Samuel Sicklick

Simple Example

Prove that: If x is an odd integer, then x + 1 is even.
Formally, you must prove that $\forall x \in Z$ (x is odd $\Rightarrow$ x + 1 is even).

1. The definition of an odd integer is $2k + 1$, for some constant $k \in Z$.
2. Thus, if we assume that x is an odd integer, then x + 1 could be rewritten as:
   $(2x + 1) + 1$
3. The equation in step 2 can be simplified to:
   $2x + 2$
4. The equation in step 3 can be factored into:
   $2(x + 1)$
5. The equation in step 4 is divisible by 2, which shows that the original equation of x + 1 (before it was rewritten in step 2) is indeed even.
6. QED

Proof By Induction (I)

First some notation and a definition for divisibility. We say that integer a divides b (or b is divisible by a), if and only if for some integer q, b = aq.
We write this as a | b.

Theorem: $\forall n \in N$, $3 \mid (n^3 - n)$
Prove the theorem using induction.

1. Let $P(n)$ be the statement $\forall n \in N$, $3 \mid (n^3 - n)$.
2. To prove $P(n)$, we must show that $(n^3 - n) = 3q$, for some $q \in N$, by the definition of divisibility.
3. A base case can be established for n = 1, as $(1^3 - 1) = 3(0)$. Thus, $P(1)$ is true.
4. Our inductive hypothesis assumes $P(k)$ to be true, meaning $(k^3 - k) = 3q$ for some $q \in N$.
5. As an inductive step, we will show that $(k + 1)^3 - (k + 1) = 3q$, for some $k \in N$.
   a. The equation in step 5 can be expanded into:
      $(k + 1)(k + 1)(k + 1) - (k - 1)$
   b. The equation in step 5.a can be simplified into:
      $(k^2 + 2k + 1)(k + 1) - (k - 1)$
   c. The equation in step 5.b can be simplified into:
      $((3k^3 + 2k^2 + k) + (k^2 + 2k + 1) - (k - 1)$
   d. The equation in step 5.c can be simplified into:
      $(3k^3 + 3k^2 + 3k + 1) - (k - 1)$
   e. The equation in step 5.d can be simplified into:
      $3k^3 + 3k^2 + 3k + 1 - k - 1$
   f. The equation in step 5.e can be simplified into:
      $3k^3 + 3k^2 + 3k - k$
   g. The equation in step 5.f can be rearranged into:
      $3k^3 - k + 3k^2 + 3k$

h. The equation from step 4 (inductive hypothesis) can be plugged in:

$$3q + 3k^2 + 3k$$

i. The equation from step 5.h can be factored into:

$$3(q + k^2 + k)$$

j. For the equation in step 5.i, let j be an integer such that:

$$j = q + k^2 + k$$

k. Thus, we have shown that $(k + 1)^3 - (k + 1) = 3j$, which establishes $P(k + 1)$ as being true since $(k + 1)^3 - (k + 1)$ is divisible by 3.

6. Accordingly, we have proven the theorem $3 \mid (n^3 - n)$ for all $n \in N$ via induction.
7. QED

Proof By Induction (II)

Theorem: $\forall n \in N$, for $n > 1$ we have $n! < n^n$.

1. Let $P(n)$ be the statement $n! < n^n$ for all $n \in N$, where $n > 1$.
2. To prove $P(n)$, we must show that $n! < n^n$ for some $n \in N$, where $n > 1$.
3. A base case can be established for $n = 2$, the first number which satisfies the condition $n > 1$.
   a. $2! = (2)(1) = 2$
   b. $2^2 = (2)(2) = 4$
   c. Thus, $2! < 2^2$ and therefore $P(2)$ is true.
4. Our inductive hypothesis assumes $P(k)$ to be true, meaning $k! < k^k$ for some $k \in N$, where $k > 1$.
5. As an inductive step, we will show that $(k + 1)! < (k + 1)^{(k + 1)}$, for some $k \in N$.
   a. By the definition of a factorial, we can simplify the equation in step 5 into:
   $$(k + 1)k! < (k + 1)^{(k + 1)}$$
   b. By the definition of a power, we can simplify the equation in step 5.q into:
   $$(k + 1)k! < (k + 1)(k + 1)^k$$
   c. Both sides can be divided by $(k + 1)$ to yield:
   $$k! < (k + 1)^k$$
   d. While there is no direct plug-in of the equation from step 4 (inductive hypothesis), it is clear that if $k! < k^k$ is true then $k! < (k + 1)^k$ is also true.
      i. The definition of a factorial is:
      $$(k)(k - 1)(k - 2) \ldots (2)(1)$$
      which is an equation of k terms, where each operand is 1 less than its predecessor.
      ii. The definition of a raising to a power *k* is:
      $$(k)(k)(k) \ldots (k)(k)(k)$$
      which is an equation of k terms, where each operand is the same as its predecessor.
      iii. Thus, the equation in step 5.c can have both sides divided by $(k + 1)^k$ to yield:
      $$k! / (k + 1)^k < 1$$
      iv. The equation in step 5.d.iii can be expanded into:
      $$\frac{k}{(k+1)} \frac{(k-1)}{(k+1)} \ldots \frac{1}{(k+1)}$$

v. Since each term in the equation in 5.d.iv is causing the number to get smaller, and the first term is already less than 1, it can be concluded that $k! / (k + 1)^k < 1$ is indeed correct.

6. Accordingly, we have proven the theorem $n! < n^n$ for all $n \in N$, where $n > 1$, via induction.
7. QED