UNIT-2

LAPLACE

LAPLACE TRANSFORMS

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L^{-1}[F(s)] = f(t)$$

Linearity Property of Laplace

1. $L[c_1 f(t) \pm c_2 g(t)] = c_1 L[f(t)] \pm c_2 L[g(t)]$
$= c_1 F(s) + c_2 G(s)$

2. $L^{-1}[c_1 F(s) \pm c_2 G(s)] = c_1 L^{-1}[F(s)] + c_2 L^{-1}[G(s)]$
$= c_1 f(t) + c_2 g(t)$

LAPLACE TRANSFORM FOR SOME STANDARD FUNC.

1. $L[K] = \frac{K}{s}$ $\qquad$ $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L[e^{at}] = \frac{1}{s-a}$ $\qquad$ $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L[e^{-at}] = \frac{1}{s+a}$ $\qquad$ $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

4. $L[\sin at] = \frac{a}{s^2+a^2}$ $\qquad$ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

5. $L[\cos at] = \frac{s}{s^2+a^2}$ $\qquad$ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6. $L[\sinh at] = \frac{a}{s^2-a^2}$ $\qquad$ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

7. $L[\cosh at] = \frac{s}{s^2-a^2}$ $\qquad$ $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

8. $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ (or) $\frac{n!}{s^{n+1}}$ $\qquad$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

(or)

$$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

(1) LT of $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}$

CW

$$L[f(t)] = \int_0^1 e^{-st} e^t \, dt + \int_1^\infty e^{-st}(0)\, dt$$

$$= \int_0^1 e^{(1-s)t} dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1$$

$$= \frac{e^{1-s}}{1-s} - \frac{1}{1-s} = \frac{e^{1-s}-1}{1-s}$$

(2) $L\{t^3 + 2 + 2e^{-4t} + 5\sin 6t\}$

$$L[f(t)] = L\{t^3\} + L\{2\} + 2L\{e^{-4t}\} + 5L\{\sin 6t\}$$

$$= \frac{3!}{s^4} + \frac{2}{s} + \frac{2}{s+4} + \frac{5(6)}{s^2+36}$$

$$= \frac{6}{s^4} + \frac{2}{s} + \frac{2}{s+4} + \frac{30}{s^2+36}$$

(3) $L\{\sin 2t \cos 2t\}$

$$L[f(t)] = \frac{1}{2} L[2\sin 2t \cos 2t] = \frac{1}{2} L[\sin 4t]$$

$$= \frac{1}{2} \cdot \frac{4}{s^2+16} = \frac{2}{s^2+16}$$

(4) $L\{\sin^3 2t\}$

$$L\{f(t)\} = L\left[\frac{3\sin 2t - \sin 6t}{4}\right]$$

$$= \frac{1}{4}\left[\frac{3 \cdot 2}{s^2+4} - \frac{6}{s^2+36}\right] = \frac{6}{4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

(5) $L[\cos^2 at]$

CW

$$L[f(t)] = L\left[\frac{\cos 2t - 1}{2}\right] = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{1}{s}\right]$$

(6) $L\{\cos(at+b)\}$

$$L[f(t)] = L[\cos at \cos b - \sin at \sin b]$$

$$= \cos b \cdot \frac{s}{s^2+a^2} - \sin b \cdot \frac{a}{s^2+a^2} = \frac{s\cos b - a\sin b}{s^2+a2}$$

(7) Find $L\{\sinh^2 2t + 5\cosh 2t\}$

$$L[f(t)] = L[\sinh^2 2t] + 5L[\cosh 2t]$$

$$= L\left[\left(\frac{e^{2t} - e^{-2t}}{2}\right)^2\right] + \frac{5s}{s^2-4}$$

$$= \frac{1}{4}L\left[e^{4t} - 2 + e^{-4t}\right] + \frac{5s}{s^2-4}$$

$$= \frac{1}{4}\left[\frac{1}{s-4} - \frac{2}{s} + \frac{1}{s+4}\right] + \frac{5s}{s^2-4}$$

(8) $L\left\{3\sqrt[3]{t} - \frac{3}{\sqrt{t}} - 2t^{3/2}\right\}$

$$L\left\{3t^{1/3} - 3t^{-1/2} - 2t^{3/2}\right\} = 3L[t^{1/3}] - 3L[t^{-1/2}] - 2L[t^{3/2}]$$

$$= 3\frac{\Gamma(1/3+1)}{s^{4/3}} - 3\frac{\Gamma(1-1/2)}{s^{1-1/2}} - 2\frac{\Gamma(3/2+1)}{s^{3/2+1}}$$

$$= \frac{3 \cdot 1/3\, \Gamma(1/3)}{s^{4/3}} - \frac{3\sqrt{\pi}}{s^{1/2}} - \frac{2 \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma(1/2)}{s^{5/2}}$$

$$= \frac{\Gamma(1/3)}{s^{4/3}} - 3\sqrt{\frac{\pi}{s}} - \frac{3}{2} \cdot \frac{\sqrt{\pi}}{s^{5/2}}$$

⑨ $L^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$

$3L^{-1}\left[\frac{s}{s^2+4}\right] - 8L^{-1}\left[\frac{1}{s^2+4}\right] - 4L^{-1}\left[\frac{s}{s^2-16}\right]$

$+24L^{-1}\left[\frac{1}{s^2-16}\right]$

$= 3\cos 2t - \frac{8\sin 2t}{2} - 4\cosh 4t + \frac{24\sinh 4t}{4}$

$= 3\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$

⑩ $L^{-1}\left\{\frac{5s-4}{s^2+8}\right\}$

$5L^{-1}\left\{\frac{s}{s^2+8}\right\} - 4L^{-1}\left\{\frac{1}{s^2+8}\right\} = 5\cos 2\sqrt{2}\,t - \frac{4\sin 2\sqrt{2}t}{2\sqrt{2}}$

$= 5\cos 2\sqrt{2}t - \sqrt{2}\sin 2\sqrt{2}t$

⑪ $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

$3L^{-1}\left\{\frac{s^4-4s^2+4}{2s^5}\right\} = 3\left[L^{-1}\left[\frac{s^4}{2s^5}\right] - 4L^{-1}\left[\frac{s^2}{2s^5}\right)\right.$

$\left.+ L^{-1}\left[\frac{4}{2s^5}\right]\right]$

$= \frac{3}{2}\left[L^{-1}\left[\frac{1}{s}\right] - 4L^{-1}\left[\frac{1}{s^3}\right] + 4L^{-1}\left[\frac{1}{s^5}\right]\right]$

$= \frac{3}{2}\left[1 - \frac{4t^2}{2} + 4\frac{t^4}{4!}\right]$

$= \frac{3}{2}\left[1 - 2t^2 + \frac{t^4}{6}\right]$

(12) $L^{-1}\left[\frac{3s-8}{4s^2+5}\right]$

$3L^{-1}\left[\frac{s}{4s^2+5}\right] - 8L^{-1}\left[\frac{1}{4s^2+5}\right]$

$= \frac{3}{4}\cos\frac{\sqrt{5}}{2}t - \frac{8}{4}\frac{\sin\frac{\sqrt{5}}{2}t}{\sqrt{5}/2}$

$= \frac{3}{4}\cos\frac{\sqrt{5}}{2}t - \frac{4}{\sqrt{5}}\sin\frac{\sqrt{5}}{2}t$

(13) $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

$2L^{-1}\left[\frac{s}{9s^2-25}\right] - 5L^{-1}\left[\frac{1}{9s^2-25}\right]$

$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{5}{9}\frac{\sinh\frac{5}{3}t}{5/3}$

$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{1}{3}\sinh\frac{5}{3}t$

(14) HW

LT of $f(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$

$L[f(t)] = L\left[t^{3/2} + t^{-3/2} + 3\sqrt{t}\cdot\frac{1}{\sqrt{t}}\left(\frac{1}{\sqrt{t}}+\sqrt{t}\right)\right]$

$= L[t^{3/2}] + L[t^{-3/2}] + L[3t^{1/2}] + 3L[t^{-1/2}]$

$= \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} - \frac{2\sqrt{\pi}}{s^{-1/2}} + \frac{3\cdot\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

$= \frac{\sqrt{\pi}}{4}\left[\frac{3}{s^{5/2}} + \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} - \frac{8}{s^{-1/2}}\right]$

(15) LT of $f(t) = \sin\sqrt{t}$

HW

$$L[\sin\sqrt{t}] = L\left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots\right]$$

$$= L[t^{1/2}] - \frac{L}{3!}[t^{3/2}] + \frac{L}{5!} L[t^{5/2}]$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{s^{7/2}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{3!}\cdot\frac{3/2}{s} + \frac{1}{5!}\cdot\frac{5/2\cdot 3/2}{s^2}\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2\cdot(4s)^2} - \dots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

(16) $L^{-1}\left[\frac{2s+5}{s^2+25}\right]$

$$L^{-1}\left[\frac{2s+5}{s^2+25}\right] = 2L^{-1}\left[\frac{s}{s^2+25}\right] + 5L^{-1}\left[\frac{1}{s^2+25}\right]$$

$$= 2\cos 5t + 5\frac{\sin 5t}{5} = 2\cos 5t + \sin 5t$$

(17) $L^{-1}\left[\dfrac{s}{s^4+16}\right]$

CW

$$L^{-1}\left[\frac{s}{(s^2+4)^2-8s^2}\right] = L^{-1}\left[\frac{s}{(s^2+4-\sqrt{8}s)(s^2+4+\sqrt{8}s)}\right]$$

$$= \frac{1}{2\sqrt{8}} L^{-1}\left[\frac{s^2+4+\sqrt{8}s-(s^2+4-\sqrt{8}s)}{(s^2+4-\sqrt{8}s)(s^2+4+\sqrt{8}s)}\right]$$

$$= \frac{1}{2\sqrt{8}} L^{-1}\left[\frac{1}{s^2+4-\sqrt{8}s} - \frac{1}{s^2+4+\sqrt{8}s}\right]$$

$$= \frac{1}{2\sqrt{8}} L^{-1}\left[\frac{1}{(s-\sqrt{2})^2+4-2} - \frac{1}{(s+\sqrt{2})^2+4-2}\right]$$

$$= \frac{1}{2\sqrt{8}}\left[\frac{e^{\sqrt{2}t}\sin\sqrt{2}t}{\sqrt{2}} - \frac{e^{-\sqrt{2}t}\sin\sqrt{2}t}{\sqrt{2}}\right]$$

$$= \frac{1}{2\times 4}\sin\sqrt{2}t\,(e^{\sqrt{2}t}-e^{-\sqrt{2}t}) = \frac{1}{4}\sin\sqrt{2}t\sinh\sqrt{2}t$$

(18) $L^{-1}\left[\dfrac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right]$

$5s^2+3s-16 = A(s-2)(s+3) + B(s-1)(s+3) + C(s-1)(s-2)$

Put $s=1 \Rightarrow 5+3-16 = -4A \Rightarrow A=2$

Put $s=2 \Rightarrow 20+6-16 = 5B \Rightarrow B=2$

Put $s=-3 \Rightarrow 45-9-16 = 20C \Rightarrow C=1$

$$L^{-1}\left[\frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s-3}\right]$$

$$= 2e^{t} + 2e^{2t} + e^{3t}$$

PROPERTIES OF LAPLACE TRANSFORM

PROPERTY 1 [FIRST SHIFTING PROPERTY]

$$L[e^{at}f(t)] = F(s-a)$$

$$L^{-1}[F(s-a)] = e^{at}f(t)$$

Note:

$$L[e^{-at}f(t)] = F(s+a)$$

$$L^{-1}[F(s+a)] = e^{-at}f(t)$$

PROPERTY 2 [DIFFERENTIATION OF LT]

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$L^{-1}[F^n(s)] = (-1)^n t^n f(t)$$

Note:

$$L[t f(t)] = \frac{-d}{ds} F(s)$$

$$L[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$$

$$L^{-1}[F'(s)] = -t f(t)$$

$$L^{-1}[F''(s)] = t^2 f(t)$$

Logarithm / Trigono

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}[F'(s)]$$

PROPERTY 3 [INTEGRATION OF LT]

$$L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \int_s^\infty \dots n \text{ times } F(s)\,ds$$

$$L^{-1}\left[\int_s^\infty \int_s^\infty \dots F(s)\,ds\right] = \frac{f(t)}{t^n}$$

Note:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$$

PROPERTY 4 [LT OF A DERIVATIVE]

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Note: (Multiplication by s)

$$L^{-1}[sF(s)] = \frac{d}{dt}f(t) \quad \text{when } f(0)=0$$

PROPERTY 5 [LT OF INTEGRAL]

$$L\left[\int_0^t \int_0^t \dots n \text{ times } f(t)\,dt\right] = \frac{F(s)}{s^n}$$

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Note: (Division by s)

$$L^{-1}\left[\frac{F(s)}{s^n}\right] = \int_0^t \int_0^t \dots n \text{ times } f(t)\,dt$$

SHIFTING THEOREM (PROPERTY I)

① LT of $e^{at}\cos bt$ and $e^{\alpha t}\sin\beta t$

cw

$$L[e^{at}\cos bt] = L[\cos bt]_{s\to s-a}$$

$$= \left(\frac{s}{s^2+b^2}\right)_{s\to s-a} = \frac{s-a}{(s-a)^2+b^2}$$

$$L[e^{\alpha t}\sin\beta t] = L[\sin\beta t]_{s\to s-\alpha}$$

$$= \frac{\beta}{s^2+\beta^2} = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

② $L^{-1}\left[\frac{1}{s^2-4s+8}\right] = L\left[\frac{1}{(s-2)^2+2^2}\right]$

cw

$$= \frac{e^{2t}\sin 2t}{2}$$

③ $L^{-1}\left[\frac{4}{s^2-s+2}\right] = L^{-1}\left[\frac{4}{s^2-s+1/4+7/4}\right]$

$$= L^{-1}\left[\frac{4}{(s-1/2)^2+7/4}\right] = 4\,\frac{\sin\frac{\sqrt{7}}{2}t}{\sqrt{7}}\cdot 2\cdot e^{\frac{1}{2}t}$$

$$= \frac{8}{\sqrt{7}}e^{\frac{1}{2}t}\sin\frac{\sqrt{7}}{2}t$$

④ $L^{-1}\left[\frac{s+6}{s^2+6s+13}\right] = L^{-1}\left[\frac{s+6}{(s+3)^2+4}\right]$

$$= e^{-3t}\cos 2t + 3e^{-3t}\frac{\sin 2t}{2}$$

⑤ $e^{2t}(3\sin 4t - 4\cos 4t)$

HW

$L[e^{2t}(3\sin 4t - 4\cos 4t)]$ $\qquad s \to s-2$

$= L[3\sin 4t - 4\cos 4t]$

$= \frac{3 \times 4}{s^2+16} - \frac{4s}{s^2+16} = \frac{12 - 4(s-2)}{(s-2)^2+16}$

$= \frac{12 - 4s + 8}{s^2 + 4 - 4s + 16} = \frac{20 - 4s}{s^2 - 4s + 20}$

⑥ $y'' + 4y' + 4y = 12t^2e^{-2t}, \ y(0) = 2, \ y'(0) = 1$

CW

$s^2L[y(t)] - sy(0) - y'(0) + 4[sL[y(t)] - y(0)]$

$+ 4L[y(t)] = L[12t^2e^{-2t}]$

$(s^2+4s+4)L[y(t)] - 2s - 1 - 8 = 12L[t^2e^{-2t}]$

$= 12[L[t^2]]_{s \to s+2}$

$(s^2+4s+4)L[y(t)] - 2s - 9 = 12 \cdot \frac{2}{(s+2)^3}$

$(s+2)^2 L[y(t)] = \frac{24}{(s+2)^3} + 2s + 9$

$L[y(t)] = \frac{24}{(s+2)^5} + \frac{2s}{(s+2)^2} + \frac{9}{(s+2)^2}$

$y(t) = \frac{24e^{-2t}t^4}{4!} + 2L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right] + 9e^{-2t}t$

$= e^{-2t}t^4 + 2e^{-2t}(1 - 2t) + 9e^{-2t}t$

$= e^{-2t}t^4 + 2e^{-2t} + 5e^{-2t}t$

⑦ $e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt$

$$L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right] = L\left[\int_0^t \frac{\sin 3t}{t}\,dt\right]_{s\to s+4}$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9}\,ds = 3\left[\frac{1}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right] = \cot^{-1}\frac{s}{3}$$

$$= L\left[\int_0^t \frac{\sin 3t}{t}\,dt\right]_{s\to s+4} = \left[\frac{\cot^{-1}\frac{s}{3}}{s}\right]_{s\to s+4}$$

$$= \frac{1}{s+4}\cot^{-1}\frac{s+4}{3}$$

PROPERTY 2

⑧ $L\{t(3\sin 2t - 2\cos 2t)\}$

$$3L[t\sin 2t] - 2L[t\cos 2t]$$

$$= 3\left(\frac{-d}{ds}\left(\frac{2}{s^2+4}\right)\right) + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -6\left[\frac{-2s}{(s^2+4)^2}\right] + 2\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{12s - 2s^2 + 8}{(s^2+4)^2}$$

⑨ $L[t^2 e^t \sin 4t]$

$$L[e^t t^2 \sin 4t] = \{L[t^2 \sin 4t]\}_{s \to s-1}$$

$$\because L[t^2 \sin 4t] = \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$$

$$= \frac{d}{ds}\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = 4\frac{d}{ds}\left(\frac{-2s}{(s^2+16)^2}\right)$$

$$= -8\left[\frac{(s^2+16)^2 - 2s(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= -8\left[\frac{s^2+16-4s^2}{(s^2+16)^3}\right] = +8\left[\frac{3s^2-16}{(s^2+16)^3}\right]$$

$$\{L[t^2 \sin 4t]\}_{s \to s-1} = 8\left[\frac{3(s-1)^2-16}{((s-1)^2+16)^3}\right]$$

⑩ $L[te^{-2t}\cos 3t]$

$$L[te^{-2t}\cos 3t] = \{L[t\cos 3t]\}_{s \to s+2}$$

$$= \left\{\frac{-d}{ds}\left(\frac{s}{s^2+9}\right)\right\}_{s \to s+2}$$

$$= \left\{\frac{-(s^2+9-2s^2)}{(s^2+9)^2}\right\}_{s \to s+2}$$

$$= \left\{\frac{s^2-9}{(s^2+9)^2}\right\}_{s \to s+2} = \frac{(s+2)^2-9}{((s+2)^2+9)^2}$$

(11) $L[t^5 e^{4t} \cosh 4t]$

$$L\left[t^5 e^{4t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\right]$$

$$=\frac{1}{2}L\left[t^5(e^{8t}+1)\right] = \frac{1}{2}\left[\frac{5!}{(s-8)^6} + \frac{5!}{s^6}\right]$$

$$= 60\left[\frac{1}{(s-8)^6} + \frac{1}{s^6}\right]$$

(12) $L[(1+te^{-t})^3]$

$$L\left[1 + (te^{-t})^3 + 3te^{-t}(1+te^{-t})\right]$$

$$= L\left[1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}\right]$$

$$= \frac{1}{s} + \frac{3!}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{3\cdot 2!}{(s+2)^3}$$

$$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3}$$

(13) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$$= \frac{-1}{2} L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = \frac{-1}{2} L^{-1}\left\{\frac{d}{ds}\frac{1}{s^2+a^2}\right\}$$

$$= \frac{-1}{2}\left(-t\,\frac{\sin at}{a}\right) = \frac{t\sin at}{2a}$$

(14) $L^{-1}\left\{\frac{2(s+1)}{(s^2+2s+2)^2}\right\}$

$$= -L^{-1}\left\{\frac{-2(s+1)}{((s+1)^2+1)^2}\right\} = -L^{-1}\left\{\frac{d}{ds}\frac{1}{(s+1)^2+1}\right\}$$

$$= -(-t\sin t \cdot e^{-t}) = e^{-t}\, t\sin t$$

(15) $L^{-1}\left\{\frac{s}{(s^2-9)^2}\right\}$

$$= \frac{-1}{2} L^{-1}\left\{\frac{-2s}{(s^2-9)^2}\right\} = \frac{-1}{2} L^{-1}\left[\frac{d}{ds}\frac{1}{s^2-9}\right]$$

$$= \frac{-1}{2}\left(-t\frac{\sinh 3t}{3}\right) = \frac{t\sinh 3t}{6}$$

ILT OF LOGORITHMIC & TRIGONO FUNC.

(16) $L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right]$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}[F'(s)]$$

$$F(s) = \log\left(\frac{s^2+1}{s^2}\right) = \log(s^2+1) - \log s^2$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2} = \frac{2s}{s^2+1} - \frac{2}{s}$$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right]$$

$$= \frac{-1}{t}\left[2L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}(1/s)\right]$$

$$= \frac{-1}{t}(2\cos t - 2) = \frac{2-2\cos t}{t}$$

(17) $L^{-1}\{\cot^{-1} s/a\}$

$$F(s) = \cot^{-1} s/a \qquad F'(s) = \frac{1}{a}\frac{-a^2}{s^2+a^2} = \frac{-a}{s^2+a^2}$$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{-a}{s^2+a^2}\right] = \frac{a}{t}\frac{\sin at}{a}$$

$$= \frac{1}{t}\sin at$$

(18) $L^{-1}\left[s\log\left(\frac{s-a}{s+a}\right)\right]$

$$\# \quad F(s) = \log\left(\frac{s-a}{s+a}\right) = \log(s-a) - \log(s+a)$$

$$= \frac{1}{s-a} - \frac{1}{s+a}$$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{-1}{t}(e^{at} - e^{-at})$$

$$= \frac{-2}{t}\sinh at$$

$$L^{-1}\left[s\log\left(\frac{s-a}{s+a}\right)\right]$$

$$F(s) = s\log\left(\frac{s-a}{s+a}\right)$$

$$F'(s) = \log\left(\frac{s-a}{s+a}\right) + s\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \log\left(\frac{s-a}{s+a}\right) + s\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \log\left(\frac{s-a}{s+a}\right) + \frac{2as}{s^2-a^2}$$

$$L^{-1}[F(s)] = \frac{-1}{t}\left[L^{-1}(F'(s))\right]$$

$$= \frac{-1}{t}\left[-\frac{2\sinh at}{t} + 2a\cosh at\right]$$

$$= \frac{2\sinh at}{t^2} + \frac{2a\cosh at}{t}$$

PROPERTY 3

(19) $L\left\{\frac{\sin t}{t}\right\}$, $\int_0^\infty \frac{\sin t}{t} = \pi/2$.

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

By definition of LT,

$$\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = L\left[\frac{\sin t}{t}\right]$$

Put $s=0$

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}0$$

$$= \pi/2$$

(20) $L\left[\frac{1-e^{-t}}{t}\right]$

$$L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty 1/s - \frac{1}{s+1}\,ds$$

$$= \left[\log s - \log(s+1)\right]_s^\infty$$

$$= \left[\log\frac{s}{s+1}\right]_s^\infty = \log\left(\frac{1}{1+1/s}\right)_s^\infty$$

$$= -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$

(21) $L\left[\dfrac{\sin 3t \cos t}{t}\right]$

$$L[\sin 3t \cos t] = \frac{1}{2} L[\sin 4t + \sin 2t]$$

$$= \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] = \frac{2}{s^2+16} + \frac{1}{s^2+4}$$

$$L\left[\frac{\sin 3t \cos t}{t}\right] = \int_s^\infty \frac{2}{s^2+16} + \frac{1}{s^2+4}\, ds$$

$$= \left[\frac{2}{4}\tan^{-1}\frac{s}{4} + \frac{1}{2}\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\pi/2 - \tan^{-1}\frac{s}{4} + \frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right]$$

$$= \frac{1}{2}\left[\pi - \tan^{-1}\frac{s}{4} - \tan^{-1}\frac{s}{2}\right]$$

(22) $L\left[\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2 + t\sin 2t\right]$

$$L\left[\frac{\sin^2 2t}{t}\right] + L[t\sin 2t]$$

$$= \frac{1}{2} L\left[\frac{1 - \cos 4t}{t}\right] + L[t\sin 2t]$$

$$= \frac{1}{2}\left[\int_s^\infty 1/s - \frac{s}{s^2+16}\, ds\right] + \left[-\frac{d}{ds}\frac{2}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log s^2+16\right]_s^\infty - 2\left[\frac{-1}{(s^2+4)^2} 2s\right]$$

$$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+16}}\right]_s^\infty + \frac{4s}{(s^2+4)^2}$$

$$= \frac{4s}{(s^2+4)^2} - \frac{1}{2}\log\frac{s}{\sqrt{s^2+16}}$$

(23) $L\left[\frac{e^{3t}}{t}\right] = \int_s^\infty \frac{1}{s-3}\,ds = \left[\log s-3\right]_s^\infty$

doesnot exist

(24) $L^{-1}\left[\int_s^\infty \frac{a}{a^2+s^2}\,ds\right]$

$= \frac{1}{t}\sin at$

(25) $L^{-1}\left[\int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\,ds\right]$

$= \frac{1}{t}\left[\cos at - \cos bt\right]$

(26) $L^{-1}\left[\int_s^\infty \log\left(\frac{s+2}{s+1}\right)ds\right]$

$F(s) = \log \frac{s+2}{s+1}$

$F'(s) = \frac{1}{s+2} - \frac{1}{s+1}$

$L^{-1}[F(s)] = \frac{-1}{t}\left[e^{-2t} - e^{-t}\right]$

$L^{-1}\left[\int_s^\infty \log\left(\frac{s+2}{s+1}\right)ds\right] = \frac{-1}{t^2}\left(e^{-2t} - e^{-t}\right)$

$= \frac{1}{t^2}\left(e^{-t} - e^{-2t}\right)$

PROPERTY 4

(27) $L[\sin^2 t]$ using $L[f'(t)]$

$f(t) = \sin^2 t \quad , f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$L[\sin 2t] = sL[f(t)] - f(0)$

$= sL[\sin^2 t] - 0$

$sL[\sin^2 t] = \dfrac{2}{s^2+4}$

$L[\sin^2 t] = \dfrac{2}{s(s^2+4)}$

(28) $f(t) = t^3$ using $L[f'''(t)]$

$f(t) = t^3 \qquad f(0) = 0$

$f'(t) = 3t^2 \qquad f'(0) = 0$

$f''(t) = 6t \qquad f''(0) = 0$

$f'''(t) = 6$

$L[f'''(t)] = s^3 L[t^3] - s^2 f(0) - sf'(0) - f''(0)$

$\dfrac{6}{s} = s^3 L[t^3] - 0$

$L[t^3] = 6/s^4$

(29) $L[\sin\sqrt{t}]$ and find $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!}$$

$$L[\sin\sqrt{t}] = L[t^{1/2}] - L\left[\frac{t^{3/2}}{3!}\right] + L\left[\frac{t^{5/2}}{5!}\right]$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}}{s^{7/2}}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2(4s)^2} - \cdots\right]$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}e^{-1/4s}$$

$f(t) = \sin\sqrt{t}$ $\qquad f(0) = 0$

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = 2\left[s\,L[\sin\sqrt{t}] - 0\right]$$

$$= 2\frac{\sqrt{\pi}}{s^{3/2}\cdot 2}e^{-1/4s}$$

$$= \frac{\sqrt{\pi}}{s^{3/2}}e^{-1/4s}$$

(30) $L[t\cos at] = ?$ $\quad L[\sin at] = a/a^2+s^2$

$f(t) = \cos at \qquad f(0) = 1$

$f'(t) = -\sin at$

$$L[\sin at] = sL[t\cos at] - 1$$

$$L[t\cos at] = \frac{a - s^2 - a^2}{(a^2 - s^2)s}$$

(31) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

$$L^{-1}\left[s \cdot \frac{s}{(s^2+a^2)s}\right] = \frac{d}{dt} L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{dt}\left(\frac{1}{2a} t\sin at\right)$$

$$= \frac{1}{2a}[\sin at + t\cos at]$$

32) $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 6$

$s^2 L[y(t)] - sy(0) - y'(0) + 4L[y(t)] = 0$

$(s^2+4) L[y(t)] - s - 6 = 0$

$L[y(t)] = \dfrac{s+6}{s^2+4}$

$y(t) = L^{-1}\left[\dfrac{s+6}{s^2+4}\right]$

$= \cos 2t + \dfrac{6\sin 2t}{2}$

$= \cos 2t + 3\sin 2t$

33) $y'' + 2y' - 3y = 3$, $y(0) = 4$, $y'(0) = -7$

$s^2 L[y(t)] - sy(0) - y'(0) + 2[sL[y(t)] - y(0)]$

$- 3L[y(t)] = L[3]$

$(s^2+2s-3) L[y(t)] = \dfrac{3}{s} + 4s - 7 + 8$

$= \dfrac{3}{s} + 4s + 1$

$L[y(t)] = \dfrac{4s^2+s+3}{s(s-1)(s-3)}$

$4s^2+s+3 = A(s-1)(s-3) + B(s)(s-3) + C(s)(s-1)$

Put $s=0$ | $3 = 3A \Rightarrow A = 1$

Put $s=1$ | $8 = -2B \Rightarrow B = -4$

Put $s=3$ | $C = 7$

$y(t) = 1 - 4e^t + 7e^{3t}$

34) $y'' - 5y' + 4y = e^{2t}, \; y(0) = \frac{19}{12}, \; y'(0) = \frac{8}{3}$

$$s^2 L[y(t)] - sy(0) - y'(0) - 5\left[sL[y(t)] - y(0)\right]$$
$$+4L[y(t)] = L[e^{2t}]$$

$$(s^2 - 5s + 4) L[y(t)] = \frac{1}{s-2} + \frac{19}{12}s - \frac{63}{12}$$

$$L[y(t)] = \frac{19s^2 - 101s + 138}{12(s-2)(s-1)(s-4)}$$

$$19s^2 - 101s + 138 = A(s-2)(s-4) + B(s-1)(s-4)$$
$$+ C(s-1)(s-2)$$

Put $s = 1$ | $A = 56/3$
Put $s = 2$ | $B = -6$
Put $s = 4$ | $C = 19/3$

$$y(t) = \frac{1}{12} L^{-1}\left[\frac{56}{3}\frac{1}{s-1} - 6\frac{1}{s-2} + \frac{19}{3}\frac{1}{s-4}\right]$$
$$= \frac{1}{12}\left[\frac{56}{3}e^{t} - 6e^{2t} + \frac{19}{3}e^{4t}\right]$$

PROPERTY 5

(35) $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$ (PT)

$$L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right]$$

$$= \frac{1}{s}\left[\int_s^\infty \frac{1}{s^2+1}\,ds\right]$$

$$= \frac{1}{s}\left[\tan^{-1}s\right]_s^\infty = \frac{1}{s}\cot^{-1}s$$

$$= \frac{1}{s}\tan^{-1}1/s$$

(36) $\int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt$

$$L\left[\int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt\right]$$

$$= \frac{1}{s^3} L[\cos at] = \frac{1}{s^3}\frac{s}{s^2+a^2} = \frac{1}{s^2(s^2+a^2)}$$

(37) $L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right]$

$s \to s+4$

$$\frac{1}{s} L\left[\frac{\sin 3t}{t}\right]$$

$$= \frac{1}{s}\left[\int_s^\infty \frac{3}{s^2+9}\,ds\right] = \frac{1}{s}\left[\frac{1}{3}\cdot 3\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right]$$

$$= \frac{1}{s}\cot^{-1}\frac{s}{3} = \frac{1}{s+4}\tan^{-1}\frac{3}{s+4}$$

(38) $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$

$$\frac{1}{3}\int_0^t \sin 3t\,dt = \frac{1}{3}\left[\frac{-\cos 3t}{3}\right]_0^t$$

$$= \frac{-1}{9}\left[\cos 3t - \cos 0\right] = \frac{1-\cos 3t}{9}$$

(39) $L^{-1}\left[\frac{1}{s^2(s^2+9)}\right]$

$$\int_0^t\int_0^t \frac{\sin 3t}{3}\,dt\,dt = \int_0^t \frac{1-\cos 3t}{9}\,dt$$

$$= \frac{t}{9} - \frac{1}{9}\left[\frac{\sin 3t}{3}\right]_0^t = \frac{1}{9}\left[t - \frac{\sin 3t}{3}\right]$$

(40) $\int_0^\infty te^{t}\sin t\,dt$

By definition of LT

$$\int_0^\infty e^{-st}\, t\sin t\,dt = L\left[t\sin t\right]$$

$$= \frac{-d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

Put $s=-1$

$$\int_0^\infty e^{t}\, t\sin t\,dt = \frac{-2}{(2)^2} = -1/2$$

41. $\int_0^\infty e^{-2t}\left[\frac{2\sin t - 3\sinh t}{t}\right]dt$

By definition of LT,

$$\int_0^\infty e^{-st}\left[\frac{2\sin t}{t} - \frac{3\sinh t}{t}\right]dt$$

$$= 2L\left[\frac{\sin t}{t}\right] - 3L\left[\frac{\sinh t}{t}\right] \quad \text{--- (1)}$$

$$= 2\left[\int_s^\infty \frac{1}{s^2+1}\,ds\right] - 3\left[\int_s^\infty \frac{1}{s^2-1}\,ds\right]$$

$$= 2\cot^{-1}s - 3\left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= 2\cot^{-1}s + \frac{3}{2}\log\left(\frac{s-1}{s+1}\right)$$

Put $s = 2$ in (1)

$$\int_0^\infty e^{-2t}\left[\frac{2\sin t - 3\sinh t}{t}\right] = 2\cot^{-1}2 + \frac{3}{2}\log 1/3$$

$$= 2\cot^{-1}2 - \frac{3}{2}\log 3$$

(42) $\displaystyle\int_0^{\infty} \frac{\cos 6t - \cos 4t}{t}\, dt$ Ans $\log 2/3$

$$\int_0^{\infty} e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = L\left[\frac{\cos 6t}{t}\right] - L\left[\frac{\cos 4t}{t}\right]$$

$$= \int_s^{\infty} \frac{s}{s^2+36}\, ds - \int_s^{\infty} \frac{s}{s^2+16}\, ds$$

$$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^{\infty}$$

$$= -\frac{1}{2}\log\frac{s^2+36}{s^2+16}$$

Put $s=0$ to above

$$= -\frac{1}{2}\log\frac{36}{16}$$

$$= -\frac{1}{2}\log 9/4 = \log 2/3$$

(43) $\int_0^\infty t^3 e^{-t} \sin t \, dt$ Ans 0

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = L[t^3 \sin t]$$

$$= \int_s^\infty \int_s^\infty \int_s^\infty \frac{1}{s^2+1} \, ds$$

$$= \int_s^\infty \int_s^\infty \cot^{-1} s \, ds$$

$$= \int_s^\infty \left[s\cot^{-1} s + \frac{\log(1+s^2)}{2} \right]_s^\infty ds$$

$$= 0$$

PERIODIC FUNCTION

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$T \rightarrow$ time period

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

BERNOULLI'S

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \ldots$$

$u' = d/dx \qquad v_1 = \int dx$

(44) $f(t) = \begin{cases} \cos t & 0 < t \le \pi \\ -1 & \pi \le t \le 2\pi \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \cos t \, dt + \int_{\pi}^{2\pi} -1 \cdot e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\left[\frac{e^{-st}}{s^2+1} [-s\cos t + \sin t] \right]_0^{\pi} - \left[\frac{e^{-st}}{-s} \right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\left[\frac{e^{-\pi s}}{s^2+1} (s+0) - \frac{1}{s^2+1} (-s+0) \right] + \frac{e^{-2\pi s}}{s} - \frac{e^{-\pi s}}{s} \right]$$

$$= \frac{1}{(1+e^{-\pi s})(1-e^{-\pi s})} \left[\frac{(e^{-\pi s}+1)s}{s^2+1} + \frac{e^{-\pi s}(e^{-\pi s}-1)}{s} \right]$$

$$= \frac{s}{(1-e^{-\pi s})(s^2+1)} - \frac{e^{-\pi s}}{s(1+e^{-\pi s})}$$

(45) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}} \left[\int_0^{2\pi} e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s\sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} (-\cos \pi) - \frac{1}{s^2+1} (-\cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2+1} - \frac{e^{-s\pi}}{s^2+1} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1-e^{-s\pi}}{s^2+1} \right]$$

46 $f(t) = \begin{cases} 1+t & 0 \le t < 1 \\ 3-t & 1 \le t < 2 \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}(1+t)\,dt + \int_1^2 e^{-st}(3-t)\,dt \right]$$

$u = 1+t$ | $v = e^{-st}$

$u' = 1$ | $v_1 = \frac{e^{-st}}{-s}$

$u'' = 0$ | $v_2 = \frac{e^{-st}}{s^2}$

$u = 3-t$ | $v = e^{-st}$

$u' = -1$ | $v_1 = e^{-st}/-s$

$v_2 = e^{-st}/s^2$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ \left[(1+t)\frac{e^{-st}}{-s} - 1\left(\frac{e^{-st}}{s^2}\right) \right]_0^1 + \left[(3-t)\frac{e^{-st}}{-s} + 1\left(\frac{e^{-st}}{s^2}\right) \right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{2e^{-s}}{s} + \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{e^{-2s}}{s} + \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1-e^{-2s}}{s} + \frac{1-2e^{-s}+e^{-2s}}{s} \right]$$

(47)

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ t-1 & 1 \le t \le 2 \end{cases}$$

$$\frac{y-0}{x-1} = \frac{1-0}{2-1}$$

$$y = x - 1$$
$$= t - 1$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_0^1 0 \cdot e^{-st}\,dt + \int_1^2 e^{-st}(t-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\int_1^2 e^{-st}(t-1)\,dt\right]$$

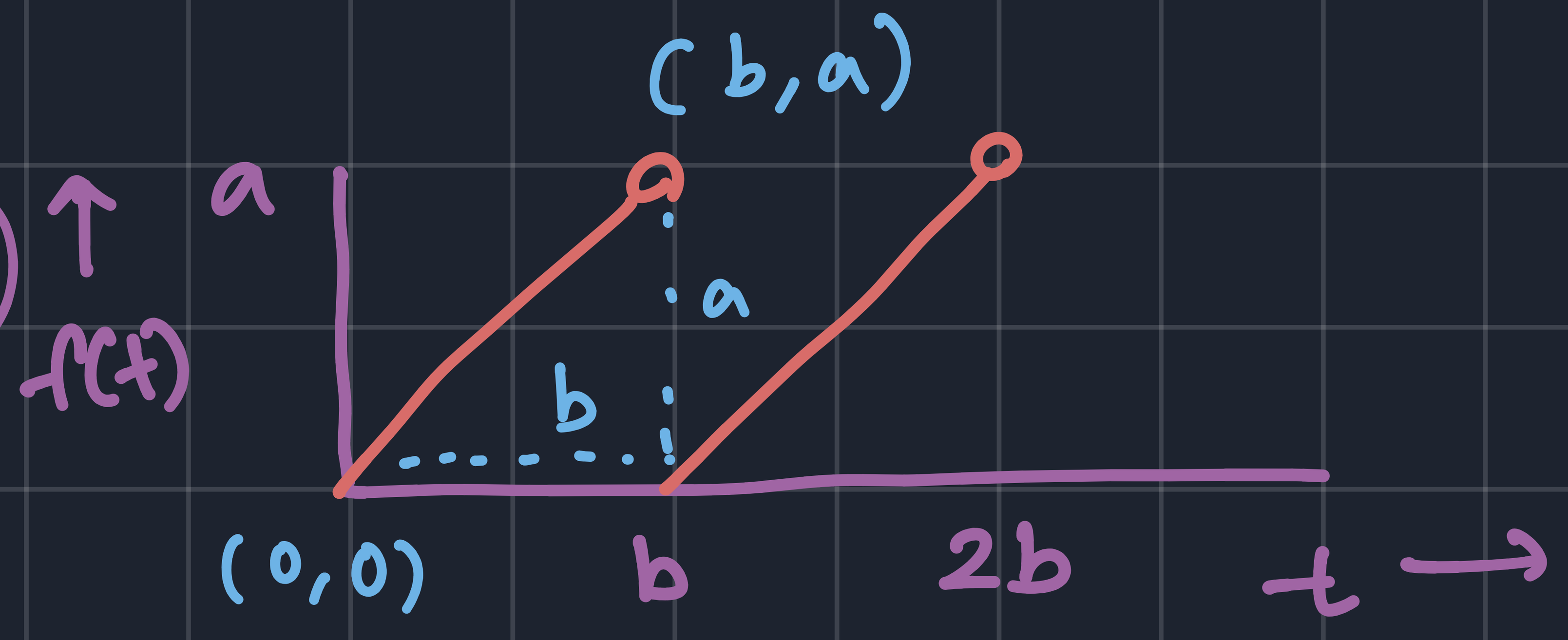
$$\begin{array}{l|l} u = t-1 & v = e^{-st} \\ u' = 1 & v_1 = e^{-st}/-s \\ u'' = 0 & v_2 = e^{-st}/s^2 \end{array}$$

$$= \frac{1}{1-e^{-2s}}\left[(t-1)\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{e^{-2s}}{s} + 0 - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-s} - e^{-2s} - se^{-2s}}{s^2}\right]$$

48



$$\frac{y-0}{a-0} = \frac{x-0}{b-0}$$

$$y = \frac{a}{b} x$$

$$f(t) = \frac{a}{b} t$$

f(t) is a periodic function with time period 'b'

$$L[f(t)] = \frac{1}{1-e^{-bs}} \int_0^b e^{-st} \frac{a}{b} \cdot t \, dt$$

$$= \frac{a}{b(1-e^{-bs})} \int_a^b e^{-st} \cdot t \, dt$$

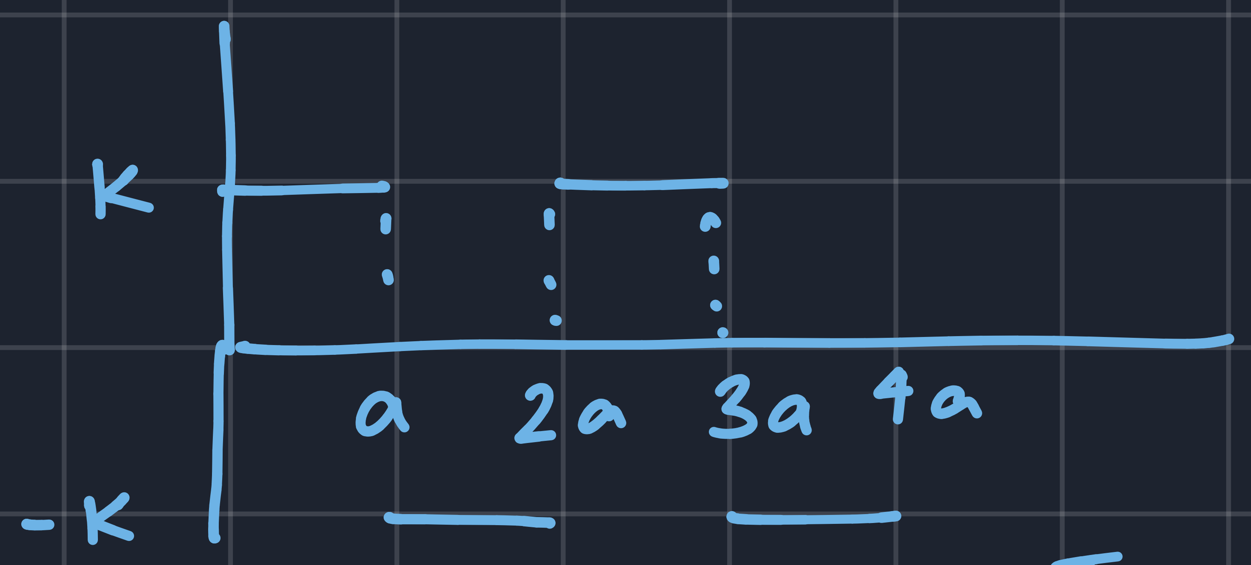
$$= \frac{a}{b(1-e^{-bs})} \left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right) \right]_a^b$$

$$= \frac{a}{b(1-e^{-bs})} \left[\frac{-b\,e^{-sb}}{s} + \frac{a\,e^{-sa}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^{-sa}}{s^2} \right]$$

(49) ST LT of Sq wave fn of $T = 2a$

$$f(t) = \begin{cases} k & 0 < t \le a \\ -k & a < t \le 2a \end{cases} \quad \text{is } \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

(Graph)



$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Here $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot k \, dt + \int_a^{2a} e^{-st} (-k)\, dt \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\frac{1}{s} - \frac{e^{-as}}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\frac{e^{-2as} - 2e^{-as} + 1}{s} \right]$$

$$= \frac{k}{s} \left[\frac{(e^{-as}-1)^2}{(1-e^{-as})(1+e^{-as})} \right]$$

$$= \frac{k}{s} \left[\frac{1-e^{-as}}{1+e^{-as}} \right]$$

$$= \frac{k}{s} \left[\frac{1-e^{-as}}{1+e^{-as}} \right] \cdot \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{k}{s} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right]$$

$$= \frac{k}{s} \tanh\left(\frac{as}{2}\right)$$

(50) $f(t) = \begin{cases} 1 & 0 \le t \le T/2 \\ 0 & T/2 \le t < T \end{cases}$ $f(t+T) = f(t)$

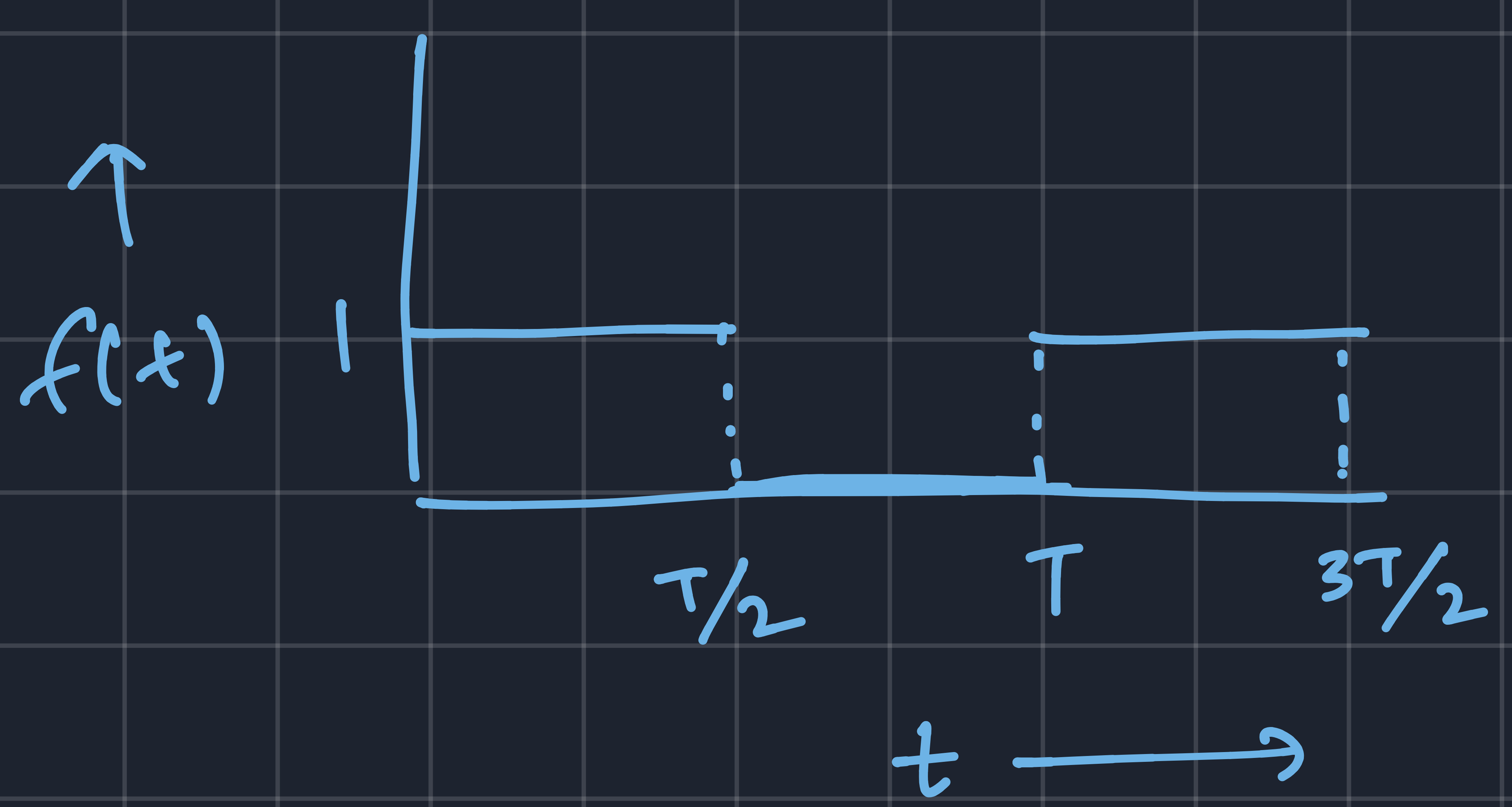
$$L[f(t)] = \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} e^{-st}\,dt + \int_{T/2}^{T} e^{-st}(0)\,dt\right]$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{e^{-st}}{-s}\right]_0^{T/2}$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{1}{s} - \frac{e^{-sT/2}}{s}\right]$$

$$= \frac{1-e^{-sT/2}}{s} \cdot \frac{1}{(1-e^{-sT/2})(1+e^{-sT/2})}$$

$$= \frac{1}{s(1+e^{-sT/2})} \quad (\text{or}) \quad \frac{1-e^{-sT/2}}{s(1-e^{-sT})}$$



51 $f(t) = t/3$, $0 < t < 3$, LT of periodic fn

(Extra)

52 $f(t) = \begin{cases} \sin \omega t & 0 < t \le \pi/\omega \\ 0 & \frac{\pi}{\omega} < t < 2\pi/\omega \end{cases}$ $T = \frac{2\pi}{\omega}$

(graph)

(Extra)

CONVOLUTION THEOREM

$$f*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}[F(s)\cdot G(s)] = f(t) * g(t)$$

53 $L^{-1}\left[\dfrac{1}{(s+1)(s^2+1)}\right]$

$$F(s) = \frac{1}{s^2+1} \qquad G(s) = 1/s+1$$

$$L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s+1}\right] = \sin t * e^{-t}$$

$$= \int_0^t \sin u \cdot e^{-(t-u)}\,du$$

$$= e^{-t}\int_0^t \sin u\; e^{u}\,du$$

$$= e^{-t}\left[\frac{e^{u}}{2}[\sin u - \cos u]\right]_0^t$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}(-1)\right]$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) + 1/2\right]$$

(54) $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

$$F(s) = \frac{s}{s^2+1} \qquad G(s) = \frac{1}{s^2+1}$$

$$L^{-1}\left[\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\right] = \cos t * \sin t$$

$$= \int_0^t \cos u \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin(t-u+u) + \sin(t-u-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin(t) + \sin(t-2u)\, du$$

$$= \frac{1}{2}\left[u \sin t - \frac{\cos(t-2u)}{-2}\right]_0^t$$

$$= \frac{1}{2}\left[t \sin t + \left(\frac{\cos(t-2u)}{2}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t \sin t - \frac{\cos t}{2} - \frac{\cos t}{2}\right]$$

(55) $L^{-1}\left[\dfrac{1}{(s+2)^2(s-2)}\right]$

$$L^{-1}\left[\frac{1}{(s^2-4)(s+2)}\right]$$

$$F(s) = \frac{1}{s^2-4} \qquad G(s) = \frac{1}{s+2}$$

$$L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right] = \frac{\sinh 2t}{2} * e^{-2t}$$

$$= \frac{1}{2}\int_0^t \frac{e^{2u}-e^{-2u}}{2} \cdot e^{-2(t-u)}\,du$$

$$= \frac{e^{-2t}}{4}\int_0^t e^{4u}-1\,du$$

$$= \frac{e^{-2t}}{4}\left[\frac{e^{4u}}{4}-u\right]_0^t$$

$$= \frac{e^{-2t}}{4}\left[\frac{e^{4t}}{4}-\frac{1}{4}-t\right]$$

(56) $L^{-1}\left[\dfrac{1}{s-a}\cdot\dfrac{1}{s-b}\right]$

$$F(s) = \frac{1}{s-a} \qquad G(s) = \frac{1}{s-b}$$

$$f(t) = e^{at} \qquad g(t) = e^{bt}$$

$$L^{-1}\left[\frac{1}{s-a}\cdot\frac{1}{s-b}\right] = e^{at} * e^{bt}$$

$$= \int_0^t e^{au}\cdot e^{b(t-u)}\,du$$

$$= e^{bt}\int_0^t e^{(a-b)u}\,du$$

$$= e^{bt}\left[\frac{e^{(a-b)t}-1}{a-b}\right]$$

$$= \frac{e^{at}-e^{bt}}{a-b}$$

57 $L^{-1}\left[\dfrac{1}{(s^2+4)(s+1)^2}\right]$ (Refer NB)

(58) $L^{-1}\left[\dfrac{s^2}{(s^2+4)(s+1)^2}\right]$

(Doubt)

(59) $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

$$F(s) = \frac{1}{s^2+a^2} \qquad G(s) = \frac{1}{s^2}$$

$$f(t) = \frac{\sin at}{a} \qquad g(t) = t$$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{\sin at}{a} * t$$

$$= \frac{1}{a}\int_0^t \sin au \,.\, (t-u)\,du$$

$$= \frac{1}{a}\left[-(t-u)\frac{\cos au}{a} - \frac{\sin au}{a^2}\right]_0^t$$

$$= \frac{1}{a}\left[t \,.\, \frac{\cos a(0)}{a} - \frac{\sin at}{a^2} + \frac{\sin a(0)}{a^2}\right]$$

$$= \frac{1}{a}\left[\frac{t}{a} - \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{a^3}\left[at - \sin at\right]$$

60 $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

$$F(s) = \frac{s}{s^2+a^2} \qquad G(s) = \frac{s}{s^2+b^2}$$

$$f(t) = \cos at \qquad g(t) = \cos bt$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \cos at * \cos bt$$

$$= \int_0^t \cos au \, \cos b(t-u) \, du$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu) \, . \, du$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \frac{\sin((a+b)-bt)}{a-b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2(a^2-b^2)}\left[a\sin at + b\sin at - a\sin bt - b\sin bt + a\sin at - b\sin at + a\sin bt - b\sin bt\right]$$

$$= \frac{1}{2(a^2-b^2)}\left[2a\sin at - 2b\sin bt\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

(61) $L^{-1}\left[\dfrac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$

Doubt

UNIT STEP FUNC

$$H(t-a) = u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Second Shifting Theorem

$$L[f(t-a)u(t-a)] = e^{-as} L[f(t)]$$

$$L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$$

---

if $\quad f(t) = \begin{cases} f_1(t) & t < a \\ f_2(t) & t > a \end{cases}$

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a)$$

if $\quad f(t) = \begin{cases} f_1(t) & t \leq a \\ f_2(t) & a < t \leq b \\ f_3(t) & t > b \end{cases}$

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a) + \{f_3(t) - f_2(t)\} u(t-b)$$

$$L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

(62) $f(t) = \begin{cases} t^2 & 0<t<2 \\ 4t & t>2 \end{cases}$ (USF)

$$f(t) = t^2 + (4t - t^2)\,u(t-2)$$

$$L[f(t)] = L[t^2] + L[(4t-t^2)\,u(t-2)]$$

$$= \frac{2}{s^3} + e^{-2s} L[4(t+2) - (t+2)^2]$$

$$= \frac{2}{s^3} + e^{-2s} L[\cancel{4t} + 8 - t^2 - 4 - \cancel{4t}]$$

$$= \frac{2}{s^3} + e^{-2s} L[4 - t^2]$$

$$= \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

(63) $f(t) = \begin{cases} \frac{k}{a}t & 0<t<a \\ \frac{k}{a}(t-a) & a<t<2a \\ \frac{k}{a}(t-2a) & t>2a \end{cases}$ (USF)

$$f(t) = \frac{k}{a}t + \left\{\frac{k}{a}(t-a) - \frac{k}{a}t\right\}u(t-a)$$
$$+ \left\{\frac{k}{a}(t-2a) - \frac{k}{a}(t-a)\right\}u(t-2a)$$

$$= \frac{k}{a}t + \left\{\frac{-k}{a}(a)\right\}u(t-a) + \left\{\frac{k}{a}(-a)\right\}u(t-2a)$$

$$= \frac{k}{a}t - k\,u(t-a) - k\,u(t-2a)$$

$$L[f(t)] = \frac{k}{a}\frac{1}{s^2} - \frac{ke^{-as}}{s} - \frac{ke^{-2as}}{s}$$

(64) $f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$

$$f(t) = 0 + \{t-1-0\}u(t-1)$$
$$+ \{3-t-t+1\}u(t-2)$$
$$+ \{0-3+t\}u(t-3)$$
$$= (t-1)u(t-1) + (4-2t)u(t-2)$$
$$+ (t-3)u(t-3)$$
$$= (t-1)u(t-1) - 2(t-2)u(t-2)$$
$$+ (t-3)u(t-3)$$

$$L[f(t)] = L[(t-1)u(t-1)] - 2L[(t-2)u(t-2)]$$
$$+ L[(t-3)u(t-3)]$$
$$= \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

65 $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

$$f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$$

$$L[f(t)] = L[\cos t] + L[(\sin t - \cos t)\, u(t-\pi)]$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L[\sin(t+\pi) - \cos(t+\pi)]$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L[-\sin t + \cos t]$$

$$= \frac{s}{s^2+1} + e^{-\pi s} \left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

(66) $L^{-1}\left[\frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= (t-1)^2 u(t-1) - 3u(t-2)$

(67) $L^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}\right]$

$= \sin(t-\pi)\,u(t-\pi) + \cos 2(t-2\pi)\,u(t-2\pi)$

$= -\sin t\,u(t-\pi) + \cos 2t\,u(t-2\pi)$

(68) $L^{-1}\left[\frac{se^{-4s}}{s^2-5s+6}\right]$

$L^{-1}\left[e^{-4s}\frac{s}{(s-2)(s-3)}\right]$

$= L^{-1}\left[e^{-4s}\frac{-2}{s-2} + \frac{3}{s-3}\right]$

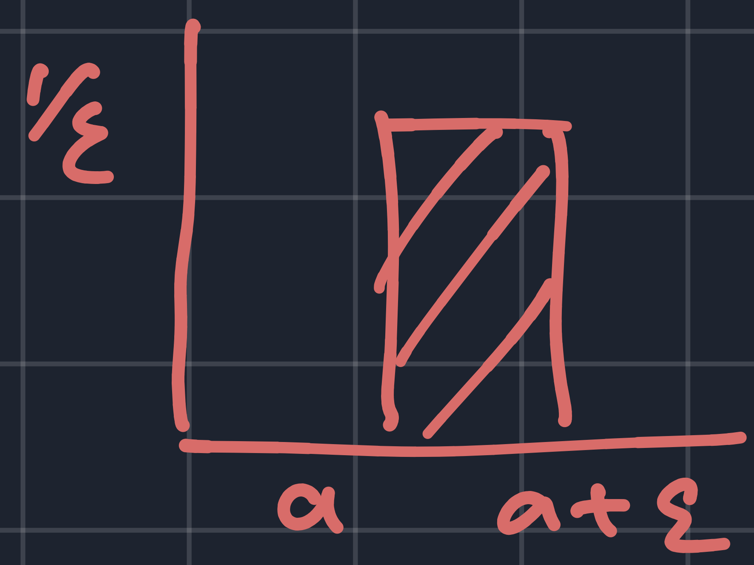
$= -2e^{2(t-4)}u(t-4) + 3e^{3(t-4)}u(t-4)$

$= \left[3e^{3(t-4)} - 2e^{2(t-4)}\right]u(t-4)$

UNIT-IMPULSE / DIRAC

$$\delta(t-a) = \lim_{\varepsilon \to 0} f_\varepsilon (t-a)$$

$$f_\varepsilon (t-a) = \begin{cases} \frac{1}{\varepsilon} & \text{for } a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



$$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Prove that $L[\delta(t-a)] = e^{-as}$

$$L[f_\varepsilon(t-a)] = \int_0^\infty e^{-st} f_\epsilon (t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^{a+\epsilon} e^{-st} \cdot \frac{1}{\epsilon}\,dt + \int_{a+\epsilon}^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \int_a^{a+\epsilon} e^{-st} \frac{1}{\epsilon}\,dt = \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon}$$

$$= \frac{1}{\epsilon s}\left[-e^{-s(a+\epsilon)} + e^{-as}\right]$$

$$= \frac{1}{\epsilon s}\left[e^{-as}(1-e^{-\epsilon s})\right]$$

L'hospital

$$\lim_{\epsilon \to 0} L[f_\epsilon(t-a)] = \lim_{\epsilon \to 0} \frac{e^{-as}(0 - e^{-\epsilon s}(-s))}{s}$$

$$= \lim_{\epsilon \to 0} \frac{e^{-\epsilon s} \cdot e^{-as} \cdot s}{s}$$

$$= e^{-as}$$

(69) $L\left[\frac{1}{t}\delta(t-a)\right] = e^{-as}\frac{1}{a}$

(70) $L[\cosh 3t\,\delta(t-2)] = e^{-2s}\cosh 6$

(71) $L[t e^{-2t}\delta(t-2)]$

$= e^{-2s}(2)e^{-4} = 2e^{-2(s-2)}$

(72) $L\left[\frac{2\delta(t-3)+3t\,\delta(t-2)}{t}\right]$

$e^{-3s}\left(\frac{2}{3}\right) + 3e^{-2s}$

(73) $L[t\,u(t-1) + t^2\delta(t-1)]$

$e^{-s}L[t+1] + e^{-s}(1)^2$

$= e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) + e^{-s}$

74) $$\frac{d^2 i}{dt^2} + 2\frac{di}{dt} = \begin{cases} 0 & 0 < t < 10 \\ 1 & 10 < t < 20 \\ 0 & t > 20 \end{cases}$$

$$i(0) = 0, \quad i'(0) = 0$$

$$\frac{d^2 i}{dt^2} + 2\frac{di}{dt} = 0 + (1-0)\,u(t-10) + (0-1)\,u(t-20)$$

$$i''(t) + 2i'(t) = u(t-10) - u(t-20)$$

$$L[i''(t)] + 2L[i'(t)] = L[u(t-10)] - L[u(t-20)]$$

$$s^2 L[i(t)] - s\,i(0) - i'(0) + 2[sL[i(t)] - i(0)] = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$(s^2+2s)\,L[i(t)] = \frac{e^{-10s} - e^{-20s}}{s}$$

$$L[i(t)] = \frac{e^{-10s} - e^{-20s}}{s(s^2+2s)} = \frac{e^{-10s} - e^{-20s}}{s^2(s+2)}$$

$$i(t) = L^{-1}\left[e^{-10s} \cdot \frac{1}{s^2(s+2)}\right] - L^{-1}\left[e^{-20s} \cdot \frac{1}{s^2(s+2)}\right]$$

$$L^{-1}\left[\frac{1}{s^2(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{(s+2)}}{s^2}\right] = \int_0^t \int_0^t e^{-2t}\,dt\,dt$$

$$= \int_0^t \left(\frac{e^{-2t}}{-2}\right)_0^t dt = \frac{-1}{2}\int_0^t e^{-2t} - 1\,dt$$

$$= -\frac{1}{2}\left[\frac{e^{-2t}}{-2} - t\right]_0^t$$

$$= \frac{e^{-2t}}{4} + \frac{t}{2} - \frac{1}{4}$$

$$i(t) = \left(\frac{e^{-2(t-10)}}{4} + \frac{(t-10)}{2} - \frac{1}{4}\right) u(t-10)$$

(75) $y'' + 2y' + 5y = \delta(t-2)$, $y(0) = 0$, $y'(0) = 0$

$$s^2 L[y(t)] - s\,y(0) - y'(0)$$
$$+ 2[s L[y(t)] - y(0)] + 5 L[y(t)]$$
$$= L[\delta(t-2)]$$

$$(s^2 + 2s + 5) L[y(t)] = e^{-2s}$$

$$y(t) = L^{-1}\left[e^{-2s} \frac{1}{s^2+2s+5}\right]$$

$$L^{-1}\left[\frac{1}{s^2+2s+5}\right] = L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$
$$= e^{-t} \frac{\sin 2t}{2}$$

$$y(t) = e^{-(t-2)} \frac{\sin 2(t-2)}{2} u(t-2)$$

76 $$\frac{di}{dt} + 2i + 5\int_0^t i\,dt = u(t) \quad , \quad i(0) = 0$$

$$s\,L[i(t)] - i(0) + 2\,L[i(t)] + 5\,L\left[\int_0^t i\,dt\right] = L[u(t)]$$

$$(s+2)\,L[i(t)] + \frac{5}{s}\,L[i(t)] = \frac{1}{s}$$

$$(s^2+2s+5)\,L[i(t)] = 1$$

$$L[i(t)] = 1/s^2+2s+5$$

$$i(t) = L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{e^{-t}\sin 2t}{2}$$