Kepler's Law

Group 2,8 and 11 project

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Mathematical prove of Kepler's Law

$$F = G\frac{m_1 m_2}{r^2} \tag{1}$$

- F:Force.
- G:Gravitational Constant.
- m_x:Mass of the object x.
- r:Distance between centres of two objects.

The two-body problem is a mathematical problem that deals with the motion of two objects in space. It was first posed by Newton, and later solved by combining his law of universal gravitation and his second law of motion.

We first write Newton's law of universal gravitation and Newton's second law of motion to express the force acting on the satellite with mass m,due to the large mass M of the earth.

$$F_m = m\frac{d^2r}{dt^2} \tag{2}$$

- F:vector sum of all forces acting on mass .
- d^2r/dt^2:vector acceleration of the mass measured relative to an inertial reference frame.

$$F_m = -G\frac{m_1 m_2}{r^2}$$

(minus sign indicates that the force is towards the origin)
The integration of equation(1)and(2) gives:

$$GM\frac{r}{r^3} + \frac{d^2 r}{dt^2} = r\frac{d^2}{dt^2} + \mu\frac{r}{r^3} = 0 \quad (3)$$

And then,this method leads to proof of kepeler's laws.
To proof the constant vector,we cross multiply the equation above or equation(3) by r:

$$r\frac{d^2 r}{dt^2} + r^2\frac{\mu}{r^3} = 0$$

Since r times r is 0,the second term will be 0 or

$$r\frac{d^2r}{dt^2} = 0 \tag{4}$$

After this we take the derivative of the angular momentum h:

$$\frac{d^2h}{dt^2} = \frac{d}{dt^2}[r\frac{dr}{dt^2}] = r\frac{d^2r}{dt^2} + \frac{dr}{dt}(\frac{dr}{dt}) = 0 \tag{5}$$

Equation (4) is zero, so the first term on the right will be zero and it makes the whole term to be zero when the two terms multiplied.This implies that h is constant vector.

Kepler's first law

Kepler's first law states that the paths of planets around the sun are ellipses. The center of the ellipse always lies where the Sun is, and the planets' orbits are ellipses, so their distance from the Sun changes continuously.

Figure: Kepler's 1st law

To proof Kepler's first law,we cross multiply equation(3) by h.

$$\frac{d^2r}{dt^2} \cdot h = -r \cdot h\frac{\mu}{r^3} = -r\frac{\mu}{r^3} \cdot r\frac{dr}{dt^2}$$

By using vector triple vector product,a$\cdot(b \cdot c) = (a \cdot c)b - (a \cdot b)c$
The final expression becomes:

$$-\frac{\mu}{r^3}[(r \cdot r\frac{dr}{dt}) - (r \cdot r)\frac{dr}{dt}] = -\frac{\mu}{r^3}[r\frac{dr}{dt}r - r^2\frac{dr}{dt}] = \mu[\frac{dr}{dtr} - \frac{dr}{dtr^2}r] = \mu\frac{d}{dt}(\frac{r}{r})$$

This equation may be integrated directly, since h is constant:

$$\frac{dr}{dt} \cdot h = \mu(\frac{r}{r}) + c$$

,

Where c is constant of integration. We write c as μe.

Then the last expression becomes:

$$\frac{dr}{dt} \cdot h = \frac{}{(}r + er) \tag{6}$$

Finally,we took the dot product of equation (6) and r, using a (b $\cdot c) = c(a \cdot b)$

$$h(\frac{dr}{dt} \cdot r = h \cdot h = h^2 = \frac{\mu}{r}(r^2 + er^2 cosv)$$

,Where

- v is angle between e and r.

Hence,

$$r = \frac{h^2/\mu}{1 + ecosv} \tag{7}$$

Equation (7) is the general equation in polar coordinates for a conic section with the origin at a focal point.
If $0 \longleftarrow e \longleftarrow 1$,
The orbit is an ellipse that proves Kepler's first law.

Kepler's second Law

Kepler's second law states that as a planet revolves, an imaginary line connecting it to the Sun passes across equal portions of space at equal intervals of time.

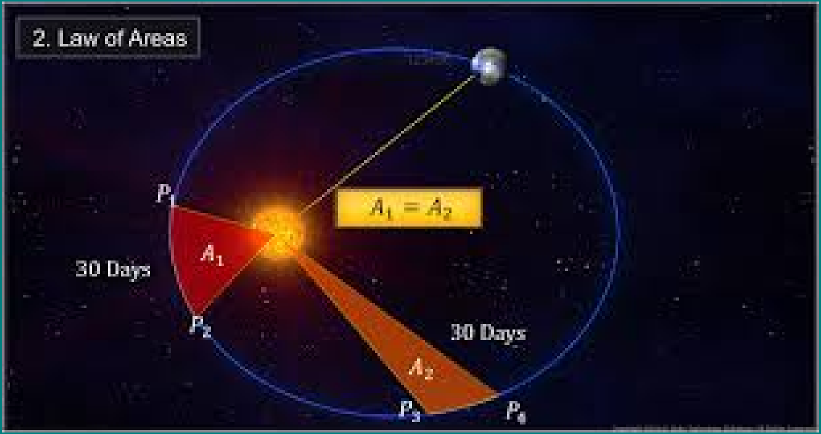

Figure: Kepler's 2nd law

Kepler's second law can be proved as: First, we rewrite the equation for h using

$$(r \cdot h)\frac{dr}{dt} = e_r\frac{dr}{dt} + re_\theta\frac{d\theta}{dt} \tag{8}$$

$$h = r \cdot \frac{dr}{dt} = r \cdot [e_r\frac{dr}{dt} + re_\theta\frac{d\theta}{dt}] = re_r\frac{dr}{dt} + r^2 e_r\frac{d\theta}{dt} \cdot e_\theta \tag{9}$$

$$h = r^2\frac{d\theta}{dt}$$

Kepler's Third Law

Kepler'sthe Third Law states that the cubes of the semi-major axes of the planets' orbits are directly proportional to the squares of the planets' orbital periods.

The numerator of equation(7) is the sami-latus rectum p.Hence,

$$\frac{h^2}{\mu} = p = a(1 - e^2), \tag{10}$$

or

$$h = [\mu a(1 - e^2)]^1/2 \tag{11}$$

By joining the two equations (10) and (11).We get:

$$h = 2\frac{dA_e}{dt} \tag{12}$$

"h" is constant so,it follows that

$$A_e(t) = t\frac{h}{2} = \frac{[\mu a(1 - e^2]]^1/2}{2}t \tag{13}$$

T=orbital period,

$$A_e = \pi a^2(1 - e^2)^1/2 \tag{14}$$

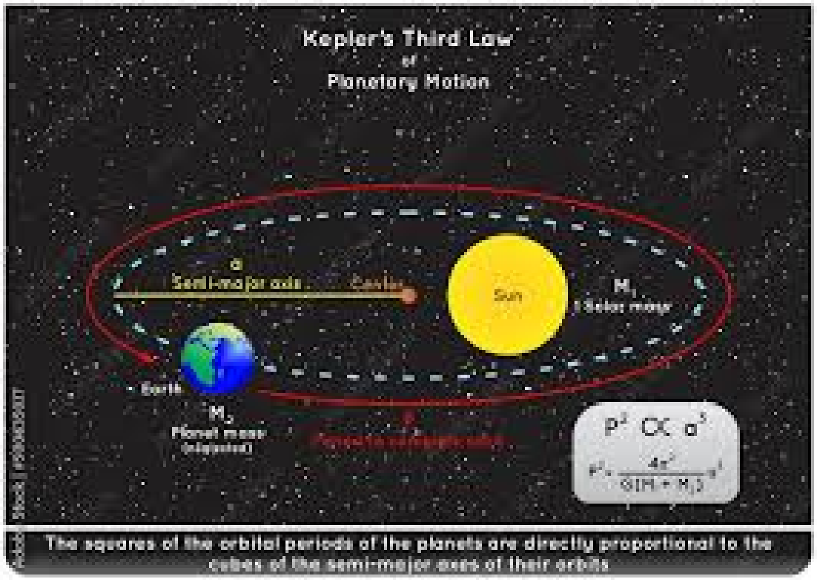

Figure: kepler's 3rd law

Then,when we join equation (13) and (14),we get:

$$\frac{[\mu a(1-e^2)]^1/2}{2}T = \pi a^2(1-e^2)^1/2 \tag{15}$$

$$h = [\mu a(1-e^2)]^1/2 \tag{16}$$

or,

$$T = 2\pi\frac{a^1/3}{\mu^1/2} \tag{17}$$

So this is a proof for Kepler's third law.