LOGICAL DEDUCTION IN AI

INFERENCING BY RESOLUTION REFUTATION

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Predicate Logic

Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.

No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.

All dancers are graceful. Ayesha is a student. Ayesha is a dancer. Therefore some student is graceful.

Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

New Additions in Proposition (First Order Logic)

Variables, Constants, Predicate Symbols and

New Connectors: ∃ (there exists), ∀(for all)

Wherever Mary goes, so does the Lamb. Mary goes to School. So the Lamb goes to School.

Predicate: goes(x,y) to represent x goes to y

New Connectors: ∃ (there exists), ∀(for all)

F1: $\forall x(goes(Mary, x) \rightarrow goes(Lamb, x))$

F2: goes(Mary, School)

G: goes(Lamb, School)

To prove: $(F1 \land F2) \rightarrow G)$ is always true

Inferencing in Predicate Logic

Domain: D

Constant Symbols: M, N, O, P, ….

Variable Symbols: x,y,z,….

Function Symbols: F(x), G(x,y), H(x,y,z)

Predicate Symbols: p(x), q(x,y), r(x,y,z),

Connectors: ~, Λ, V, →, ∃, ∀

Terms:

Well-formed Formula:

Free and Bound Variables:

Interpretation, Valid, Non-Valid, Satisfiable, Unsatisfiable

What is an Interpretation? Assign a domain set D, map constants, functions, predicates suitably. The formula will now have a truth value

Example:

F1: ∀x(g(M, x) → g(L, x))

F2: g(M, S)

G: g(L, S)

Interpretation 1: D = {Akash, Baby, Home, Play, Ratan, Swim}, etc.,

Interpretation 2: D = Set of Integers, etc.,

How many interpretations can there be?

To prove Validity, means (F1 Λ F2) → G) is true under all interpretations

To prove Satisfiability means (F1 Λ F2) → G) is true under at least one interpretation

Resolution Refutation for Propositional Logic

To prove validity of

F = ((F1 ∧ F2 ∧ … ∧ Fn) → G)

we shall attempt to prove that

~F = (F1 ∧ F2 ∧ … ∧ Fn ∧ ~G)

is unsatisfiable

Steps for Proof by Resolution Refutation:

1. Convert of Clausal Form / Conjunctive Normal Form (CNF, Product of Sums).
2. Generate new clauses using the resolution rule.
3. At the end, either False will be derived if the formula ~F is unsatisfiable implying F is valid.

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

a b c

C1 ¬a ∨ (b∧c)

F1: (a → (b ∧ c)) = (~a V b) ∧ (~a V c)

C2

F2: ~b, G: ~a, ~G: a

C3 C4

Clauses of Clause Form: ~F = (C1 ∧ C2 ∧ C3 ∧ C4)

where: C1: (~a V b)

C2: (~a V c)

C3: ~b

C4: a

To prove that ~F is False

Let C1 = a V b and C2 = ~a V c

then a new clause C3 = b V c can be derived.

(a∨b) ∧ (b∨c)

∧ E ∧ (¬a ∨ a)

(Proof by showing that ((C1 ∧ C2) → C3) is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form C1 = a and C2 = ~a from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

Applying Resolution Refutation

Let C1 = a V b and C2 = ~a V c then a new clause C3 = b V c can be derived.

(Proof by showing that ((C1 Λ C2) → C3) is a valid formula).

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C_1 C_2

F1: (a → (b Λ c)) = (~a V b) Λ (~a V c)

$\neg a \vee b$

F2: ~b

C_3 $\boxed{(F_1 \wedge F_2) \rightarrow G \rightarrow \text{valid}}$

$\neg b$

G: ~a

$\neg a$

~G: a

C_4 $F \rightarrow \neg F$ is unsatisfiable

$C_5 = \neg a$

$= \neg a$

Clauses of Clause Form: ~F = (C1 Λ C2 Λ C3 Λ C4)

where: C1: (~a V b)

C2: (~a V c)

C3: ~b

C4: a

$C_5: \neg a$

To prove that ~F is False

New Clauses Derived

C5: ~a (Using C1 and C3)

C6: False (using C4 and C5)

$C_2 \wedge C_1 \wedge C_3 \wedge C_4 \wedge C_5$

a $\neg a$

False

$\neg F \Rightarrow F_1 \wedge F_2 \wedge \neg G$

Solve Using Resolution Refutation

Suppose we know that: “if Arjun is thin, then Mohit is not bearded or Julia is not tall” and “if Julia is tall then Devika is graceful” and “if Devika is graceful and Mohit is bearded then Arjun is thin” and “Mohit is bearded”. Can we deduce that “Julia is not tall” ?

$F_1 : a \rightarrow (\neg b \vee \neg c)$

$F_2 : c \rightarrow d$

$F_3 : (d \wedge b) \rightarrow a$

$F_4 : b$

$G \rightarrow \neg c$

$a \rightarrow$ Arjun is thin

$b \rightarrow$ Mohit is bearded

$c \rightarrow$ Julia is tall

$d \rightarrow$ Devika is graceful

$a \overset{\vee}{+} (b \overset{\vee}{+} c)$

$(a \overset{\vee}{+} b) \overset{\vee}{+} c$

$\{F_1 \wedge F_2 \wedge F_3 \wedge F_4 \rightarrow G\}$ F is valid

$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge \neg G \rightarrow$ FALSE

$c_1 : \neg a \vee (\neg b \vee \neg c)$

$c_2 : \neg c \vee d$

$c_3 : \neg d \vee (\neg b \vee a)$

$c_4 : b$

$c_5 : c$

c_1 & c_2

$c_6 : (\neg b \vee \neg c) \vee (\neg b \vee \neg d)$

$c_7 : d$

$c_8 : a \vee \neg b$

$c_9 : a$

$c_{10} : \neg b \vee \neg c$

$c_{11} : \neg c$

FALSE

Example

Let C1 = a V b and C2 = ~a V c then a new clause C3 = b V c can be derived.

(Proof by showing that ((C1 Ʌ C2) → C3) is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form C1 = a and C2 = ~a from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

Rajesh either took the bus or came by cycle to class. If he came by cycle or walked to class he arrived late. Rajesh did not arrive late. Therefore he took the bus to class.

bus , cycle , walk , late

$C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6 \wedge C_7 \wedge C_8$

$G \to bus$

$F_2 \to (cycle \vee walk) \to late$

$F_3 \to \neg late \} C_5$

$\neg G \to \neg bus \} C_6$

$F_1 \to \underbrace{(\neg bus \vee \neg cycle)}_{C_1} \wedge \underbrace{(bus \vee cycle)}_{C_2}$

$C_2 \,\&\, C_6 \to cycle$ C_7

$F_2 \to \neg(cycle \vee walk) \vee late$

$\to (\neg cycle \wedge \neg walk) \vee late$

$\to \underbrace{(\neg cycle \vee late)}_{C_3} \wedge \underbrace{(\neg walk \vee late)}_{C_4}$

$C_3 \,\&\, C_7$ late C_8

Resolution Refutation for Predicate Logic

Given a formula F which we wish to check for validity, we first check if there are any free variables. We then quantify all free variables universally.

Create F' = ~F and check for unsatisfiability of F'

STEPS:

Conversion to Clausal (CNF) Form:

- Handling of Variables and Quantifiers, Ground Instances

Applying the Resolution Rule:

- Concept of Unification
- Principle of Most General Unifier (mgu)
- Repeated application of Resolution Rule using mgu

CONVERSION TO CLAUSAL FORM IN PREDICATE LOGIC

1. Remove implications and other Boolean symbols converting to equivalent forms using ~, V, Λ
2. Move negates (~) inwards as close as possible
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function /constant symbol taking into account the variables dependent on the quantifier (Skolemization)
5. Drop Universal Quantifiers
6. Distribute V over Λ and convert to CNF

$\forall y(\exists x(f(x,y)))$ A

$\cancel{\forall y}\ f(F(y),y)$ F(y)

value

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Conversion to Clausal (CNF) Form:

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F1: ~~∀~~x(goes(Mary, x) → goes(Lamb, x))

F2: goes(Mary, School)

G: goes(Lamb, School)

To prove: (F1 ∧ F2) → G) is valid

CONVERSION TO CLAUSAL FORM IN PREDICATE LOGIC

1. Remove implications and other Boolean symbols converting to equivalent forms using ~, V, Λ
2. Move negates (~) inwards as close as possible
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5. Drop Universal Quantifiers
6. Distribute V over Λ and convert to CNF

C_1 : $\cancel{\forall x}$ ($\neg$goes(Mary, x) $\vee$ goes(lamb, x))

C_2 : goes (Mary, School)

C_3 : $\neg$goes (lamb, school)

C_4 : goes (lamb, School) $\quad x = \text{School}$

FALSE

Conversion to Clausal Form

1. Remove implications and other Boolean symbols converting to equivalent forms using ~, V, Λ
2. Move negates (~) inwards as close as possible
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function /constant symbol taking into account the variables dependent on the quantifier (Skolemization)
5. Drop Universal Quantifiers
6. Distribute V over Λ and convert to CNF

$\forall x(\forall y(\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow (\exists z(\text{likes}(z,x)))$

$\forall x(\neg(\forall y(\text{student}(y) \rightarrow \text{likes}(x,y)) \vee \exists z(\text{likes}(z,x))$

$\forall x(\neg(\forall y(\neg \text{student}(y) \vee \text{likes}(x,y)) \vee \exists z(\text{likes}(z,x))$

$\forall x(\exists y(\text{student}(y) \wedge \neg \text{likes}(x,y)) \vee \exists z(\text{likes}(z,x))$

$\cancel{\forall x}(\text{student}(F(x)) \wedge \neg \text{likes}(x, F(x)))$

$\vee \ \text{likes}(G(x), x)$

$(\text{stu} \ \vee \text{likes}(G..) \wedge (\neg \text{lik}(x). \ \vee \text{likes}..)$

Substitution, Unification, Resolution

Consider clauses:

- C1: ~studies(x,y) V passes(x,y)
- C2: studies(Madan,z)
- C3: ~passes(Chetan, Physics)
- C4: ~passes(w, Mechanics)

w m

What new clauses can we derive by the resolution principle?

Ground Clause and a more general clause

Concept of substitution / unification and the Most General Unifier (mgu)

Resolution Rule for Predicate Calculus: Repeated Application of Resolution using mgu

c1 , c2

x = Madan
y = z

z = Physics

c5 :

passes (Madan, z)

General

passes (Madan, Physics)

c1 & c4

x = w y = mech

c5 : ~studies (w, mech)

False

c2

w = Madan
z = mech

Examples

F1: ∀x(contractor(x) → ~dependable(x))

F2: ∃x(engineer(x) ∧ contractor(x))

G: ∃x(engineer(x) ∧ ~dependable(x))

$C_1: \cancel{\forall x} (\neg c(x) \vee \neg d(x))$

$C_2: \cancel{\exists x} \; e(x) \wedge c(x)$ A = const

$e(A) \wedge c(A)$

$C_2 \qquad C_3$

$\sim G: \neg \exists x (e(x) \wedge \neg d(x))$

$\cancel{\forall x} (\neg e(x) \vee d(x))$

C_4

$C_1 \& C_4 \; (\neg c(x) \vee \neg e(x))$

$C_5 \Rightarrow$

$x = A$

$C_5 \& C_2 \; C_6 \Rightarrow \neg c(A) \Leftrightarrow C_3$

FALSE

F1: ∀x(dancer(x) → graceful (x))

F2: student(Ayesha), F3: dancer(Ayesha)

G: ∃x(student (x) ∧ graceful(x))

$C_1: \cancel{\forall x} \; \neg d(x) \vee g(x)$

$C_2: s(A)$

$C_3: d(A)$

$\sim G: \neg \exists x (s(x) \wedge g(x))$

$C_4: \cancel{\forall x} (\neg s(x) \vee \neg g(x))$

$x = A$

$C_1 \& C_3 \rightarrow C_5 \Rightarrow g(A)$

$C_4 \& C_5 \Rightarrow C_6 \Rightarrow \neg s(A)$

FALSE

Example to be worked out in class

Every passenger is either in first class or second class. Each passenger is in second class if and only if the passenger is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

Thank you