# ST.JOHN BAPTIST DE LA SALLE

# CATHOLIC SCHOOL

## CLASS:10C

## Physics Group Project Work

## On

# Mathematical Proof Of Kepler's Law

## SUBMITTED TO: Teacher Aaron GK

Submitted by:

Group 2 members

Group 8 members and

Group 11 members

Due Date: December 15, 2022

Submitted: January 12, 2023

# Abstract

Kepler's law of planetary motion revolutionized the field of astronomy and classical mechanics.They make a great contribution to the field of celestial mechanics.Kepler was the first to introduce elliptical motion.Although he rejected Aristotle's model of the theory of motion,he understood that his model lacks the dynamics of motion and he was unable to explain why the planets had to move elliptically.He also doubted that the planetary motion was due to a magnetic force emanating from the sun in,which became a passionate intellectual problem in the following century. These laws can be applied to any celestial body that revolves around a star, such as exoplanet orbiting star.With a few necessary modifications,these laws can be used to describe binary systems, or how a star orbits a compact object or any compact object orbiting another compact object.

# Contents

# List of Figures

1 Acknowledgement

First and for most, we would like to praise and thank the almighty God for giving us strength to accomplish this project and also we would like to express our special thanks of gratitude for our physics teacher,teacher Aaron GK,who gave us this golden opportunity to do this assignment as well as our parents who helped us a lot in doing this project.

Thank You!!!

Group 2,8 and 11 members,

10C

## 2 Introduction

If we go outside every clear night and observe Mars' position in relation to the stars,we will find that its apparent motion across the sky is quite complex ,instead of moving in a simple curved path,it occasionally changes direction.

The Danish astronomer Tycho Brahe (1546-1601) followed the paths of the planets and Mars in particular for many years,even though the telescope have not been yet.Instead,he used a sophisticated observation device to determine the exact position of the planets.Brahe linked his work to Johannes Kepeler (1571-1630). In 1600 and after Brahe's death Kepeler inherited his astronomical observations.Kepeler took advantage of Brahe's work by extracting from his carefully collected data the three laws of orbital motion that we now know as Kepeler's laws.Of course,the sun and planets do not revolve around the earth like Ptolemy did ancient Greek astronomer-rather stated that the earth orbited the sun along with the other planets,as suggested by Copernicus (1473-1543).

No one knew why the planets obey Kepeler's laws,not even Kepeler,until Newton considered the problem decades after Kepeler's death.Newton was able to show that every Kepeler law follows as a direct consequence of the universal law of gravitation.

The three Kepeler law describe the motion of the planets around the sun.Moreover,They are valid in describing motion of artificial satelite around earth by ignoring the perturbation of orbit.

First Kepeler law =Law of ellipses

Second Kepeler Law=Law of equal area

Third Kepeler Law=Law of distance

Figure 1: motion of stars and planets around the sun

# 3 Mathematical Proof Of Kepeler's Law

By using vector calculus,we can derive Kepeler's as a result of Newton's inverse square law of gravitational acceleration.

In addition of the 3 laws of motion,Newton published law of universal gravitation.It stated that two objects attract each other with a force proportional to the product of their mass and inversely proportional to square of distance among them.
It can be expressed in vector notations as:

$$F = G\frac{m_1 m_2}{r^2} \tag{1}$$

- $F$:Force.
- $G$:Gravitational Constant.
- $m_x$:Mass of the object x.
- $r$:Distance between centres of two objects.

The two body problem was first posed and solved by Newton.The importance of this problem depends on two facts:
First,the problem of spheres,where mass is distributed in spherical shells.
Second,practical problems of orbital motion can be seen as approximate two-body problems,that is,the two body solution can be used to provide first approximation of orbital motion and,is therefore used as starting point of more accurate solutions.
The mathematical formulation of the two- body problems comes from combination of Newtons law of universal gravitation and from Newton's second law of motion which state that

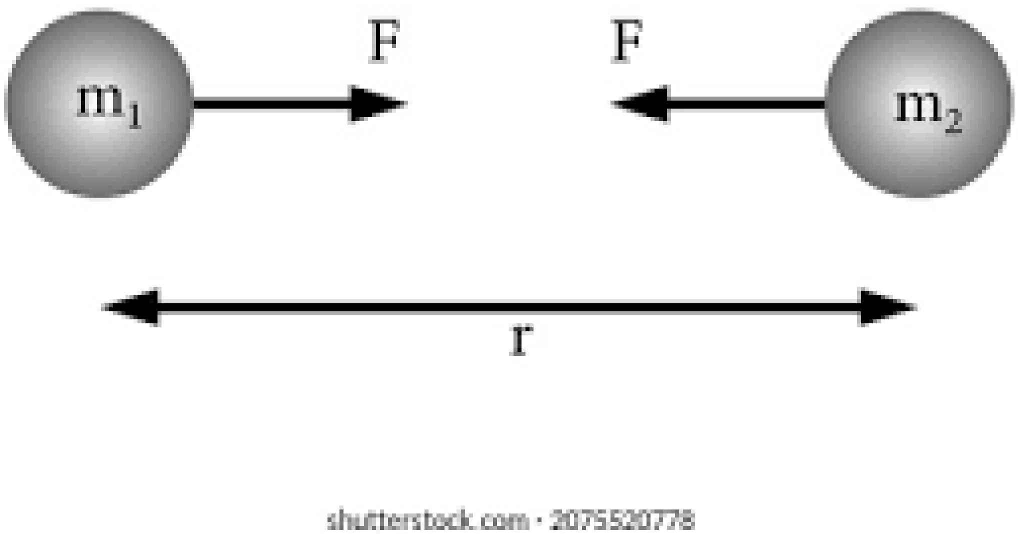


Figure 2: gravitational acceleration

the rate of change of momentum measured relative to an inertial reference frame,is proportional to the force impressed and is in the same direction of that force.

We use center of earth as the origin of our coordinate system,and we define the radius vector "r" as the direction away from the origin.We first write Newton's law of universal gravitation and Newton's second law of motion to express the force acting on the satellite with mass m,due to the large mass M of the earth.

$$F_m = m\frac{d^2r}{dt^2} \tag{2}$$

- $F$:vector sum of all forces acting on mass .
- $d^2r/dt^2$:vector acceleration of the mass measured relative to an inertial reference frame.

$$F_m = -G\frac{m_1 m_2}{r^2}$$

(minus sign indicates that the force is towards the origin)

The integration of equation(1)and(2) gives:

$$GM\frac{r}{r^3} + \frac{d^2r}{dt^2} = r\frac{d^2}{dt^2} + \mu\frac{r}{r^3} = 0 \tag{3}$$

And then,this method leads to proof of kepeler's laws.

To proof the constant vector,we cross multiply the equation above or equation(3) by r:

$$r\frac{d^2r}{dt^2} + r^2\frac{\mu}{r^3} = 0$$

Since r times r is 0,the second term will be 0 or

$$r\frac{d^2r}{dt^2} = 0 \tag{4}$$

After this we take the derivative of the angular momentum h:

$$\frac{d^2h}{dt^2} = \frac{d}{dt^2}[r\frac{dr}{dt^2}] = r\frac{d^2r}{dt^2} + \frac{dr}{dt}(\frac{dr}{dt}) = 0 \tag{5}$$

Equation (4) is zero, so the first term on the right will be zero and it makes the whole term to be zero when the two terms multiplied.This implies that h is constant vector.

## 3.1 Proof Of Kepler's 1st Law

Kepler's first law states that planets move the sun in elliptical orbits.An ellipse is a curve surrounding two points,called foci,so that the total distance from one focus to a point on the ellipse and back to the other focus is constant for every point on the curve.This law mark an essential turning point in transformation from geocentrism to heliocentrism.Aphelion and perihelion describes the farthest and shortest distance,respectively,between the earth and the sun,as shown in the figure below.Earth is furthest from the sun(aphelion) about

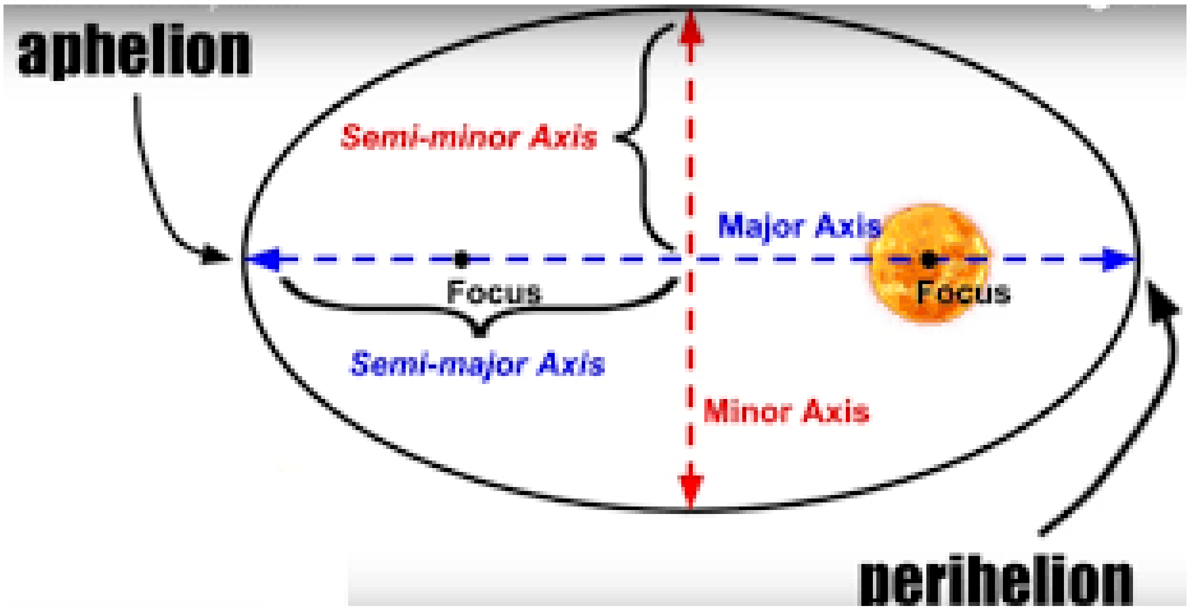


Figure 3: aphelion and perihelion

two weeks after the June solstice and closest to the sun (perihelion) about two weeks after the December solstice.

To proof Kepler's first law,we cross multiply equation(3) by h.

$$\frac{d^2r}{dt^2}\cdot h = -r\cdot h\frac{\mu}{r^3} = -r\frac{\mu}{r^3}\cdot r\frac{dr}{dt^2}$$

By using vector triple vector product,a$\cdot(b\cdot c) = (a\cdot c)b-(a\cdot b)c, thefinalexpressionbecomes:$ $-\frac{\mu}{r^3}[(r\cdot r\frac{dr}{dt})-(r\cdot r)\frac{dr}{dt}] = -\frac{\mu}{r^3}[r\frac{dr}{dt}r - r^2\frac{dr}{dt}] = \mu[\frac{dr}{dtr} - \frac{dr}{dtr^2}r] = \mu\frac{d}{dt}(\frac{r}{r})$ This equation may be integrated directly, since h is constant:

$$\frac{dr}{dt}\cdot h = \mu(\frac{r}{r}) + c$$

, Where c is constant of integration. We write c as $\mu e.$

$Thenthelastexpressionbecomes:$

$\frac{dr}{dt}\cdot h = \frac{\mu}{r}(r+er)(6)$ Finally,we took the dot product of equation (6) and r, using a (b $\cdot c) = c(a\cdot b)$

$h(\frac{dr}{dt}\cdot r = h\cdot h = h^2 = \frac{\mu}{r}(r^2 + er^2cosv)$,Where

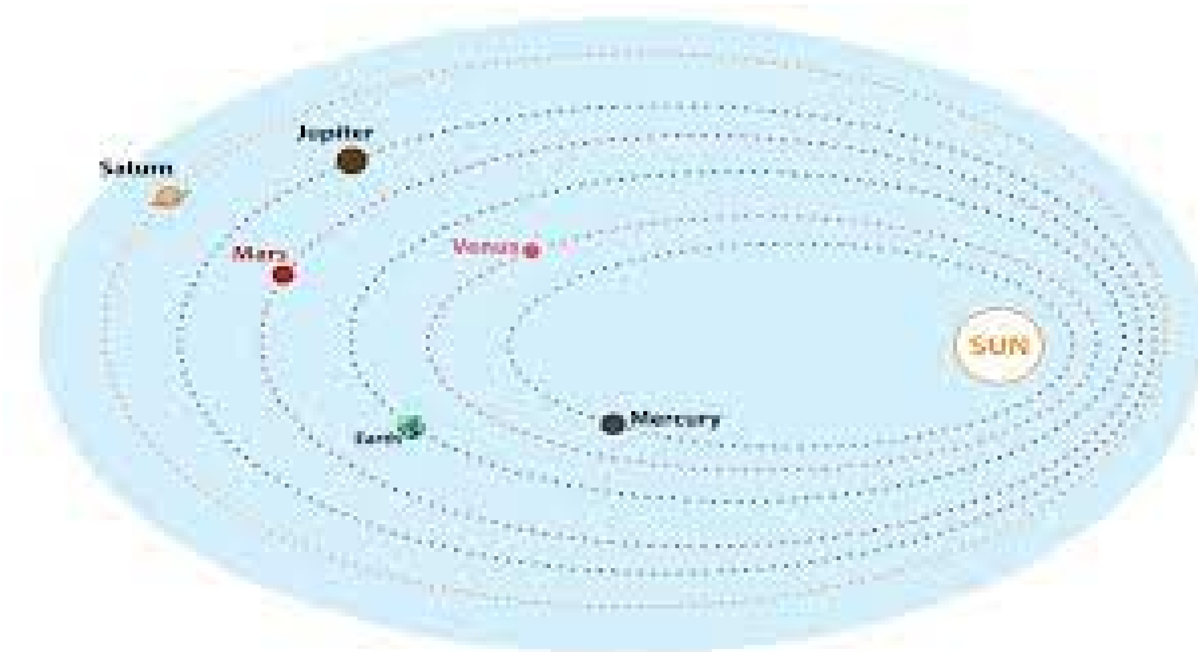


Figure 4: orbits are ellipses

- $v$ is angle between e and r.

Hence,

$$r = \frac{h^2/\mu}{1 + ecosv} \tag{7}$$

Equation (7) is the general equation in polar coordinates for a conic section with the origin at a focal point.

If $0 \longleftarrow e \longleftarrow 1$,

$Theorbitisanellipsethatproofs Kepler's firstlaw.$

## 3.2 Proof Of Kepeler's 2nd Law

Kepler's second law characterizes the speed of a planet alongside its elliptical path.Kepler's second law states that a line jogging from the solar to the planet sweeps out the same regions of ellipse in same times.

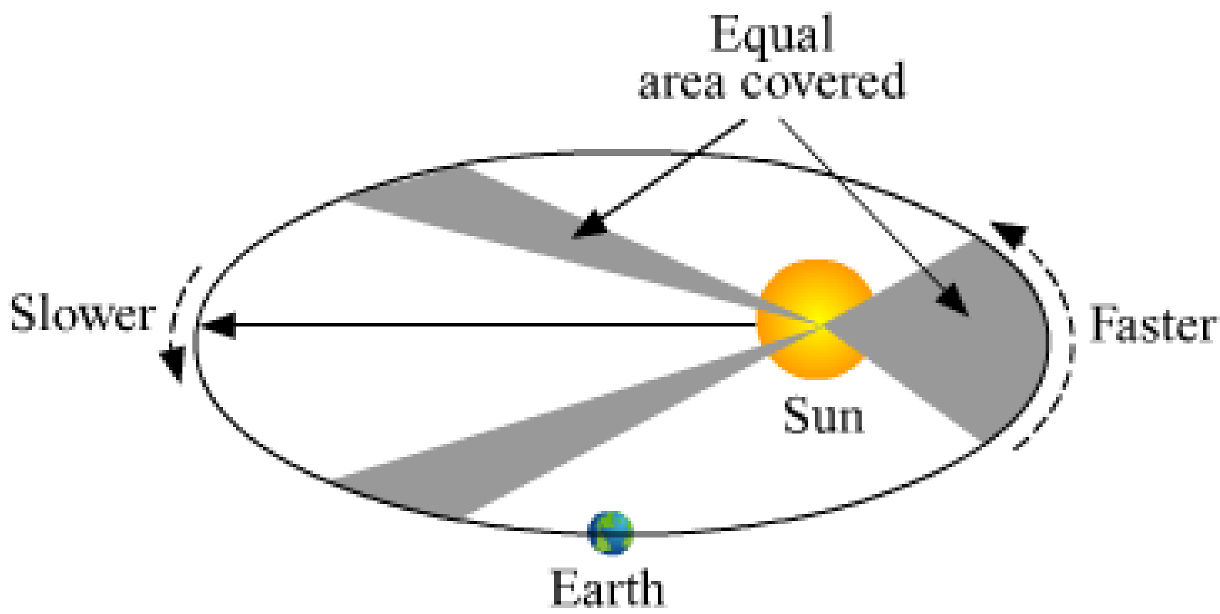


Figure 5: sweeping of area

This shows that the planet quickens as it becomes near to the sun and slows down when it becomes far from the sun.

Kepler's second law can be proved as: First,we rewrite the equation for h using

$$(r \cdot h)\frac{dr}{dt} = e_r\frac{dr}{dt} + re_\theta\frac{d\theta}{dt} \tag{8}$$

(Refer to the fig.6)

$$h = r \cdot \frac{dr}{dt} = r \cdot [e_r\frac{dr}{dt} + re_\theta\frac{d\theta}{dt}] = re_r\frac{dr}{dt} + r^2e_r\frac{d\theta}{dt} \cdot e_\theta \tag{9}$$

$$h = r^2\frac{d\theta}{dt}$$

$$dr^2\theta = 2dA_e$$

This shows that twice the area is swept by the radius vector per unit time."h" is constant so the satellite sweeps out equal areas in equal area in equal period of time.

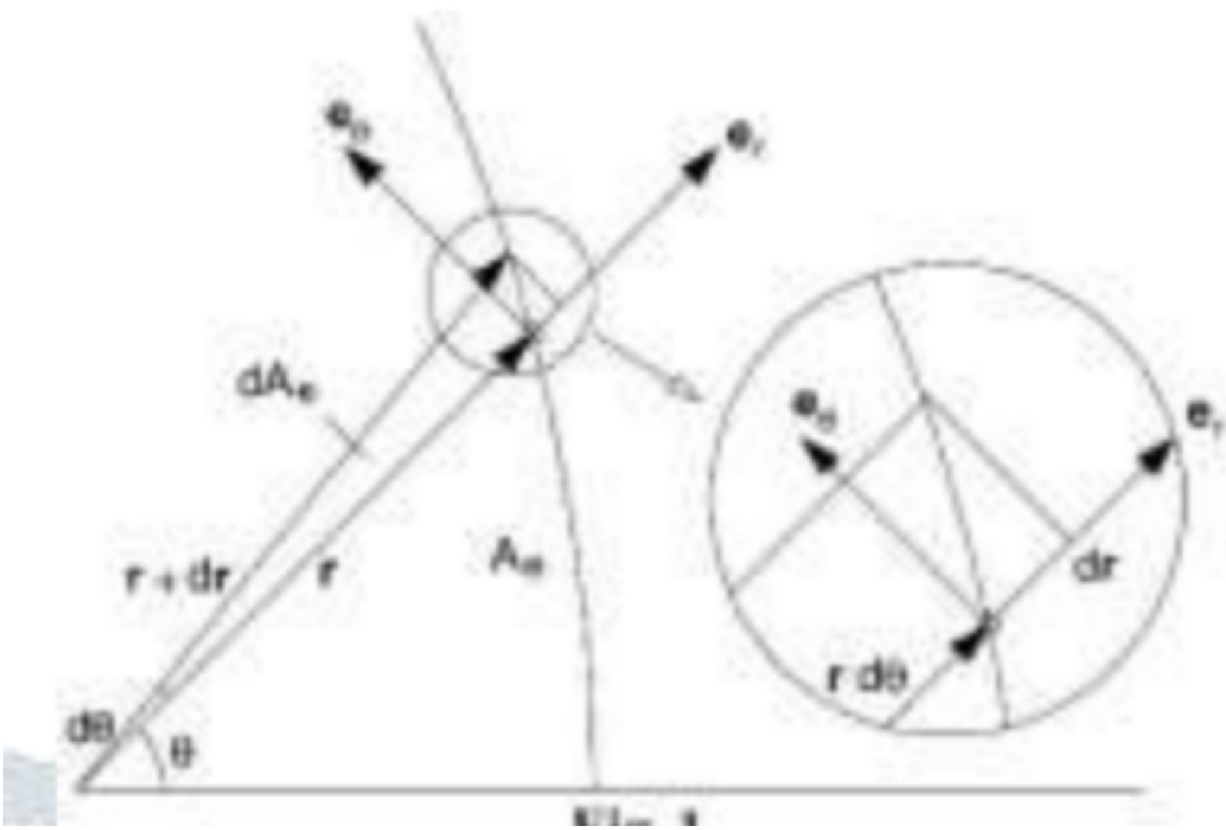


Figure 6: figure 6

## 3.3 Proof Of Kepler's 3rd Law

Kepler's Third Law, sometimes called the law of harmonies,compares the orbital period and radius of a planet's orbit to those of other planets.

In contrast to Kepler's first and second laws,which describe the motion properties of a single planet,the third law makes a comparison between the motion properties of different planets.The comparison that is made is that the ratio of the squares of the periods to the cubes of their average distances from the sun is the same for each of the planets.

The numerator of equation(7) is the sami-latus rectum p.Hence,

$$\frac{h^2}{\mu} = p = a(1 - e^2), \tag{10}$$

or

$$h = [\mu a(1 - e^2)]^1/2 \tag{11}$$

By joining the two equations (10) and (11).We get:

$$h = 2\frac{dA_e}{dt} \tag{12}$$

"h" is constant so,it follows that

$$A_e(t) = t\frac{h}{2} = \frac{[\mu a(1 - e^2]]^1/2}{2}t \tag{13}$$

T=orbital period,

$$A_e = \pi a^2(1 - e^2)^1/2 \tag{14}$$

Then,when we join equation (13) and (14),we get:

$$\frac{[\mu a(1 - e^2)]^1/2}{2}T = \pi a^2(1 - e^2)^1/2 \tag{15}$$

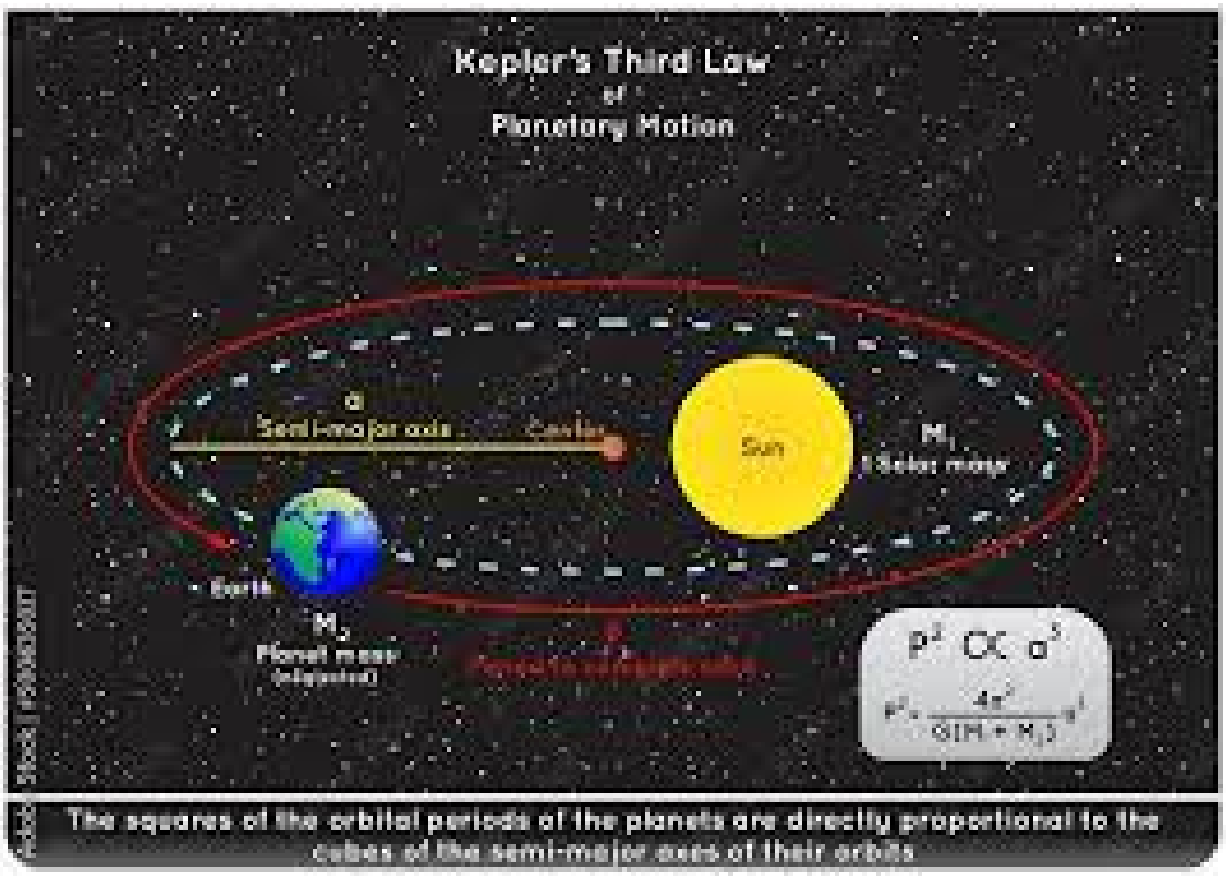


Figure 7: kepler's 3rd law

$$h = [\mu a(1 - e^2)]^1/2 \tag{16}$$

or,

$$T = 2\pi \frac{a^1/3}{\mu^1/2} \tag{17}$$

So this is a proof for Kepler's third law.

# 4 Conclusion

In general,in this project we tried to do our best to discuss about mathematical proof of Kepler's three laws by using vector calculus.Further more we included different diagrams for better understanding,so after reading this project we hope that you get a better knowledge about Kepler's law.

# 5 Literature Cited

## References

(1) Ronald L.Lipsman and Jonathan M.Rosenberg (2014) *with application to geometry and physics*, Multivariable Calculus with MATLAB

(2) Dr.G.D.Singh,2 J.S.Singh(2019) , Journal Of Emerging Technologies and Innovative Research(JETIR),volume 6

(2) Evans,J.C.(1998);"Newton's Laws and Planetary Motion." *http://physics.gmu.edu"*

(3) Danby,J.M.A. and Burkhardt,T.M.(1983):'The Solution of Kepler's Equation,I', *http://Clestial Mechanics.*