UNIT - 1

INTEGRAL

CALCULAS

① CW $$\int_0^1\int_0^3 x^3y^3\,dx\,dy = \int_0^1 y^3\,dy \int_0^3 x^3\,dx = \left[\frac{y^4}{4}\right]_0^1 \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{1}{4}\cdot\frac{81}{4} = \frac{81}{16}$$

② CW $$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}} = \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \left[\sin^{-1}y\right]_0^1 \left[\sin^{-1}x\right]_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

③ HW $$\int_1^4\int_0^{\sqrt{4-x}} xy\,dy\,dx = \int_1^4 x\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx = \int_1^4 x\left[\frac{4-x}{2}\right]dx$$

$$= \frac{1}{2}\int_1^4 4x-x^2\,dx = \frac{1}{2}\left[\left[\frac{4x^2}{2}\right]_1^4 - \left[\frac{x^3}{3}\right]_1^4\right]$$

$$= \frac{1}{2}\left[2[16-1] - \left[\frac{64-1}{3}\right]\right] = \frac{1}{2}\left[2\times15 - \frac{63}{3}\right]$$

$$= 15-\frac{1}{2}\times21 = 15-10.5 = 4.5$$

④ CW $$\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dx\,dy = \int_0^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx = \int_0^a x^2\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x^2\cdot\frac{1}{2}\cdot a^2-x^2\,dx = \frac{1}{2}\int_0^a a^2x^2-x^4\,dx$$

$$= \frac{1}{2}\left[a^2\left[\frac{x^3}{3}\right]_0^a - \left[\frac{x^5}{5}\right]_0^a\right] = \frac{1}{2}\left[\frac{a^5}{3}-\frac{a^5}{5}\right]$$

$$= \frac{2a^5}{15}\times\frac{1}{2} = \frac{a^5}{15}$$

(5) $$\int_1^2\int_3^4 xy+e^y\,dy\,dx = \int_1^2\left[\frac{xy^2}{2}+e^y\right]_3^4 dx$$

$$= \int_1^2 \frac{x}{2}(16-9) + e^4 - e^3\,dx$$

$$= \left[\frac{7x^2}{4}\right]_1^2 + e^4(2-1) - e^3(2-1)$$

$$= \frac{7}{4}[4-1] + e^4 - e^3 = \frac{21}{4} + e^4 - e^3$$

(6) $$\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\left(\frac{y}{x}\right) dx\,dy = \int_{\pi/2}^{\pi}\int_0^{x^2}\frac{1}{x}\cos(y/x)\,dy\,dx$$

$$= \int_{\pi/2}^{\pi}\frac{1}{x}\cdot x\left[\sin(y/x)\right]_0^{x^2} dx = \int_{\pi/2}^{\pi}\sin x\,dx$$

$$= \left[-\cos x\right]_{\pi/2}^{\pi} = -[-1-0] = 1$$

(7) $$\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dy\,dx}{1+x^2+y^2}$$ 0.69

EVALUATION OF DOUBLE INTEGRALS OVER R

① $\iint_R xy\,dx\,dy \longrightarrow$ coordinate axes & line $x+y=1$

$$\int_0^1\int_0^{1-y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}\right]_0^{1-y} y\,dy$$

$$= \int_0^1 \frac{(1-y)^2 y}{2}\,dy = \frac{1}{2}\int_0^1 (1+y^2-2y)\,y\,dy$$

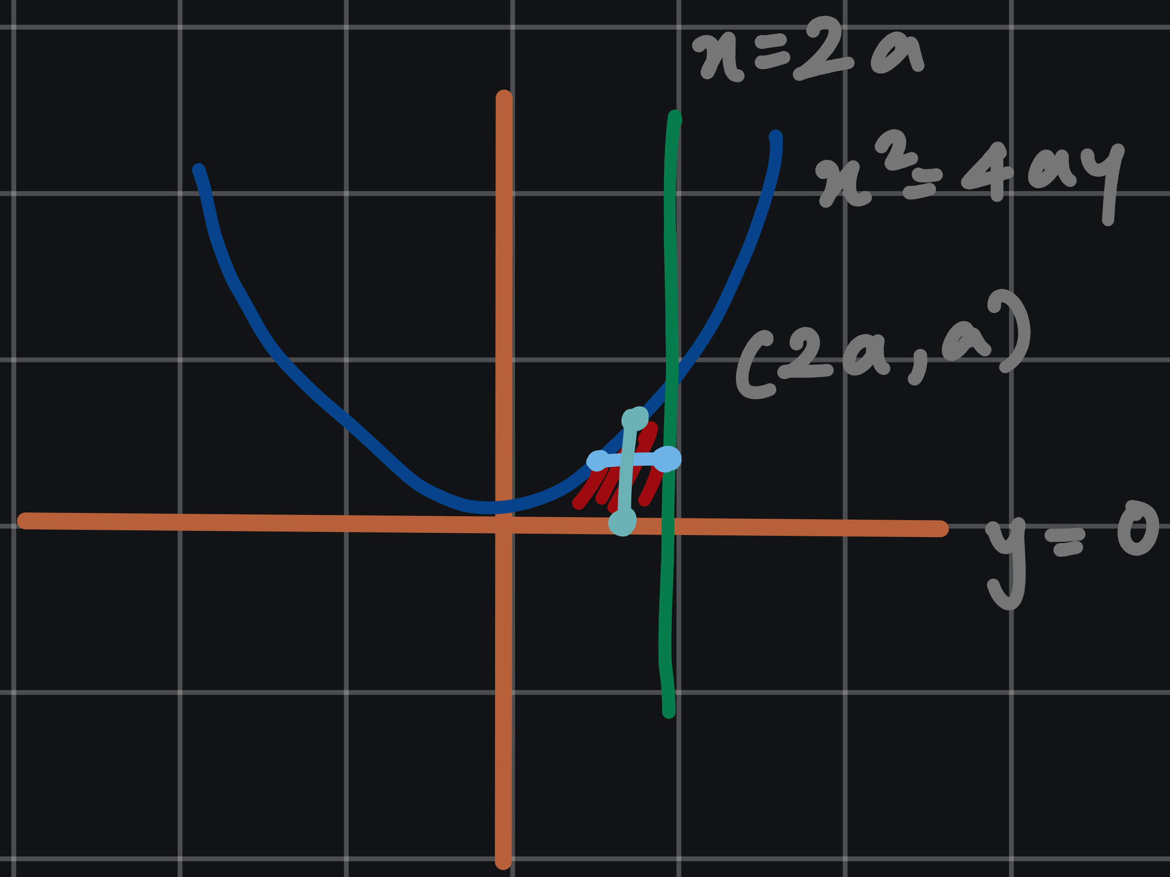
$$= \frac{1}{2}\int_0^1 y + y^3 - 2y^2\,dy = \frac{1}{2}\left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right] = 0.041$$

$$\int_0^1\int_0^{1-x} xy\,dy\,dx = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = \frac{1}{2}\int_0^1 (1+x^2-2x)\,x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{3}\right] = 0.041$$

② $\iint_R dx\,dy \longrightarrow$ x-axis, $x = 2a$, parabola $x^2 = 4ay$



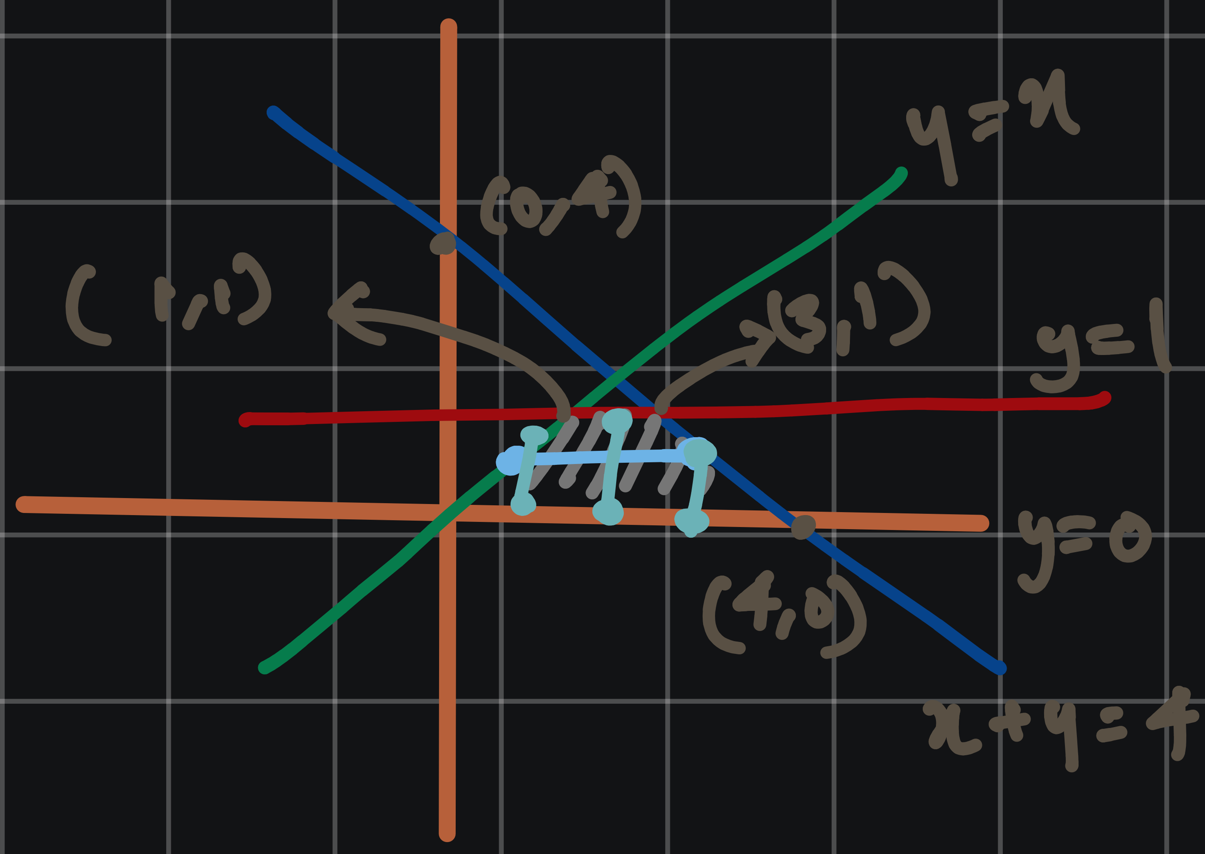
$$\int_0^a\int_{2\sqrt{ay}}^{2a} dx\,dy = \int_0^a \left(2a - 2\sqrt{ay}\right) dy$$

$$= (2ay)_0^a - 2a^{1/2}\left(\frac{y^{1/2+1}}{1/2+1}\right)_0^a$$

$$= 2a^2 - 2a^{1/2}\,\frac{a^{3/2}}{3}\cdot 2 = 2a^2 - \frac{4}{3}a^2 = \frac{2a^2}{3}$$

$$\int_0^{2a}\int_0^{x^2/4a} dy\,dx = \int_0^{2a} \frac{x^2}{4a}\,dx = \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{2a} = \frac{1}{4a}\,\frac{8a^3}{3} = \frac{2a^2}{3}$$

(3) $\iint_R dx\,dy \longrightarrow$ line $x=y$, $x+y=4$, $y=1$ & $y=0$



$$\int_0^1\int_{y}^{4-y} dx\,dy = \int_0^1 (4-y-y)\,dy = \int_0^1 4-2y\,dy$$

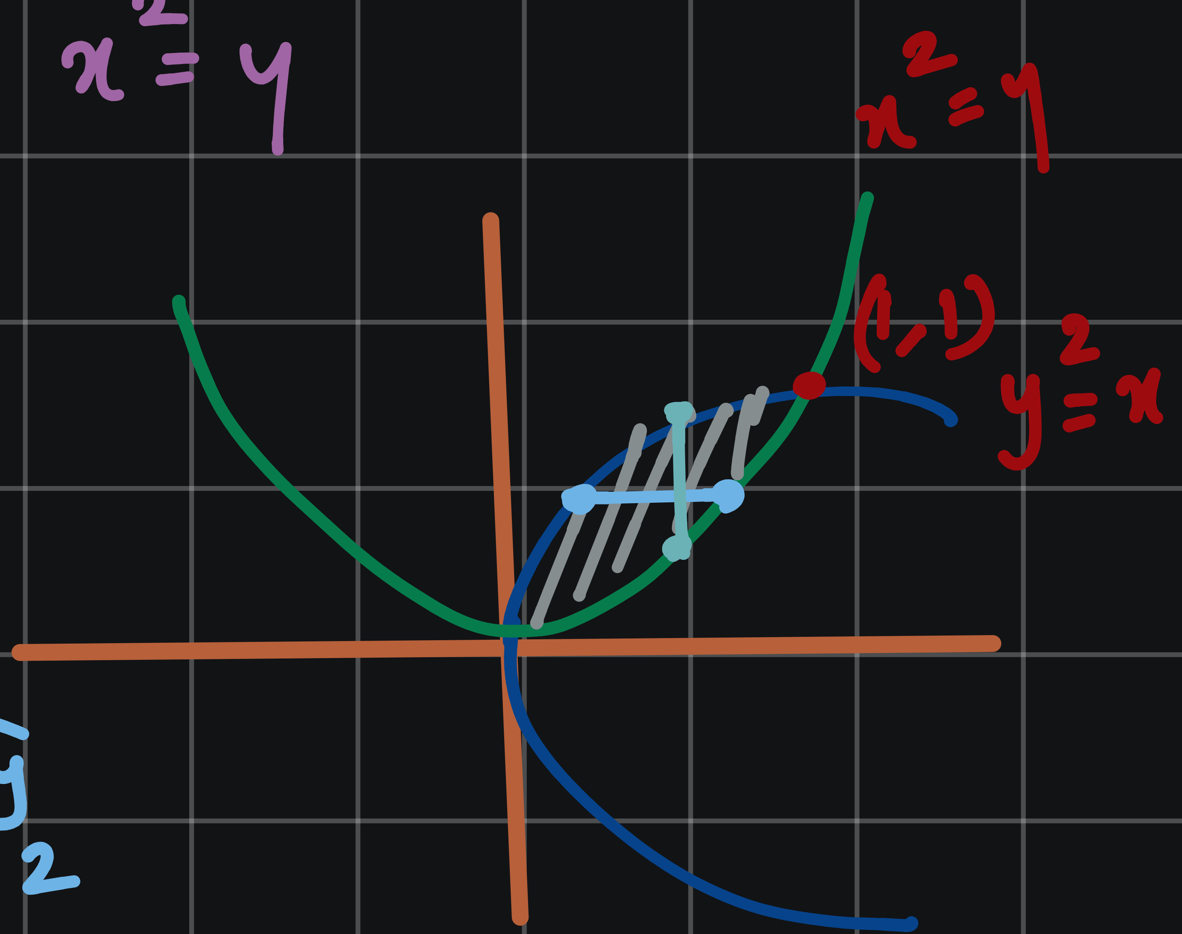
$$= \left[4y - \frac{2y^2}{2}\right]_0^1 = 4-1 = 3$$

$$\int_0^1\int_0^x dy\,dx + \int_1^3\int_0^1 dy\,dx + \int_3^4\int_0^{4-x} dy\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + (3-1) + \left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{2} + 2 + 4 - \frac{7}{2} = 6-3 = 3$$

(4) $\iint_R xy(x+y)\,dx\,dy = 3/28 \longrightarrow y^2=x$ & $x^2=y$



$$\int_0^1\int_{y^2}^{\sqrt{y}} x^2y + xy^2\,dx\,dy = \int_0^1\left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_{y^2}^{\sqrt{y}} dy$$

$$= \frac{1}{3}\int_0^1 y\,[x^3]_{y^2}^{\sqrt{y}} + \frac{1}{2}\int_0^1 y^2\,[x^2]_{y^2}^{\sqrt{y}}$$

$$= \frac{1}{3}\int_0^1 y\,[y^{3/2} - y^6]\,dy + \frac{1}{2}\int_0^1 y^2\,[y - y^4]\,dy$$

$$= \frac{1}{3}\int_0^1 (y^{5/2} - y^7)\,dy + \frac{1}{2}\int_0^1 (y^3 - y^6)\,dy$$

$$= \frac{1}{3}\left[\frac{y^{5/2+1}}{5/2+1} - \frac{y^8}{8}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^7}{7}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{2}{7} - \frac{1}{8}\right] + \frac{1}{2}\left[\frac{1}{4} - \frac{1}{7}\right] = 3/28$$

(5) $\iint_R \sqrt{xy - y^2}\, dx\,dy$ over Δ $(0,0)$ $(10,1)$ $(1,1)$

$y = \frac{1}{10}x \Rightarrow x = 10y$

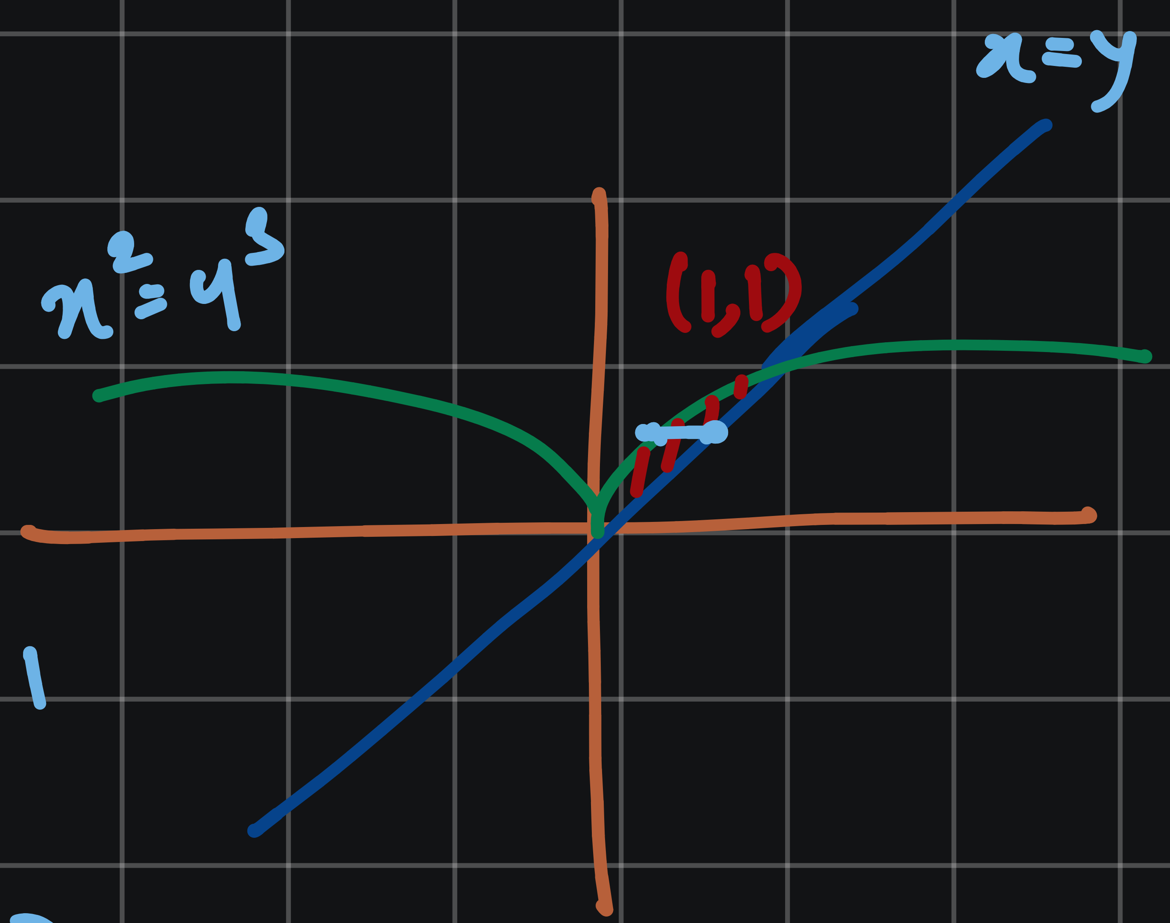
$$\int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\,dy = \int_0^1 \int_y^{10y} (xy - y^2)^{1/2}\, dx\,dy$$

(6) $x^2 = y^3$ & $x = y$ → Area bounded

$$A = \iint_R dx\,dy = \int_0^1 \int_{y^{3/2}}^{y} dx\,dy$$

$$= \int_0^1 y - y^{3/2}\, dy = \left[\frac{y^2}{2} - \frac{y^{5/2}}{5} \cdot 2 \right]_0^1$$

$$= \frac{1}{2} - \frac{2}{5} = \frac{3}{10}$$



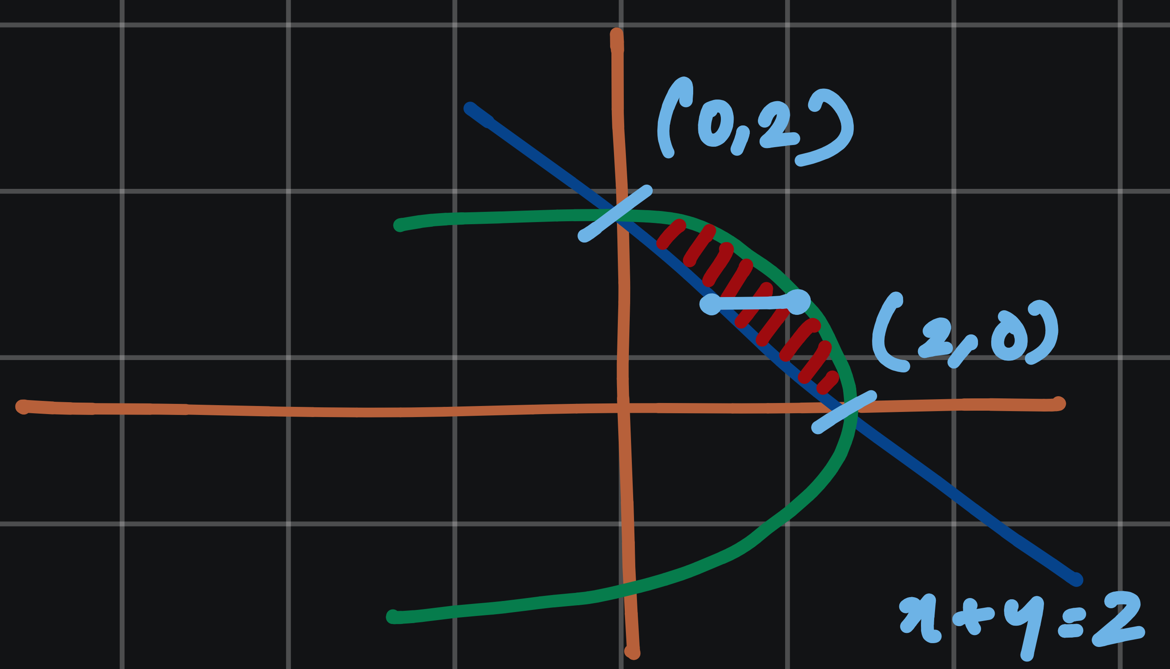
CW

(7) Area → $y = 2 - x$ & $y^2 = 4 - 2x$

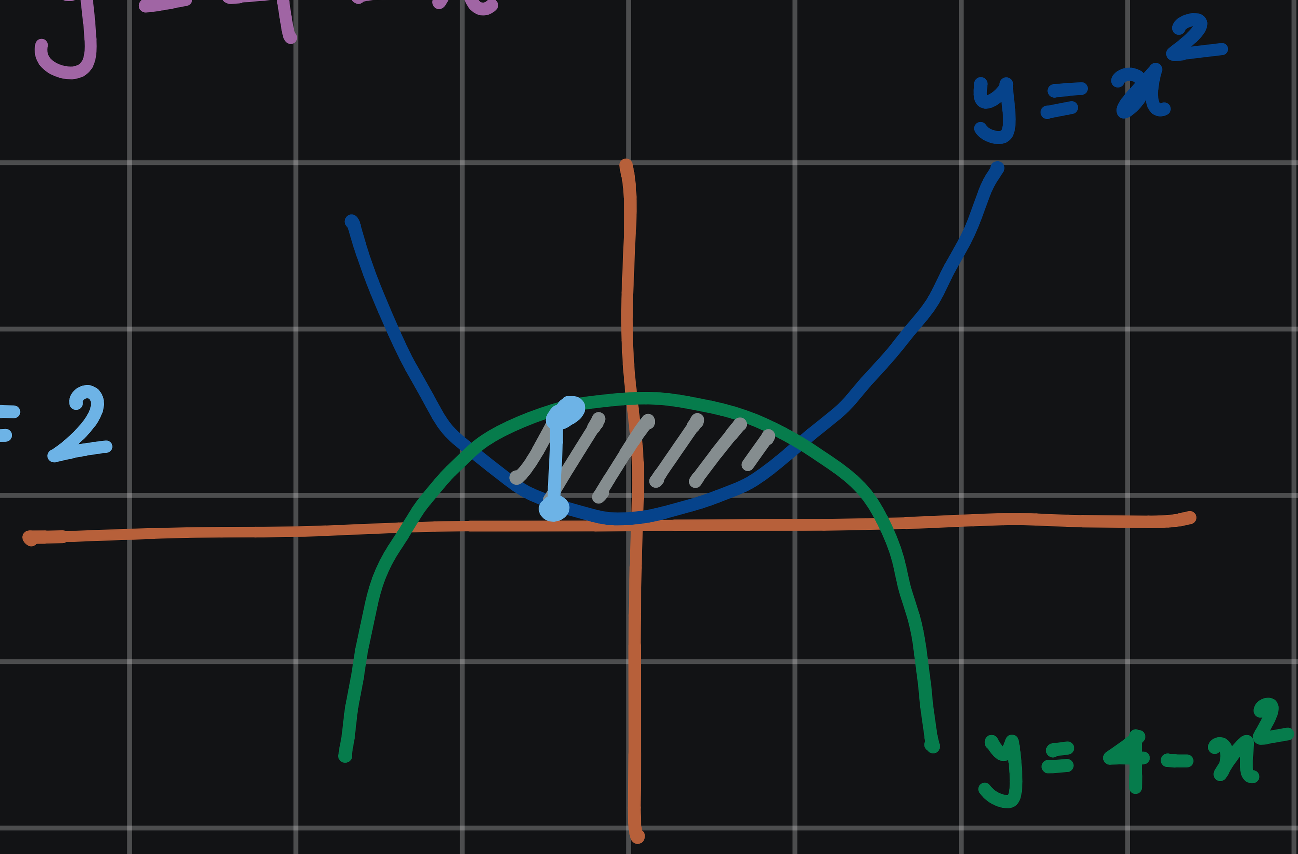
$$A = \int_0^2 \int_{2-y}^{4-y^2/2} dx\,dy = \int_0^2 \frac{4 - y^2 - 4 + 2y}{2}\, dy$$

$$= \frac{1}{2}\int_0^2 2y - y^2\, dy = \frac{1}{2}\left[\frac{2y^2}{2} - \frac{y^3}{3} \right]_0^2$$

$$= \frac{1}{2}\left[4 - \frac{8}{3} \right] = \frac{2}{3}$$



⑧ Area bounded $\longrightarrow$ $y = x^2$ & $y = 4 - x^2$



$y = x^2$ & $y = 4 - x^2$

$x^2 + x^2 = 4 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$

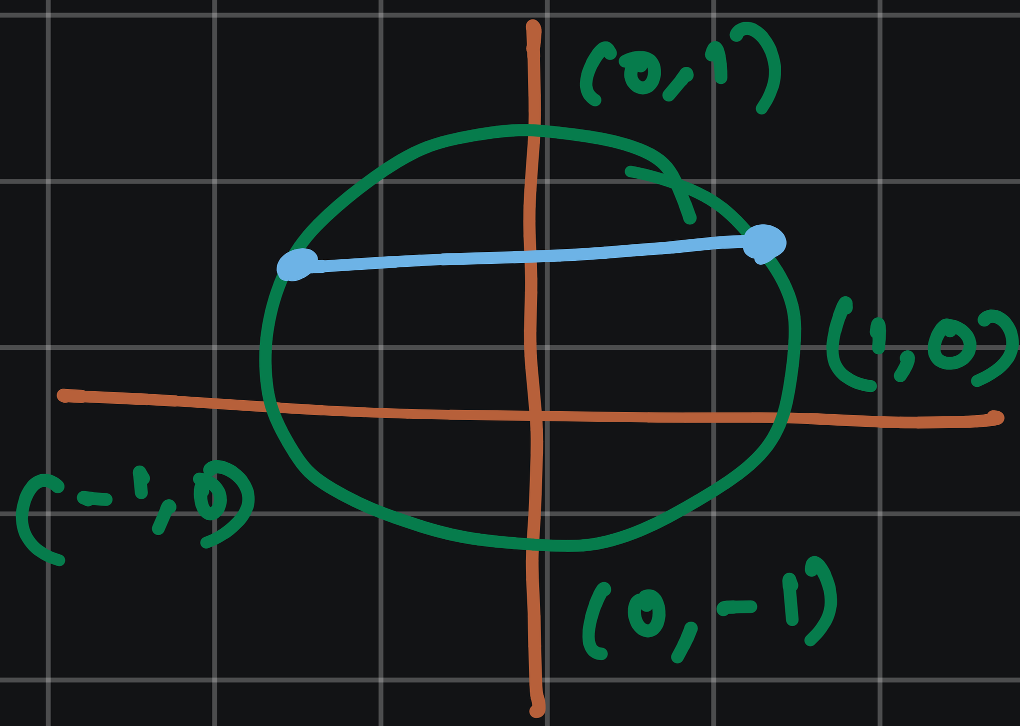
$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^{4-x^2} dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - x^2 - x^2 \, dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2 \, dx = \left[4x - \frac{2x^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{16\sqrt{2}}{3} \text{ sq.units}$$

⑨ Volume of solid $\Longrightarrow$ cylinder $x^2 + y^2 = 1$, plane $y + z = 1$ and $z = 0$

$$V = \iint_R f(x,y)\, dx\, dy \qquad f(x,y) \Rightarrow z = 1 - y$$



$$V = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (1-y)\, dx\, dy$$

$x^2 + y^2 = 1$

$x = \pm\sqrt{1 - y^2}$

$$= \int_{-1}^{1} (1-y) \left[2\sqrt{1-y^2} \right] dy$$

$$= 2 \int_0^1 (1-y^2)^{1/2 + 1} dy$$

$$= 2 \int_0^1 (1 - y^2)^{3/2} dy = \pi$$

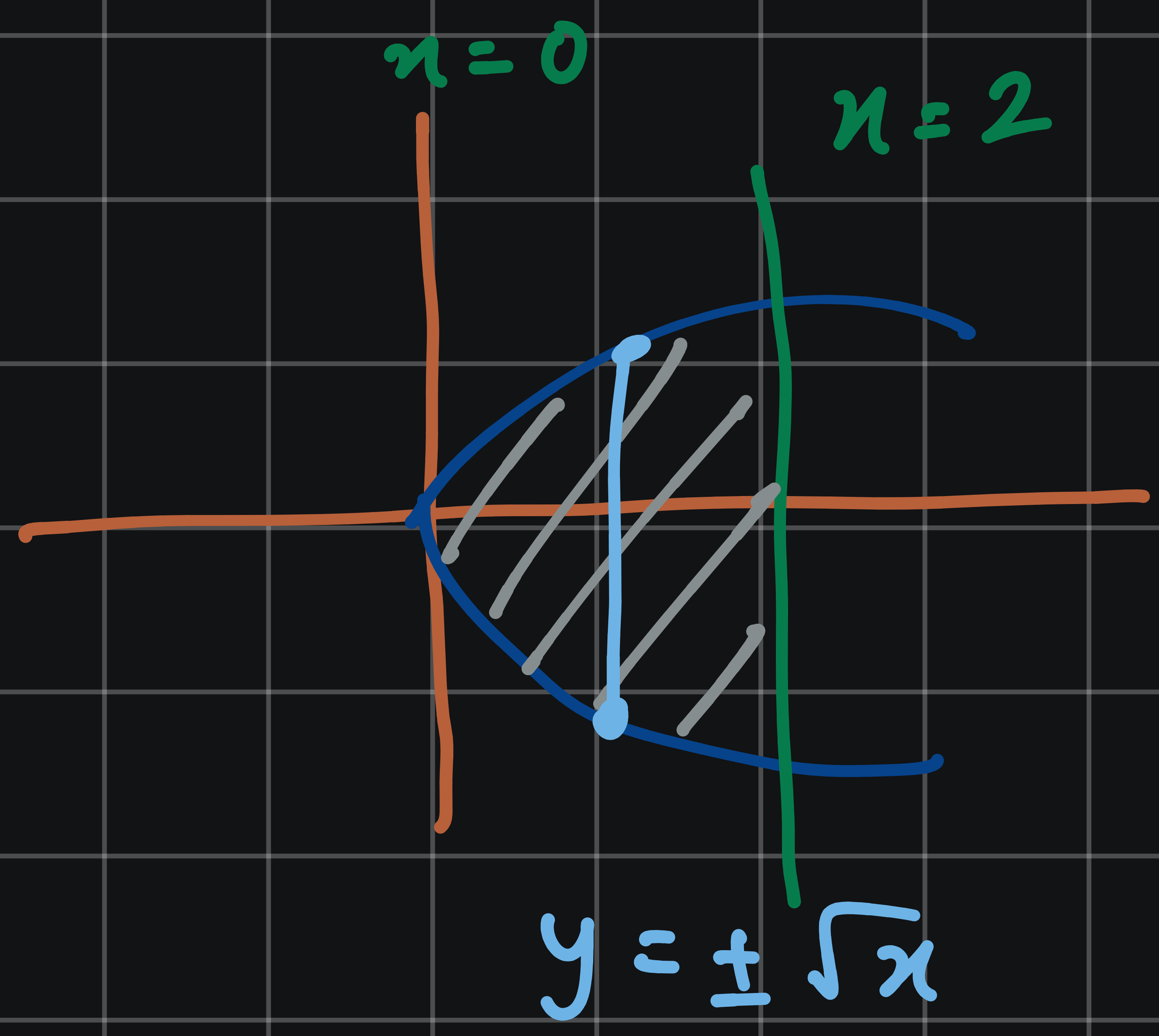
(10) Volume $\Longrightarrow$ below plane $z = 2x+3$, above x-y
$y^2 = x$, $x = 0$, $x = 2$

$$V = \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} 2x+3 \, dy\,dx$$

$$= \int_0^2 2x(2\sqrt{x}) + 3(2\sqrt{x})\,dx$$

$$= \int_0^2 4x^{3/2} + 6x^{1/2}\,dx$$

$$= 20.36$$



JACOBIAN

cartesian to polar

$x = r\cos\theta, \quad y = r\sin\theta$

$$J\left(\frac{x,y}{r,\theta}\right) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

cartesian to cylindrical coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

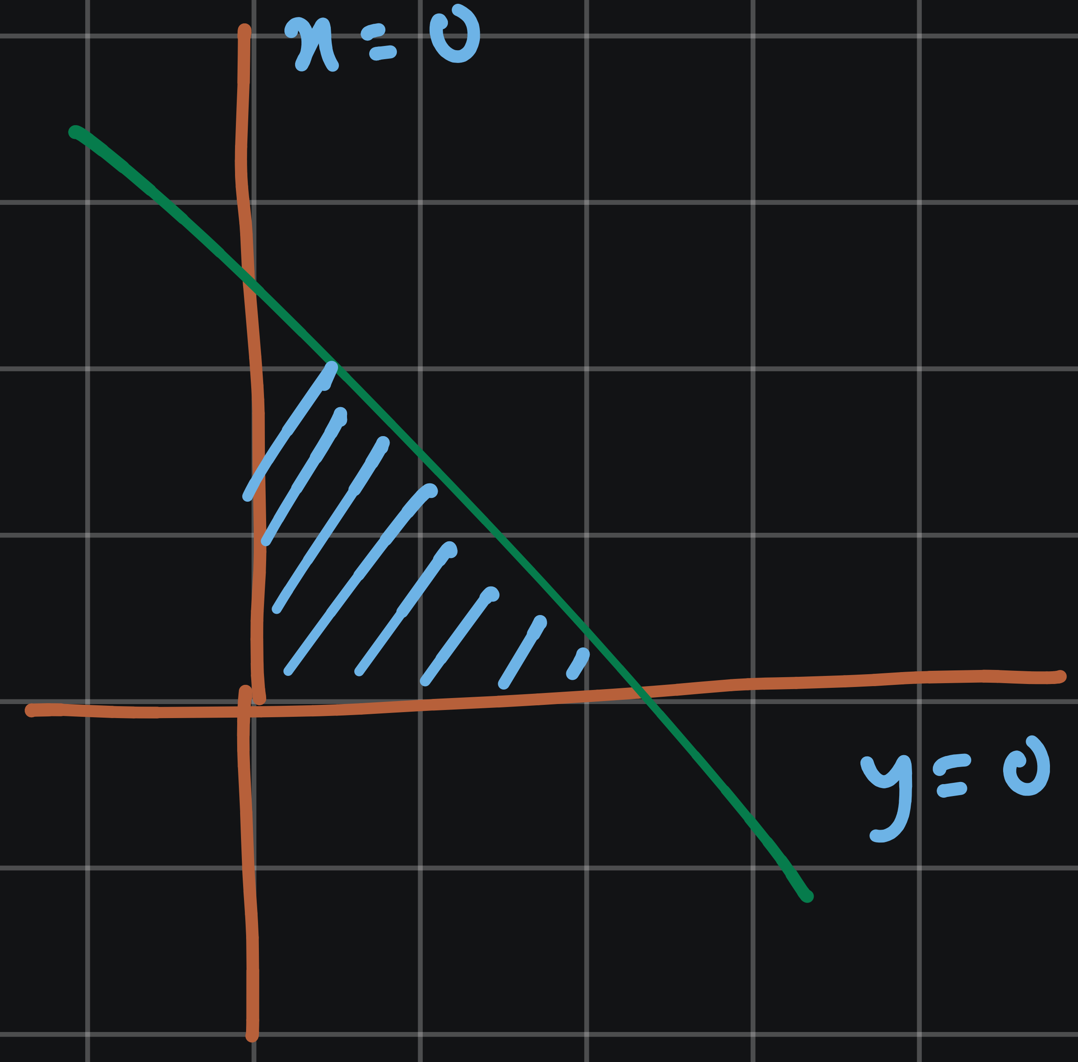
cartesian to polar coordinates

$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta$

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

(1) cw $\displaystyle\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx = \frac{1}{2}(e-1) \longrightarrow x+y=u\ ,\ y=uv$



$x+y=u \qquad y=uv$

$x=u-y$

$x = u(1-v)\ ,\ y=uv$

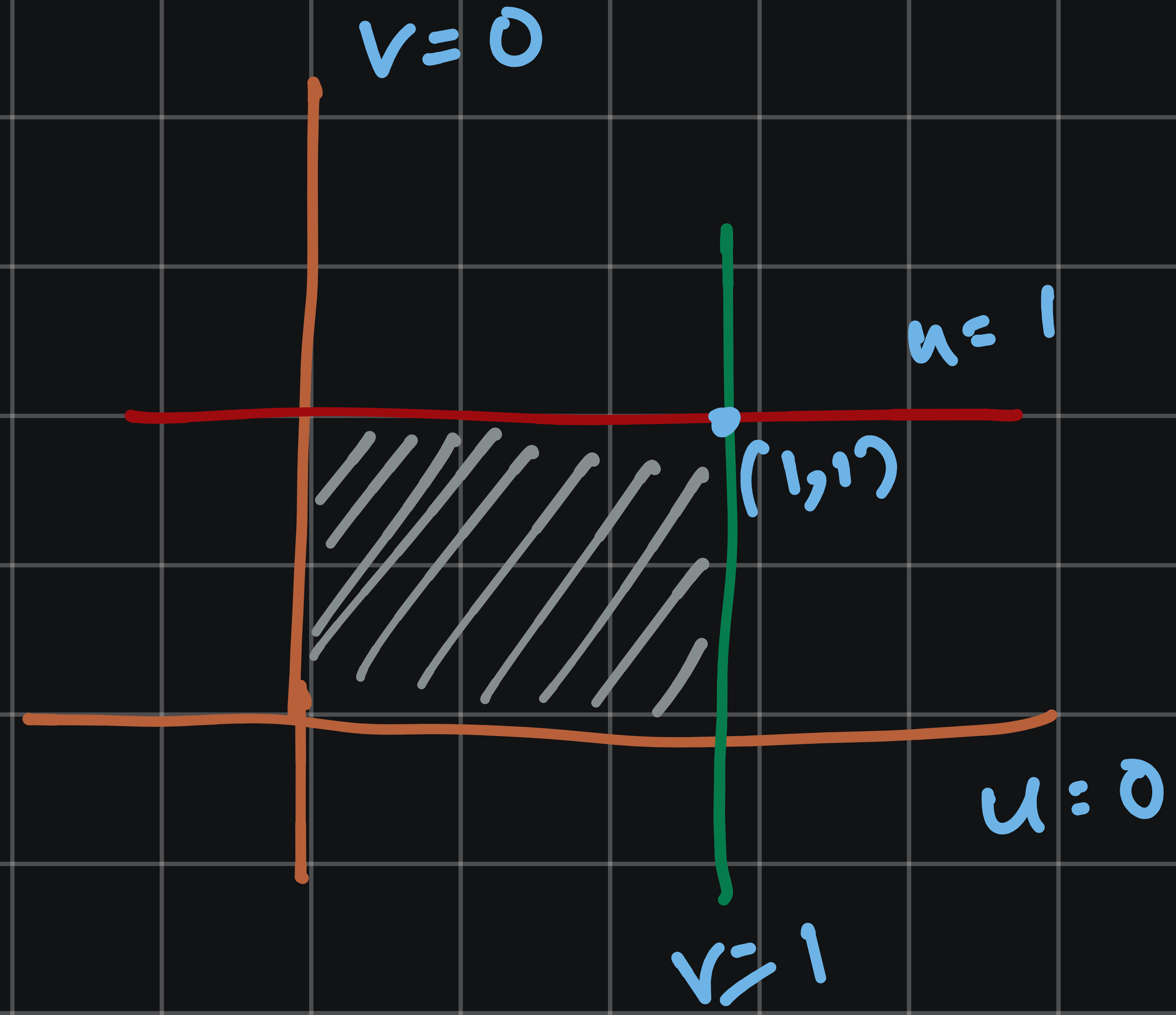
Bounded by $y=1-x \Rightarrow y+x=1$

$y=0,\ x=0\ ,\ x=1$

If $x=0 \Rightarrow u=0$ or $1-v=0 \Rightarrow v=1$

If $y=0 \Rightarrow u=0$ or $v=0$

If $y+x=1 \Rightarrow u=1$



$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u-uv+uv = u$$

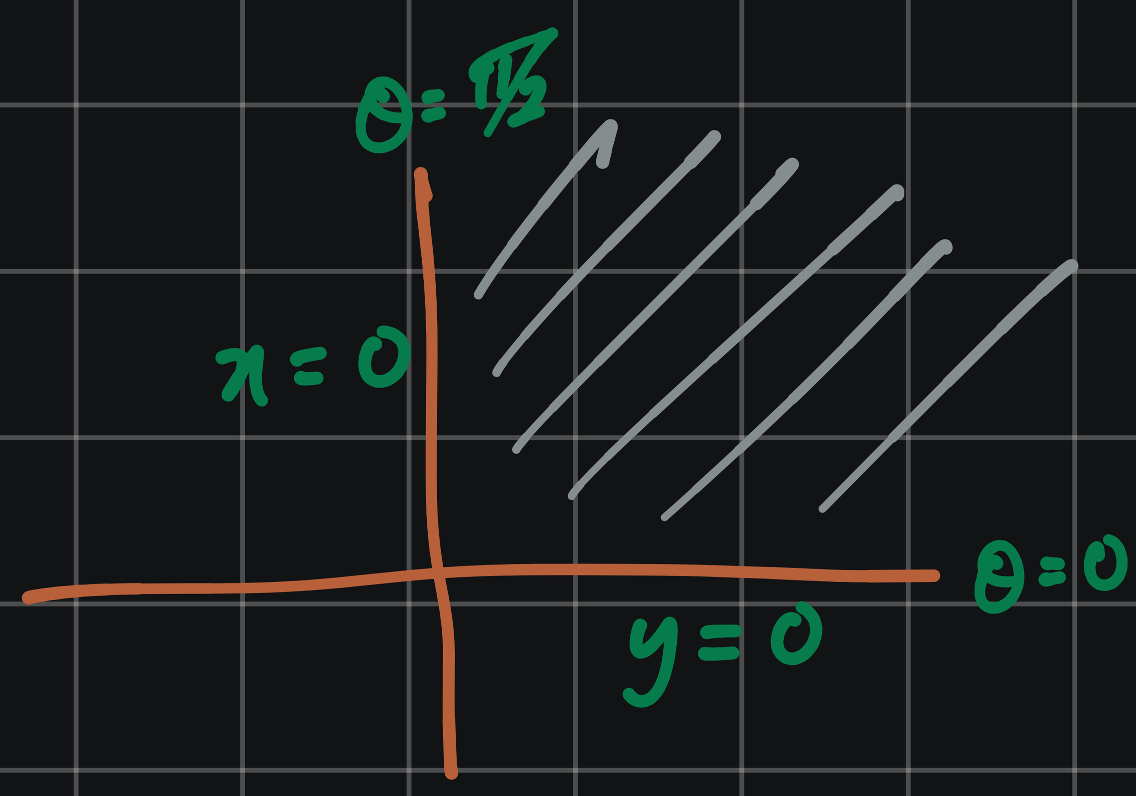
$$I = \int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx = \int_0^1\int_0^1 e^{\frac{uv}{-uv+u+uv}}\cdot u\cdot dv\,du$$

$$= \int_0^1\int_0^1 e^{v}\cdot u\,dv\,du = \int_0^1 u(e^1-e^0)\,du = \int_0^1 (ue-u)\,du$$

$$= \left[\frac{u^2e}{2} - \frac{u^2}{2}\right]_0^1 = \frac{e}{2} - \frac{1}{2} = \frac{1}{2}(e-1)\ //$$

② CW

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx \longrightarrow \text{polar coordinates}$$



$x=0, x=\infty$
$y=0, y=\infty$

$$\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\infty} e^{-r^2}\cdot r\,dr = \int_0^{\pi/2} d\theta \int_0^{\infty} e^{-t}\frac{dt}{2}$$

$r^2 = t$
$2r\,dr = dt$
$r\,dr = dt/2$
$r=0, t=0$
$r=\infty, t=\infty$

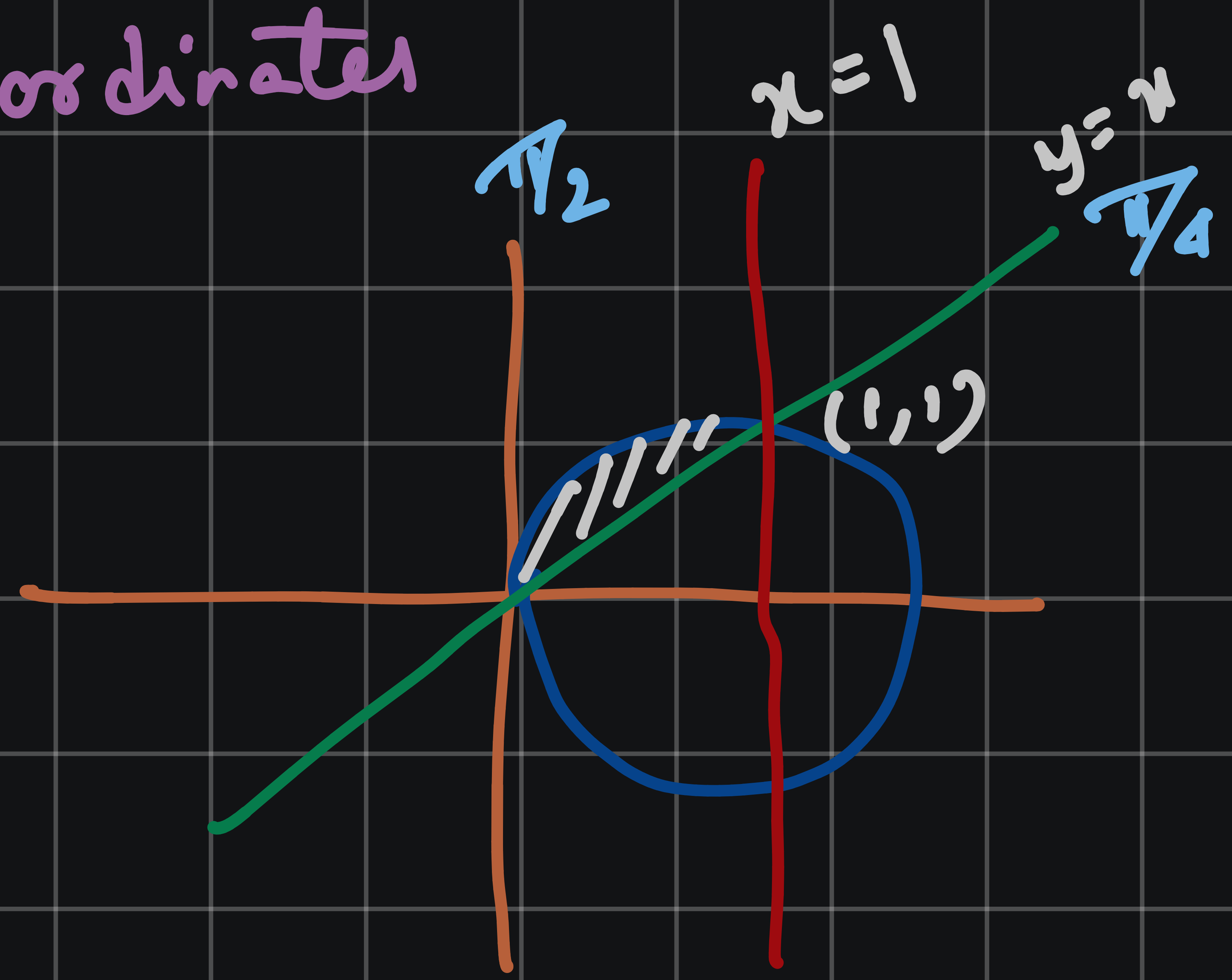
$$= \frac{1}{2}\cdot\frac{\pi}{2}\int_0^{\infty} e^{-t} = \frac{\pi}{4}\left[-e^{-t}\right]_0^{\infty}$$

$$= \frac{\pi}{4}\left[-(-1)\right] = \pi/4$$

③ CW

$$\int_0^1\int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx \longrightarrow \text{polar coordinates}$$

Bounded by $x=0, x=1$
$y=x$, $y^2 = 2x - x^2$
$x^2+y^2-2x+1-1=0$
$(x-1)^2+y^2=1$



$x = r\cos\theta$, $y = r\sin\theta$

$r^{\cancel{2}} = 2\cancel{r}\cos\theta$
$r = 2\cos\theta$

$$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^2\cdot r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{16\cos^4\theta}{4}\,d\theta = 4\int_{\pi/4}^{\pi/2}\cos^4\theta\,d\theta = 0.178$$

④ Find the area inside the circle $r = 2a\cos\theta$
CW and outside the circle $r = a$

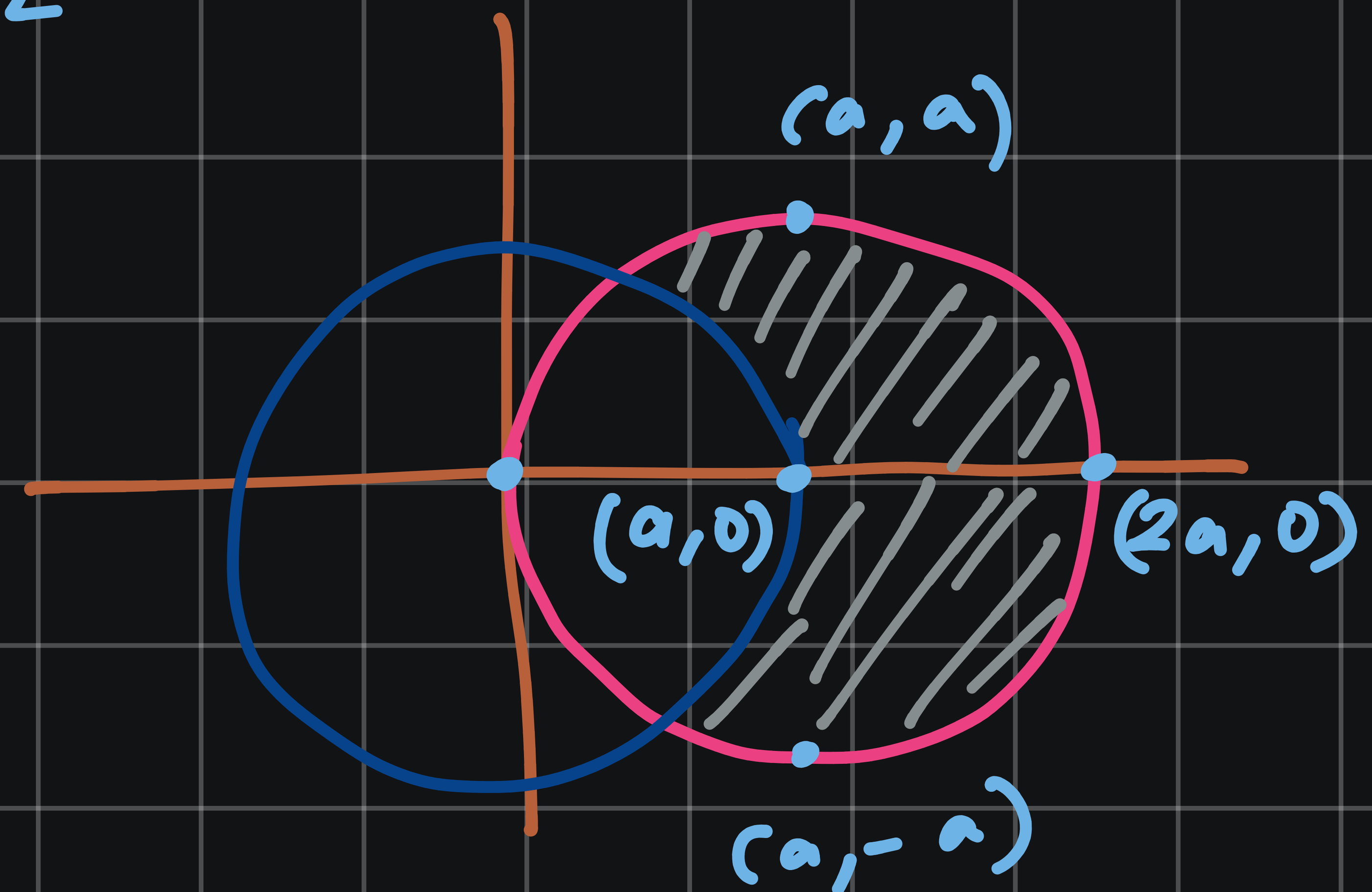
$r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$

$x^2 + y^2 - 2ax + a^2 - a^2 = 0$

$(x-a)^2 + y^2 = a^2$

$x^2 + y^2 = a^2$



$$A = \iint dx\,dy = \int_{-\pi/3}^{\pi/3} \int_{a}^{2a\cos\theta} r\,dr\,d\theta$$

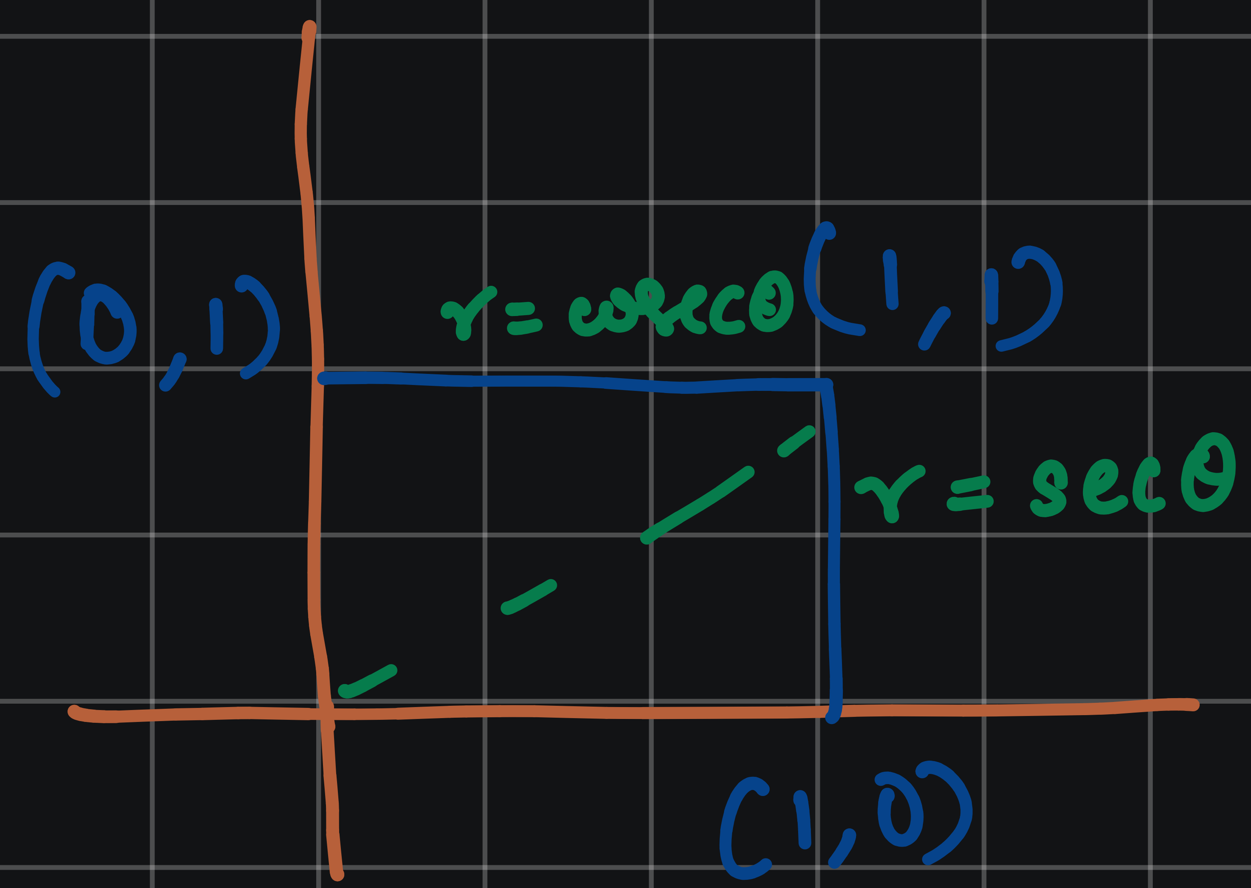
$$= \int_{-\pi/3}^{\pi/3} \left[\frac{r^2}{2}\right]_a^{2a\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3} \frac{4a^2\cos^2\theta - a^2}{2}\, d\theta$$

$$= \frac{a^2}{2} \int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1\, d\theta$$

⑤ $\int_0^1 dx \int_0^1 f(x,y)\, dy \longrightarrow$ polar coordinates
CW

$r : 0 \longrightarrow \sec\theta$ | $r : 0 \longrightarrow \operatorname{cosec}\theta$

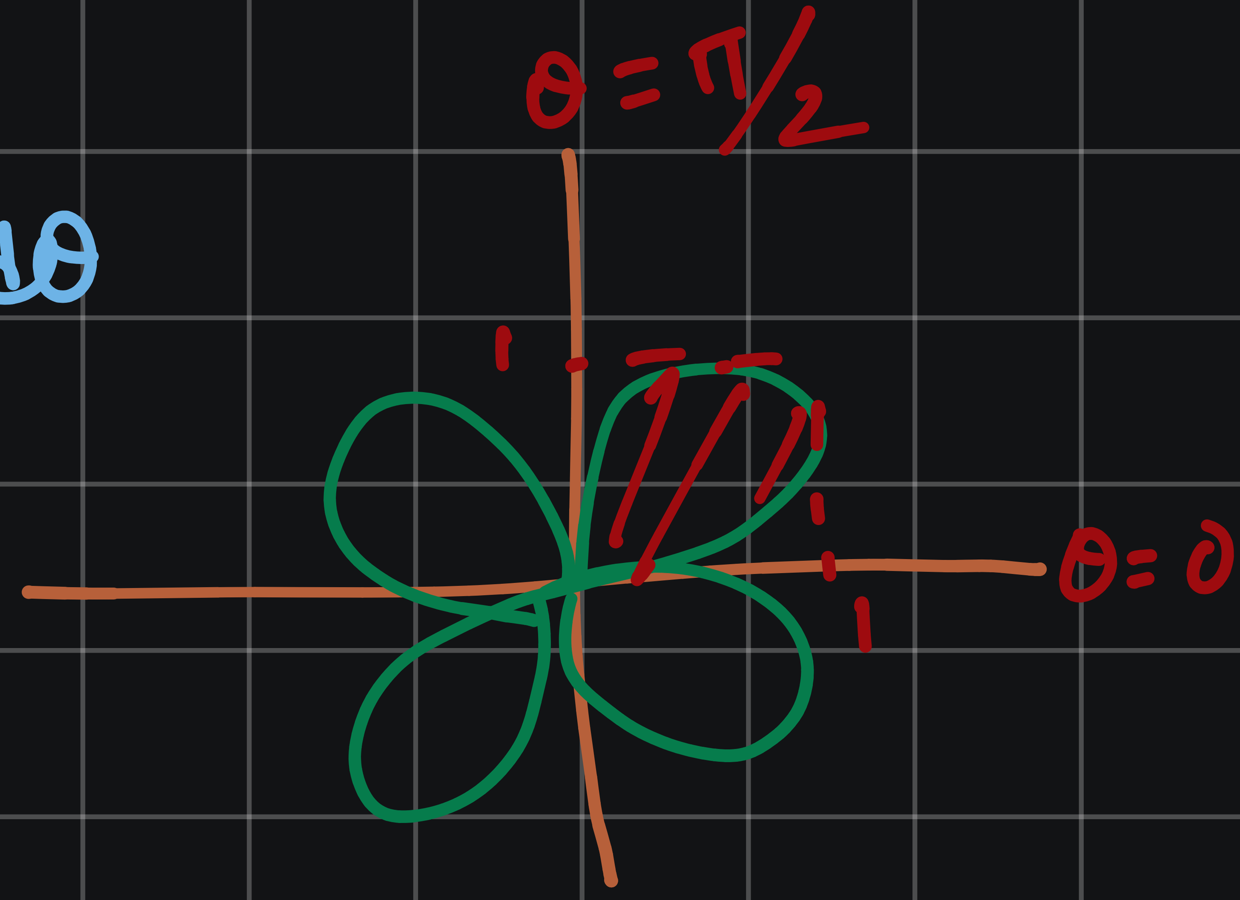
$\theta : 0 \rightarrow \pi/4$ | $r : \pi/4 \rightarrow \pi/2$



$$\int_0^{\pi/4} \int_0^{\sec\theta} \phi(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$+ \int_{\pi/4}^{\pi/2} \int_0^{\operatorname{cosec}\theta} \phi(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(6) HW $\iint_R xy\,dx\,dy \Rightarrow \begin{array}{l} r=\sin 2\theta \\ 0\le\theta\le\pi/2 \end{array}$



$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\sin 2\theta}\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{\sin^4 2\theta}{4}\cdot\sin 2\theta\,d\theta = \frac{1}{8}\int_0^{\pi/2}\sin^5 2\theta\,d\theta$$

$$= \frac{1}{15}$$

(7) HW $\displaystyle\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy \Rightarrow$ polar coordinates

Bounded by $x=y,\ x=a$

$y=0,\ y=a \;\big|\; x=r\cos\theta \Rightarrow a\sec\theta = r$

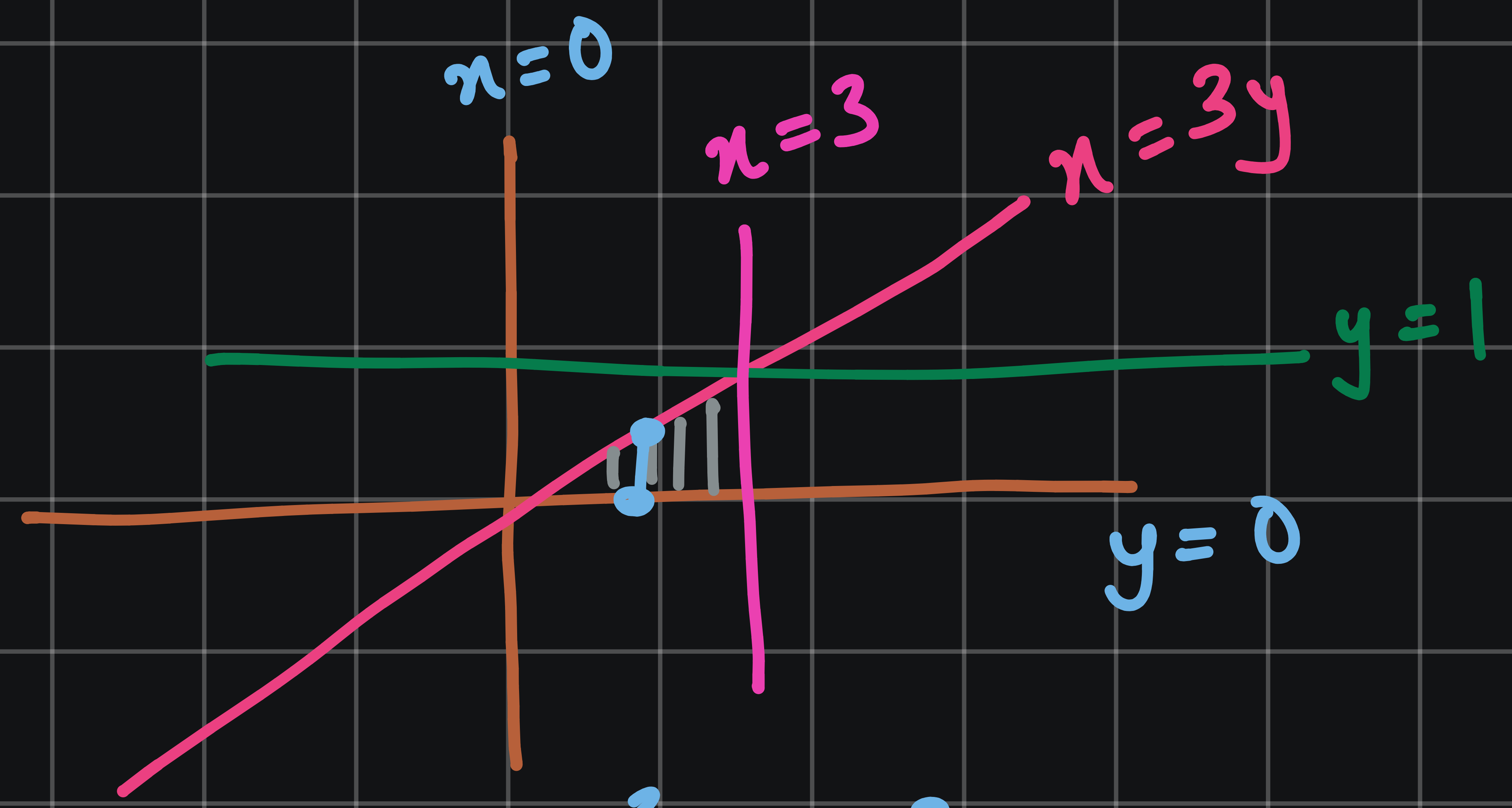
$$\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r^2\cos^2\theta}{r}\cdot r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\cos^2\theta\left[\frac{r^3}{3}\right]_0^{a\sec\theta}d\theta = \int_0^{\pi/4}\cos^2\theta\,\frac{a^3\sec^3\theta}{3}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta = \frac{a^3}{3}\log(\sqrt{2}+1)$$

CHANGING ORDER OF INTEGRATION

① cw $\int_0^1 \int_{3y}^{3} e^{x^2}\,dx\,dy$



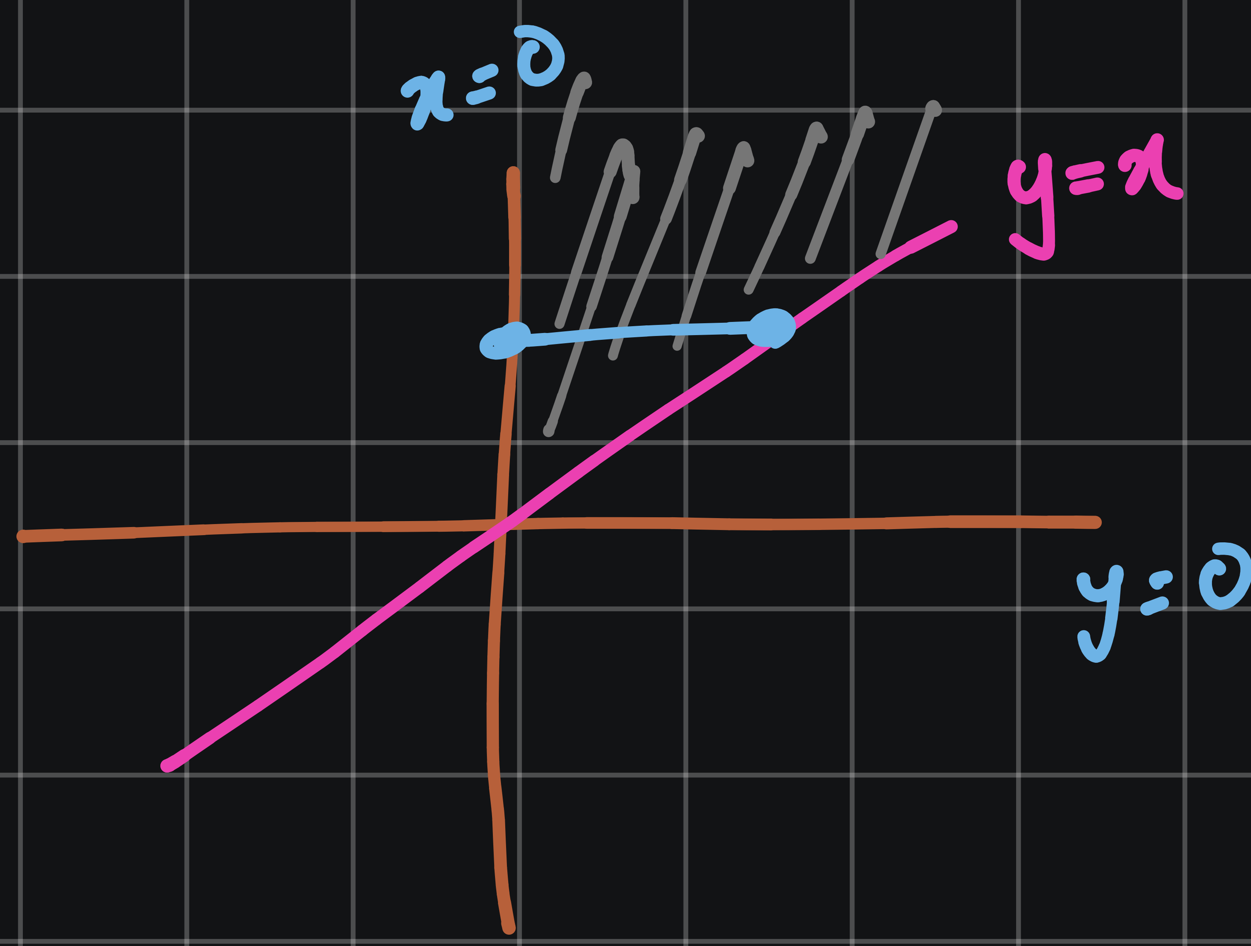
$x = 3y \qquad x = 3$

$y = 0 \qquad y = 1$

$$\int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 e^{x^2}\,\frac{x}{3}\,dx = \frac{1}{3}\int_0^3 x e^{x^2}\,dx$$

$$= \frac{1}{6}\left(e^9 - 1\right)$$

② cw $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$



$y = x \quad , y = \infty$

$x = 0 \quad , x = \infty$

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy = \left(-e^{-y}\right)_0^\infty = -(-1) = 1$$

③ cw $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

$y = x^2 \qquad y = 2 - x$

$x = 0 \qquad x = 1$

$$I = \int_1^2 \int_0^{2-y} xy\,dx\,dy + \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$=$

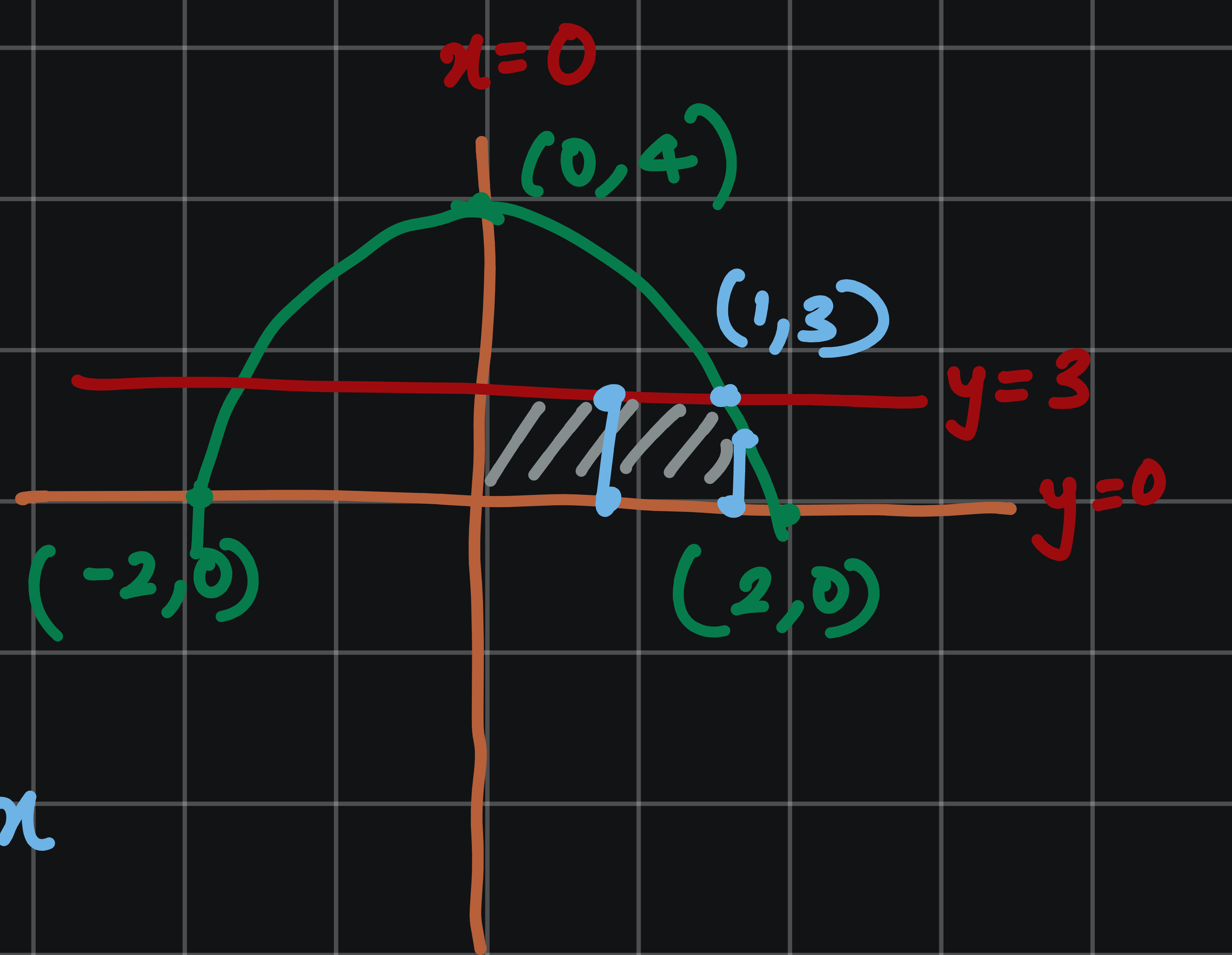
4

$$\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\,dx\,dy$$

$x=0$ $\quad x^2 = 4-y$

$y=0$ $\quad y=3$

x	0	2	-2
y	4	0	0



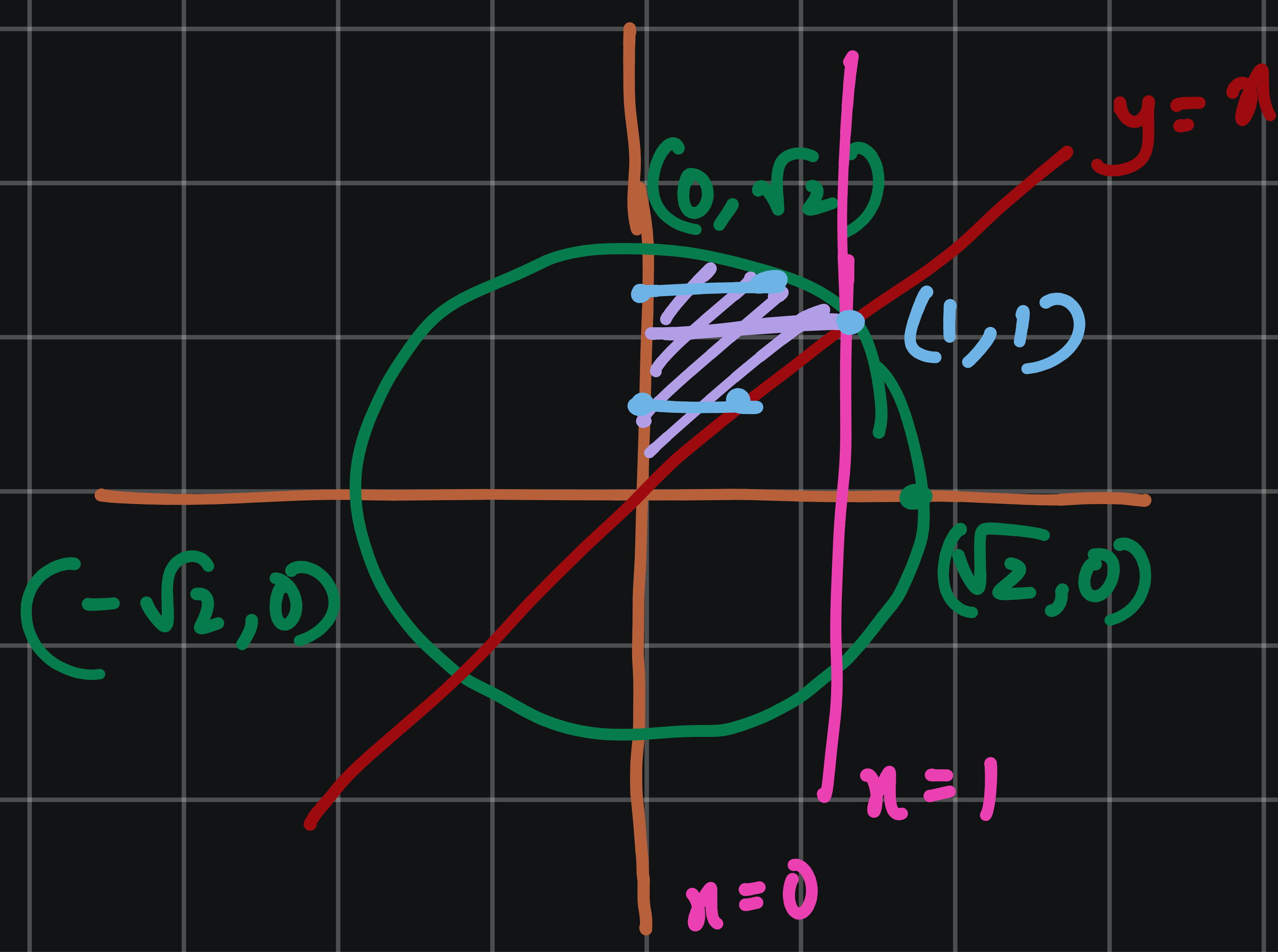
$$I = \int_0^1 \int_0^3 (x+y)\,dy\,dx + \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^3 dx + \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_0^1 3x + \frac{9}{2}\,dx$$

$$+ \int_1^2 x(4-x^2) + \frac{(4-x^2)^2}{2}\,dx$$

$$= \frac{3}{2} + \frac{9}{2} + \frac{241}{60} = 6 + \frac{241}{60} = \frac{601}{60}$$

⑤ CW $\displaystyle\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$

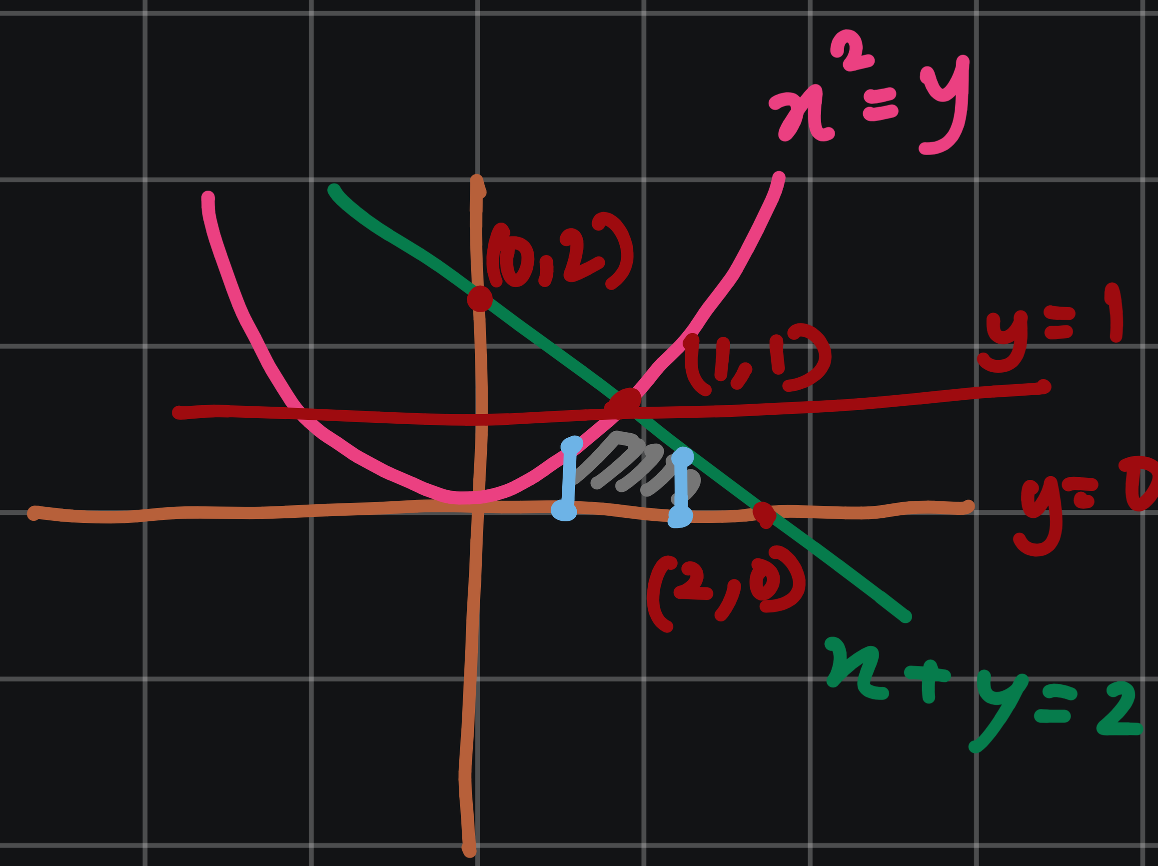


$y=x$ $\quad y^2+x^2=2$

$x=0$ $\quad x=1$

$$I=\int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy+\int_1^{\sqrt2}\int_0^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

⑦ CW $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-y} dx\,dy$



$x^2=y$ $\quad x=2-y$

$y=0$ $\quad y=1$

$$I=\int_0^1\int_0^{x^2} dy\,dx+\int_1^2\int_0^{2-x} dy\,dx$$

$$=\int_0^1 x^2\,dx+\int_1^2 2-x\,dx=\frac13+2-2+\frac12$$

$$=5/6$$

TRIPLE INTEGRATION

① CW $\int_2^3\int_1^2\int_2^5 xy^2\,dz\,dy\,dx$

$$I = \int_2^3\int_1^2 3xy^2\,dy\,dx = \int_2^3 \left(\frac{3xy^3}{3}\right)_1^2 dx = \int_2^3 7x\,dx$$

$$= 7\left(\frac{x^2}{2}\right)_2^3 = \frac{35}{2}$$

② CW $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz$ Ans = 0

$$\int_{-1}^{1}\int_0^z \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_0^z (2zx + 2zx + 2z^2)\,dx\,dz$$

$$= \int_{-1}^{1}\int_0^z (4zx + 2z^2)\,dx\,dz$$

$$= \int_{-1}^{1} \left(\frac{4x^2}{2}z + 2z^2x\right)_0^z dz$$

$$= \int_{-1}^{1} (2z^3 + 2z^3)\,dz = 4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$= 1 - 1 = 0$$

③ Volume of a solid bounded by surfaces
CW $z=0$, $z=1-x^2-y^2$, $y=0$, $y=1-x$, $x=0$ and $x=1$

$$V=\int_0^1\int_0^{1-x}\int_0^{1-x^2-y^2} dz\,dy\,dx = \int_0^1\int_0^{1-x} 1-x^2-y^2\,dy\,d\cancel{z}$$

$$=\int_0^1\left(y-x^2y-y^3/3\right)_0^{1-x} dx$$

$$=\int_0^1 1-x-x^2+x^3-\frac{(1-x)^3}{3}\,dx = 1/3$$

AVERAGE VALUE OF A FUNCTION OF 3 VAR

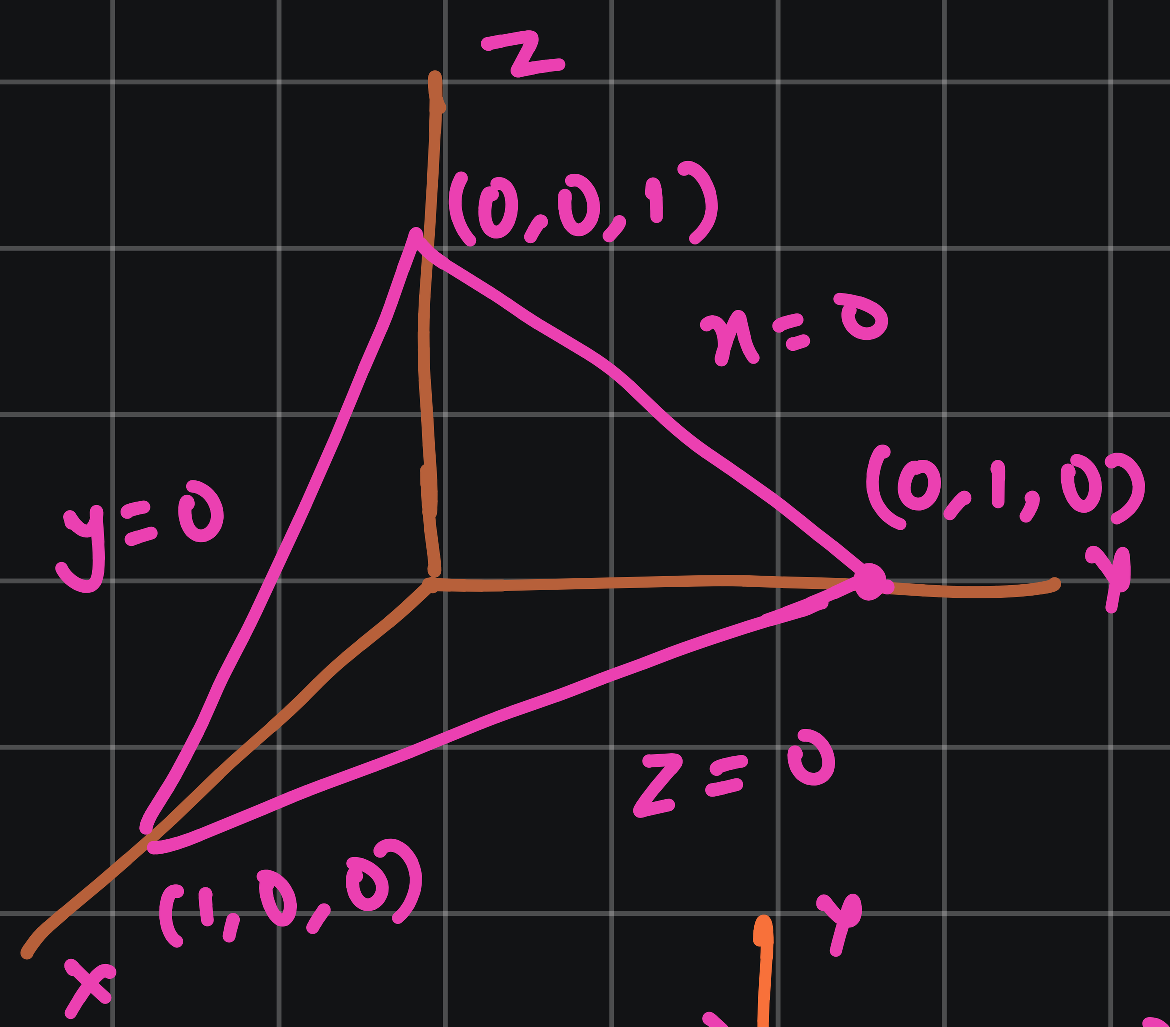
$$\text{Avg value} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV$$

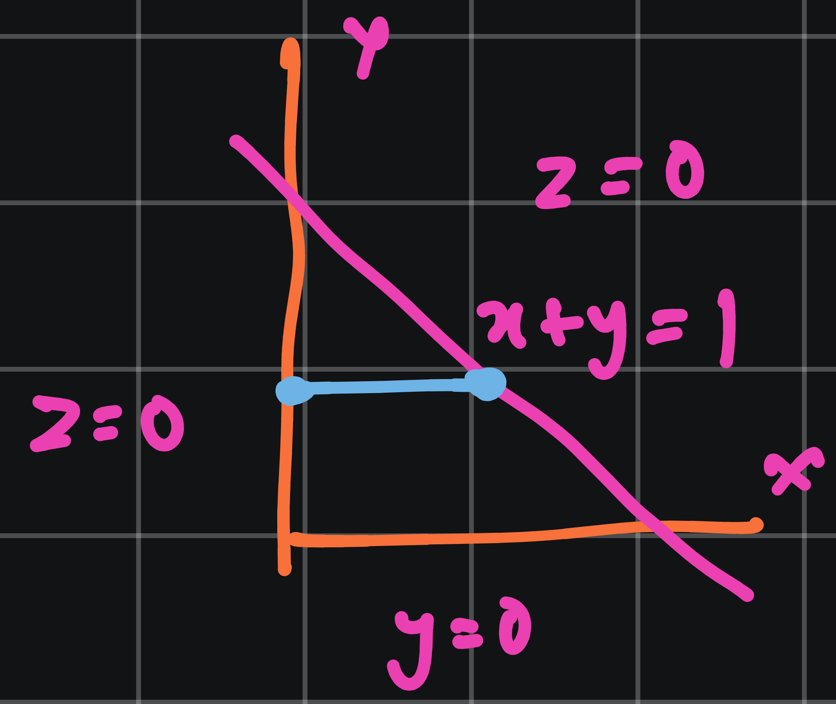
where $V(E) = \iiint 1\, dV$

Q. The temperature at a point (x,y,z) of a solid
cw E bounded by the plane $x+y+z=1$ is
$\frac{1}{(1+x+y+z)^3}$ degree celcius. Find the average T over the solid.

$x+y+z=1 \Rightarrow z = 1-x-y$

$x+y=1 \Rightarrow y = 1-x$





$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 1-x-y \; dy\, dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 1 - x - x + x^2 - \frac{(1-x)^2}{2} \, dx$$

$$= \frac{1}{2} \int_0^1 2 - 4x + 2x^2 - x^2 - 1 + 2x \, dx$$

$$= \frac{1}{2} \int_0^1 x^2 - 2x + 1 \, dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{2x^2}{2} + x \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - 1 + 1 \right] = 1/6 \qquad \boxed{V = 1/6}$$

$$
\begin{aligned}
\text{Avg } T &= \frac{1}{1/6} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} \, dz\, dy\, dx \\
&= 6 \int_0^1 \int_0^{1-x} \left[\frac{-1/2}{(1+x+y+z)^2} \right]_0^{1-x-y} dy\, dx \\
&= -3 \int_0^1 \int_0^{1-x} \frac{1}{2^2} - \frac{1}{(1+x+y)^2} \, dy\, dx \\
&= -3 \int_0^1 \left[\frac{1}{4}(1-x) \right] + \left[\frac{1}{1+x+y} \right]_0^{1-x} dx \\
&= -3 \int_0^1 \frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x} \, dx \\
&= -3 \int_0^1 \frac{3}{4} - \frac{x}{4} - \frac{1}{1+x} \, dx \\
&= -3 \left[\frac{3}{4}x - \frac{x^2}{8} - \log(1+x) \right]_0^1 \\
&= -3 \left[\frac{3}{4} - \frac{1}{8} - \log 2 + \log 1 \right] \\
&= \frac{-9}{4} + \frac{3}{8} + 3 \log 2 \\
&= 3 \log 2 - \frac{15}{8}
\end{aligned}
$$

TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES (r, θ, z)

$x = r\cos\theta \quad , y = r\sin\theta \quad , \quad z = z$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

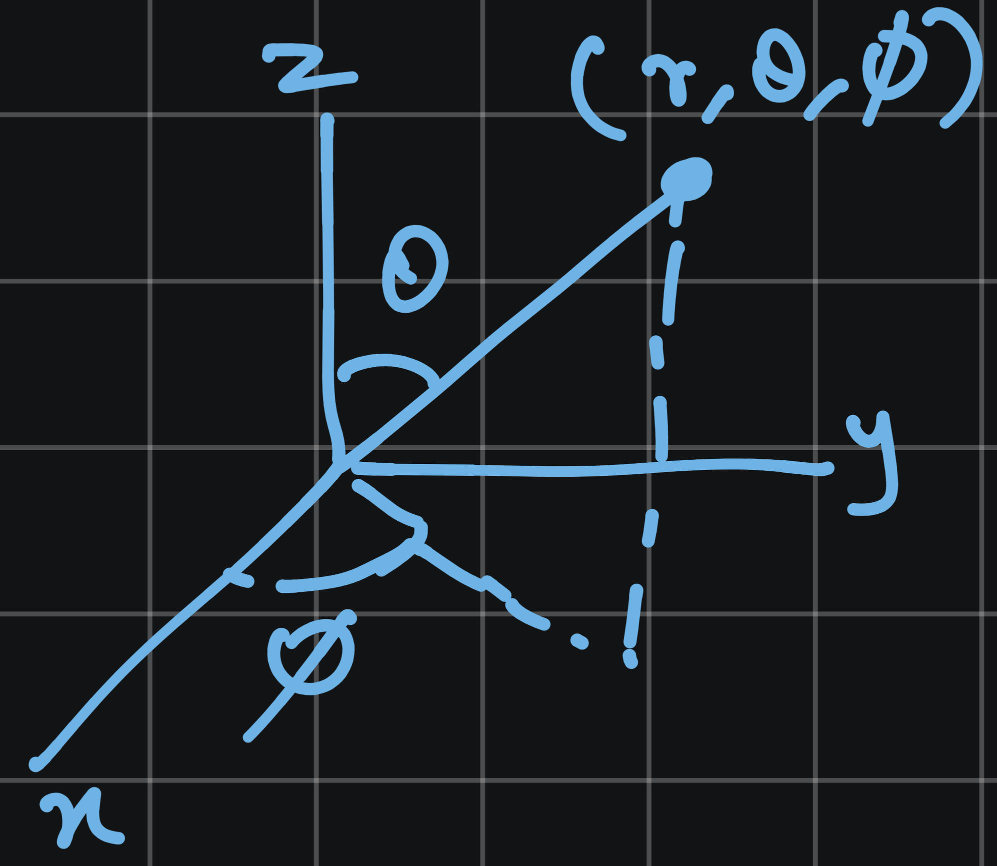
* bounded by cylinders and paraboloids
planes $\perp$ to z-axis, through z-axis

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint f(r\cos\theta, r\sin\theta, z)\, dr\, d\theta\, dz$$

TRIPLE INTEGRATION IN SPHERICAL COORDINATES (r, θ, ϕ)

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$J: \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$



$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\iiint f(x,y,z)\,dz\,dy\,dx = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

+ve octant of sphere

r: $r = 0$ to $r = a$

θ: $\theta = 0$ to $\theta = \pi/2$

ϕ: $\phi = 0$ to $\phi = \pi/2$

For hemisphere

r: $r = 0$ to $r = a$

θ: $\theta = 0$ to $\theta = \pi/2$

ϕ: $\phi = 0$ to $\phi = 2\pi$

For complete sphere

r: $r = 0$ to $r = a$

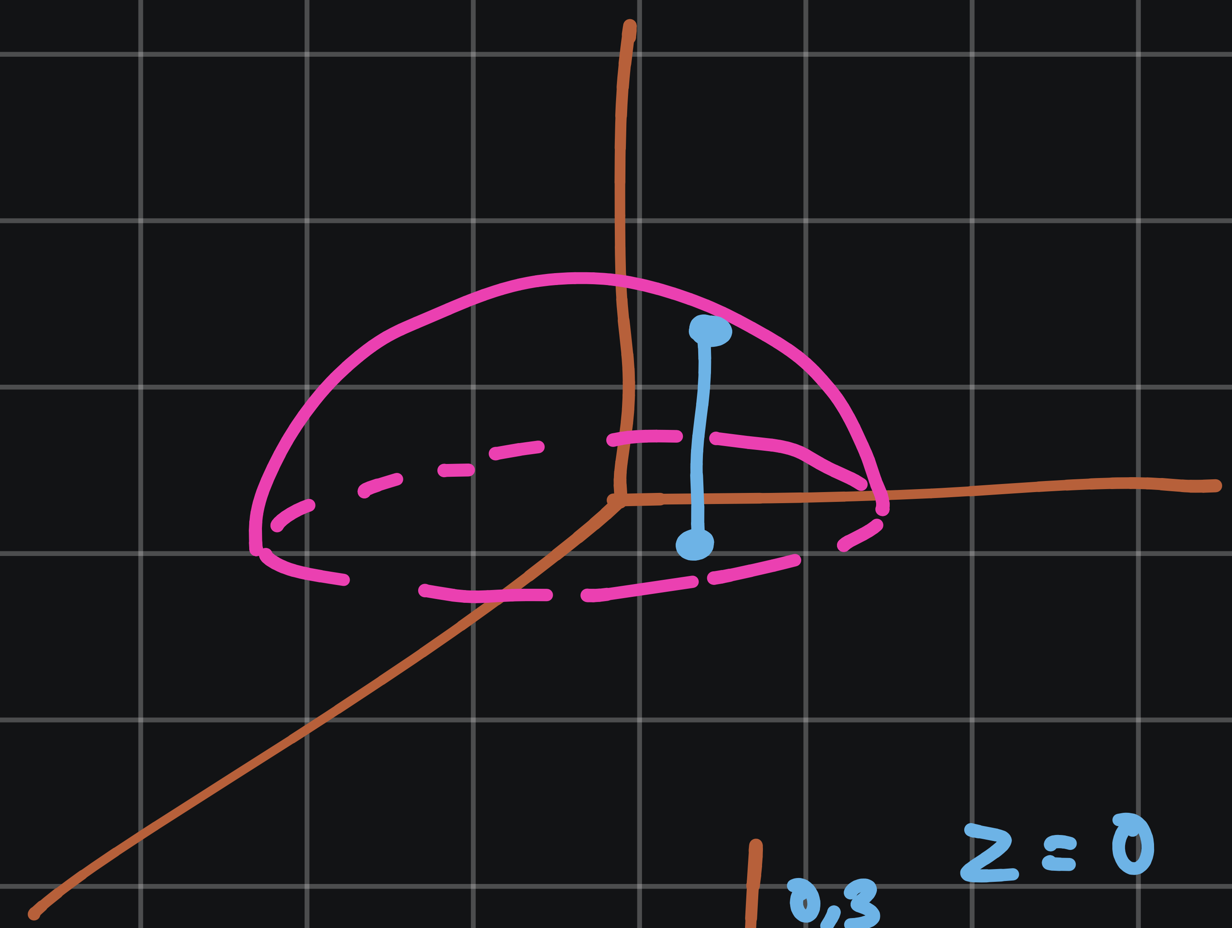
θ: $\theta = 0$ to $\theta = \pi$

ϕ: $\phi = 0$ to $\phi = 2\pi$

① Use cylindrical coordinates to evaluate

CW

$\iiint_V (x^2+y^2)\,dx\,dy\,dz$ taken over the region V bounded by paraboloid $z = 9 - x^2 - y^2$ and the plane $z = 0$

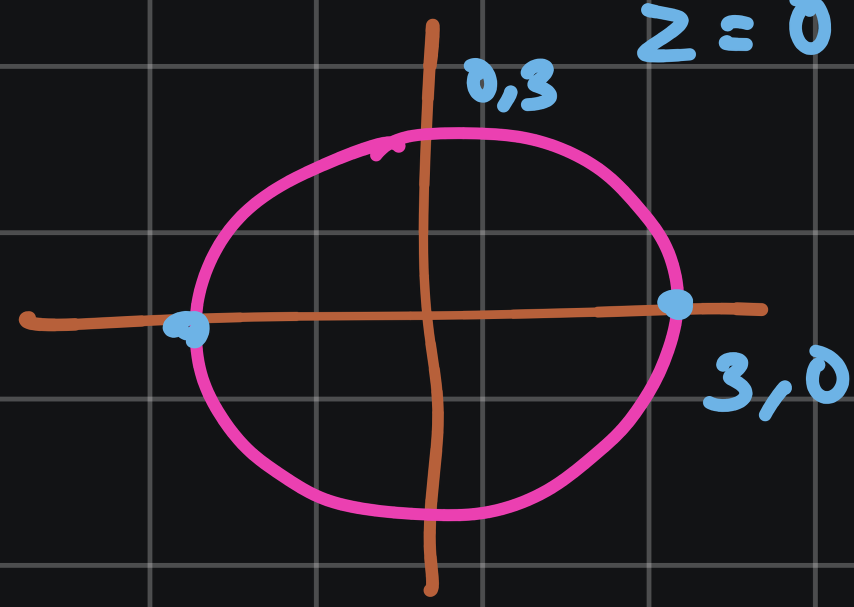


changing to cylindrical coordinates

$z: \; z = 0$ to $z = 9 - r^2$

$r: \; r = 0$ to $r = 3$

$\theta: \; \theta = 0$ to $\theta = 2\pi$



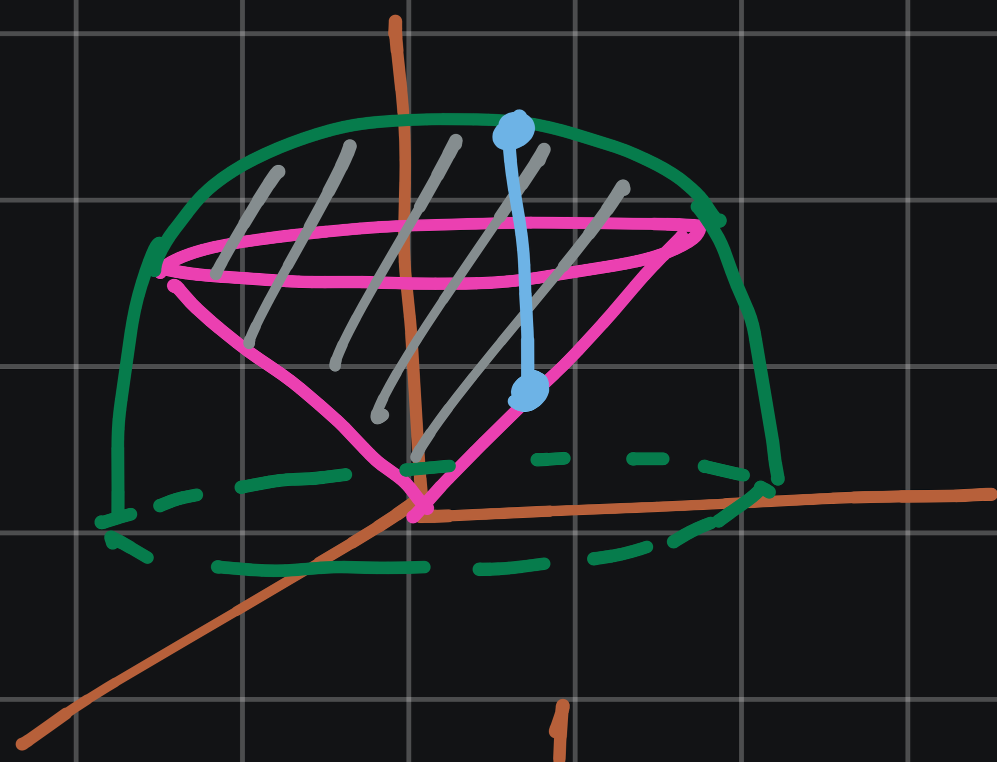
$$I = \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2 \cdot r \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r^3(9-r^2)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{9r^4}{4} - \frac{r^6}{6}\right)_0^3 d\theta$$

$$= 2\pi\left[\frac{9(81)}{4} - \frac{9(81)}{6}\right] = \frac{243\pi}{2}$$

(2) Calculate the volume of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and conic surface $z = \sqrt{x^2+y^2}$

CW



By changing to cylindrical coordinates

$z: \quad z = r$ to $z = 2 - r^2$

$r: \quad r = 0$ to $r = 1$

$\theta: \quad \theta = 0$ to $\theta = 2\pi$

$$V = \int_0^{2\pi}\int_0^{1}\int_r^{2-r^2} r\,dz\,dr\,d\theta$$

$r = 2 - r^2$

$r^2 + r - 2 = 0$

$r = -2, 1$

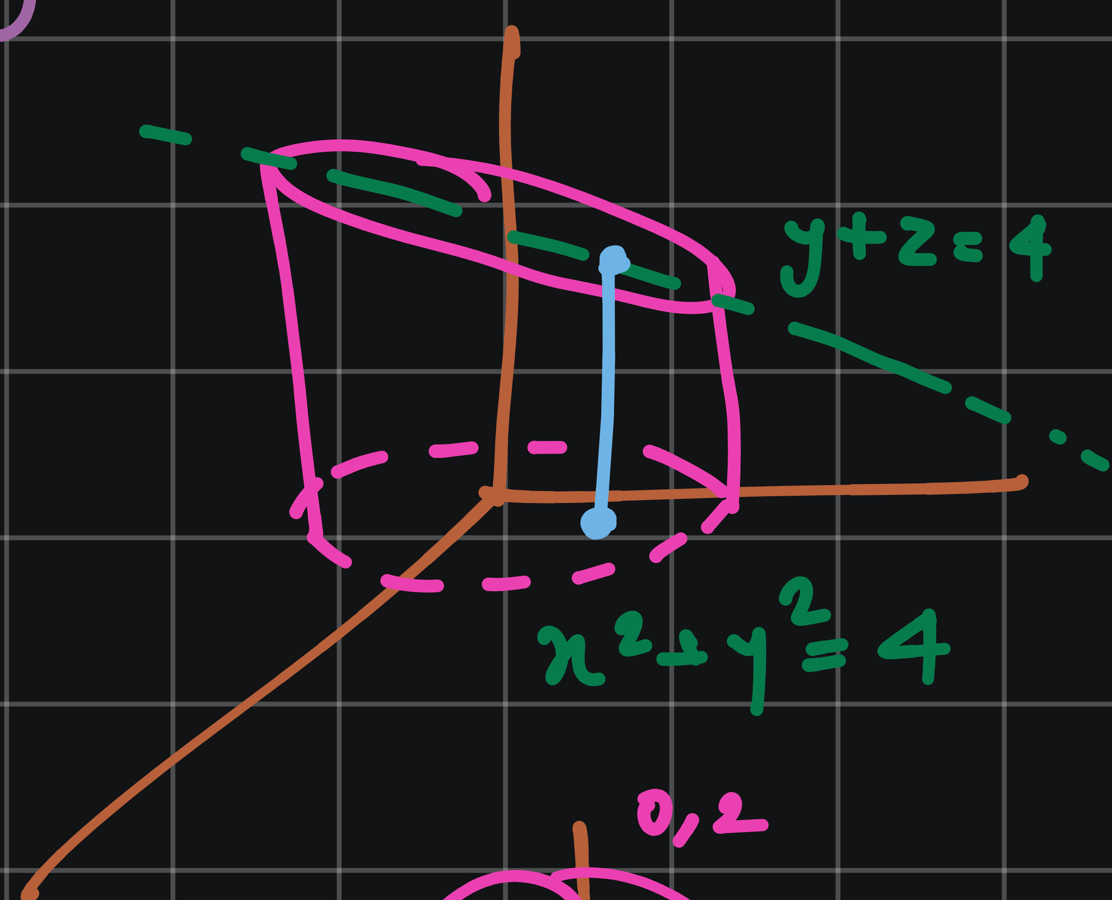
$$= \int_0^{2\pi}\int_0^{1} (2r - r^3 - r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{2r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3}\right)\Bigg|_0^1 d\theta = 2\pi\left[1 - \frac{1}{4} - \frac{1}{3}\right]$$

$$= 2\pi\left[\frac{12-3-4}{12}\right] = 5\pi/6$$

(3) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

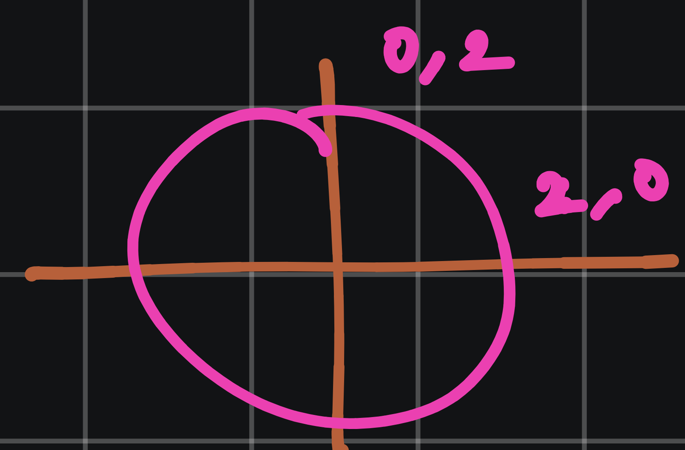
CW



$z: \quad z = 0$ to $z = 4 - r\sin\theta$

$r: \quad r = 0$ to $r = 2$

$\theta: \quad \theta = 0$ to $\theta = 2\pi$



$$V = \int_0^{2\pi}\int_0^{2}\int_0^{4-r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 4r - r^2\sin\theta\,dr\,d\theta$$

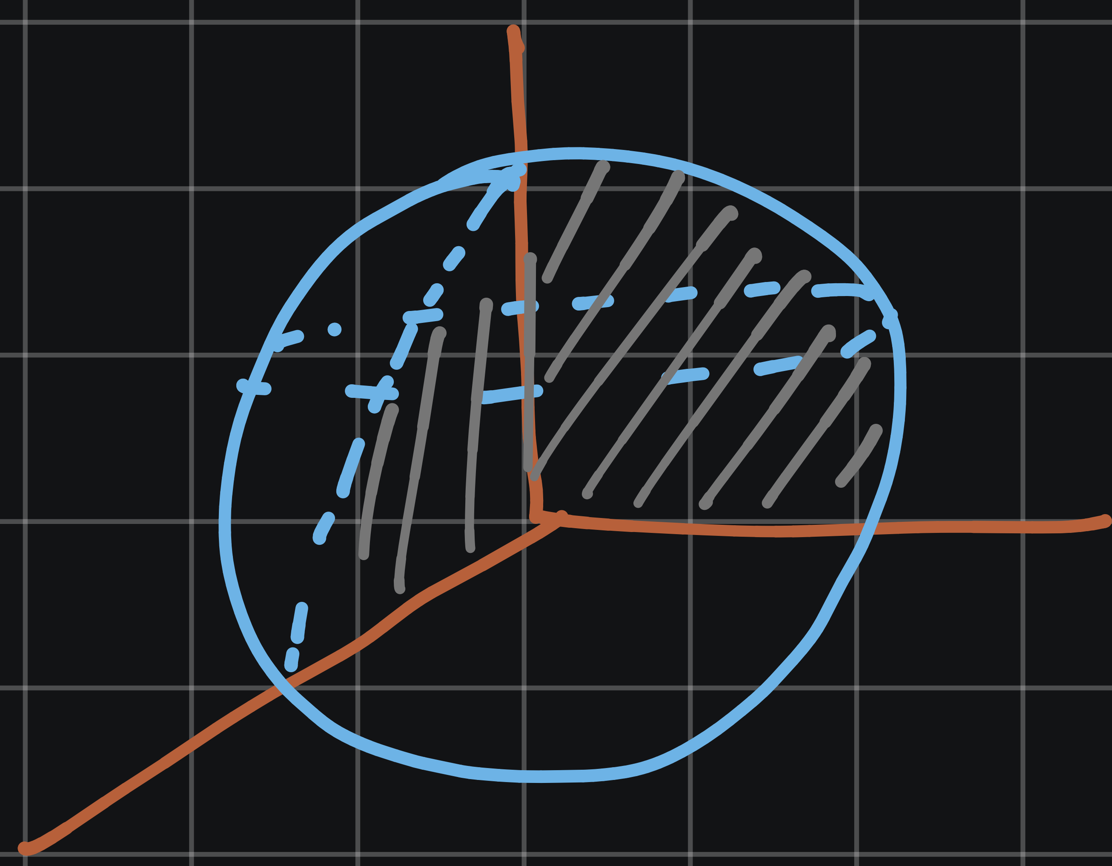
$$= \int_0^{2\pi}\left[\frac{4r^2}{2} - \frac{r^3}{3}\sin\theta\right]_0^2$$

$$= \int_0^{2\pi} 8 - \frac{8}{3}\sin\theta \, d\theta$$

$$= 16\pi + \frac{8}{3}\left[\cos\theta\right]_0^{2\pi} = 16\pi$$

(4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical coordinates

$x^2+y^2+z^2=1$
$x^2+y^2=1$



r : $r=0$ to $r=1$
θ : $\theta=0$ to $\theta=\pi/2$
ϕ : $\phi=0$ to $\phi=\pi/2$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta \, dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

(5) Find the total mass of the region in the cube $0 \le x \le 1$; $0 \le y \le 1$; $0 \le z \le 1$ with density at any point given by xyz

$\rho = xyz$

$$\text{Mass} = \int_0^1 \int_0^1 \int_0^1 xyz \, dz \, dy \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot 1/2 = 1/8$$

(6) Find the mass of sphere of radius b if the density varies inversely as the square of the distance from the center.

$$\rho \propto \frac{1}{x^2+y^2+z^2} \Rightarrow \rho = \frac{K}{x^2+y^2+z^2}$$

$$\text{Mass} = K\int_0^{2\pi}\int_0^{\pi}\int_0^{b} \frac{1}{r^2} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= K\int_0^{2\pi}\int_0^{\pi}\int_0^{b} \sin\theta \, dr\, d\theta\, d\phi$$

$$= Kb(2\pi)(-\cos\theta)_0^{\pi}$$

$$= -2\pi Kb[\cos\pi - \cos 0]$$

$$= -2\pi Kb[-1-1] = 4K\pi b$$

(7) Find the moment of inertia of a right circular cylinder of altitude 2h and radius b, relative to the diameter of its median section with density equals k

MOI relative to x-axis $= \iiint \rho(y^2+z^2)\,dz\,dy\,dx$

$$= k\iiint (y^2+z^2)\,dz\,dy\,dx$$

By changing into cylindrical coordinate system

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dz\,dy\,dx = r\,dz\,dr\,d\theta$

z : $-h$ to h
r : 0 to b
θ : 0 to 2π

$$\text{MOI} = k\int_0^{2\pi}\int_0^{b}\int_{-h}^{h} (r^2\sin^2\theta + z^2)\cdot r\,dz\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{b}\int_{-h}^{h} (r^3\sin^2\theta + rz^2)\,dz\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{b} \left(r^3\sin^2\theta + r\left(\frac{z^3}{3}\right)_{-h}^{h}\right)dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{b} r^3\sin^2\theta + \frac{2rh^3}{3}\,dr\,d\theta$$

$$= k\int_0^{2\pi}\left[\frac{r^4\sin^2\theta}{4} + \frac{h^3r^2}{3}\right]_0^b d\theta$$

$$= k\int_0^{2\pi} \frac{b^4\sin^2\theta}{4} + \frac{b^2h^3}{3}\,d\theta$$

$$= hk\pi b^2\left(\frac{b^2}{2} + \frac{2h^2}{3}\right)$$