PROBABILITY

INTRODUCTION

Mutually exhaustive : $A \cap B = \emptyset$

- Discrete Random Variable (Counting) — Binomial, Bernoulli, Poisson
- Continuous Random Variable (Measuring) — Exponential, Normal

DISCRETE PROBABILITY DISTRIBUTION

$p(x_i) \geq 0$

$\sum_i p(x_i) = 1$

$p(x) = P(X = x)$

probability mass function

$[x_i, p(x_i)]$

Graph → bar

CUMULATIVE DISTRIBUTION FUNC.

OF DISCRETE RAN. VAR. (cdf)

$$F(x) = P(X \leq x) = \sum_{i=1}^{n} p(x_i)$$

MEAN (μ) $= \sum_i x_i\, p(x_i)$

VARIANCE (σ^2) $= \sum_i (x_i - \mu)^2 p(x_i)$ (or)

$= \sum_i x_i^2 p(x_i) - \mu^2$

STANDARD DEVIATION (σ) $= \sqrt{\sigma^2}$

1. In a voice communication system with 50 lines, the random variable is the number of lines in use at a particular time.

$$X = \{0, 1, 2, 3, 4 \ldots 50\}$$

2. A die is tossed thrice. A success is getting 1 or 6" on a toss. Find the mean and variance of the number of successes

$$X = \{0, 1, 2, 3\}$$

$$P(\text{success}) = \frac{2}{6} = \frac{1}{3} \; ; \; P(\text{failure}) = 2/3$$

$$P(X=0) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$P(X=1) = 3\left(\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}\right) = \frac{4}{27} \times 3 = \frac{12}{27}$$

$$P(X=2) = 3\left(\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}\right) = \frac{2}{27} \times 3 = \frac{6}{27}$$

$$P(X=3) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

X	0	1	2	3
$P(X=x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

$$\text{Mean } (\mu) = \left(0 \times \frac{8}{27}\right) + \left(1 \times \frac{12}{27}\right) + \left(2 \times \frac{6}{27}\right) + \left(3 \times \frac{1}{27}\right)$$

$$= \frac{12}{27} + \frac{12}{27} + \frac{3}{27} = 1$$

$$\text{Variance} = \frac{12}{27} + \left(4 \times \frac{6}{27}\right) + \left(9 \times \frac{1}{27}\right) - 1$$

$$= \frac{4}{9} + \frac{8}{9} + \frac{3}{9} - 1 = \frac{15}{9} - 1 = \frac{6}{9}$$

$$= 2/3$$

3. A coin is tossed three times. Let X denote the number of heads showing up. Find it's mean and variance.

$$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$$

X = Number of heads $= \{0, 1, 2, 3\}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\text{Mean } (\mu) = \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

$$\text{Variance } (\sigma^2) = \left(0^2 \times \frac{1}{8}\right) + \left(1^2 \times \frac{3}{8}\right) + \left(2^2 \times \frac{3}{8}\right) + \left(3^2 \times \frac{1}{8}\right) - \left(3/2\right)^2$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} - \frac{9}{4}$$

$$= \frac{24}{8} - \frac{18}{8} = \frac{6}{8} = 3/4$$

4. Consider the toss of a two fair dice. Let X be the random variable defined as the sum of toss of 2 fair dice. Find probability mass function

$$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$$
$$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$$
$$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$$
$$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$$
$$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$$
$$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$$

$X = \{2, 3, 4 \dots 12\}$

$P(X=2) = \frac{1}{36}$	$P(X=4) = \frac{3}{36} = \frac{1}{12}$	$P(X=6) = \frac{5}{36}$
$P(X=3) = \frac{2}{36} = 1/18$	$P(X=5) = \frac{4}{36} = \frac{1}{9}$	$P(X=7) = \frac{6}{36} = \frac{1}{6}$

$P(X=8) = \frac{5}{36}$ $\quad P(X=10) = \frac{3}{36} = \frac{1}{12}$ $\quad P(X=12) = \frac{1}{36}$

$P(X=9) = \frac{4}{36} = \frac{1}{9}$ $\quad P(X=11) = \frac{2}{36} = \frac{1}{18}$

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$1/6$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

$p(x_i) \geq 0$ & $\Sigma_i \, p(x_i) = 1$

It is PMF

5.

The probability density function of a random variable X is given as follows:

x:	0	1	2	3	4	5	6
p(x):	k	3k	5k	7k	9k	11k	13k

Find the value of k. Evaluate,
(i) $P(X < 4)$; $P(X \geq 5)$; and $P(3 < X \leq 6)$.
(ii) What will be the maximum value of k so that $P(X \leq 2) > 0.3$?

$\Sigma p(x_i) = 1 \Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$49k = 1 \Rightarrow \boxed{k = 1/49}$

(i) $P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= k + 3k + 5k + 7k = 16k = 16/49$

$P(X \geq 5) = P(X=5) + P(X=6) = 11k + 13k = 24k$

$= 24/49$

$P(3 < X \leq 6) = P(X=4) + P(X=5) + P(X=6)$

$= 9k + 11k + 13k = 33k = 33/49$

(ii) $P(X \leq 2) > 0.3 \Rightarrow P(X=0) + P(X=1) + P(X=2) > 0.3$

$k + 3k + 5k > 0.3 \Rightarrow 9k > 0.3$

$k > \frac{3}{10} \times \frac{1}{9} \Rightarrow k > \frac{1}{30} = 0.033$

6.

Home work problem: The probability density function of a random variable X is given as follows:

x:	0	1	2	3	4	5	6	7
p(x):	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find the value of k. Evaluate $P(X<6)$; $P(X\geq 6)$; and $P(0<X<5)$.

$\Sigma p(x_i)=1 \Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$9k+10k^2=1 \quad \Rightarrow k=\frac{1}{10}$ or $k=-1$

$\boxed{k=\frac{1}{10}}$

$P(X<6)=0+k+2k+2k+3k+k^2=8k+k^2$

$=\frac{8}{10}+\frac{1}{100}=\frac{81}{100}=0.81$

$P(X\geq 6)=1-P(X<6)=1-0.81=0.19$

$P(0<X<5)=k+2k+2k+3k=8k=0.8$

7.

The Sample space of a random experiment is {a, b, c, d, e, f} and each outcome is equally likely. A random variable is defined as folows:

Outcome	a	b	c	d	e	f
X	0	0	1.5	1.5	2	3

Determine the probability mass function of X. Use the probability mass function to determine the following probabilities. *a)* $P(X=1.5)$ *b)* $P(0.5<X<2.7)$ *c)* $P(X>3)$ *d)* $P(0\leq X<2)$ *e)* $P(X=0$ *or* $X=2)$.

x	0	1.5	2	3
$P(X=x)$	$\frac{2}{6}=\frac{1}{3}$	$\frac{1}{3}$	$1/6$	$1/6$

$\Sigma p(x_i)=1$ and $p(x_i)\geq 0$ $\therefore$ It is a PMF

(a) $P(X=1.5)=\frac{1}{3}$

(b) $P(0.5<X<2.7)=P(X=1.5)+P(X=2)$

$=1/3+1/6=1/2$

(c) $P(X>3)=0$

(d) $P(0\leq X<2)=P(X=0)+P(X=1.5)=2/3$

(e) $P(X=0 \text{ or } X=2)=1/3+1/6=1/2$

8.

The space shuttle flight control system called PASS[primary Avionics software set] uses four independent computers working in parallel. At each critical step, the computers 'vote' to determine the appropriate step. The probability that a computer will ask for roll to the left when a roll to the right is appropriate is 0.0001. let X denote the number of computers that vote for a left roll when a right roll is appropriate. what is the probability mass function of X? what is mean variance of X?
Answer: $P(X = 0) = 0.9996$, $P(X = 1) = 0.0003999$, $P(X = 2) = 5.999 \times 10^{-8}$, $P(X = 3) = 3.9996 \times 10^{-12}$, $P(X = 4) = 10^(-16)$
$Mean = 0.0004$, $Variance = 4 \times 10^{-4}$

$X = \{0, 1, 2, 3, 4\}$ $\quad p = 0.0001$

$\quad q = 0.9999$

$P(X = x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

$P(X = 0) = 0.9996$

$P(X = 1) = 3.9988 \times 10^{-4}$

$P(X = 2) = 5.998 \times 10^{-8}$

$P(X = 3) = 3.9996 \times 10^{-12}$

$P(X = 4) = 10^{-16}$

X	0	1	2	3	4
$P(X = x)$	0.9996	4×10^{-4}	6×10^{-8}	4×10^{-12}	10^{-16}

Mean $= np = 0.0004$

Variance $= np(1-p) = 3.9996 \times 10^{-4} = 4 \times 10^{-4}$

(Binomial)

CONTINUOUS PROBABILITY DISTRI.

$$\left.\begin{aligned} & f(x) \geq 0 \\ & \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{aligned}\right\} \text{conditions}$$

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx = \text{Area} \rightarrow \text{Probability}$$

CUMULATIVE DISTRIBUTION FUNCTION

$$P(x \geq r) = \int_{r}^{\infty} f(x)\,dx$$

$$P(x < r) = 1 - \int_{r}^{\infty} f(x)\,dx$$

$$\text{MEAN } (\mu) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\text{VARIANCE } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

9.

Is the function defined as follows a density function?

$$f(x) = e^{-x} \quad (x \geq 0)$$
$$= 0 \quad (x < 0) \tag{0.1}$$

If so, determine the probability that random variable X having this density will fall in the interval $(1, 2)$.

$f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$\therefore$ It follows a density function

$$P(1 \leq X \leq 2) = \int_1^2 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^2$$

$$= -e^{-2} + e^{-1} = \frac{1}{e} - \frac{1}{e^2}$$

$$= 0.2325$$

10.

Let X be a continuous random variable with probability density function given by

$$f(x) = kx \quad (0 \leq x \leq 2)$$
$$= 2k \quad (2 \leq x < 4)$$
$$= -kx + 6k \quad (4 \leq x < 6)$$

Find the value of k and the mean value of X.

Rewriting the function as

$$f(x) = \begin{cases} 0 & x \leq 0 \\ kx & 0 \leq x \leq 2 \\ 2k & 2 \leq x < 4 \\ -kx + 6k & 4 \leq x < 6 \\ 0 & x \geq 6 \end{cases}$$

For a density function, $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx\,dx + \int_{2}^{4} 2k\,dx$$

$$+ \int_{4}^{6} -kx + 6k\,dx + \int_{6}^{\infty} 0\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[x\right]_2^4 - k\left[\frac{x^2}{2}\right]_4^6 + 6k\left[x\right]_4^6$$

$$\Rightarrow k(2) + 2k(2) - k(10) + 6k(2) = 1$$

$$\Rightarrow \quad 8k = 1 \quad \Rightarrow \boxed{k = 1/8}$$

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{2} x \cdot kx\,dx + \int_{2}^{4} x \cdot 2k\,dx$$

$$+ \int_{4}^{6} x(-kx + 6k)\,dx$$

$$= k\left[\frac{x^3}{3}\right]_0^2 + 2k\left[\frac{x^2}{2}\right]_2^4$$

$$- k\left[\frac{x^3}{3}\right]_4^6 + 6k\left[\frac{x^2}{2}\right]_4^6$$

$$= k\left[\frac{8}{3} + 16 - 4 - \frac{6^3}{3} + \frac{4^3}{3} + 3(36)\right.$$

$$\left. - 3(16)\right]$$

$$= k[24] = \frac{1}{8} \times 24 = 3$$

BERNOULLI DISTRIBUTION

$$P(X=x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases} \quad \text{for } 0<p<1$$

$X \sim \text{Bernoulli}(p)$

11.

A coin has a probability of 0.5 of landing heads when tossed. Let X = 1 if the coin comes up heads, and X = 0 if the coin comes up tails. What is the distribution of X?
Answer: $X \sim Bernoulli(0.5)$.

$p = 0.5$ = prob of success $\quad X=1 \downarrow$ heads, $X=0 \downarrow$ tails

X	0	1
P	0.5	0.5

$$P(X) = \begin{cases} 0.5 & x=1 \\ 0.5 & x=0 \end{cases}$$

$X \sim \text{Bernoulli}(0.5)$

12.

A die has a probability $\frac{1}{6}$ of coming up 6 when rolled. Let X = 1 if the die comes up 6, and X = 0 otherwise. What is the distribution of X?
Answer: The success probability is $p = P(X = 1) = \frac{1}{6}$. Therefore $X \sim Bernoulli\ (\frac{1}{6})$.

p = prob of getting a 6 = $1/6$

$$P(X) = \begin{cases} 1/6 & x=1 \\ 5/6 & x=0 \end{cases}$$

$X \sim \text{Bernoulli}(1/6)$

13.

Ten percent of the components manufactured by a certain process are defective. A component is chosen at random. Let X = 1 if the component is defective, and X = 0 otherwise. What is the distribution of X?

$p = \frac{10}{100} = \frac{1}{10} = 0.1$ = prob of getting a defective piece

$$P(X) = \begin{cases} 0.1 & x=1 \\ 0.9 & x=0 \end{cases}$$

$X \sim \text{Bernoulli}(0.1)$

BINOMIAL DISTRIBUTION

$X \sim Bin(n, p)$

$$P(X) = P(X=x) = \begin{cases} {}^{n}C_{x}\, p^{x}(1-p)^{n-x} & x = 0, 1, 2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

Mean $= \mu_x = np$

Variance $= \sigma_x^2 = np(1-p) = npq$

14. The prob that a pen manufactured by the company will be defective is $\frac{1}{10}$. If 12 pens are manufactured. Find the prob that
(i) exactly 2 pens are defective
(ii) atleast 2 pens are defective
(iii) None of the pens are defective

No. of successes = No. of defective pens
$= X = \{0, 1, 2 \ldots 12\}$
p = prob of getting defective pen
$= \frac{1}{10} = 0.1$

$q = 1 - p = 0.9$

$X \sim Bin(12, 0.1)$

* Use Binomial PD in calci

$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

(i) $P(X=2) = {}^{12}C_{2}(0.1)^{2}(0.9)^{10}$
$= 0.2301$

(ii) $P(X \geq 2) = 1 - P(X < 2)$
$= 1 - [P(X=0) + P(X=1)]$
$= 1 - [0.2824 + 0.3765]$
$= 0.342$

(iii) $P(X=0) = 0.2824$

15. Out of 800 families with 5 children each, how many have
(i) 3 boys (ii) 5 girls (iii) either 2 or 3 boys

Let getting a girl be a success
$p = 0.5$ $\quad q = 0.5$ $\quad X \sim Bin(5, 0.5)$

(i) P(getting 3 boys) = P(getting 2 girls) = $P(X=2)$
$= {}^{5}C_{2}(0.5)^{2}(0.5)^{3} = 0.3125$

No. of families having 3 boys
$= 800 \times 0.3125 = 250$

(ii) $P(X=5) = {}^5C_5 (0.5)^5 (0.5)^0$

$= 0.03125$

No. of families having 5 girls $= 800 \times 0.03125$

$= 25$

(iii) P(getting 2 or 3 boys) $= P(X=3) + P(X=2)$

$= 0.3125 + 0.3125$

$= 0.6250$

No. of families having 2 or 3 boys

$= 800 \times 0.6250 = 500$

16.

The phone lines to an airline reservation system are occupied 40% of the time. Assume that the events that the lines are occupied on succesive calls are independent. Assume that 10 calls are placed to the airline.
(i) what is the probability that for exactly three calls the lines are occupied?
(ii) what is the probability that for atleast one call the lines are not occupied?
(iii) what is the expected number of calls in which the lines are all occupied.
Answer: (i)$P(X = 3) = 0.2149$, (ii)$P(z \geq 1) = 0.9999$, (iii)$E(X) = 4$

Let line occupied be success

$X = \{0, 1, 2 \ldots 10\}$ $\quad n = 10$

$p = 0.4$ $\quad q = 0.6$

$X \sim Bin(10, 0.4)$

(i) $P(X=3) = {}^{10}C_3 (0.4)^3 (0.6)^7$

$= 0.2149$

(ii) $P(X \leq 9) = 1 - P(X=10) = 1 - 1.048 \times 10^{-4}$

$= 0.9999$

(iii) $E(X) = np = 10(0.4) = 4$

17.

. Heart failure is due to either natural occuerence(87%) or outside factors(13%). outside factors are releated to induce substances or foreign objects. Natural occurence are caused by arterial blockage, disease and infection. Suppose that 20 patients will visit an emergency room with heart failure. Assume that cause of heart failure between individuals are independent.
(a) what is the probability that three individuals have conmditions caused by outtside factors?
(b) what is the probability that three or more individuals have conditions caused by outside factors?
(c) what is the mean and standard deviation of the number of individuals with conditions caused by outside factors?
Answer: (a) 0.235, (b) 0.492, (c) 2.6,
$V(X) = 2.262$, Standard deviation$=1.504$

Let outside factor be success

$p = 0.13 \qquad q = 0.87 \qquad n = 20$

$X \sim Bin(20, 0.13)$

(a) $P(X=3) = {}^{20}C_3 (0.13)^3 (0.87)^{17} = 0.2347$

(b) $P(X \geq 3) = 1 - P(X < 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - [0.0617 + 0.1844 + 0.2618]$

$= 0.4921$

(c) Mean $(\mu) = np = 20 \times 0.13 = 2.6$

Variance $(\sigma^2) = npq = 2.262$

S.D $(\sigma) = \sqrt{\sigma^2} = 1.504$

18.

In eight throws of a fair dies, 5 or 6 is considered a success. Find the mean of the number of success and the standard deviation. Answer: Mean$=\frac{8}{3}$, standard deviation$=\frac{4}{3}$

$n = 8 \qquad X =$ getting 5 or 6

$= \{0, 1, 2 \ldots 8\}$

$p = \frac{2}{6} = 1/3 \qquad X \sim Bin(8, 1/3)$

Mean $= np = 8/3 \qquad$ Variance $= npq = 16/9$

$S.D = 4/3$

19.

The probability that a man hits a target is $\frac{1}{3}$ how many times must he fire so that the probability of hitting the target atleast once is more than 90% ? Answer: Man should fire 6 times.

p = hits a target = $1/3$ $\quad q = 2/3$

$P(X \geq 1) > 0.90$

$1 - P(X < 1) > 0.9$

$1 - P(X = 0) > 0.9$

$1 - 0.9 > P(X = 0)$

$0.1 > {}^nC_0 (1/3)^0 (2/3)^n$

$0.1 > (2/3)^n$

$n > 5.67 \Rightarrow \boxed{n = 6}$

if he fires 5 times, what is the probability of hitting the target atleast 2 times?

$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$

$= 1 - [0.1316 + 0.3292]$

$= 0.5392$

20. In a sampling, large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain atleast 3 defective parts

$np = 2 \quad , n = 20 \Rightarrow p = 0.1$

$q = 0.9$

$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 0.3233$

No. of samples that contain atleast 3 defects $= 1000 \times 0.3233 = 323$

POISSON DISTRIBUTION

$n \rightarrow$ large $\qquad \Longrightarrow \qquad n \rightarrow \infty$
$p \rightarrow$ small $\qquad\qquad\qquad p \rightarrow 0$
$np = \lambda$

Mean $= \lambda = np$

Variance $= \lambda = np$

$X \sim \text{Poisson}(\lambda)$

$$p(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & x \text{ is non-negative integer} \\ 0 & \text{otherwise} \end{cases}$$

* Use Poisson PD in calci

Proof : Given $n \to \infty$, $p \to 0$ and $np = \lambda$

$$p = \lambda/n \qquad q = 1 - p = 1 - \lambda/n$$

For binomial distribution

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} \left(\frac{\lambda}{n}\right)^{x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} \cdot \frac{\lambda^{x}}{n^{x}} \cdot \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{x}}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(x-1))(n-x)!}{(n-x)!\,x!} \cdot \frac{\lambda^{x}}{n^{x}} \cdot \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{x}}$$

$$= \frac{n^{x}\left[(1-1/n)(1-2/n)\cdots\left(1-\left(\frac{x-1}{n}\right)\right)\right]}{x!} \cdot \frac{\lambda^{x}}{n^{x}} \; \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{x}}$$

$$= \frac{(1-1/n)\ldots\left(1-\left(\frac{x-1}{n}\right)\right)}{x!} \; \frac{\lambda^{x}(1-\lambda/n)^{n}}{(1-\lambda/n)^{x}}$$

$$\lim_{n \to \infty} P(X = x) = \frac{1}{x!}\,\lambda^{x} \lim_{n \to \infty} \frac{(1-1/n)\cdots\left(1-\left(\frac{x-1}{n}\right)\right)}{(1-\lambda/n)^{x}} \cdot (1-\lambda/n)^{n}$$

$$= \frac{\lambda^{x}}{x!} e^{-\lambda}$$

21.

Suppose that X has poisson distribution with a mean of 4. Determine the following probabilities:
(a) $P(X = 0)$ (b) $P(X \leq 2)$, (c) $P(X = 4)$, (d) $P(X = 8)$
Answer: (a) 0.0183, (b) 0.2375, (c) 0.1954, (d) 0.0298

$\lambda = 4$

(a) $P(x=0) = \dfrac{e^{-4}(4)^0}{0!} = 0.0183$

(b) $P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$
$= 0.0183 + 0.07326 + 0.1465$
$= 0.238$

(c) $P(x=4) = 0.1953$

(d) $P(x=8) = 0.02977$

22.

The number of telephone calls that arrive at a phone exchange is often modelled as a Poisson random variable. Assume that on the average there are 10 calls per hour.
(a) What is the probability that there are exactly 5 calls in one hour?
(b) What is the probability that there are 3 or fewer calls in one hour?
(c) What is the probability that there are exactly 15 calls in two hour?
(d) What is the probability that there are exactly 5 calls in 30 minutes?
Answer: (a) 0.0378, (b) 0.0103, (c) 0.0516, (d) 0.1755

$\lambda = 10$

(a) $P(X=5) = \dfrac{e^{-10}(10)^5}{5!} = 0.0378$

(b) $P(X \leq 3) = P(x=0) + P(X=1) + P(X=2) + P(X=3$
$= 4.539 \times 10^{-5} + 4.539 \times 10^{-4}$
$+ 2.269 \times 10^{-3} + 7.566 \times 10^{-3}$
$= 0.0103$

(c) $\lambda = 10$ for 1 hour
$\lambda_n = 20$ for 2 hour
$P(x=15) = \dfrac{e^{-20}(20)^{15}}{15!} = 0.0516$

(d) $\lambda = 10$ for 1 hour
$\lambda_n = 5$ for ½ hour
$P(x=5) = \dfrac{e^{-5}(5)^5}{5!} = 0.1754$

23.

Fit a Poisson distribution for the following data and calculate the theoretical frequencies

X	0	1	2	3	4
f	122	60	15	2	1

Answer: The theoritical frequencies are 121, 61, 15, 3, 0

X	f	f.x
0	122	0
1	60	60
2	15	30
3	2	6
4	1	4
	$\Sigma f = 200$	$\Sigma fx = 100$

$$\lambda = \frac{\Sigma fx}{\Sigma f} = \frac{100}{200} = 0.5$$

$$\boxed{\lambda = 0.5}$$

Theoretical frequencies is given by

$$F(x) = \Sigma f \times P(X=x)$$

$$F(0) = 200 \times \frac{e^{-0.5}(0.5)^0}{0!} = 200 \times 0.6065 = 121$$

$$F(1) = 200 \times \frac{e^{-0.5}(0.5)^1}{1!} = 60.64 = 61$$

$$F(2) = 200 \times \frac{e^{-0.5}(0.5)^2}{2!} = 15.16 = 15$$

$$F(3) = 200 \times \frac{e^{-0.5}(0.5)^3}{3!} = 2.52 = 3$$

$$F(4) = 200 \times \frac{e^{-0.5}(0.5)^4}{4!} = 0.3159 = 0$$

24.

In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing
(i) no defective (ii) one defective (iii) two defective blades, in a consignment of 10000 packets.
Answer: (i) $f(0)=9802$, (ii) $f(1)=196$, (iii) $f(2)=2$

$$p = \frac{1}{500} = 0.002 \qquad n = 10$$

$$\lambda = np = 0.02$$

(i) $$P(X=0) = \frac{e^{-0.02}(0.02)^0}{0!} = 0.9801$$

No. of packets containing no defects
$$= 10000 \times 0.9802 = 9802$$

(ii) $$P(x=1) = \frac{e^{-0.02}(0.02)}{1!} = 0.0196$$

No. of packets $= 196$

(iii) $$P(X=2) = \frac{e^{-0.02}(0.02)^2}{2!} = 1.96 \times 10^{-4}$$

No. of packets $= 1.96 \approx 2$

25. Particles are suspended in a liquid medium at a concentration of 6 particles per ml. A large volume of the suspension is thoroughly agitated & then 3ml are withdrawn. Probability that 15 particles are withdrawn

$\lambda = 6$ per ml $\qquad$ For, 3 ml

$$\lambda_n = 18$$

$$P(X=15) = \frac{(18)^{15} e^{-18}}{15!} = 0.0785$$

26. Assume that the no. of cars that pass through certain intersection during a fixed time interval follows a Poisson dist. Assume that the mean rate is 5 cars per min. Find the prob that 17 cars will pass through intersection in nxt 3 mins

$\lambda = 5$ per minute

$\lambda = 15$ for 3 min

$$P(X=17) = \frac{e^{-15}(15)^{17}}{17!} = 0.0847$$

27. A communication channel receives independent pulses at the rate of 12 per microsecond. The probability of transmission error is 0.001 for each microsecond. Compute prob

(i) No error (ii) One error

(iii) Atleast 1 error (iv) Two error

(v) Almost 2 error

$n = 12 \quad p = 0.001$

$\lambda = np = 0.012$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

(i) $P(X=0) = 0.9880$

(ii) $P(X=1) = 0.118$

(iii) $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$
$= 0.012$

(iv) $P(X=2) = 7.1141 \times 10^{-5}$

(v) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$
$= 0.9999$

28. An avg of 0.61 soldiers died of horse kicks per year in Prussian army corps. Calculate the prob that 2 soldiers died of horse kicks in 1898.

(Poisson D)

$\lambda = 0.61$

$P(X = 2) = 0.1010$

29. Suppose 2 % of the items produced by a factory are defective. Find the prob that there are 5 defective items in a sample of 100 items?

$p = 0.02 \quad n = 100 \quad \lambda = 2$

$P(X = 5) = 0.0360$

30. Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the prob that a given book contains

(a) Exactly 2 misprints (b) 2 or more misprints

$\lambda = \frac{300}{500} = 0.6$ per page

(a) $P(X = 2) = 0.0987$

(b) $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$

$= 0.1219$

31. If the prob of bad reaction from a certain injection is 0.001. Determine the chance that out of 2000 individuals more than 2 will get a bad reaction

$p = 0.001 \quad n = 2000 \quad \lambda = 2$

$P(X > 2) = 1 - P(X \leq 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$= 1 - 0.135 - 0.270 - 0.270 = 0.325$

EXPONENTIAL DISTRIBUTION

$$f(x) = \begin{cases} e^{-\lambda x} \cdot \lambda & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean / Expected value $= \mu = 1/\lambda$

Variance $= 1/\lambda^2$

SD $= 1/\lambda$

$$P(x) = \int_0^{\infty} f(x)\,dx$$

32.

If X is an exponential variate with mean 3 find
(i) $P(X > 1)$, (ii) $P(X < 3)$
Answer: (i) 0.7165, (ii) 0.6321

$$\mu = 3 \Rightarrow 1/\lambda = 3 \Rightarrow \boxed{\lambda = 1/3}$$

$$f(x) = e^{-1/3 x} \cdot 1/3$$

(i) $$P(X>1) = \int_1^\infty e^{-1/3 x} \cdot 1/3 \, dx = \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_1^\infty$$

$$= e^{-1/3} = 0.7165$$

(ii) $$P(X<3) = \int_0^3 e^{-1/3 x} \cdot 1/3 \, dx = \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_0^3$$

$$= -e^{-1} + e^{0} = 0.6321$$

33.

The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth
(i) ends less than 5 minutes
(ii) ends between 5 and 10 minutes.
Answer: (i) 0.6321, (ii) 0.2325.

$$\mu = 5 \Rightarrow \lambda = 1/5 \qquad f(x) = e^{-1/5 x} \cdot 1/5$$

(i) $$P(X<5) = \int_0^5 1/5 \, e^{-x/5} dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^5$$

$$= e^{0} - e^{-1} = 0.6321$$

(ii) $$P(5<X<10) = \int_5^{10} \frac{1}{5} e^{-x/5} dx = \frac{1}{5}\left\{\frac{e^{-x/5}}{-1/5}\right]_5^{10}$$

$$= e^{-1} - e^{-2} = 0.2325$$

34.

In a certain town the duration of a shower is exponential distribution with mean 5 minutes. What is the probability that a shower will last for
(i) 10 minutes or more
(ii) less than 10 minutes
(iii) between 10 and 12 minutes.
Answer: (i) 0.1353, (ii) 0.8647, (iii) 0.0446.

$\mu = 5 \Rightarrow \lambda = 1/5 \qquad f(x) = e^{-x/5} \cdot 1/5$

(i) $P(X \geq 10) = \int_{10}^{\infty} \frac{1}{5} e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{10}^{\infty}$

$= e^{-2} = 0.1353$

(ii) $P(X \leq 10) = \int_{0}^{10} 1/5\, e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{0}^{10}$

$= 1 - e^{-2} = 0.8646$

(iii) $P(10 < X < 12) = \int_{10}^{12} 1/5\, e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{10}^{12}$

$= -e^{-12/5} + e^{-2} = 0.0446$

35. At a certain city between bus stop, 3 buses arrive per hour on an average. Assuming that the time between successive arrival is exponentially distributed, find the probability that the time between arrival of successive bus is
i) less than 10 mins
ii) atleast 30 mins

$\mu = \frac{60}{3} = 20$ minutes per successive arrival

$\lambda = 1/20$

(i) $P(X < 10) = \int_{0}^{10} e^{-x/20} \cdot 1/20\,dx = \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{0}^{10}$

$= 1 - e^{-1/2} = 0.3934$

(ii) $P(X \geq 30) = \int_{30}^{\infty} e^{-x/20} \cdot 1/20\,dx = \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{30}^{\infty}$

$= e^{-3/2} = 0.2231$

36. The lifetime of a compressor manufactured by a company is 200 months on an average. Find the probability that the lifetime of a compressor is

i) less than 200 months

ii) between 100 months and 25 years

$\mu = 200 \Rightarrow \lambda = 1/200$

$f(x) = e^{-x/200} \cdot 1/200$

(i) $P(X < 200) = \int_0^{200} \frac{1}{200} e^{-x/200} dx = \frac{1}{200} \left[\frac{e^{-x/200}}{-1/200} \right]_0^{200}$

$= 0.6321$

(ii) $P(100 < X < 300) = \int_{100}^{300} \frac{1}{200} e^{-x/200} dx$

$= \frac{1}{200} \left[\frac{e^{-x/200}}{-1/200} \right]_{100}^{300} = 0.3834$

37. The sales per day in a shop is exponential distribution with average sales amounting to 100 rupees and net profit is 8%. Find the probability that the net profit exceeds 30 rupees on 2 consecutive days

Intially, $\mu = 100 \Rightarrow \lambda = 10^{-2}$

but for 2 consecutive days, $\lambda = \frac{2}{100}$

$A \times \frac{8}{100} = 30 \Rightarrow A = 375$

$P(X > 375) = \int_{375}^{\infty} e^{-2x/100} \cdot \frac{2}{100} dx = \frac{2}{100} \left[\frac{e^{-2x/100}}{-2/100} \right]_{375}^{\infty}$

$= e^{-2(375)/100}$

$= 5.5308 \times 10^{-4}$

NORMAL DISTRIBUTION

Bell curve / Gaussian distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$X \sim N(\mu, \sigma^2)$$

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z$

$dx = \sigma\,dz$

$X = a \Rightarrow z_1 = \frac{a-\mu}{\sigma}$ & $X = b \Rightarrow z_2 = \frac{b-\mu}{\sigma}$

$$P(z_1 < z < z_2) = \int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\,dz$$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Standard normal curve

$\mu = 0$

$\sigma^2 = 1$

$P(-\infty \le z \le \infty) = 1$

$P(-\infty \le z \le 0) = 0.5$

$P(0 \le z \le \infty) = 0.5$

* Use Normal CD in calci
& Inverse Normal

38.

If X is normally distributed with mean 6 and standard deviation 5, Find.
(i) $P(0 \le X \le 8)$
(ii) $P(|X-6| < 10)$
Answer: (i) 0.5403, (ii) 0.9544.

$\mu = 6$, $\sigma = 5$

(i) $P(0 \le X \le 8) = 0.5403$

(ii) $P(|X-6| < 10)$

$= P(-4 < X < 16)$

$= 0.9544$

$|x-6| < 10$

$x - 6 < 10$ $\quad$ $x - 6 > -10$

$x < 16$ $\quad$ $x > -4$

39.

The life of army shoes is normally distributed with mean 8 months and Standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months?
Answer: 4886.

$\mu = 8 \quad \sigma = 2 \quad x = 12$

$P(x > 12) = P(z > 2)$ $\quad$ $z = \frac{12-8}{2} = 2$

$= 0.5 - P(0 < z < 2)$

$= 0.5 - 0.4772 = 0.0228$

For 5000 pairs, $0.0228 \times 5000 = 114$ pairs don't need replacement

Require replacement $= 5000 - 114 = 4886$

40.

If the height of 300 students is normally distributed with 64.5 inches and Standard deviation 3.3 inches. Find the height below which 99% of the students lie.
Answer: $X = 72.25$

$\mu = 64.5 \quad \sigma = 3.3$

z-score for 99% or 0.99 area $= 2.3263$

$2.3263 = \frac{x - 64.5}{3.3} \Rightarrow x = 72.17$

41.

In a Normal distribution 30.85% of the items are over 64 and 8% are under 45. Find the mean and Standard deviation

Answer: Mean=59 and standard deviation=10

$P(X > 64) = 0.3085$

$P(X < 45) = 0.08$

$z = \frac{x - \mu}{\sigma}$ $\quad P(z_1 < z) = 0.3085$

$P(z < z_2) = 0.08$

$z_1 = -0.5001$

$z_2 = +1.4050$

$-0.5001 = \frac{64 - \mu}{\sigma} \Rightarrow \mu = 64 + 0.5001\sigma$

$+1.405 = \frac{45 - \mu}{\sigma} \Rightarrow \mu = 45 - 1.405\sigma$

$\mu = 59$ and $\sigma = 10$

42. Lifetimes of batteries in a certain application are normally distributed with mean 50 hours & S.D 5 hours. Find prob that a randomly chosen battery lasts b/w 42 and 52 hours. Find the 40th percentile of the battery lifetimes

$\mu = 50, \sigma = 5$

$A = 0.40$

$P(42 < X < 52) = 0.6006$

Can be done by converting to z-scores

$P(42 < X < 52) = P(-1.6 < z < 0.4)$

$= 0.6006$

$A = 0.40, \mu = 50, \sigma = 5$

$X = 48.73$

$z = -0.2533$

43. In a test on 2000 electric bulbs, it was found that the life of a particular bulb with an average life of 2040 hours and SD of 60 hours. Find no. of bulbs likely to burn for

(i) More than 2150 hours

(ii) Less than 1950 hours

(iii) More than 1920 hours & less than 2160 hours

$\mu = 2040 \qquad \sigma = 60$

$$z = \frac{X - 2040}{60}$$

(i) $X = 2150$

$z = 1.833$

$P(X > 2150) = 0.5 - P(0 < z < 1.833)$

$= 0.5 - 0.466 = 0.034$

No. of bulbs = 67 bulbs

(ii) $P(X < 1950)$ $\qquad X = 1950$, $z = -1.5$

$P(z < 1.5) = 0.5 - P(-1.5 < z < 0)$

$= 0.0669$

No. of bulbs = 134

(iii) $P(1920 < X < 2160) = P(-2 < z < 2)$

$= 0.9545$

No. of bulbs = 1909

NORMAL APPROX. TO BINOMIAL

$$X \sim B(n, p) \implies X \sim N(\mu, \sigma^2)$$

n should be large
and p is close to 0.5

$np \geq 5$ and $nq \geq 5$

$\mu = np$ & $\sigma^2 = np(1-p)$

To make it a continuous interval

$$P(A < x < B)$$

$A \downarrow A + 0.5$ $\quad$ $B \downarrow B - 0.5$

44.

The manufacturing of semiconductor chips produces 2 % defective chips. Assume the chips are independent and that a lot contains 1000 chips.
(a) Approximate the probability that more than 25 chips are defective.
(b) Approximate the probability that between 20 and 30 chips are defective.
Answer: (a) $P(X > 25) = 0.107485$.
(b) $P(20 < X < 30) = 0.4401$.

$p = 0.02 \qquad q = 1 - p = 0.98$

$n = 1000$

$\mu = 20 \qquad \mu = \sqrt{npq} = 4.427$

(a) $P(X > 25) = P(X > 25.5)$

$z = \frac{25.5 - 20}{4.427} = 1.2423$

$P(z > 1.2423) = 0.5 - P(0 < z < 1.2423)$
$= 0.5 - 0.3929$
$= 0.1071$

(b) $P(20 < X < 30) = P(20.5 < X < 29.5)$
$= P(0.1129 < z < 2.1458)$
$= 0.4391$

45.

There were 49.7 million people with some type of long-lasting condition or disability living in the United States in 2000. This represented 19.3 percent of the majority of civilians aged five and over. A sample of 1000 persons is selected at random.
(a) Approximate the probability that more than 200 persons in the sample have a disability.
(b) Approximate the probability that between 180 and 300 people in the sample have a disability.
Answer: (a) $P(X > 200) = 0.2743$. (b) $P(180 < X < 300) = 0.84134$.

$n = 1000 \qquad p = 0.193$

$\mu = np = 193 \qquad \sigma = \sqrt{np(1-p)} = 12.48$

(a) $P(X > 200) = P(X > 200.5)$

$z = \frac{200.5 - 193}{12.48} = 0.6009$

$P(z > 0.6009) = 0.5 - P(0 < z < 0.6009)$
$= 0.2739$

(b) $P(180 < X < 300) = P(180.5 < X < 299.5)$
$= P(-1.0016 < z < 8.5337)$
$= 0.8417$

46. In a sack of mixed grass seeds, the prob that a seed is ryegrass is 0.35. find the prob that in a random sample of 400 seeds from this sack
(a) less than 120 are ryegrass seeds
(b) more than 160 are ryegrass seeds
(c) b/w 120 & 150 are ryegrass
(d) 150 are ryegrass

$p = 0.35$, $n = 400$
$\mu = np = 140$ $\sigma = 9.539$

(a) $P(X < 120) = P(X < 119.5)$
$= P(z < -2.149)$
$= 0.0158$

(b) $P(X > 160) = P(X > 160.5) = P(z > 2.1490)$
$= 0.0158$

(c) $P(120 < X < 150) = P(120.5 < X < 149.5)$
$= P(-2.0442 < z < 0.9959)$
$= 0.8199$

(d) $P(X = 150) = P(149.5 < X < 150.5)$
$= P(0.9959 < z < 1.007)$
$= 0.0241$

NORMAL APPROX. TO POISSON

$\lambda = np$ (where it is more than 15)

47.

Suppose that X is a Poisson random variable with $\lambda = 6$.
(a) Compute the exact probability that X is less than 4.
(b) Approximate the probability that X is less than 4 and compare to the result in part(a).
(c) Approximate the probability that $8 < X < 12$.

$\lambda = 6$ $\quad \sigma^2 = 6$ $\quad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$\mu = 6$

$\sigma = 2.449$

(a) $P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.0025 + 0.0148 + 0.0446 + 0.0892$

$= 0.151$

(b) $P(X<4) = P(X<3.5) = P(z < -1.0206)$

$= 0.5 - P(-1.0206 < z < 0)$

$= 0.1537$

(c) $P(8 < X < 12) = P(8.5 < X < 11.5)$

$= P(1.0206 < z < 2.2454)$

$= 0.1414$

48.

Hits to a high-volume web site are assumed to follow a Poisson distribution with a mean of 10000 per day. Approximate each of the following:
(a) The Probability of more than 20000 hits in a day.
(b) The Probability of less than 9900 hits in a day.
(c) The value such that the probability that the number of hits in a day exceed the value is 0.01.
Answer: (a) $P(X > 20000) = 0$.
(b) $P(X < 9900) = 0.1562$.
(c) $X = 10233.5$.

$\lambda = \mu = 10000 \qquad \sigma^2 = 10000$

$\sigma = 100$

(a) $P(X > 20000) = P(X > 20000.5)$

$= P(Z > 100.005)$

$= 0$

(b) $P(X < 9900) = P(X < 9899.5)$

$= P(Z < -1.005)$

$= 0.1575$

(c) $P(X > x) = 0.01$

$P(X > x + 0.5) = 0.01$

$P(Z > z) = 0.01$

$1 - A(z) = 0.01$

$A(z) = 0.99 \Rightarrow z = 2.3263$

$X = (2.3263)(100) + 10000.5$

$= 10233.5$