FOURIER SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

even → cos
odd → sin

1a) Even function in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

1b) Odd function in $(0, 2\pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$$

1c) Even function in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

1d) Odd function in $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$$

NOTE:

① If $f(x) = \begin{cases} \phi(x) & \text{in } -\pi < x < 0 \\ \psi(x) & \text{in } 0 < x < \pi \end{cases}$

$f(x) \rightarrow$ even if $\phi(-x) = \psi(x)$
$f(x) \rightarrow$ odd if $\phi(-x) = -\psi(x)$

② Discontinuity at x_0

$$f(x_0) = \frac{1}{2}\left[f(x_0-0) + f(x_0+0)\right]$$

③ At end point $f(x)$ converges to

$$\frac{1}{2}\left[f(c) + f(c+2\pi)\right]$$

Things to remember

1. $\cos n\pi = (-1)^n$
   $\sin n\pi = 0$
   $e^{-\infty} = 0$

2. $\sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \rightarrow \text{even} \\ 1 & n = 1, 5, 9 \\ -1 & n = 3, 7, 11 \end{cases}$

   $\cos \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & n = 0, 4, 8 \\ -1 & n = 2, 6, 10 \end{cases}$

3. $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

   $$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

1. Obtain the FS of the periodic fn $f(x) = e^x$ in $-\pi < x < \pi$, $f(x+2\pi) = f(x)$. Hence find the sum of series : $\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \dots + \frac{(-1)^n}{1+n^2}$

$f(x)$ is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—} ①$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{1}{\pi} \left[e^x \right]_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$= \frac{2 \sinh \pi}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx = \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{\pi}}{1+n^2} \left[(-1)^n + 0 \right] - \frac{e^{-\pi}}{1+n^2} \left[(-1)^n + 0 \right] \right]$$

$$= \frac{(-1)^n}{\pi(1+n^2)} \cdot 2 \sinh \pi$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx = \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\sin nx - n \cos nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{\pi}}{1+n^2} (0 - n(-1)^n) - \frac{e^{-\pi}}{1+n^2} (0 - n(-1)^n) \right]$$

$$= \frac{n(-1)^n}{\pi(1+n^2)} \cdot (-2 \sinh \pi)$$

① becomes,

$$f(x) = \frac{1}{\pi} \sinh \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi \cos nx - \frac{2n(-1)^n}{\pi(1+n^2)} \sinh \pi \sin nx \right]$$

$$\Rightarrow e^x = \frac{1}{\pi} \sinh \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi \cos nx - \frac{2n(-1)^n}{\pi(1+n^2)} \sinh \pi \sin nx \right]$$

Deduction

Put $x=0$

$$1 = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{\pi(1+n^2)}\sinh\pi\right]$$

$$1 = \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right]$$

$$1 = \frac{\sinh\pi}{\pi}\left[1 + 2\left(\frac{-1}{2}+\frac{1}{5}-\frac{1}{10}+\cdots\right)\right]$$

$$\frac{\pi}{\sinh\pi} = \left[1-1+2\left(\frac{1}{5}-\frac{1}{10}+\cdots\right)\right]$$

$$\frac{\pi}{2\sinh\pi} = \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \cdots$$

2.

$f(x) = x - x^2$ from $-\pi$ to π. Hence deduce that (i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

Answer: $\frac{a_0}{2} = \frac{-\pi^2}{3}; a_n = \frac{-4(-1)^n}{n^2}; b_n = \frac{-2(-1)^n}{n}$

Also deduce (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$f(x)$ is neither odd nor even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) \quad — (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^3}{3}\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(-\frac{\cos nx}{n^2}\right) - 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[(1-2x)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - (1+2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{\pi n^2}(-4\pi) = -\frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-\pi^2)\frac{(-1)^n}{n} + (-\pi-\pi^2)\frac{(-1)^n}{n} - \frac{2(-1)^n}{n^3} + \frac{2(-1)^n}{n^3}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[-\pi+\pi^2-\pi-\pi^2\right] = -\frac{2(-1)^n}{n}$$

(1) becomes

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[-\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right] \quad \text{--- (2)}$$

Deduction

(i) Put $x = 0$

$$0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(-\frac{4(-1)^n}{n^2}\right)$$

$$\frac{\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = -\left(\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{—} (3)$$

(ii) Put $x = \pi$ and since its the end point $f(x)$ converges to

$$f(x) = \frac{1}{2}\left[f(-\pi) + f(\pi)\right]$$

$$= \frac{1}{2}\left[-\pi - \pi^2 + \pi - \pi^2\right] = -\pi^2$$

(2) becomes

$$-\pi^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[-\frac{4(-1)^n}{n^2}(-1)^n\right]$$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{—} (4)$$

Adding (3) & (4).

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

3. $f(x) = x + x^2$ from $-\pi$ to π. Hence deduce that (i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

Also deduce (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

(similar to previous)

4.

$$f(x) = \begin{cases} -\pi & \text{for } -\pi < x < 0 \\ x & \text{for } 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Answer: $\frac{a_0}{2} = \frac{-\pi}{4}; a_n = \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right); b_n = \frac{1}{n}\left(1 - 2(-1)^n\right)$

$f(x)$ is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \quad \text{— (1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\pi/2$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_0^{\pi} x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_0^{\pi} x \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] - \frac{\pi(-1)^n}{n}\right]$$

$$= \frac{1 - 2(-1)^n}{n}$$

(1) becomes

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{1 - 2(-1)^n}{n}\sin nx\right]$$

Deduction

Put $x = 0$ but $f(x)$ is discontinuous at $x = 0$ so

$f(x)$ converges to $f(x) = \frac{1}{2}\left[f(0^+) + f(0^-)\right]$

$$f(x) = \frac{1}{2}\left[0 - \pi\right] = -\pi/2$$

$$\frac{-\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\right]$$

$$\pi\left(\frac{-\pi}{2} + \frac{\pi}{4}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}$$

$$\frac{-\pi^2}{4} = \frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 \cdots$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

5.

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Answer: $\frac{a_0}{2} = \frac{\pi}{2}$; $a_n = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$; $b_n = 0$

$$\phi(x) = x \qquad \psi(x) = 2\pi - x$$

$\phi(2\pi - x) = 2\pi - x = \psi(x) \Rightarrow$ func is even

$$\therefore \boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{— (1)}$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi\left[2\pi - \pi\right] - \frac{4\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{2} + 2\pi - 2\pi + \pi/2 = \pi$$

(since even fn)

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2((-1)^n - 1)}{\pi n^2}$$

(1) becomes

$$f(x) = \pi/2 + \sum_{n=1}^{\infty}\left[\frac{2((-1)^n-1)}{\pi n^2}\cos nx\right]$$

Deduction

Put $x = 0$

$$0 = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{\pi n^2}$$

$$\frac{-\pi^2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}$$

$$\frac{-\pi^2}{4} = \frac{-2}{1^2} + 0 - \frac{2}{3^2} + \dots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

6.

$f(x) = x^2$ in $(-\pi, \pi)$. Hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$
Answer: $\frac{a_0}{2} = \frac{\pi^2}{3}; a_n = \frac{4(-1)^n}{n^2}; b_n = 0$

$f(x)$ is an even function

$$\therefore \boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—(1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

① becomes

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

Deduction

Put $x = 0$

$$-\frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

7.

$f(x) = x\cos x$ in $(-\pi, \pi)$.

Answer: $a_0 = 0; a_n = 0; b_n = \frac{2n(-1)^n}{n^2-1}$ for $n \neq 1$; $b_1 = -\frac{1}{2}$

$f(x)$ is an odd function

$$\therefore a_0 = a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} x\left(\frac{\sin(n+1)x + \sin(n-1)x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} x\sin(n-1)x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi}\right.$$

$$+\left[x\left(-\frac{\cos(n-1)x}{n-1}\right)-\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{-\pi(-1)^{n+1}}{n+1}+\frac{(-\pi)(-1)^{n-1}}{n-1}\right]$$

$$=-\left[\frac{(-1)^n(-1)}{n+1}+\frac{(-1)^n}{(-1)(n-1)}\right]$$

$$=(-1)^n\left[\frac{1}{n+1}-\frac{1}{n-1}\right]=\frac{2n(-1)^n}{n^2-1}$$

for $n \geq 2$

For $n=1$

$$b_1=\frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx=\frac{2}{\pi}\int_0^{\pi} x\,\frac{\sin 2x}{2}\,dx$$

$$=\frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{2}\right]=-1/2$$

(1) becomes

$$f(x)=-\tfrac{1}{2}\sin x+\sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

8.

Home work $f(x)=|cos x|$ in $(-\pi,\pi)$.
Answer: $\frac{a_0}{2}=\frac{2}{\pi}$; $a_n=\frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}$ for $n\neq 1$; $a_1=0$; $b_n=0$

$f(x)$ is even function

$\therefore\ b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad \text{—— (1)}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}|\cos x|\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}-\cos x\,dx\right]$$

$$=\frac{2}{\pi}\left[1-0-(0-1)\right]=4/\pi$$

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\,dx-\int_{\pi/2}^{\pi}\cos x\cos nx\,dx\right]$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\cos(n+1)x + \cos(n-1)x}{2}\,dx - \int_{\pi/2}^{0} \frac{\cos(n+1)x + \cos(n-1)x}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - \left[-\frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2\cos n\pi/2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= -\frac{4\cos(n\pi/2)}{\pi(n^2-1)} \qquad \text{for } n \geq 2$$

For $n=1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\pi/2 - \pi/2\right] = 0$$

(1) becomes

$$f(x) = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} -\frac{4\cos(n\pi/2)}{\pi(n^2-1)}\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)\cos nx}{n^2-1}$$

9. Find F.S for $f(x) = \begin{cases} x & 0 \le x \le \pi \\ x - 2\pi & \pi \le x \le 2\pi \end{cases}$

Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

(Extra Qs)

FOURIER SERIES EXPANSION OF $f(x)$ OVER AN ARBITARY INTERVAL $(0, 2l)$ & $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

INTERVAL $(0, 2l)$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

INTERVAL $(-l, l)$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

10.

Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.
Answer: $\frac{a_0}{2} = \frac{sinhl}{l}$; $a_n = \frac{2l(-1)^n sinhl}{l^2+n^2\pi^2}$; $b_n = \frac{2n\pi(-1)^n sinhl}{l^2+n^2\pi^2}$

f(x) is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—} (1)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l} = \frac{e^{l}-e^{-l}}{l}$$

$$= \frac{2\sinh l}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{l^2}}\left[-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{-e^{-l}l^2}{l^2+n^2\pi^2}(-1)^n + \frac{e^{l}l^2}{l^2+n^2\pi^2}(-1)^n\right]$$

$$= \frac{2l(-1)^n\sinh l}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{l^2}}\left[-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right]\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{l^2e^{-l}}{l^2+n^2\pi^2}\left(\frac{-n\pi}{l}\right)(-1)^n - \left(\frac{l^2e^{l}}{l^2+n^2\pi^2}\left(\frac{-n\pi}{l}\right)(-1)^n\right)\right]$$

$$= \frac{n\pi}{l^2+n^2\pi^2}(-1)^n \cdot 2\sinh l$$

(1) becomes

$$f(x) = \frac{\sinh l}{l} + \sum_{n=1}^{\infty}\left[\frac{2l(-1)^n\sinh l}{l^2+n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{2n\pi(-1)^n\sinh l}{l^2+n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]$$

11.

$$f(x) = \begin{cases} \pi x & \text{for } 0 < x < 1 \\ \pi(2-x) & \text{for } 1 < x < 2 \end{cases}$$

Answer: $\frac{a_0}{2} = \frac{\pi}{2}$; $a_n = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)$; $b_n = 0$

$\phi(x) = \pi x \quad , \quad \psi(x) = \pi(2-x)$ $\qquad (0, 2l) \longrightarrow (0, 2)$

$\boxed{l = 1}$

$\phi(2-x) = \pi(2-x) = \psi(x)$

$\therefore$ f(x) is an even fn

$\boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l \pi x \, dx = 2\int_0^1 \pi x \, dx$$

$$= 2\frac{\pi}{2} = \pi$$

$$a_n = 2\int_0^1 \pi x \cos(n\pi x) \, dx$$

$$= 2\pi\left[x\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}\left((-1)^n - 1\right)$$

(1) becomes

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2\pi^2}\left((-1)^n - 1\right)\cos n\pi x$$

12.

Home work problem: $f(x) = x^2$ in $(-l, l)$. Answer: $\frac{a_0}{2} = \frac{l^2}{3}; a_n = \frac{4l^2(-1)^n}{n^2\pi^2}; b_n = 0$

$f(x)$ is an even fn

$$\therefore b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{--- (i)}$$

$$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\cdot\frac{l^3}{3} = \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\Bigg[x^2\left(\sin\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right) - 2x\left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l^2}{n^2\pi^2}\right) + 2\left(-\sin\frac{n\pi x}{l}\right)\left(\frac{l^3}{n^3\pi^3}\right)\Bigg]_0^l$$

$$= \frac{2}{l}\left[2l(-1)^n\frac{l^2}{n^2\pi^2}\right] = \frac{4l^2(-1)^n}{n^2\pi^2}$$

(i) becomes

$$f(x) \quad \frac{l^2}{3} + \sum_{n=-1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

13. Expand $f(x) = 2x - x^2$ as a F.S. in the interval $(0,3)$

(extra Q.S.)

HALF RANGE FOURIER SERIES

HALF RANGE COSINE SERIES (EVEN)

INTERVAL $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

INTERVAL $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l} \int_0^{l} f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

HALF RANGE

INTERVAL $(0, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

INTERVAL $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l} \int_0^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

14. Obtain the Half-range sine and cosine series for the function $f(x) = x^2$ in $(0, \pi)$

HRCS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

HRCS is given by $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2(-1)^n}{n^3} - \frac{\pi^2(-1)^n}{n} - \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}\left((-1)^n - 1\right) - \frac{\pi^2}{n}(-1)^n\right]$$

HRSS is given by $f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{2}{n^3}\left((-1)^n - 1\right) - \frac{\pi^2}{n}(-1)^n\right] \sin nx$

15.

Find the half-range Fourier sine and cosine series of $f(x) = \begin{cases} x & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$

Answer: $\frac{a_0}{2} = \frac{\pi}{4}$; $a_n = \frac{2}{\pi n^2}\left(2cos\frac{n\pi}{2} - 1 - (-1)^n\right)$; $b_n = \frac{4}{\pi n^2} sin\frac{n\pi}{2}$

and the S.T. $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$

HRCS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} \pi - x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi\left[\pi - \frac{\pi}{2}\right] - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \pi/2\,\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos n\pi/2 - (-1)^n - 1\right]$$

HRCS

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[2\cos n\pi/2 - (-1)^n - 1\right] \cdot \cos nx$$

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) + \left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\pi/2 \frac{\cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} + \pi/2 \frac{\cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2 \sin n\pi/2}{n^2}\right] = \frac{4}{\pi n^2} \sin n\pi/2$$

HRSS is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin n\pi/2 \, \sin nx$$

$$= \frac{4}{\pi}\left[\frac{1}{1^2} \sin \pi/2 \sin x + \frac{1}{2^2} \sin \pi \sin 2x \; \cdots\right]$$

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \cdots\right]$$

16.

Find the half-range Fourier sine series of $f(x) = \begin{cases} \frac{1}{4} - x & \text{for } 0 \le x \le \frac{1}{2} \\ x - \frac{3}{4} & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$

Answer: $b_n = \frac{1}{2n\pi}(1-(-1)^n) - \frac{4}{n^2\pi^2} sin \frac{n\pi}{2}$

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (0,l) \to (0,1) \quad \boxed{l=1}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin(n\pi x)\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin(n\pi x)\,dx\right]$$

$$= 2\left[\left[\left(\frac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2} + \left[\left(x-\frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right]$$

$$= 2\left[\frac{1}{4n\pi} - \frac{(-1)^n}{4n\pi} - \frac{2\sin(n\pi/2)}{n^2\pi^2}\right]$$

HRSS

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4\sin n\pi/2}{n^2\pi^2}\right]\sin n\pi x$$

17.

Home work Find the half-range Fourier cosine series of $f(x) = \begin{cases} kx & \text{for } 0 \le x \le \frac{l}{2} \\ k(l-x) & \text{for } \frac{l}{2} \le x \le l \end{cases}$

Answer: $\frac{a_0}{2} = \frac{kl}{4}; a_n = \frac{2kl}{\pi^2 n^2}\left(2cos\frac{n\pi}{2} - 1 - (-1)^n\right)$

HRCS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\cdot 2\int_0^{l/2} kx\,dx = \frac{4}{l}k\frac{l^2}{8} = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\cdot 2\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{4k}{l}\left[x\left(\sin\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l^2}{n^2\pi^2}\right)\right]_0^{l/2}$$

$$= \frac{4K}{l}\left[\frac{l}{2}\sin\frac{n\pi}{2}\cdot\frac{l}{n\pi} + \cos\frac{n\pi}{2}\cdot\frac{l^2}{n^2\pi^2} - \frac{l^2}{n^2\pi^2}\right]$$

* Doubt.

PARSEVAL'S IDENTITY

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

then

$$\int_{-l}^{l} [f(x)]^2 \, dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

Period of $f(x)$

$(-l, l)$ $$\int_{-l}^{l} [f(x)]^2 \, dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

$(0, 2l)$ $$\int_{0}^{2l} [f(x)]^2 \, dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

$(-l, l)$ $$2\int_{0}^{l} [f(x)]^2 \, dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

(even fn)

$(-l, l)$ $$2\int_{0}^{l} [f(x)]^2 \, dx = l\left[\sum_{n=1}^{\infty} b_n^2\right]$$

(odd fn)

18.

Obtain the Fourier series for $y = x^2$ in $-\pi < x < \pi$ and hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots = \frac{\pi^4}{90}$. Answer: $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

$f(x)$ is an even function

Here $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{— (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2\,dx = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{2\pi(-1)^n}{n^2} \right] = \frac{4(-1)^n}{n^2}$$

(1) becomes

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Parseval's identity for even fn :

$$2 \int_0^{\pi} x^4\,dx = \pi \left[\frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4} \right]$$

$$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16\pi} \left[\frac{2\pi^5}{5} - \frac{2\pi^5}{9} \right]$$

$$= \frac{\pi^4}{90}$$

19.

Expand $f(x) = x - \frac{x^2}{2}$ in $(0, 2)$ as Fourier sine series and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$ Answer: $f(x) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} sin \frac{(2n-1)\pi x}{2}$; $\frac{\pi^4}{960}$

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad a_0 = a_n = 0$$

$$(0, l) \Rightarrow (0, 2) \qquad \boxed{l = 2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \quad \text{—} ①$$

$$b_n = \frac{2}{2}\int_0^2 \left(x - \frac{x^2}{2}\right) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \Bigg[\left(x - \frac{x^2}{2}\right)\left(\frac{2}{n\pi}\right)\left(-\cos\frac{n\pi x}{2}\right) - \left(1 - \frac{2x}{2}\right)\left(-\sin\left(\frac{n\pi x}{2}\right)\right)\left(\frac{4}{n^2\pi^2}\right) + (-1)\left(\cos\left(\frac{n\pi x}{2}\right)\right)\left(\frac{16}{n^2\pi^2}\right)\Bigg]_0^2$$

$$= \frac{-8(-1)^n}{n^3\pi^3} - \left(\frac{-8}{n^3\pi^3}\right)$$

$$= \frac{8}{n^3\pi^3}\left(1 - (-1)^n\right)$$

① becomes

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^3\pi^3}\left(1 - (-1)^n\right) \sin\left(\frac{n\pi x}{2}\right)$$

Deduction

$$f(x) = \frac{8}{\pi^3}\left[\sum_{n=1}^{\infty} \frac{1}{n^3}\left(1 - (-1)^n\right) \sin\left(\frac{n\pi x}{2}\right)\right]$$

$$= \frac{8}{\pi^3}\left[\frac{2}{1^3}\sin\frac{n\pi}{2} + \frac{2}{3^3}\sin\frac{3n\pi}{2} + \dots\right]$$

$$= \frac{16}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin (2n-1)\pi/2^x$$

Parseval's identity :

$$2\int_0^2 \left(x - \frac{x^2}{2}\right)^2 dx = 2\sum_{n=1}^{\infty} \frac{16^2}{\pi^6} \frac{1}{(2n-1)^6}$$

$$\frac{4}{15} = \frac{16^2}{\pi^6} \sum \frac{1}{(2n-1)^6}$$

$$\sum \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$$

20.

Home work Using the Fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$ show that:

$(i) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ $\quad$ $(ii) \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

(i) $f(x)$ is an even fn $\quad \therefore b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |x|\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

① becomes

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos (2n-1)x$$

Parseval's identity for even fn

$$2\int_0^{\pi} x^2\, dx = \pi\left[\frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\right]$$

$$\sum \frac{1}{(2n-1)^4} = \left[\frac{2\pi^2}{3} - \frac{\pi^2}{2}\right]\frac{\pi^2}{16}$$

$$= \frac{\pi^4}{96}$$

(ii) Let $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$= \left[\frac{1}{1^4} + \frac{1}{3^4} + \cdots\right] + \frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \cdots\right]$$

$$S = \frac{\pi^4}{96} + \frac{1}{2^4} S$$

$$S - \frac{1}{16} S = \frac{\pi^4}{96} \Rightarrow \frac{15S}{16} = \frac{\pi^4}{96}$$

$$\boxed{S = \frac{\pi^4}{90}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

21. Prove that $0 < x < 2$,

$$x = 1 - \frac{8}{\pi^2}\left\{\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \cdots\right\}$$

and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^2}{96}$

(Extra QS)

COMPLEX FORM OF F.S.

INTERVAL $(c, c+2\pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where $$C_n = \frac{1}{2\pi} \int_{c}^{c+2\pi} f(x)\, e^{-inx}\, dx$$

INTERVAL $(c, c+2l)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where $$C_n = \frac{1}{2\pi} \int_{c}^{c+2l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

22.

Home work problem: $f(x) = e^{ax}$ in $-\pi \le x \le \pi$. Answer: $e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n sinha\pi}{\pi(a-in)} e^{inx}$

The complex form of FS is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{— (1)}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} \, dx = \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)} \left[e^{(a-in)\pi} - e^{-(a-in)\pi} \right]$$

$$= \frac{1}{2\pi(a-in)} \left[e^{a\pi}[\cos n\pi - i\sin n\pi] - e^{-a\pi}[\cos n\pi + i\sin n\pi] \right]$$

$$= \frac{(-1)^n (e^{a\pi} - e^{-a\pi})}{2\pi(a-in)} = \frac{(-1)^n \sinh a\pi}{\pi(a-in)}$$

(1) becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh a\pi}{\pi(a-in)} e^{inx}$$

23.

$f(x) = cosax$ in $-\pi \leq x \leq \pi$. Answer: $cosax = \sum_{n=-\infty}^{\infty} \frac{(-1)^n asina\pi}{\pi(a^2-n^2)} e^{inx}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax \, . \, e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{(-in)^2 + a^2} (-in\cos ax + a\sin ax) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2-n^2)} \Big[e^{-in\pi} (-in\cos a\pi + a\sin a\pi) - e^{in\pi}(-in\cos a\pi - a\sin a\pi) \Big]$$

$$= \frac{1}{2\pi(a^2-n^2)} (-1)^n (2a\sin a\pi) \quad \begin{pmatrix} \text{expand} \\ e^{-in\pi} \; \& \\ e^{in\pi} \end{pmatrix}$$

$$= \frac{(-1)^n (a\sin a\pi)}{\pi(a^2-n^2)}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a\sin a\pi)}{\pi(a^2-n^2)} e^{inx}$$

24.

$f(x) = e^{-x}$ in $-1 \leq x \leq 1$. Answer: $e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n sinh1}{(1+in\pi)} e^{in\pi x}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \qquad l = 1$$

$$C_n = \frac{1}{2l} \int_{-l}^{l} e^{-x} . e^{-in\pi x/l} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-x} e^{-in\pi x} \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2(1+in\pi)}\left[-e^{-(1+in\pi)} + e^{(1+in\pi)}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[-e^{-1}e^{-in\pi} + e^{1}e^{in\pi}\right]$$

$$= \frac{(-1)^n\,(e - e^{-1})}{2(1+in\pi)}$$

$$= \frac{(-1)^n \sinh 1}{(1+in\pi)}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh 1}{1+in\pi}\, e^{in\pi x}$$

FOURIER TRANSFORM

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega)$$

INVERSE FOURIER TRANSFORM

$$F^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

25. Find the Fourier Transform for the function

$$f(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$$ & hence evaluate

$$\int_0^\infty \frac{\sin w}{w}\, dw$$

$$f(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & t < -1 \text{ \& } t > 1 \end{cases}$$

By definition of FT,

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \int_{-1}^{1} e^{-iwt}\, dt = \left[\frac{e^{-iwt}}{-iw}\right]_{-1}^{1}$$

$$= \frac{-e^{-iw}}{iw} + \frac{e^{iw}}{iw} = \frac{e^{iw} - e^{-iw}}{iw}$$

$$= \frac{2i \sin w}{iw} = \frac{2 \sin w}{w} = F(w)$$

By inverse FT,

$$f(t) = F^{-1}[F(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dw$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin w}{w}\, e^{iwt}\, dw$$

Put $t = 0$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w}\, dw$$

$$2 \int_0^\infty \frac{\sin w}{w}\, dw = \pi$$

$$\int_0^\infty \frac{\sin w}{w}\, dw = \pi/2$$

26. Find the Fourier Transform for the function

$$f(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$$ and hence

evaluate $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$

By FT,

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \int_{-1}^{1} (1 - |t|)(\cos wt - i \sin wt)\, dt$$

$$= 2\int_0^1 (1 - t) \cos wt \; dt$$

$$= 2\left[(1-t)\frac{\sin wt}{w} - (-1)\left(-\frac{\cos wt}{w^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos w}{w^2} + \frac{1}{w^2} \right] = \frac{2}{w^2}(1 - \cos w)$$

$$= \frac{4}{w^2} \sin^2 w/2$$

By IFT,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4 \sin^2 w/2}{w^2}\, e^{iwt}\, dw$$

Put $t = 0$

$$1 = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 w/2}{w^2}\, dw$$

$$\int_{-\infty}^{\infty} \frac{\sin^2 w/2}{w^2} = \frac{\pi}{2}$$

Let $w/2 = t$

$dw = 2dt$

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{4t^2} \cdot 2dt = \pi/2 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} = \pi$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} = \pi/2$$

27. Find the FT for the function

$$f(t) = \begin{cases} 1 - t^2 & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$ and hence

evaluate $\displaystyle\int_0^\infty \left(\frac{\sin w - w\cos w}{w^3}\right) \cdot \cos\frac{w}{2}\, dw$

$$F(w) = F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \int_{-1}^{1} (1-t^2)\, e^{-iwt}\, dt$$

$$= \left[(1-t^2)\frac{e^{-iwt}}{-iw} + 2t\,\frac{e^{-iwt}}{(-iw)^2} + (-2)\frac{e^{-iwt}}{(-iw)^3} \right]_{-1}^{1}$$

$$= \frac{2e^{-iw}}{-w^2} - \frac{2e^{-iw}}{iw^3} - \left(\frac{-2e^{iw}}{-w^2} - \frac{2e^{iw}}{iw^3} \right)$$

$$= \frac{-2}{w^2}\left(e^{iw} + e^{-iw}\right) + \frac{2}{iw^3}\left(e^{iw} - e^{-iw}\right)$$

$$= \frac{-2}{w^2}(2\cos w) + \frac{2}{iw^3}(2i\sin w)$$

$$F(w) = 4\left(\frac{\sin w - w\cos w}{w^3}\right)$$

By IFT,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin w - w\cos w}{w^3}\right) e^{iwt}\, dw$$

Put $t = 1/2$

$$f(1/2) = 1 - 1/4 = 3/4$$

$$\frac{3}{4} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin w - w\cos w}{w^3}\, e^{iw/2}\, dw$$

$$\frac{3\pi}{8} = \int_{-\infty}^{\infty} \left(\frac{\sin w - w\cos w}{w^3}\right)\left(\cos w/_2 + i\sin w/_2\right) dw$$

Comparing real parts

$$\frac{3\pi}{8} = \int_{-\infty}^{\infty} \left(\frac{\sin w - w\cos w}{w^3}\right) \cos w/_2 \, dw$$

$$\int_0^{\infty} \frac{\sin w - w\cos w}{w^3} \cdot \cos w/_2 \, dw = \frac{3\pi}{16}$$

FOURIER SINE TRANSFORM

$$F_s[f(t)] = \int_0^\infty f(t)\sin\omega t \, dt = F_s(\omega)$$

INVERSE FOURIER SINE TRANSFORM

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t \, d\omega$$

FOURIER COSINE TRANSFORM

$$F_c[f(t)] = \int_0^\infty f(t)\cos\omega t \, dt = F_c(\omega)$$

INVERSE FOURIER COSINE TRANSFORM

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t \, d\omega$$

28. Find the Fourier sine transform of $e^{-|t|}$ and hence evaluate $\int_0^\infty \frac{w \sin mw}{1+w^2}\, dw$, $m > 0$

$$F_s[f(t)] = \int_0^\infty f(t) \sin wt \, dt$$

$$= \int_0^\infty e^{-t} \sin wt \, dt$$

$$= \left[\frac{e^{-t}}{1+w^2} (-\sin wt - w \cos wt) \right]_0^\infty$$

$$= 0 - \left[\frac{1}{1+w^2} (-w) \right] = \frac{w}{1+w^2}$$

By IFST,

$$f(t) = \frac{2}{\pi} \int_0^\infty \frac{w}{1+w^2} \sin wt \, dw$$

$$e^{-|t|} \cdot \frac{\pi}{2} = \int_0^\infty \frac{w \sin wt}{1+w^2} \, dw$$

Put $t = m$

$$\int_0^\infty \frac{w \sin mw}{1+w^2} \, dw = e^{-|m|} \pi/2$$

27. Find the Fourier sine transform of $\frac{e^{-at}}{t}$

$t \neq 0$, $a > 0$

$$F_s[f(t)] = \int_0^\infty f(t) \sin wt \, dt$$

$$= \int_0^\infty \frac{e^{-at}}{t} \sin wt \, dt \quad \text{— ①}$$

Differentiating on both sides with 'w'

$$\frac{d(F_s(w))}{dw} = \int_0^\infty \frac{e^{-at}}{t} \cos wt \, . \, t \, dt$$

$$= \int_0^\infty e^{-at} \cos wt \, dt$$

$$= \left[\frac{e^{-at}}{a^2+w^2} \left[-a\cos wt + w \sin wt \right] \right]_0^\infty$$

$$= \frac{-1}{a^2+w^2}(-a) = \frac{a}{a^2+w^2}$$

$$\frac{d F_s(w)}{dw} = \frac{a}{a^2+w^2}$$

Integrating on both sides wrt 'w'

$$F_s(w) = \int \frac{a}{a^2+w^2} dw$$

$$= a \cdot \frac{1}{a} \tan^{-1}\left(\frac{w}{a}\right) + c$$

$$F_s(w) = \tan^{-1}\left(\frac{w}{a}\right) + c \quad \text{— ②}$$

To find c

From (1) & (2)

$$\int_0^\infty \frac{e^{-at}}{t} \sin wt \, dt = \tan^{-1}\left(\frac{w}{a}\right) + c$$

Put $w = 0$

$$0 = 0 + c \quad \Rightarrow c = 0$$

$$\therefore F_s(w) = \tan^{-1}\left(\frac{w}{a}\right)$$

30. Find the Fourier sine transform of e^{-at} and hence show that $\displaystyle\int_0^\infty \frac{w \sin kw}{a^2+w^2} dw = \frac{\pi}{2} e^{-ak}$

$$F_s(w) = \int_0^\infty e^{-at} \sin wt \, dt$$

$$= \left[\frac{e^{-at}}{a^2+w^2} \left[-a \sin wt - w \cos wt \right] \right]_0^\infty$$

$$= -\frac{1}{a^2+w^2}(-w) = \frac{w}{a^2+w^2}$$

By IFST,

$$f(t) = \frac{2}{\pi} \int_0^\infty \frac{w}{a^2+w^2} \sin wt \, dw$$

$$\int_0^\infty \frac{w \sin wt}{a^2+w^2} dw = e^{-at} \cdot \pi/2$$

Put $t = k$

$$\int_0^\infty \frac{w \sin wk}{a^2+w^2} dw = e^{-ak} \cdot \pi/2$$

31. Find the Fourier cosine transform of $\frac{1}{1+t^2}$

$$F_c[f(t)] = \int_0^\infty \frac{1}{1+t^2} \cos wt \, dt = I \quad \text{—} (1)$$

Differentiating wrt 'w'

$$\frac{dI}{dw} = \frac{d}{dw} F_c(w) = \int_0^\infty \frac{1}{1+t^2} (-\sin wt \cdot t) \, dt$$

$$= -\int_0^\infty \frac{t}{1+t^2} \sin wt \, dt$$

$$= -\int_0^\infty \frac{t^2}{t(1+t^2)} \sin wt \, dt$$

$$= -\int_0^\infty \frac{t^2+1-1}{(1+t^2)t} \sin wt \, dt$$

$$= -\left[\int_0^\infty \frac{\sin wt}{t} dt - \int_0^\infty \frac{\sin wt}{t(1+t^2)} dt\right]$$

$$= \frac{-\pi}{2} + \int_0^\infty \frac{\sin wt}{t(1+t^2)} dt \quad \text{—} (2)$$

Again differentiating wrt 'w'

$$\frac{d^2I}{dw^2} = 0 + \int_0^\infty \frac{1}{t(1+t^2)} \cos wt \cdot t \, dt$$

$$\frac{d^2I}{dw^2} = I \Rightarrow \frac{d^2I}{dw^2} - I = 0$$

Auxillary :

$$D^2 - I = 0 \Rightarrow m = \pm 1$$

$$I = c_1 e^{w} + c_2 e^{-w} \quad \text{—} (3)$$

To find c_1 & c_2

From ① & ③

$$\int_0^\infty \frac{1}{1+t^2} \cos wt \, dt = c_1 e^{w} + c_2 e^{-w}$$

Put $w = 0$

$$\int_0^\infty \frac{1}{1+t^2} \, dt = c_1 + c_2$$

$$c_1 + c_2 = (\tan^{-1} t)_0^\infty = \pi/2$$

$$\boxed{c_1 + c_2 = \pi/2} \quad \text{— ④}$$

Differentiating ③ wrt 'w'

$$\frac{dI}{dw} = c_1 e^{w} - c_2 e^{-w}$$

$$-\frac{\pi}{2} + \int_0^\infty \frac{\sin wt}{t(1+t^2)} \, dt = c_1 e^{w} - c_2 e^{-w}$$

Put $w = 0$

$$\boxed{-\frac{\pi}{2} = c_1 - c_2} \quad \text{— ⑤}$$

From ④ & ⑤

$$c_1 = 0 \quad \& \quad c_2 = \pi/2$$

$$\therefore \quad F_c[f(t)] = \frac{\pi}{2} e^{-w}$$

32. Find the Fourier cosine transform of e^{-t^2}

$$F_c[f(t)] = \int_0^\infty f(t) \cos wt \, dt$$

$$I = \int_0^\infty e^{-t^2} \cos wt \, dt \quad — (1)$$

$$\frac{dI}{dw} = \int_0^\infty e^{-t^2} (-\sin wt) \, t \cdot dt$$

$$\frac{d\left(e^{-t^2}/2\right)}{dt} = \frac{-2t\, e^{-t^2}}{2}$$

$$= \int_0^\infty \sin wt \; d\left(\frac{e^{-t^2}}{2}\right)$$

$u = \sin wt$, $dv = d\left(\frac{e^{-t^2}}{2}\right)$

$du = \cos wt \cdot w\,dt$

$v = e^{-t^2}/2$

$\int u\,dv = uv - \int v\,du$

$$\frac{dI}{dw} = \left(\sin wt \cdot \frac{e^{-t^2}}{2}\right)_0^\infty - \int_0^\infty \frac{e^{-t^2}}{2} \cos wt \cdot w \cdot dt$$

$$\frac{dI}{dw} = 0 - \frac{w}{2} \int_0^\infty e^{-t^2} \cos wt \, dt$$

$$\frac{dI}{dw} = -\frac{w}{2} I \Rightarrow \frac{dI}{I} = -\frac{w}{2} dw$$

On integrating

$$\log I = \frac{-w^2}{4} + c$$

$$I = e^{-w^2/4 + c} = k e^{-w^2/4} \quad — (2)$$

To find K

From (1) & (2)

$$\int_0^\infty e^{-t^2}\cos wt\,dt = k e^{-w^2/4}$$

Put $w=0$

$$\int_0^\infty e^{-t^2}\,dt = K \Rightarrow K = \frac{\sqrt{\pi}}{2}$$

$t^2 = x$

$2t\,dt = dx$

$dt = \frac{dx}{2\sqrt{x}}$

$\Gamma_n = \int_0^\infty e^{-x}x^{n-1}dx$

$= \int_0^\infty e^{-x}\frac{1}{2}x^{-1/2}dx$

$= \frac{1}{2}\Gamma_{1/2} = \frac{\sqrt{\pi}}{2}$

$$F_c[f(t)] = \frac{\sqrt{\pi}}{2}e^{-w^2/4}$$

33. Find the Fourier cosine transform of

$$f(t) = \begin{cases} t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$$

$$F_c[f(t)] = \int_0^\infty f(t)\cos wt\,dt$$

$$= \int_0^1 t\cos wt\,dt + \int_1^2 (2-t)\cos wt\,dt$$

$$= \left[t\left(\frac{\sin wt}{w}\right) - 1\left(-\frac{\cos wt}{w^2}\right)\right]_0^1$$

$$+ \left[(2-t)\left(\frac{\sin wt}{w}\right) + 1\left(-\frac{\cos wt}{w^2}\right)\right]_1^2$$

$$= \frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2}$$

$$- \frac{\sin w}{w} - \frac{\cos 2w}{w^2} + \frac{\cos w}{w^2}$$

$$F_c(w) = \frac{2\cos w - \cos 2w - 1}{w^2}$$

34. Find the Fourier sine transform of

$$f(t) = \begin{cases} \sin t & 0 < t < a \\ 0 & t > a \end{cases} \quad \text{for } w = 1$$

$$F_s[f(t)] = \int_0^a \sin t \sin wt \, dt$$

$$= \int_0^a \left(\frac{\cos(w-1)t - \cos(w+1)t}{2} \right) dt$$

$$= \frac{1}{2} \int_0^a (\cos(w-1)t - \cos(w+1)t) \, dt$$

$$= \frac{1}{2} \left[\frac{\sin(w-1)t}{w-1} - \frac{\sin(w+1)t}{w+1} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{\sin(w-1)a}{w-1} - \frac{\sin(w+1)a}{w+1} \right]$$

For $w=1$

$$F_s[f(t)] = \frac{1}{2} \left[\underset{\substack{\downarrow \\ \text{indeterminate form}}}{\frac{0}{0}} - \frac{\sin 2a}{2} \right]$$

$\therefore$ differentiate wrt 'w'

$$= \frac{1}{2} \left[\frac{\cos(w-1)a \cdot a}{1} - \frac{\sin 2a}{2} \right]$$

$$= \frac{1}{2} \left[a - \frac{\sin 2a}{2} \right] = \frac{2a - \sin 2a}{4}$$

PROPERTIES OF FOURIER T.

① LINEARITY PROPERTY

$$F\{a f(t) + b g(t)\} = a F[f(t)] + b F[g(t)]$$

② SHIFTING ON t-AXIS

$F[f(t)] = F(\omega)$ & $t_0 \rightarrow$ real no.

then
$$F[f(t - t_0)] = e^{-i\omega t_0} F(\omega)$$

$$F^{-1}[e^{-i\omega t_0} F(\omega)] = f(t - t_0)$$

③ FREQUENCY SHIFTING

$$F[e^{i\omega_0 t} f(t)] = f(\omega - \omega_0)$$

④ CHANGE OF SCALE PROPERTY

$$F[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

⑤ MODULATION THEOREM

$$F[f(t) \cos \omega_0 t] = \frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$$

$$F[f(t) \sin \omega_0 t] = \frac{1}{2}[F(\omega + \omega_0) - F(\omega - \omega_0)]$$

⑥ FOURIER TRANSFORM OF DERIVATIVES

$$F[f^n(t)] = (i\omega)^n F(\omega)$$

⑦ SYMMETRY PROPERTY OF FT

$$F[f(t)] = 2\pi f(\omega)$$

FOURIER & INV. FOURIER OF SOME STANDARD FUNCTIONS

① $F\left[e^{-a|t|}\right] = \frac{2a}{a^2+\omega^2}$

$$F^{-1}\left[\frac{2a}{a^2+\omega^2}\right] = e^{-a|t|} \quad *$$

② (i) $F\left[e^{-at} f(t)\right] = \frac{1}{a+i\omega}$

$$F^{-1}\left[\frac{1}{a+i\omega}\right] = e^{-at} H(t) \quad *$$

(ii) $F\left[e^{at} H(-t)\right] = \frac{1}{a-i\omega}$ *

$$F^{-1}\left[\frac{1}{a-i\omega}\right] = e^{at} H(-t)$$

③ $F[\delta(t-a)] = e^{-i\omega a}$

$$F[\delta(t)] = 1$$

④ $F(k) = 2\pi k\, \delta(\omega)$

$$F(1) = 2\pi\, \delta(\omega)$$

⑤ $F\left[e^{iat}\right] = 2\pi\, \delta(\omega - a)$

$$\rightarrow \quad F\left[e^{-a^2t^2}\right] = \frac{\sqrt{\pi}}{a} e^{-w^2/4a^2}$$

$$\rightarrow \quad F\left[e^{-a|t|}\right] = \frac{2a}{a^2+w^2}$$

$$\rightarrow \quad F\left[e^{-t^2/2}\right] = \sqrt{2a}\; e^{-w^2/2}$$

$$F\left[\delta(t-a)\right] = e^{-iwa}$$

35. Find the inverse Fourier transform of

$$\frac{e^{3i\omega}}{2-i\omega}$$

$\because F^{-1}\left[\frac{1}{a-i\omega}\right] = e^{at}H(-t)$

$\because F\left[e^{-i\omega t_0}F(\omega)\right] = f(t-t_0)$

$$F^{-1}\left[e^{3i\omega}\cdot\frac{1}{2-i\omega}\right]$$

$\boxed{t_0 = -3}$ (Shifting on t-axis)

$$F^{-1}\left[\frac{1}{2-i\omega}\right] = e^{2t}H(-t)$$

$$F\left[e^{3i\omega}\frac{1}{2-i\omega}\right] = e^{2t}H(-(t-(-3)))$$
$$= e^{2t}H(-(t+3))$$

36. Find the Fourier Transform of

(i) $e^{-2(t-3)^2}$

$\because F\left[e^{-a^2t^2}\right] = \frac{\sqrt{\pi}}{a}e^{-\omega^2/4a^2}$

$$F[f(t-t_0)] = e^{-i\omega t_0}F(\omega)$$

$\boxed{t_0 = 3}$ (Shifting on t-axis)

$$F\left[e^{-2t^2}\right] = \frac{\sqrt{\pi}}{\sqrt{2}}e^{-\omega^2/4(\sqrt{2})^2}$$
$$= \sqrt{\frac{\pi}{2}}e^{-\omega^2/8}$$

$$F\left[e^{-2(t-3)^2}\right] = e^{-3i\omega}\cdot\sqrt{\frac{\pi}{2}}e^{-\omega^2/8}$$
$$= \sqrt{\frac{\pi}{2}}e^{-(\omega^2/8+3i\omega)}$$

(ii) $e^{-t^2}\cos 3t$

$\because F[f(t)\cos w_0 t] = \frac{1}{2}[F(w+w_0) + F(w-w_0)]$

$$F\left[e^{-t^2}\cos 3t\right]$$

$\boxed{w_0 = 3}$ (Frequency shifting)

$$F\left[e^{-t^2}\right] = \sqrt{\pi}\, e^{-w^2/4}$$

$\because F\left[e^{-a^2t^2}\right] = \frac{\sqrt{\pi}}{a} e^{-w^2/4a^2}$ $\boxed{a=1}$

$$F\left[e^{-t^2}\cos 3t\right] = \frac{1}{2}\left[\sqrt{\pi}e^{-(w+3)^2/4} + \sqrt{\pi}e^{-(w-3)^2/4}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-(w+3)^2/4} + e^{-(w-3)^2/4}\right]$$

37. Find the inverse Fourier Transform of $\frac{1}{(w^2+9)(w^2+4)}$

$$F^{-1}\left[\frac{2a}{a^2+w^2}\right] = e^{-a|t|}$$

$$F^{-1}\left[\frac{1}{(w^2+9)(w^2+4)}\right]$$

$$\frac{1}{(w^2+9)(w^2+4)} = \frac{A}{w^2+9} + \frac{B}{w^2+4}$$

$1 = A(w^2+4) + B(w^2+9)$

$A+B=0 \Rightarrow A=-B$

$4A+9B=1$

$-4B+9B=1$ $B=1/5$

$A=-1/5$

$$= F^{-1}\left[\frac{-1}{5(w^2+9)} + \frac{1}{5(w^2+4)}\right]$$

$$= \frac{-1}{5}F^{-1}\left[\frac{1}{w^2+9}\right] + \frac{1}{5}F^{-1}\left[\frac{1}{w^2+4}\right]$$

$$= \frac{-1}{5}\cdot\frac{1}{2(3)}F^{-1}\left[\frac{2(3)}{w^2+3^2}\right] + \frac{1}{5}\cdot\frac{1}{2(2)}F^{-1}\left[\frac{2(2)}{w^2+2^2}\right]$$

$$= \frac{-1}{30}e^{-3|t|} + \frac{1}{20}e^{-2|t|}$$

38. Find the Fourier Transform of the function $f(t) = e^{-a^2t^2}$, $a>0$. Hence deduce that $e^{-t^2/2}$ is self reciprocal in respect of F.T.

$$F\left[e^{-a^2t^2}\right] = \int_{-\infty}^{\infty} e^{-a^2t^2} e^{-i\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-(a^2t^2+i\omega t)}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\left(a^2t^2+i\omega t+\left(\frac{i\omega}{2a}\right)^2-\left(\frac{i\omega}{2a}\right)^2\right)}\,dt$$

$$= e^{-\omega^2/4a^2}\int_{-\infty}^{\infty} e^{-\left(at+\frac{i\omega}{2a}\right)^2}\,dt$$

$$= e^{-\omega^2/4a^2}\int_{-\infty}^{\infty} e^{-x^2}\,\frac{dx}{a}$$

$$at+\frac{i\omega}{2a} = x$$

$$a\,dt = dx$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\Gamma_n = 2\int_{0}^{\infty} e^{-x^2}x^{2n-1}\,dx$$

$$= \frac{e^{-\omega^2/4a^2}}{a}\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{e^{-\omega^2/4a^2}}{a}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{a}\,e^{-\omega^2/4a^2}$$

Deduction

Put $a = 1/\sqrt{2}$

$$F\left[e^{-t^2/2}\right] = \frac{\sqrt{\pi}}{1/\sqrt{2}}\,e^{-\omega^2/4(1/\sqrt{2})^2}$$

$$= \sqrt{2\pi}\,e^{-2\omega^2/4}$$

$$= \sqrt{2\pi}\,e^{-\omega^2/2}$$

* $e^{-t^2/2}$ is constant times $e^{-\omega^2/2}$ and therefore is a self reciprocal

39. Find the Fourier Transform of the dirac-delta function $\delta(t)$

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt$$

$$\delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t-a)$$

where

$$f_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon} & a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & 0 \le t \le \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$= \lim_{\epsilon \to 0} \int_0^\epsilon \frac{1}{\epsilon} e^{-i\omega t}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon e^{-i\omega t}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_0^\epsilon$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{e^{-i\omega\epsilon}}{-i\omega} - \frac{1}{-i\omega} \right] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{1 - e^{-i\omega\epsilon}}{i\omega} \right]$$

(L'hospital's)

$$= \lim_{\epsilon \to 0} \left[\frac{+i\omega e^{-i\omega\epsilon}}{i\omega} \right] = \lim_{\epsilon \to 0} e^{-i\omega\epsilon} = 1$$

$$F[\delta(t)] = 1$$

$$F^{-1}[1] = \delta(t)$$

40. Solve $y' - 4y = H(t)\, e^{-4t}$, $-\infty < t < \infty$

$$y'(t) - 4y(t) = H(t)\, e^{-4t}$$

$$F[e^{-at} H(t)] = \frac{1}{a + i w}$$

$$F[f^n(t)] = (iw)^n F(w)$$

Taking Fourier T. on both sides

$$F[y'(t)] - 4F[y(t)] = F[e^{-4t} H(t)]$$

$$(iw)F(w) - 4F(w) = \frac{1}{4 + iw}$$

$$F(w)[iw - 4] = \frac{1}{4 + iw}$$

$$F(w) = \frac{1}{(4 + iw)(iw - 4)}$$

$$F(w) = \frac{-1}{16 + w^2}$$

Taking inverse

$$F^{-1}[F(w)] = -F^{-1}\left[\frac{1}{16 + w^2}\right]$$

$$y(t) = -\frac{1}{2(4)} \cdot F^{-1}\left[\frac{2(4)}{4^2 + w^2}\right]$$

$$= \frac{-1}{8} e^{-4|t|}$$

$$\boxed{y(t) = \frac{-1}{8} e^{-4|t|}}$$

41. Solve $y' + 3y = H(t)\, e^{-2t}$, $-\infty < t < \infty$

$$y'(t) + 3y(t) = H(t) \cdot e^{-2t}$$

$F[f^n(t)] = (i\omega)^n F(\omega)$

$F[e^{-at} H(t)] = \frac{1}{a + i\omega}$

$$F[y'(t)] + 3F[y(t)] = F[e^{-2t} H(t)]$$

$$i\omega F(\omega) + 3F(\omega) = \frac{1}{2 + i\omega}$$

(Derivative property)

$$F(\omega)[3 + i\omega] = \frac{1}{2 + i\omega}$$

$$F(\omega) = \frac{1}{(2 + i\omega)(3 + i\omega)}$$

$$F(\omega) = \frac{1}{2 + i\omega} - \frac{1}{3 + i\omega}$$

$\frac{1}{(2+i\omega)(3+i\omega)} = \frac{A}{2+i\omega} + \frac{B}{3+i\omega}$

$1 = A(3+i\omega) + B(2+i\omega)$

$A + B = 0$

$3A + 2B = 1$

$-3B + 2B = 1 \Rightarrow B = -1$

$A = 1$

Taking inverse,

$$F^{-1}[F(\omega)] = F^{-1}\left[\frac{1}{2 + i\omega}\right] - F^{-1}\left[\frac{1}{3 + i\omega}\right]$$

$$y(t) = e^{-2t} H(t) - e^{-3t} H(t)$$

42. Solve $y'' + 5y' + 4y = \delta(t-2)$, $-\infty < t < \infty$

$$y''(t) + 5y'(t) + 4y(t) = \delta(t-2)$$

$F[f^n(t)] = (i\omega)^n F(\omega)$

$F[\delta(t-a)] = e^{-i\omega a}$

$F^{-1}\left[e^{-i\omega t_0} F(\omega)\right] = f(t-t_0)$

$$F[y''(t)] + 5F[y'(t)] + 4F[y(t)] = F[\delta(t-2)]$$

$$(i\omega)^2 F(\omega) + 5(i\omega)F(\omega) + 4F(\omega) = e^{-i\omega(2)}$$

$$F(\omega)\left[(i\omega)^2 + 5i\omega + 4\right] = e^{-2i\omega}$$

$$F(\omega)\left[(i\omega)^2 + i\omega + 4i\omega + 4\right] = e^{-2i\omega}$$

$$F(\omega)\left[(i\omega+1)(4+i\omega)\right] = e^{-2i\omega}$$

$$F(\omega) = e^{-2i\omega} \cdot \frac{1}{(1+i\omega)(4+i\omega)}$$

$$\frac{1}{(1+i\omega)(4+i\omega)} = \frac{A}{1+i\omega} + \frac{B}{4+i\omega}$$

$$1 = A(4+i\omega) + B(1+i\omega)$$

$A + B = 0$

$4A + B = 1 \Rightarrow B = -1/3$

$A = 1/3$

$$F(\omega) = e^{-2i\omega}\left[\frac{1}{3(1+i\omega)} - \frac{1}{3(4+i\omega)}\right]$$

$$F^{-1}[F(\omega)] = \frac{1}{3}\left[F^{-1}\left(e^{-2i\omega}\frac{1}{1+i\omega}\right) - F^{-1}\left(e^{-2i\omega}\frac{1}{4+i\omega}\right)\right]$$

$$= \frac{1}{3}\left[e^{-(t-2)}H(t-2) - e^{-4(t-2)}H(t-2)\right]$$

FINITE FOURIER COSINE T.

$$F_c[f(t)] = F_c(n) = \int_0^{\pi} f(t)\cos nt \, dt$$

INVERSE FINITE FOURIER COSINE T.

$$f(t) = \frac{1}{\pi}\left[F_c(0) + 2\sum_{n=1}^{\infty} F_c(n)\cos nt\right]$$

FINITE FOURIER SINE T.

$$F_s[f(t)] = F_s(n) = \int_0^{\pi} f(t)\sin nt \, dt$$

INVERSE FINITE FOURIER SINE T.

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n)\sin nt$$

43. Find the Finite Fourier sine transform of

$$f(t) = \begin{cases} -t & 0 < t < c \\ \pi - t & c < t < \pi \end{cases}$$

$$F_s[f(t)] = F_s(n) = \int_0^{\pi} f(t) \sin nt \, dt$$

$$= \int_0^{c} -t \sin nt \, dt + \int_c^{\pi} (\pi - t) \sin nt \, dt$$

$$= -\left[t\left(-\frac{\cos nt}{n}\right) - \left(-\frac{\sin nt}{n^2}\right) \right]_0^c$$

$$+ \left[(\pi - t)\left(-\frac{\cos nt}{n}\right) - (-1)\left(-\frac{\sin nt}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{c \cos nc}{n} - \frac{\sin nc}{n^2} + 0$$

$$- \left(-(\pi - c)\frac{\cos nc}{n} - \frac{\sin nc}{n^2}\right)$$

$$= \frac{c \cos nc}{n} + \frac{\pi \cos nc}{n} - \frac{\cos nc}{n}$$

$$= \frac{\pi \cos nc}{n}$$

44. Find the inverse finite fourier sine transform of $\frac{1 - \cos n\pi}{n^2 \pi^2}$

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n^2 \pi^2} \sin nt$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2 \pi^2} \sin nt$$

$$= \frac{2}{\pi^3}\left[\frac{2}{1^2}\sin t + 0 + \frac{2}{3^2}\sin 3t + \cdots\right]$$

$$= \frac{4}{\pi^3}\left[\frac{1}{1^2}\sin t + \frac{1}{3^2}\sin 3t + \cdots\right]$$

$$= \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)^2}$$

45. Find the finite Fourier sine transform of

$f(t) = t(\pi - t)$ in $0 < t < \pi$

$$F_s[f(t)] = F_s(n) = \int_0^{\pi} t(\pi-t)\sin nt\, dt$$

$$= \left[(\pi t - t^2)\left(\frac{-\cos nt}{n}\right) - (\pi - 2t)\left(-\frac{\sin nt}{n^2}\right) + (-2)\left(\frac{\cos nt}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{-2(-1)^n}{n^3} - \left(\frac{-2}{n^3}\right)$$

$$F_s(n) = \frac{2}{n^3}\left(1 - (-1)^n\right)$$

46. Find the finite Fourier cosine transform of

$f(t) = t(\pi - t)$ in $0 < t < \pi$

$$F_c[f(t)] = F_c(n) = \int_0^{\pi}(\pi t - t^2)\cos nt\, dt$$

$$= \left[(\pi t - t^2)\left(\frac{\sin nt}{n}\right) - (\pi - 2t)\left(\frac{-\cos nt}{n^2}\right) + (-2)\left(\frac{-\sin nt}{n^3}\right)\right]_0^{\pi}$$

$$= \left[-(\pi - 2t)\left(\frac{-\cos nt}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{-\pi(-1)^n}{n^2} - \left(\frac{\pi}{n^2} \right) = \frac{-\pi}{n^2}\left(1 + (-1)^n\right)$$

$$F_c(n) = \frac{-\pi}{n^2}\left(1 + (-1)^n\right)$$

47. Find the finite Fourier cosine transform of $f(t) = 2t$ in $0 < t < 4$

$$F_c[f(t)] = F_c(n) = \int_0^{l} f(t) \cos\left(\frac{n\pi t}{l} \right) dt$$

$$\boxed{l = 4}$$

$$= \int_0^{4} 2t \cos\left(\frac{n\pi t}{4} \right) dt$$

$$= \left[2t\left(\sin\left(\frac{n\pi t}{4} \right) \cdot \frac{4}{n\pi} \right) - 2\left(-\cos\left(\frac{n\pi t}{4} \right) \cdot \frac{16}{n^2\pi^2} \right) \right]_0^4$$

$$= \frac{32}{n^2\pi^2}(-1)^n - \frac{32}{n^2\pi^2}$$

$$= \frac{32}{n^2\pi^2}\left[(-1)^n - 1 \right]$$

$$F_c[f(t)] = \frac{32}{n^2\pi^2}\left((-1)^n - 1 \right)$$

48. Find the finite fourier cosine transform of $f(t) = e^{at}$ in $0 < t < l$

$$F_c[f(t)] = F_c(n) = \int_0^l f(t)\cos\left(\frac{n\pi t}{l}\right) dt$$

$$= \int_0^l e^{at}\cos\left(\frac{n\pi t}{l}\right) dt$$

$$= \left[\frac{e^{at}}{a^2+\left(\frac{n\pi}{l}\right)^2}\left(a\cos\frac{n\pi t}{l} + \frac{n\pi}{l}\sin\frac{n\pi t}{l}\right)\right]_0^l$$

$$= \frac{e^{al}\cdot l^2}{l^2a^2+n^2\pi^2}\left(a(-1)^n\right) - \frac{l^2}{l^2a^2+n^2\pi^2}(a)$$

$$= \frac{al^2}{l^2a^2+n^2\pi^2}\left(e^{al}(-1)^n - 1\right)$$

49. Find the finite Fourier cosine transform of $f(t) = \sinh at$, $a > 0$ in $(0, \pi)$. Also find its inverse

$$F_c[f(t)] = F_c(n) = \int_0^\pi \sinh at \cos nt \, dt$$

$$= \int_0^\pi \frac{e^{at}-e^{-at}}{2} \cdot \cos nt \, dt$$

$$= \frac{1}{2}\int_0^\pi \left(e^{at}\cos nt - e^{-at}\cos nt\right) dt$$

$$= \frac{1}{2}\left[\int_0^\pi e^{at}\cos nt \, dt - \int_0^\pi e^{-at}\cos nt \, dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{e^{at}}{a^2+n^2}(a\cos nt - n\sin nt)\right]_0^{\pi} - \left[\frac{e^{-at}}{a^2+n^2}(-a\cos nt + n\sin nt)\right]_0^{\pi}\right]$$

$$= \frac{1}{2(a^2+n^2)}\left[e^{\pi a}\cdot a(-1)^n - a - e^{-\pi a}(-a)(-1)^n - a\right]$$

$$= \frac{a}{2(a^2+n^2)}\left[(-1)^n\left[e^{\pi a}+e^{-\pi a}\right] - 2\right]$$

$$= \frac{a}{2(a^2+n^2)}\left[(-1)^n 2\cosh a\pi - 2\right]$$

$$= \frac{a}{a^2+n^2}\left[(-1)^n\cosh a\pi - 1\right]$$

Inverse,

$$f(t) = \frac{1}{\pi}\left[F_c(0) + 2\sum_{n=1}^{\infty} F_c(n)\cos nt\right]$$

$$= \frac{1}{\pi}\left[\frac{a}{a^2}\left[\cosh a\pi - 1\right] + 2\sum_{n=1}^{\infty}\frac{a}{a^2+n^2}\left[(-1)^n\cosh a\pi - 1\right]\cdot\cos nt\right]$$

$$= \frac{1}{a\pi}(\cosh a\pi - 1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{a}{a^2+n^2}\left[(-1)^n\cosh a\pi - 1\right]\cdot\cos nt$$