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On Comparing Correlated Means in the Presence of Incomplete Data

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Summary

Two statistics are proposed for testing the hypothesis of equality of the means of a bivariate normal distribution with unknown common variance and correlation coefficient when observations are missing on both variates. One of the statistics reduces to the one proposed by BHOJ (1978, 1984) when the unpaired observations on the variates are equal. The distributions of the statistics are approximated by well known distributions under the null hypothesis. The empirical powers of the tests are computed and compared with those of some known statistics. The comparison supports the use of one of the statistics proposed in this paper.

Key words: Bivariate normal; Combination of independent tests; Empirical size and power; Equality of means; Incomplete data.

1. Introduction

Suppose that a sample of n independent pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ has been drawn from the bivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2)'$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \varrho\sigma_1\sigma_2 \\ \varrho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

In addition, there are n_1 observations on x only, and n_2 observations on y only. Without loss of generality, the data may be arranged as follows:

$$\begin{array}{ll} x_1, \ldots, x_n; & x_{n+1}, \ldots, x_{n+n_1}; \\ y_1, \ldots, y_n; & y_{n+1}, \ldots, y_{n+n_2}, \end{array}$$

where x_{n+j} for $j=1, \ldots, n_1$ and y_{n+k} for $k=1, \ldots, n_2$ are unpaired observations. It is assumed that (x_i, y_i), x_{n+j} and y_{n+k} are mutually independent for $i=1, \ldots, n$, $j=1, \ldots, n_1$ and $k=1, \ldots, n_2$. We wish to use all available data to test the null hypothesis $H_0: \delta = \mu_1 - \mu_2 = 0$. LIN and STIVERS (1974) and EKBOHM (1976, 1981) proposed some statistics for testing the equality of means for the above incomplete data pattern, and they carried out empirical investigations of the size and power of some of the tests. Woolson, Leeper, Cole and Clarke (1976) used the theory of the general linear hypothesis to derive the statistics for testing the null hypothesis. Hamdan, Khuri and Crews (1978) modified Morrison's (1973) test statistic which was proposed for missing data on one response. Bhoj (1978, 1984) proposed several statistics for testing $\delta=0$, and he compared the empirical powers of the proposed and known statistics. He recommended a statistic which was derived by using the convex combination of two independent statistics based on complete and incomplete data, respectively. The purpose of the present investigation is to propose two new test statistics for testing $\delta=0$ when $\sigma_1^2=\sigma_2^2$, and to compare them with some of the known statistics by Monte Carlo study.

The following notation is used in this paper:

$$n\bar{x}_1=\sum_{i=1}^{n} x_i, \quad n\bar{y}_1=\sum_{i=1}^{n} y_i, \quad n_1\bar{x}_2=\sum_{j=1}^{n_1} x_{n+j}, \quad n_2\bar{y}_2=\sum_{k=1}^{n_2} y_{n+k}$$

$$(n+n_1)\,\bar{x}=\sum_{i=1}^{n+n_1} x_i, \quad (n+n_2)\,\bar{y}=\sum_{i=1}^{n+n_2} y_i, \quad a_{11}=\sum_{i=1}^{n} (x_i-\bar{x}_1)^2$$

$$a_{22}=\sum_{i=1}^{n} (y_i-\bar{y}_1)^2, \quad a_{12}=\sum_{i=1}^{n} (x_i-\bar{x}_1)\,(y_i-\bar{y}_1)\,,$$

$$b_1=\sum_{j=1}^{n_1} (x_{n+j}-\bar{x}_2)^2\,,$$

$$b_2=\sum_{k=1}^{n_2} (y_{n+k}-\bar{y}_2)^2, \quad c_1=\sum_{i=1}^{n+n_1} (x_i-\bar{x})^2, \quad c_2=\sum_{i=1}^{n+n_2} (y_i-\bar{y})^2\,,$$

$$r=a_{12}/\sqrt{(a_{11}a_{22})}, \qquad u=2a_{12}/(a_{11}+a_{22})\,.$$

2. Test Statistics

In this paper, we propose two statistics for testing the equality of two means of a bivarite normal distribution with unknown common variance σ^2 and unknown correlation coefficient, ϱ based on a sample with missing data on both responses. The usual procedure for testing H_0 based on only complete data is a paired t test

$$t_1=\frac{(\bar{x}_1-\bar{y}_1)\sqrt{n}}{[(a_{11}+a_{22}-2a_{12})/(n-1)]^{1/2}}\,,$$

which is distributed according to Student's t distribution with $f_1=n-1$ degrees of freedom. If paired data are discarded and $\sigma_1^2=\sigma_2^2$, the usual procedure is to use a two-sample t-test

$$t_2=\frac{\bar{x}_2-\bar{y}_2}{[\{(b_1+b_2)/(n_1+n_2-2)\}\,(1/n_1+1/n_2)]^{1/2}}\,,$$

which is distributed as t with $f_2=n_1+n_2-2$ degrees of freedom. One of the purposes of the present investigation is to propose a statistic, which is independent of t_1, and is better than t_2. This statistic is then combined with t_1 to derive a new statistic for testing the null hypothesis.

When $\sigma_1^2=\sigma_2^2$, it is known that $v_i=x_i-y_i$ and $w_i=x_i+y_i$, $i=1, \ldots, n$ are independently distributed. The statistic t_1 is based on the variable, v. Now we consider the three independent samples $2x_{n+1}, \ldots, 2x_{n+n_1}$; $2y_{n+1}, \ldots, 2y_{n+n_2}$ and $w_1, \ldots, w_n$ drawn from three normal populations with means $2\mu_1$, $2\mu_2$ and $\mu_1+\mu_2$, variances $4\sigma^2$, $4\sigma^2$ and $2\sigma^2(1+\varrho)$, respectively. These samples provide two dependent unbiased estimators for δ, namely, $\delta_2=2\bar{x}_2-\bar{x}_1-\bar{y}_1$ and $\delta_3=\bar{x}_1+\bar{y}_1-2\bar{y}_2$. Note that the third unbiased estimator, $\bar{x}_2-\bar{y}_2$, does not provide any additional information on δ since it is a function of δ_2 and δ_3. δ_2 and δ_3 are independent of $\delta_1=\bar{x}_1-\bar{y}_1$ used in t_1. Now we propose an unbiased estimator

$$\delta=\omega\delta_2+(1-\omega)\,\delta_3\,, \tag{2.1}$$

where ω is determined by minimizing the variance of δ. The variance of δ can be written as

$$\sigma^2(\delta)=\sigma^2\,\{4\omega^2/n_1+4\,(1-\omega)^2/n_2\}+2\sigma^2\,(1+\varrho)\,(1-2\omega)^2/n\,.$$

It can be easily verified that the value of ω that minimize $\sigma^2(\delta)$ is

$$\omega=n_1\,\{n+(1+\varrho)\,n_2\}\,[n\,(n_1+n_2)+2\,(1+\varrho)\,n_1n_2]^{-1}, \tag{2.2}$$

and the minimum variance is given by

$$\sigma^2(\delta)=\sigma^2\left[\frac{1}{n_1}+\frac{1}{n_2}-\frac{n\,(n_1-n_2)^2}{n_1n_2\,\{n\,(n_1+n_2)+2\,(1+\varrho)\,n_1n_2\}}\right]. \tag{2.3}$$

Note that when $n_1 \neq n_2$, this variance is always less than $\sigma^2\,(1/n_1+1/n_2)$, which is the variance of $\bar{x}_2-\bar{y}_2$. Therefore one can expect that the test based on δ will be more powerful than the usual two sample t-test. Now we propose the statistic t_3, based on δ, for testing the null hypothesis

$$t_3=\delta\Big/\left[\left\{\frac{4s\,(b_1+b_2)+a_{11}+a_{22}+2a_{12}}{(n+n_1+n_2-3)}\right\}\left\{\frac{\omega^2}{sn_1}+\frac{(1-\omega)^2}{sn_2}+\frac{(1-2\omega)^2}{n}\right\}\right]^{1/2},$$

where $s=(1+u)/2$, and ω is given by (2.2). If s is replaced by $S=(1+\varrho)/2$, the exact distribution of t_3 is Student's t with $f_3=n+n_1+n_2-3$ degrees of freedom. If S is estimated by s, the error in estimation will be small at least for large n. Then we claim that t_3 is approximately distributed as t with f_3 degrees of freedom. This assertion is confirmed by a Monte Carlo study to be discussed in section 3.

Bhoj (1978) proposed the linear combination of t_1 and t_2 as the combined test statistic for testing $\delta=0$. The percentage points of this statistic can be obtained by using an approximation due to Patil (1965). Walker and Saw (1978) showed that the exact percentage points of the statistic can be computed by using only tables of t distribution if f_1 and f_2 are odd. However, this method is laborious and it has limited applications since it requires that f_1 and f_2 be odd. Therefore, Bhoj (1984) suggested to transform each t_p into a new variable so that the distribution of the convex combination of these variables has approximately a standard normal distribution. This was achieved first by transforming t_p to a symmetric F_p via CACOULLOS's (1965) result, and then to U_p $(p=1, 2)$ by using PAULSON's (1942) approximation. The statistic proposed by BHOJ (1984) is

$$(2.4) \qquad Z=\{\lambda U_1+(1-\lambda)\, U_2\}\, \{\lambda^2+(1+\lambda)^2\}^{-1/2}\,,$$

where

$$(2.5) \qquad U_p=\left(1-\frac{2}{9f_p}\right)(F_p^{1/3}-1)\left\{\frac{2}{9f_p}\,(F_p^{2/3}+1)\right\}^{-1/2},$$

$$(2.6) \qquad F_p=1+(2t_p^2/f_p)+(2t_p/\sqrt{f_p})\,\sqrt{(1+t_p^2/f_p)}\,,$$

and

$$\lambda=[1+\sqrt{\{2n_1n_2\,(1-\varrho)/n\,(n_1+n_2)\}}]^{-1}\,.$$

When the means are equal, Z is approximately normally distributed with zero mean and unit variance. The transformation (2.5) works satisfactorily provided $F_p>1$ and $f_p>3$. If $F_p<1$, find U_p corresponding to the reciprocal of F_p. Now we can use this method to derive a new test statistic by using t_1 and t_3.

Our new statistic is

$$(2.7) \qquad Z_b=\{\lambda_b U_1+(1-\lambda_b)\, U_3\}\, \{\lambda_b^2+(1-\lambda_b)^2\}^{-1/2}\,,$$

where U_3 is obtained from t_3 by using the transformation (2.5). It is difficult to determine the optimal value of λ_b in the above statistic. However, an adequate formula for λ_b can be derived if σ^2 and ϱ are assumed to be known. In this case, the two independent statistics t_1 and t_3 are to be replaced, respectively, by

$$Z_1=\hat{\delta}_1/\sigma(\hat{\delta}_1) \quad \text{and} \quad Z_3=\hat{\delta}/\sigma(\hat{\delta})\,,$$

where $\sigma(\hat{\delta}_1)=\sqrt{2\sigma^2\,(1-\varrho)/n}$, and $\hat{\delta}$ and $\sigma(\hat{\delta})$ are given by (2.1) and (2.3). When the null hypothesis holds, Z_1 and Z_3 are distributed as unit normal. If these statistics are combined as in (2.7) then the value of λ_b which maximizes the power of the resulting statistic is given by

$$(2.8) \qquad \lambda_b=\frac{\sigma(\hat{\delta})}{\sigma(\hat{\delta}_1)+\sigma(\hat{\delta})}$$

$$=[1+\sqrt{\{n_1n_2\,(1-\varrho)\}/\{2nn_2\omega^2+2nn_1\,(1-\omega)^2+n_1n_2\,(1-2\omega)^2\,(1+\varrho)\}}]^{-1},$$

where ω is given by (2.2). Now λ_b is estimated by replacing ϱ by u. The resulting estimated λ_b is used in (2.7). The formula (2.8) for λ_b is derived for the convex combination of unit normal variables, and one might expect that the estimated λ_b would give satisfactory results for Z_b. In section 3, we show that Z_b retains the significance level close to the nominal level which demonstrates that the approximation to the distribution of Z_b with estimated λ_b is adequate. It may be noted that when $n_1=n_2$, the statistics t_3 and Z_b proposed in this paper reduce, respectively, to the statistics t_2 and Z used by BHOJ (1984). When $n_2=0$, Z_b reduces to the statistic suggested by BHOJ (1987) for the case of missing data on one response.

Our second statistic is based on the simple mean estimator $\bar{x}-\bar{y}$, which is based on all available observations. This estimator can be expressed as

$$\bar{x}-\bar{y}=m_1\bar{x}_1-m_2\bar{y}_1+(1-m_1)\,\bar{x}_2-(1-m_2)\,\bar{y}_2\,,$$

where $m_1=n_1/(n+n_1)$ and $m_2=n_2/(n+n_2)$. Now we derive the statistic based on the three independent samples $m_1x_i-m_2y_i$, $(1-m_1)\,x_{n+j}$ and $(1-m_2)\,y_{n+k}$ for $i=1,\ldots,n$, $j=1,\ldots,n_1$ and $k=1,\ldots,n_2$. The proposed statistic is

$$T=\frac{\bar{x}-\bar{y}}{\left[\left\{\dfrac{(m_1^2a_{11}+m_2^2a_{22}-2m_1m_2a_{12})+K_1\,(1-m_1)^2\,b_1+K_2\,(1-m_2)^2\,b_2}{n+n_1+n_2-3}\right\}\left(\dfrac{1}{n}+\dfrac{1}{K_1n_1}+\dfrac{1}{K_2n_2}\right)\right]^{1/2}},$$

where

$$K_1=\frac{m_1^2+m_2^2-2m_1m_2u}{(1-m_1)^2}\quad\text{and}\quad K_2=\frac{m_1^2+m_2^2-2m_1m_2u}{(1-m_2)^2}\,.$$

If u is replaced by ϱ, the exact distribution of T is Student's t with $n+n_1+n_2-3$ degrees of freedom. We claim that the distribution of T is approximately t with $n+n_1+n_2-3$ degrees of freedom. This claim is well supported by a simulation study in section 3.

The statistic T is similar to the statistic T_{1s} proposed by Lin and Stivers (1974), which is given by

$$T_{1s}=\frac{\bar{x}-\bar{y}}{\left[\left\{\dfrac{1}{n+n_1}+\dfrac{1}{n+n_2}-\dfrac{2nr}{(n+n_1)\,(n+n_2)}\right\}\left(\dfrac{c_1+b_2}{n+n_1+n_2-2}\right)\right]^{1/2}}.$$

They approximated the distribution of T_{1s} by a Student's t with $n+n_1+n_2-4$ degrees of freedom. The advantage of the statistic T over T_{1s} is that it uses all available data in estimating the variance of $\bar{x}-\bar{y}$. Ekbohm (1976) also proposed the statistic based on the simple mean estimator, $\bar{x}-\bar{y}$, for the case of equal variances. However, we have excluded that statistic since the powers of all statistics based on $\bar{x}-\bar{y}$ are almost the same.

Ekbohm (1976, 1981), Hamden et al. (1978) and Woolson et al. (1976) proposed statistics which are based on the modified maximum likelihood estimators of the mean difference and utilizing the homoscedasticity assumption. Ekbohm (1981) and Bhoj (1984) compared several tests in this group and concluded that they are almost equal in size and power. Ekbohm favores the statistic due to Hamden et al. (1978) since this statistic is a little less complicated than the others. Therefore, we have included, in our comparison study, the statistic Z_h proposed by Hamden et al. (1978), which is given by

$$Z_h=\frac{a\bar{x}_1+(1-a)\,\bar{x}_2-b\bar{y}_1-(1-b)\,\bar{y}_2}{[\hat{\sigma}^2\,\{a^2+m_1\,(1-a)^2/(1-m_1)+b^2+m_2\,(1-b)^2/(1-m_2)-2abu\}/n]^{1/2}},$$

where

$$a=[m_1+uf_2\,(1-f_1)]\,[1-u^2\,(1-m_1)\,(1-m_2)]^{-1}\,,$$
$$b=[m_2+uf_1\,(1-f_2)]\,[1-u^2\,(1-m_1)\,(1-m_2)]^{-1}\,,$$

and

$$\hat{\sigma}^2=(a_{11}+a_{22}+b_1+b_2)\,(2n+n_1+n_2-4)^{-1}\,.$$

They approximated the distribution of Z_h by a Student's t with $(n-1)$ degrees of freedom. Note that when n_1 or n_2 is zero, Z_h reduces to the MORRISON'S (1973) statistic. Z_h also reduces to the paired t-statistic when $n_1=n_2=0$.

In this paper, although we have focused on the equal variance case, we include in our comparison in section 3 the statistic which was derived under the heteroscedasticity assumption. The test statistic was suggested by LIN & STIVERS (1974), and was compared with the other statistics by BHOJ (1984) and EKBOHM (1976, 1981). The statistic is obtained by finding the maximum likelihood estimator for δ with known σ_1^2, σ_2^2 and ϱ, and then replacing them by their estimators based on n complete pairs of observations. The statistic is

$$Z_{1s}=\{g\bar{x}_1+(1-g)\,\bar{x}_2-h\bar{y}_1-(1-h)\,\bar{y}_2\}/\sqrt{V_1}\,,$$

where

$$g=n\,(n+n_2+n_1a_{12}/a_{11})/\{(n+n_1)\,(n+n_2)-n_1n_2r^2\}\,,$$
$$h=n\,(n+n_1+n_2a_{12}/a_{22})/\{(n+n_1)\,(n+n_2)-n_1n_2r^2\}$$

and

$$V_1=[\{g^2/n+(1-g)^2/n_1\}\,a_{11}+\{h^2/n+(1-h)^2/n_2\}\,a_{22}-2hga_{12}/n]/f_1\,.$$

When the hypothesis of equal means holds, Z_{1s} is approximately distributed as t with n degrees of freedom.

3. Comparison of Tests

In this section we compare the various test statistics from the point of view of size and power for small to moderately large samples by using a simulation study. Some simulation results are available for the statistics t_1, T_{1s}, Z_{1s}, Z_h and Z. However, further study was needed for the statistics T and Z_b proposed in section 2. For this reason, using the BOX-MULLER (1958) technique, one thousand random numbers were generated from a bivariate normal distribution with various values of σ_1^2, σ_2^2, δ ($\mu_2=0$), and the combinations of n, n_1 and n_2. The hypothesis $H_0:\delta=0$ was tested against $H_1:\delta>0$ at the 5 % level of significance. The relative frequencies of the nominal α level, that is, the proportions of tests for which H_0 was rejected when $\delta=0$, were recorded. These proportions are called the empirical α levels. The empirical powers of the tests, that is, the proportions of test for which H_0 was rejected when $\delta>0$, were also recorded. Some of the results of simulations are given in Tables 1–3. Tables 1 and 2 deal with small samples and moderately large sam-

Table 1

Empirical levels and powers (×1000) for t_1, T_{1s}, T, Z_{1s}, Z_h, Z and Z_b, when $\sigma_1^2=\sigma_2^2=1$ for $n=10$, $n_1=5$, $n_2=10$.

ϱ	δ	t_1	T_{1s}	T	Z_{1s}	Z_h	Z	Z_b
−.9	0	49	54	58	53	47	60	54
	.4	152	248	258	250	240	258	269
	.8	321	591	593	621	595	608	642
	1.2	545	863	859	875	877	871	898
−.5	0	52	59	60	53	52	64*	57
	.4	160	267	271	261	249	268	273
	.8	366	651	649	645	626	650	656
	1.2	638	905	905	897	903	902	914
−.1	0	57	59	58	54	48	55	54
	.4	186	300	310	283	268	300	294
	.8	449	703	706	685	670	690	696
	1.2	759	955	955	932	936	937	947
.1	0	55	48	51	50	49	54	51
	.4	204	323	328	298	292	311	304
	.8	519	748	754	726	713	726	744
	1.2	832	969	971	962	963	956	966
.5	0	51	46	47	56	46	56	45
	.4	290	371	362	399	375	382	395
	.8	760	847	837	870	864	864	869
	1.2	964	992	993	990	992	995	995
.9	0	46	49	52	57	49	47	46
	.4	840	471	488	860	839	848	853
	.8	999	946	942	1000	1000	1000	1000
	1.2	1000	999	1000	1000	1000	1000	1000

ples, respectively, when $\sigma_1^2=\sigma_2^2$ while Table 3 deals with the situation when $\sigma_1^2 \neq \sigma_2^2$.

An approximate test is considered satisfactory if its nominal α level is within two standard deviations of the empirical α level. The empirical α levels marked $*$ and $**$ in the tables indicate that the empirical α levels are different from the nominal α levels by more than $2\sqrt{\alpha(1-\alpha)/1000}$ and $3\sqrt{\alpha(1-\alpha)/1000}$, respectively. When $\sigma_1^2=\sigma_2^2$ and $n \geqq 10$, the empirical α levels of all tests using both complete and incomplete data are fairly close to the nominal $\alpha=.05$. When $n<10$, the empirical α levels of Z_{1s} are erratic. Table 3 shows that in many cases they are different from the nominal levels by more than three standard deviations. This, however, is expected since LIN & STIVERS (1976) have noted that no satisfactory approximation to the distribution of Z_{1s} has been found for $n<10$.

The test statistics T and T_{1s} are almost equal in power for the sample sizes given in the tables. However, for some sample sizes T seems to be more robust when the homoscedasticity assumption is violated. These tests do well when ϱ^2 is small. However, Table 2 shows that the gain in power of these tests compared to Z_b is neglegible to none, even for small ϱ^2, when n is moderately large. Moreover, these tests might be inferior to the complete paired t test when ϱ exceeds 0.5.

As is expected, the powers of Z_b are higher than those of Z when $n_1 \neq n_2$. A

Table 2

Empirical levels and powers (×1000) of t_1, T_{1s}, T, Z_{1s}, Z_h, Z and Z_b for $n=20$, $n_1=30$, $n_2=10$, when $\sigma_1^2=\sigma_2^2=1$.

ϱ	δ	t_1	T_{1s}	T	Z_{1s}	Z_h	Z	Z_b
−.9	0	60	55	55	51	48	58	52
	.2	129	176	178	188	178	182	189
	.4	224	395	400	438	428	403	450
	.8	554	887	889	913	911	869	921
−.5	0	59	58	57	59	54	55	59
	.2	142	188	189	191	183	183	188
	.4	255	437	438	440	436	427	455
	.8	633	922	921	922	927	906	936
−.1	0	59	51	53	53	54	58	58
	.2	161	211	210	204	193	189	203
	.4	303	489	490	480	475	456	485
	.8	752	963	965	961	963	946	966
.1	0	55	51	51	53	47	51	54
	.2	172	208	206	221	209	218	222
	.4	346	514	513	509	511	498	517
	.8	821	974	975	973	974	964	975
.5	0	57	52	49	51	43	47	47
	.2	219	242	238	283	268	276	270
	.4	538	606	605	637	641	626	645
	.8	968	991	992	993	991	989	992
.9	0	51	43	42	49	42	45	43
	.2	613	308	311	651	628	623	632
	.4	988	729	724	989	987	991	991
	.8	1000	1000	1000	1000	1000	1000	1000

critical study of Tables 1 and 2 shows that the statistic Z_b is preferred to Z_{1s}, Z_h and Z when we take into consideration the differences in the empirical α levels of these statistics. Note that the statistic Z_h is somewhat conservative particularly for small departures from the homoscedasticity assumption, and this results in a loss of power.

The proposed statistics are fairly robust for departures from the homoscedaticity assumption when the variable with a larger variance has a larger number of observations. Table 3 shows that Z_{1s} has an unfair advantage in the comparison of powers with the other test statistic, as its empirical α is very high. However, even in these cases Z_b has slightly higher power than that of Z_{1s} when δ is relatively large.

To summarize, there is no single statistic which is superior to all other test statistics over the entire parameter space. Therefore we should choose the statistic which performs well from the significance level and power point of views over the wide range of values of ϱ. The use of Z_b is recommended at least for the homoscedastic cases, and when nothing is known about ϱ. The statistic Z_b is also preferred for $n<10$ for small departures from the homoscedastic assumption when the variable with larger variance has the larger number of observations.

Table 3

Empirical levels and powers (×1000) of t_1, T_{1s}, T, Z_{1s}, Z_h, Z and Z_b for $n=7$, $n_1=10$ and $n_2=5$ when $\sigma_1^2=1$, $\sigma_2^2=.5$.

ϱ	δ	t	T_{1s}	T	Z_{1s}	Z_h	Z	Z_b
−.9	0	66*	52	48	58	38	51	49
	.4	151	266	252	284	227	264	281
	.8	307	651	626	644	589	629	654
	1.2	521	922	910	911	886	908	917
−.5	0	62	56	52	62	40	51	52
	.4	172	299	274	296	232	278	283
	.8	350	697	678	671	611	676	688
	1.2	613	947	939	936	923	934	939
−.1	0	61	60	53	73**	40	56	54
	.4	207	326	303	313	252	298	309
	.8	436	763	742	729	687	730	747
	1.2	708	965	961	953	941	956	954
.1	0	55	59	49	75**	47	60	55
	.4	223	338	321	335	274	303	319
	.8	495	799	778	777	729	765	781
	1.2	773	974	969	963	953	965	965
.5	0	59	46	39	79**	47	59	60
	.4	301	391	372	440	346	380	402
	.8	688	866	855	875	839	868	870
	1.2	927	990	990	992	990	990	995
.9	0	59	45	48	76**	46	61	57
	.4	642	477	500	811	653	704	717
	.8	983	934	938	998	993	992	995
	1.2	1000	999	999	1000	1000	1000	1000

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