$ loops$\}$.

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$$A = \{x : \exists y_1 \forall y_2 \exists y_3 \forall y_4 \dots y_n\ R(x, y_1, y_2, \dots, y_n)\}.$$

- ⋆ A set A is Π^0_n if there is a n-ary relation R such that

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- A set A is Σ_n^0 if there is a n-ary relation R such that

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Task: Show that

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- ⋆ Tot is Π^0_2.

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- ⋆ Tot is Π^0_2.
- ⋆ Cof is Σ^0_3.