Grade 11 Unit 6 - Rotational Motion - Part One

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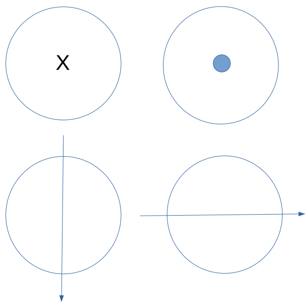
Rotational Kinematics

Until now, we have seen how we can summarize motion about a straight line whether it was uniform motion or uniformly accelerated motion. However, it is not always that we get to experience such motions in nature. Look at the skies for instance. Birds may travel in a curved path and even literally looking t the sky, the moon *orbits* the Earth. Such motions are not motions about a straight line. Such motions about a curve are called rotational motion.

Understanding Rotation

To first understand circular motion(a type of rotational motion), we have to understand the type of motion. In linear motion, for instance, we say an object moves if it changes its position linearly within a span of time. In rotational motion, it is no different. An object is said to rotate, if it changes its relative motion from the reference point of its motion. In rotational motion, our reference point is called the **axis of rotation**. Thus, when we study if a point is rotating we study how far it has traveled away from its axis of rotation.



By convention, let's define the direction of the rotational motion using the Right Hand Rule. To do this, we line up the thumb of our right hand along the line of the axis and the direction where the rest of the fingers curl is the direction of our rotation.

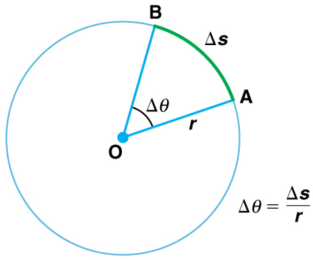
For example;

- On the top left rotational motion in the figure above, the axis is oriented **into the page**, that means, when we use the right hand rule, we see that it rotates about the Z-axis(-$\hat{\boldsymbol{k}}$ specifically) and it rotates on the XY plane(*clockwise*).
- On the top right rotational motion in the figure above, the axis is oriented **out of the page**, that means, when we use the right hand rule, we see that it rotates about the Z-axis($\hat{\boldsymbol{k}}$ specifically) and it rotates on the XY plane(*counterclockwise*).
- On the bottom left rotational motion in the figure above, the axis is oriented **downwards**, that means, when we use the right hand rule, we see that it rotates about the Y-axis(-$\hat{\boldsymbol{j}}$ specifically) and it rotates on the XZ plane(*goes into the page on the left and out of the page on the right*).
- On the bottom right rotational motion in the figure above, the axis is oriented **to the right**, that means, when we use the right hand rule, we see that it rotates about the X-axis($\hat{\boldsymbol{j}}$ specifically) and it rotates on the YZ plane(*goes out of the page on the top and into the page on the bottom*).

Thus, now we can see that we only need to see the orientation of our axis to see how the object in a rotational motion moves.

Circular Motion

It is one of the simplest forms of rotational motion. It is a motion of an object about a circle(that is, the axis of rotation passes through the center.) Let's take a point on the circumference of the circle and study it as it rotates. Its distance from the axis doesn't change at all(it is **r**-radius of the circle at all times), however, we see it moving. Thus, to signify this type of rotation, we instead use *angular measures*. Look at the figure below, for instance:



As the object moves from point A to point B on the circumference of the circle, it has deflected its from its initial position by an angle of $\varDelta\theta$, while it is still equally as far from its axis of rotation. Thus, to say that our object has rotated is the same as saying that it changed its relative orientation from the axis. Thus, as we discussed displacement as being an objects change in position linearly, we call these change of orientation relative to the axis an **angular displacement**($\varDelta\theta$). To explain how fast an object is rotating, our how fast it is changing its orientation, we study the rate of its change in orientation that is:

$$\text{rate of } \varDelta\theta = \frac{\varDelta\theta}{\varDelta t}$$

The above quantity is called angular velocity and it is the time rate of change in angular displacement, and we denote it using the Greek letter Omega(ω).

$$\omega = \frac{\varDelta\theta}{\varDelta t}$$

The SI-unit of angular displacement is Radian(rad) and the SI-unit of angular velocity is Radian per second(rad/sec). However, we can use multiple other units to describe these quantities. For example, we can use degrees and revolutions to describe angular displacement.

$$1 \text{ rev} = 2\pi rad$$

$$1 \text{ rad} = \frac{180^0}{\pi} \text{ ... thus,}$$

$$1^0 = \frac{\pi\ rad}{180}$$

Similarly as linear velocity changes, we can also have a situation in which the angular velocity is changing. We call this rate of change of angular velocity the angular acceleration(α).

Uniform Circular Motion and Uniformly Accelerated Circular Motion

Uniform Circular Motion is a type of circular motion in which the angular velocity stays constant($\varDelta\omega = 0$), that is, the angular acceleration of the object in motion is zero($\alpha = 0$). In that case, we have the following:

$$\omega = \frac{\varDelta\theta}{t}$$

We have seen above that $\theta = \dfrac{S}{r}$, let's try to see the relationship between v and ω.

$$\omega = \frac{\varDelta\theta}{t}$$

$$\omega = \frac{\varDelta\frac{S}{r}}{t} = \frac{\varDelta\frac{S}{t}}{r}$$

$$\omega = \frac{v}{r}$$

Similarly, for acceleration:

$$\alpha = \frac{\varDelta\omega}{t}$$

$$\alpha = \frac{\Delta \frac{v}{r}}{t} = \frac{\Delta \frac{v}{t}}{r}$$

$$\alpha = \frac{a}{r}$$

Now, we can discuss Uniformly Accelerated Circular Motion - similarly as in *uniformly accelerated motion*, the angular acceleration of an object in rotational motion stays constant($\Delta\alpha = 0$). In that case, we can use the equations of uniformly accelerated motion by substituting with the angular equivalents of the physical quantities.

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \frac{\alpha t^2}{2}$$

$$\theta = \omega_f t - \frac{\alpha t^2}{2}$$

$$2\alpha\theta = \omega_f^2 - \omega_i^2$$

$$\theta = \frac{(\omega_f + \omega_i)}{2}$$

When discussing motion, we have seen that we can use calculus to study physical quantities associated with it.

For example,

$$\omega = \frac{\Delta\theta}{\Delta t}$$

When using calculus, we have the following:

$$\omega = \frac{d\theta}{dt}$$

Thus, to find θ in terms of ω, we use integration:

$$d\theta = \omega dt$$

$$\theta = \int \omega dt$$

Doing the same for the acceleration, we have the following:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

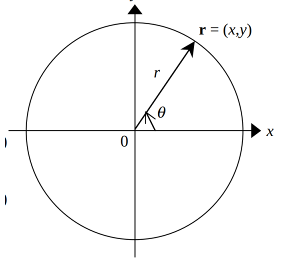
Thus,

$$d\omega = \alpha dt$$

$$\omega = \int \alpha dt$$

Centripetal Acceleration

For an object moving about a circle, the velocity changes instantaneously even when it is rotating in a uniform circular motion. That is, because as an object rotates although its speed may be the same, it changes its direction instantaneously, thus we can safely assume that it is accelerating. Whenever an object is rotating with a constant angular speed, the net force acting on it is called **centripetal force** and the acceleration associated with it is called centripetal acceleration. Understanding the proof for centripetal acceleration here is optional, but **highly recommended** to be read.



Let's consider an object is rotating on the XY plane as shown above. The position of this object at any point on its motion is given by:

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

$$\mathbf{r} = rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}}$$

We have seen earlier that:

$$v = \frac{d\mathbf{r}}{dt} = \frac{d(rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}})}{dt} = \frac{d(rcos(\omega t)\hat{\boldsymbol{i}} + rsin(\omega t)\hat{\boldsymbol{j}})}{dt}$$

When we derivate the above equation, we get the following:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -r\omega sin(\omega t)\hat{\boldsymbol{i}} + r\omega cos(\omega t)\hat{\boldsymbol{j}}$$

To find the acceleration, we derivate the above expression one more time with respect to t.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(-r\omega sin(\omega t)\hat{\boldsymbol{i}} + r\omega cos(\omega t)\hat{\boldsymbol{j}})}{dt}$$

We then get the following:

$$\mathbf{a} = -r\omega^2 cos(\omega t)\hat{\boldsymbol{i}} - r\omega^2 sin(\omega t)\hat{\boldsymbol{j}}$$

For a unit vector $\hat{\boldsymbol{r}}$, we know from our our knowledge of vectors that:

$$\hat{\boldsymbol{r}} = \frac{\mathbf{r}}{r} = \frac{rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}}}{r} = cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}}$$

$$\hat{\boldsymbol{r}} = cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}} = cos(\omega t)\hat{\boldsymbol{i}} + sin(\omega t)\hat{\boldsymbol{j}}$$

Going back to our equation of acceleration:

$$\mathbf{a} = -r\omega^2 cos(\omega t)\hat{\boldsymbol{i}} + r\omega^2 sin(\omega t)\hat{\boldsymbol{j}}$$

$$\mathbf{a} = -r\omega^2(cos(\omega t)\hat{\boldsymbol{i}} + sin(\omega t)\hat{\boldsymbol{j}})$$

$$\mathbf{a} = -r\omega^2(\hat{\boldsymbol{r}})$$

Or if we would like to express this in terms of tangential velocity, we have the following(since $v = \omega r$):

$$\mathbf{a} = -\frac{v^2}{r}(\hat{\boldsymbol{r}})$$

What does the negative sign indicate?

We defined our vector **r** to be outwards from the center and hence the negative of that implies that it is towards the center. Thus, centripetal acceleration is **always** acted towards the center.

For an object moving in a uniform circular motion, we know that it is not accelerating about its axis, that is, $\alpha = 0$. However, it has an acceleration towards the center(the centripetal acceleration - $\boldsymbol{a_c}$). Thus, when we speak of the acceleration of an object while rotating *constantly*, we only talk about the centripetal acceleration.

Let's instead consider an object in a *uniformly accelerated circular motion*, in this case, we have a changing centripetal acceleration at every instant while the angular acceleration is constant($\varDelta\alpha = 0$). Thus, if talk about an object in such motion, we are actually talking about the resultant acceleration on the object:

$$\boldsymbol{a_c} = \frac{v^2}{r} = \omega^2 r$$

While we have centripetal acceleration given by the above equation, it is also important to know that we have tangential acceleration as a result of angular acceleration

$$\boldsymbol{a_t} = r\alpha$$

Thus, if we talk about the acceleration of such an object in motion, we talk about the resultant. Since tangential acceleration(tangent) and centripetal acceleration(along the diameter) are always perpendicular, the resultant could be found using the following:

$$\boldsymbol{a} = \sqrt{\boldsymbol{a_c}^2 + \boldsymbol{a_t^2}}$$

$$\boldsymbol{a} = \sqrt{(\omega^2 r)^2 + (r\alpha)^2}$$

When simplified, we get the following:

$$\boldsymbol{a} = r\sqrt{\omega^4 + \alpha^2}$$

Rotational Dynamics

Torque

In this section, we will see interaction of bodies with other objects while rotating and effects of those interactions. Let's start with the simplest case: turning. While studying linear motion, we have seen that the cause of motion is force and it is a result of interaction between objects. The rotational equivalent of force is called **torque** and it depends on three things, one - the amount of force used, two - the distance from the axis of rotation, and three - the inclination of the force on the object(the angle between the force and the axis).

$$\tau = rFsin\theta$$

We can also use vector product to define force. It is the vector product between **r** and **F**(vector product, hence the order is important). It is also important to notice why the equation below is in boldface while the above is not. Recall that we represent vectors using boldface.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Rotation as Seen by Newton's Second Law

In linear motion, Newton's second law states that if an object is accelerating, there is a net force on it or vice versa. It is a bi-implication between force and acceleration.

$$\mathbf{F} \iff \mathbf{a}$$

And we have the following as well:

$$\mathbf{F} = m\mathbf{a}$$

We have seen that Torque is the rotational equivalent of Force and angular acceleration is the rotational equivalent of acceleration(linear). What is then, the rotational equivalent of mass?

$$\boldsymbol{\tau} =? \times \boldsymbol{\alpha}$$

This unknown physical quantity is called the moment of inertia(I)

Moment of Inertia

In the simplest case possible, we have the following be true:

$$\boldsymbol{\tau} = rF$$

$$\boldsymbol{\tau} = r(m\mathbf{a})$$

$$\boldsymbol{\tau} = r(mr\boldsymbol{\alpha})$$

$$\boldsymbol{\tau} = mr^2\boldsymbol{\alpha}$$

Thus, our unknown in the above section is $\boldsymbol{mr^2}$. This physical quantity is called the rotational inertia of a point mass rotating about an axis **r** meters far from it and with a mass **m**. We use the symbol I to denote moment of inertia. Thus, for a point mass m rotating about an axis at a distance r, we have the moment of inertia be:

$$I = mr^2$$

Thus, our torque equation becomes:

$$\tau = I\alpha$$

Moment of inertia for a point mass and a mass-system with a simple structure is easy to compute. We just add the individual moments to get the total moment of inertia of the system.

$$I = \sum_{1}^{n} m_i r_i^2$$

However, for objects such as a ball and a rod, we can use simple calculus to compute their moments of inertia. Let's start with a rod of mass M and length L that has a uniform linear mass density lambda.

It is imperative to know which axis we are using to rotate the rod to compute its moment of inertia.

Case 1 - rotation about its midpoint

$$\lambda = \frac{M}{L} = \frac{dm}{dr}$$

We have seen that

$$I = \sum_{1}^{n} m_i r_i^2$$

That means, it is integration(since we sum up individual elements)

$$I = \int_{\frac{-L}{2}}^{\frac{L}{2}} r^2 dm$$

$$\lambda = \frac{dm}{dr} \iff dm = \lambda dr$$

$$I = \int_{\frac{-L}{2}}^{\frac{L}{2}} r^2(\lambda dr)$$

$$I = \lambda \frac{r^3}{3}\Bigg|_{\frac{-L}{2}}^{\frac{L}{2}}$$

$$I = \lambda\frac{(\frac{L}{2})^3}{3} - \lambda\frac{(\frac{-L}{2})^3}{3}$$

$$I = \lambda\frac{L^3}{12}$$

Since $\lambda = \dfrac{M}{L}$, we then get:

$$I = \frac{ML^2}{12}$$

Case 2 - rotation about one end

$$I = \int_0^L r^2 dm$$

$$\lambda = \frac{dm}{dr} \iff dm = \lambda dr$$

$$I = \int_0^L r^2(\lambda dr)$$

$$I = \lambda\frac{r^3}{3}\Bigg|_0^L$$

$$I = \lambda\frac{L^3}{3} - \lambda\frac{0^3}{3}$$

$$I = \lambda\frac{L^3}{3}$$

Since $\lambda = \dfrac{M}{L}$, we then get:

$$I = \frac{ML^2}{3}$$

We see that the axis of rotation of an object actually matters and affects the moment of inertia of an object.

0.1 Work Done and Rotational Kinetic Energy

We have seen in linear cases how to calculate work done and kinetic energy. While studying work, we had:

$$W = \mathbf{F} \cdot \mathbf{r}$$

We can calculate work by substituting the rotational elements of F and r which are τ and θ respectively. Thus,

$$W = \boldsymbol{\tau} \cdot \boldsymbol{\theta}$$

Similarly for kinetic energy, we have seen that:

$$\text{KE} = \frac{1}{2}mv^2$$

When we substitute the rotational equivalents of m and v into the equation, we get the following:

$$KE_{rot} = \frac{1}{2}I\omega^2$$

An object rolling down an inclined plane, for example, has both a rotational kinetic energy and rotational kinetic energy. The existence of one doesn't imply or depend on the existence of the other.

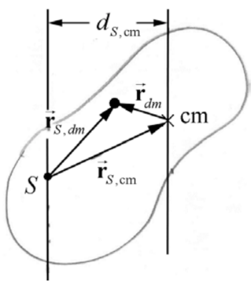
Work-KE Theorem for Rotational Motion?

We have seen above that:

$$\text{W} = \tau\theta$$
$$\text{W} = I\alpha\theta$$
$$\text{W} = I(\frac{w_f^2 - w_i^2}{2})$$
$$\text{W} = \frac{Iw_f^2}{2} - \frac{Iw_i^2}{2}$$
$$\text{W} = \Delta KE_{rot}$$

Parallel Axis Theorem and Rotational Dynamics

Usually, it is common to do the math while rotating bodies about an axis passing through their centers of masses. However, it might not always be the case. We have seen previously how to find the moment of inertia while being rotated through different axes, but now, we will see how we can easily determine the moment of inertia of a body through an axis parallel to the one passing through the center of mass.



$$\overrightarrow{\mathbf{r}}_{S,dm} = \overrightarrow{\mathbf{r}}_{S,\text{cm}} + \overrightarrow{\mathbf{r}}_{dm}$$
$$\left|\overrightarrow{\mathbf{r}}_{\text{cm},\perp,dm}\right| = r_{\perp,dm}$$
$$\left|\overrightarrow{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm}$$
$$\left|\overrightarrow{\mathbf{r}}_{S,\perp,\text{cm}}\right| = d_{S,\text{cm}}$$
$$\overrightarrow{\mathbf{r}}_{S,\perp,dm} = \overrightarrow{\mathbf{r}}_{S,\perp,\text{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}$$
$$\overrightarrow{\mathbf{r}}_{S,\parallel,dm} = \overrightarrow{\mathbf{r}}_{S,\parallel,\text{cm}} + \overrightarrow{\mathbf{r}}_{\parallel,dm}$$
$$I_S = \int_{\text{body}} dm\,(r_{S,\perp,dm})^2$$

$$
\begin{aligned}
(r_{S,\perp,dm})^2 &= \overrightarrow{\mathbf{r}}_{S,\perp,dm} \cdot \overrightarrow{\mathbf{r}}_{S,\perp,dm} \\
&= \left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}\right) \cdot \left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}\right) \\
&= d_{S,\mathrm{cm}}^2 + (r_{\perp,dm})^2 + 2\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}
\end{aligned}
$$

$$I_S = \int_{\mathrm{body}} dm d_{S,\mathrm{cm}}^2 + \int_{\mathrm{body}} dm\,(r_{\perp,dm})^2 + 2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right)$$

$$d_{S,\mathrm{cm}}^2 \int_{\mathrm{body}} dm = m d_{S,\mathrm{cm}}^2$$

$$I_{\mathrm{cm}} = \int_{\mathrm{body}} dm\,(r_{\perp,dm})^2$$

$$2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right) = \overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot 2\int_{\mathrm{body}} dm \overrightarrow{\mathbf{r}}_{\perp,dm}$$

$$2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right) = 0$$

$$I_S = I_{\mathrm{cm}} + m d_{S,\mathrm{cm}}^2$$

This proof tells us that if we know the moment of inertia of a rigid body rotating through its center of mass, we can determine its moment of inertia through any axis parallel to the axis passing through the center of mass.

For example, let's see a rod rotating on one end instead of its center of mass. We have seen that for a rod rotating through its center of mass, its moment of inertia is given by:

$$I_{cm} = \frac{ML^2}{12}$$

If we choose its rotation axis to be one end of the rod, that makes the distance from the axis through the center of mass to the axis through one end:

$$d = \frac{L}{2}$$

Using parallel axis theorem, we can determine that the moment of inertia is:

$$I_s = I_{cm} + Md^2$$

$$I_s = \frac{ML^2}{12} + M(\frac{L}{2})^2$$

$$I_s = \frac{ML^2}{3}$$

Angular Momentum

The angular momentum of a rigid object is defined as the product of the moment of inertia and the angular velocity(or the cross product between **r** and linear momentum). It is analogous to linear momentum and is subject to the fundamental constraints of the conservation of angular momentum principle if there is no external torque on the object.

$$\mathbf{L} = \mathbf{r} \times \mathbf{B} \text{ or}$$

$$\mathrm{L} = \mathrm{I}\omega$$

We have seen earlier that for an object rotating with a net torque acting on it,

$$\tau_{net} = I\alpha$$

We also know that a net torque bi-implies angular acceleration, thus:

$$\tau_{net} = I(\frac{\omega_f - \omega_i}{\Delta t})$$

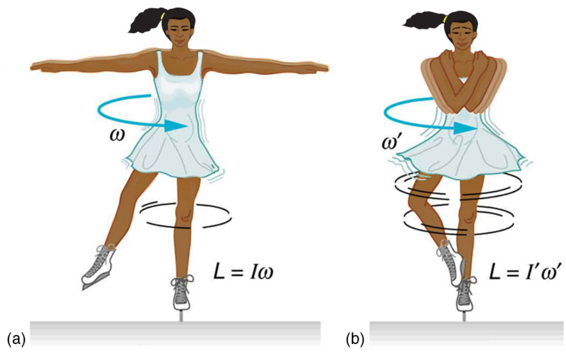
$$\tau_{net}\Delta t = I(\omega_f - \omega_i)$$

$$\tau_{net}\Delta t = \Delta I\omega$$

Thus,

$$\tau_{net}\Delta t = \Delta L$$

The quantity $\tau_{net}\Delta t$ is called the **angular impulse** of a body and is the rotational equivalent of impulse. We see that if the net torque on a rigid body is 0, then its change in angular momentum is 0 meaning angular momentum is conserved. Conservation of angular momentum is applicable in real life in multiple places. One case is with ballet-dancers where they can change how they are dancing and that, as a result, affecting how fast they are rotating.



An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is very small that it is negligible. In image (b), her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.