2. Linear Algebra (Exercises Only)

Exercises

2.1:

$$(\mathbb{R}\backslash\{-1\}, *)$$

$$a * b = ab + a + b$$

a. Show that this is an Abelian group

First, we must show that it is a group.

1. Closure: Since the set is closed under both addition and multiplication, and $*$ is just a combination of these operators, it follows that the set is closed under the $*$ operator as well.
2. Associativity:
   - $a * (b * c) = a * (bc + b + c)$
   - $a * (b * c) = a(bc + b + c) + a + (bc + b + c)$
   - $a * (b * c) = abc + ab + ac + a + bc + b + c$
   - $a * (b * c) = abc + ab + ac + bc + a + b + c$
   - $(a * b) * c = (ab + a + b) * c$
   - $(a * b) * c = c(ab + a + b) + (ab + a + b) + c$
   - $(a * b) * c = abc + ab + ac + bc + a + b + c$
   - Therefore:
   - $(a * b) * c = a * (b * c)$ and thus we've demonstrated associativity
3. Neutral Element: $x * e = x$
   - Consider 0:
   - $x * 0 = 0x + x + 0 = x$
   - Therefore 0 is the neutral element
4. Inverse Element:
   - Need to find an element that gives zero.
   - $a * y = ay + a + y$
   - $0 = ay + a + y$
   - $0 = y(a + 1) + a$
   - $-a = y(a + 1)$
   - $-\dfrac{a}{a + 1} = y$

- a cannot be equal to -1 or this is undefined, which is lucky because our set excludes -1

Now that we have shown that it is a group, we can say that it Abelian. The $*$ operator is made up of multiplications and additions on the real numbers which are commutative, therefore the $*$ operator is commutative and the group is Abelian

b. Solve:

$$3 * x * x = 15$$

$$(3x + 3 + x) * x = 15$$

$$x(3x + 3 + x) + (3x + 3 + x) + x = 15$$

$$3x^2 + 3x + x^2 + 3x + 3 + x + x = 15$$

$$4x^2 + 8x + 3 = 15$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = \{1, -3\}$$

2.2:

- Let n bet in $\mathbb{N} \backslash \{0\}$. Let k, x be in $\mathbb{Z}$
- We define a congruence class $\bar{k}$ of the integer k as set

$$\begin{aligned}\bar{k} &= \{x \in \mathbb{Z} | x - k = 0(\mathrm{mod} n)\} \\ &= \{x \in \mathbb{Z} | (\exists a \in \mathbb{Z}) : (x - k = n \cdot a)\}\end{aligned}$$

- We now define $\mathbb{Z}/n\mathbb{Z}$ as the set of all congruence classes modulo n
- Euclidian division implies that this set is a finite set containing n elements
- This implication comes from the idea that
  1. Every integer has a remainder when divided by n. (The remainder may be 0
  2. All integers belong to exactly one congruence class
  3. The remainder r falls within these bounds: $0 \leq r < n$ (this is just the definition of the remainder)
  4. Therefore, there are exactly n congruence classes
- For all $\bar{a}$, $\bar{b} \in \mathbb{Z}$ we define:

$$\bar{a} \oplus \bar{b} = \overline{a + b}$$

a. Show that $\mathbb{Z}_{\mathbb{m}}, \otimes$ is a group:

- Closure: This operation is closed by definition because, whatever the result of $\overline{a+b}$, you're modding it by n to put it in a congruence class. Therefore, the result will always be a congruence class.
- Associativity: Addition of integers is associative, therefore this operator is associative
- Neutral element: The neutral element would be 0: $\bar{a} \otimes 0 = \overline{a+0} = \bar{a}$
- Inverse element: Need to find some element y such that: $a \oplus y = 0$:
  - $0 = \overline{a+y}$
  - $\bar{y} = -\bar{a}$

Is it Abelian? I think that it is because addition is commutative.

b. We now define another operation: $\otimes$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z}_n$ as:

$$\bar{a} \otimes \bar{b} = \overline{a \times b}$$

- Let $n = 5$
- Draw the times table for $\mathbb{Z}_5 \backslash \{\bar{0}\}$ under $\otimes$, i.e. calculate the products $\bar{a} \otimes \bar{b}$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z} \backslash \{\bar{0}\}$:

$\otimes$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

- This demonstrates closure and shows that the neutral element is $\bar{1}$
- The inverse elements are as follows:
  - $\bar{1}$: $\bar{1}$
  - $\bar{2}$: $\bar{3}$

- $\bar{3}$: $\bar{2}$
    - $\bar{4}:\bar{4}$
- We've shown that it is a group. It is also Abelian because multiplication is commutative.

c. $(\mathbb{Z}_8 \backslash \{\bar{0}\}, \otimes)$ is not a group because it is not closed. For example $\bar{4} \otimes \bar{2} = \bar{0}$ which is not in the set.

d.

- $au + nv = 1$
- $\overline{au} \oplus \overline{nv} = \bar{1}$
- Obviously, $\overline{nv}$ is some multiple v of n so $\overline{nv} = \bar{0}$ and this simplifies to:
- $\bar{a} \oplus \bar{u} = \bar{1}$
- In other words, $\bar{u}$ is the inverse of $\bar{a}$
- This shows the inverse element property holds
- There is closure because, if we take two integers a and b which are relatively prime to n and multiply them together we can never get a multiple of n. The product will never be outside the set.
- If n is not prime, then $\bar{ab} = \bar{n} = \bar{0}$ is possible therefore the set is not closed

  2.3:

$$\mathcal{G} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

- $x, y, z \in \mathbb{R}$
- Is $(\mathcal{G}, \cdot)$ a group?
    - It has the property of closure -- matrix product will be a 3x3 matrix
    - It is associative because matrix multiplication is associative
    - It has a neutral element, the standard 3x3 I matrix
    - It has an inverse element because the matrix is always regular. It is already in row echelon form by this definition and has 1s in each pivot.
- Is it Abelian? No. Matrix multiplication is not commutative.

  2.4:
- a. Can't multiply
- b.

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}4 & 3 & 5\\10 & 9 & 11\\16 & 15 & 17\end{bmatrix}$$

- c.

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}5 & 7 & 9\\11 & 13 & 15\\8 & 10 & 12\end{bmatrix}$$

- d.

$$\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix}\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix} = \begin{bmatrix}14 & 6\\-21 & 2\end{bmatrix}$$

- e.

$$\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix} = \begin{bmatrix}12 & 3 & -3 & -12\\-3 & 1 & 2 & 6\\6 & 5 & 1 & 0\\13 & 12 & 3 & 2\end{bmatrix}$$

2.5:

- a. After preforming Gaussian Elimination:

$$\begin{bmatrix}1 & 1 & -1 & -1 & | & 1\\0 & 3 & -5 & -3 & | & -4\\0 & 0 & -2 & 2 & | & 2\\0 & 0 & 0 & 0 & | & 1\end{bmatrix}$$

- the last row indicates the the solution set is empty: $\mathbb{S} = \emptyset$. The last row presents a contradiction. All coefficients are 0, but the right side is nonzero.
- b. After preforming Gaussian Elimination:

$$\begin{bmatrix}1 & 0 & 0 & 0 & -1 & | & 3\\0 & 1 & 0 & 0 & -2 & | & 0\\0 & 0 & 0 & 1 & -1 & | & -1\\0 & 0 & 0 & 0 & 0 & | & 0\end{bmatrix}$$

- The pivot columns are $1, 2, 4$
- The free columns are $3, 5$
- A particular solution is:

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

- Now on to the general solution.
- For the first free variable, λ_3 our job is easy. The third column is all zeros so to produce a vector in the null space with linear combinations of the columns of the matrix we just take one of the third column:

$$\lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- The second free variable λ_5 isn't quite that trivial, but is also straightforward. A linear combination of the columns of the matrix including column 5 that produces the 0 vector is:

$$\lambda_5 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

  Therefore the solution set S is:

$$S = x \in \mathbb{R}^5 : x = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_5 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \lambda_3, \lambda_5 \in \mathbb{R}$$

2.6:

- a. After Gaussian Elimination:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & | & 1 \end{bmatrix}$$

- particular solution:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

- free variables:
- λ_1:

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- λ_3:

$$\lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- λ_6:

$$\lambda_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S = x \in \mathbb{R}^6 : x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \lambda_1, \lambda_3, \lambda_6 \in \mathbb{R}$$

2.7:
Why is it asking me about eigenvalues before we've talked about them? This book sucks.

2.8:

- a. not invertible
- b.

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

2.9: To show that something is a vector space we must show that the following properties hold:

1. Contains the zero vector
2. Closed under addition
3. Closed under multiplication

a.

1. Yes, if λ and μ are both 0, we reach the zero vector
2. Yes. This follows from the real numbers being closed under addition
3. Yes. Scalar multiplication cannot take it out of $\mathbb{R}^3$

b.

1. Yes, if $\lambda = 0$, we reach the zero vector
2. Not closed under addition:

$$\begin{bmatrix} \lambda_1^2 \\ -\lambda_1^2 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_2^2 \\ -\lambda_2^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ -\lambda_1^2 + \lambda_2^2 \\ 0 \end{bmatrix}$$

So no

c. Yes
d. Not necessarily closed under scalar multiplication if the scalar is not an integer because ζ must be an integer.

2.10:

- a. no
- b. no

2.11:

$$y = -6x_1 + 3x_2 + 2x_3$$

2.12:

$$\begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}$$

2.13:

- a. the dimensions of both are 2, they represent the null space, and they map from $\mathbb{R}^3$, so their dimension is 1
- b & c:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

2.14:

- a. Consider two subspaces U_1 and U_2 where U_1 is spanned by the columns of A_1 and U_2 is spanned by the columns of A_2 with:

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

a. Determine the dimension of U_1 and U_2

- $\dim(U_1) = \mathrm{rk}(A_1) = 2$ because the row echelon form is:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\dim(U_2) = \mathrm{rk}(A_2) = 2$ because the row echelon form is:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. Determine the bases of U_1 and U_2

- U_1 basis (read from the pivot columns of A_1:

$$\begin{bmatrix}1\\1\\2\\1\end{bmatrix}\begin{bmatrix}0\\-2\\1\\0\end{bmatrix}$$

- U_2 basis (read from the pivot columns of the row echelon form matrix of A_2:

$$\begin{bmatrix}3\\1\\7\\3\end{bmatrix}\begin{bmatrix}-3\\2\\-5\\-1\end{bmatrix}$$

c. Determine the basis of $U_1 \cap U_2$

- $U_1 \cap U_2$ consists of all vectors that belong to both U_1 and U_2
- Any vector in $U_1 \cap U_2$ must be a linear combination of both basis sets:

$$\lambda_1 \begin{bmatrix}1\\1\\2\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}0\\-2\\1\\0\end{bmatrix} = \mu_1 \begin{bmatrix}3\\1\\7\\3\end{bmatrix} + \mu_2 \begin{bmatrix}-3\\2\\-5\\-1\end{bmatrix}$$

- λs and μs are some scalars $\lambda, \mu \in \mathbb{R}$
- u_1, u_2 are the basis vectors for U_1
- v_1, v_2 are the basis vectors for U_2

$$\lambda_1 \begin{bmatrix}1\\1\\2\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}0\\-2\\1\\0\end{bmatrix} - \mu_1 \begin{bmatrix}3\\1\\7\\3\end{bmatrix} - \mu_2 \begin{bmatrix}-3\\2\\-5\\-1\end{bmatrix} = 0$$

- This is a homogenous system of equations that can be written in the form $Ax = 0$:

$$\begin{bmatrix}1 & 0 & -3 & 3\\1 & -2 & -1 & -2\\2 & 1 & -7 & 5\\1 & 0 & -3 & 1\end{bmatrix}$$

- Now we can solve for the null space of this matrix to find the coefficients which satisfy this equation. This matrix is full rank therefore the null space only contains the zero vector and there is no basis for $U_1 \cap U_2$

2.15:

- $F = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$
- For a set to be a subspace, it must satisfy three conditions:
  1. Contain the zero vector

2. Closed under addition
3. Closed under scalar multiplication

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4. if $x = 0$, $y = 0$, $z = 0$ the $F$ contains the zero vector
5. Consider two vectors:
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$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

- Given that $x_1 + y_1 - z_1 = 0$ and $x_2 + y_2 - z_2 = 0$ we can consider their sum:

$$(x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) = 0$$

  So this sum sill satisfies the equation $x + y - z = 0$ therefore F is closed under addition.
  3. Now multiplication by a scalar $\lambda(x + y - z)$
  4. If $x + y - z = 0$ then $\lambda(0) = 0$ which is still in the subspace therefore its closed under scalar multiplication and is a subspace of $\mathbb{R}^3$
- $G = \{(a - b, a + b, a - 3b) | a, b \in \mathbb{R}\}$
  1. Once again it contains the zero vector if a and b are both 0
  2. Closure under addition:

$$(a_1 - b_1) + (a_2 - b_2), (a_1 + b_1) + (a_2 + b_2), (a_1 - 3b_1) + (a_2 - 3b_2)$$

- grouping like terms:

$$(a_1 + a_2) - (b_1 + b_2), (a_1 + a_2) + (b_1 + b_2), (a_1 + a_2) - 3(b_1 + b_2)$$

  Let $a' = a_1 + a_2$ and $b' = b_1 + b_2$:

$$a' - b', a' + b', a' - 3b'$$

Which matches the form of G therefore it is closed under addition
3. Closure under scalar multiplication:

$$\lambda(a - b, a + b, a - 3b) = \lambda a - \lambda b, \lambda a + \lambda b, \lambda a - 3\lambda b$$

- Let $a' = \lambda a$ and $\lambda b = b'$:

$$\lambda b, \lambda a + \lambda b, \lambda a - 3\lambda b = a' - b', a' + b', a' - 3b'$$

  Which has the form of G so it's closed under scalar multiplication

b.

$$(a - b) + (a + b) - (a - 3b) = 0$$

$$a + 3b = 0$$

$$a = -3b$$

$$x = -3b - b = -4b$$

$$y = -3b + b = -2b$$

$$z = -3b - 3b = -6b$$

$$F \cap B = \begin{bmatrix} -4b \\ -2b \\ -6b \end{bmatrix}$$

2.16

- Are the following mappings linear?
  - a.

$$\Phi : L^1([a, b]) \to \mathbb{R}$$

$$f \to \Phi(f) = \int_a^b f(x)dx$$

- Where $L^1([a, b]$ denotes the set of integrable functions on $[a, b]$
- Basically, Φ is a function that takes as an argument a function and maps it to the definite integral from $[a, b]$
- A mapping is linear if it satisfies the following conditions:

1. $$\Phi(x + y) = \Phi(x) + \Phi(y)$$

2. $$\Phi(\lambda x) = \lambda\Phi(x)$$

- Linearity in addition:

$$\Phi(f(x) + g(x)) = \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx = \Phi(f(x)) + \Phi(g(x))$$ ✅

- Via the linearity of integration. In the same way, scalar multiplication:

$$\Phi(\lambda f(x)) = \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx = \lambda\Phi(f(x))$$ ✅

- So Φ is a linear mapping
- b.

$$\Phi : C^1 \to C^0$$

$$f \to \Phi(f) = f'$$

- Where$k \geq 1, C^k$ denotes the set of k times continuously differentiable functions and C^0 denotes the set of continuous functions
- Linearity of addition:

$$\Phi(f(x)+g(x)) = f'(x)+g'(x) = \Phi(f(x))+\Phi(g(x))$$ ✅

- Scalar multiplication again via the linearity of differentiation:

$$\Phi(\lambda f(x)) = \lambda f(x) = \lambda \Phi(f(x))$$ ✅

- So Φ is a linear mapping
- c.

$$\Phi : \mathbb{R} \to \mathbb{R}$$
$$x \to \Phi(x) = \cos(x)$$

- Linearity of addition:

$$\Phi(x+y) = \cos(x+y) \neq \cos(x)+\cos(y)$$ ❌

- Cosine is not linear therefore Φ is not linear. For example:

$$\cos(\pi+0) = -1$$

- but

$$\cos(\pi)+\cos(0) = 0$$

d.

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^2$$
$$x \to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$$

- Linearity of addition:

$$\Phi(x+y) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1+2x_2+2y_2+3x_3+3y_3 \\ x_1+y_1+3x_2+3y_2+4x_3+4y_3 \end{bmatrix}$$

$$\Phi(x) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1+2x_2+3x_3 \\ x_1+3x_2+4x_3 \end{bmatrix}$$

$$\Phi(y) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1+2y_2+3y_3 \\ y_1+3y_2+4y_3 \end{bmatrix}$$

$$\Phi(x)+\Phi(y)=\begin{bmatrix}x_1+y_1+2x_2+2y_2+3x_3+3y_3\\x_1+y_1+3x_2+3y_2+4x_3+4y_3\end{bmatrix}=\Phi(x+y)$$ ✅

- Linearity of scalar multiplication:

$$\Phi(\lambda x)=\begin{bmatrix}1&2&3\\1&4&3\end{bmatrix}\begin{bmatrix}\lambda x_1\\\lambda x_2\\\lambda x_3\end{bmatrix}=\begin{bmatrix}\lambda x_1+2\lambda x_2+3\lambda x_3\\\lambda x_1+3\lambda x_2+4\lambda x_3\end{bmatrix}=\lambda\Phi(x)$$ ✅

- So Φ is a linear mapping

e.

- Let θ be in $[0,2\pi]$ and:

$$\Phi:\mathbb{R}^2\rightarrow\mathbb{R}^2$$
$$x\rightarrow\begin{bmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{bmatrix}x$$

- Linearity of addition:

$$\Phi(x+y)=\begin{bmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}x_1+y_2\\x_2+y_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)x_1+\cos(\theta)y_1+\sin(\theta)x_2+\sin(\theta)y_2\\-\sin(\theta)x_1-\sin(\theta)y_1+\cos(\theta)x_2+\cos(\theta)y_2\end{bmatrix}$$
$$\Phi(x)=\begin{bmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)x_1+\sin(\theta)x_2\\-\sin(\theta)x_1+\cos(\theta)x_2\end{bmatrix}$$
$$\Phi(y)=\begin{bmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)y_1+\sin(\theta)y_2\\-\sin(\theta)y_1+\cos(\theta)y_2\end{bmatrix}$$
$$\Phi(x)+\Phi(y)=\begin{bmatrix}\cos(\theta)x_1+\cos(\theta)y_1+\sin(\theta)x_2+\sin(\theta)y_2\\-\sin(\theta)x_1-\sin(\theta)y_1+\cos(\theta)x_2+\cos(\theta)y_2\end{bmatrix}=\Phi(x+y)$$ ✅

- Linearity of scalar multiplication:

$$\Phi(\lambda x)=\begin{bmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}\lambda x_1\\\lambda x_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)\lambda x_1+\sin(\theta)\lambda x_2\\-\sin(\theta)\lambda x_1+\cos(\theta)\lambda x_2\end{bmatrix}=\lambda\begin{bmatrix}\cos(\theta)x_1+\sin(\theta)x_2\\-\sin(\theta)x_1+\cos(\theta)x_2\end{bmatrix}=\lambda\Phi(x)$$

2.17

- Consider the linear mapping:

$$\Phi:\mathbb{R}^3\rightarrow\mathbb{R}^4$$
$$\Phi\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}3x_1+2x_2+x_3\\x_1+x_2+x_3\\x_1-3x_2\\2x_1+3x_2+x_3\end{bmatrix}$$

- Find the transformation matrix A_Φ
- To find the transformation matrix we just look at the coefficients:

$$A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

- Determine $\mathrm{rk}(A_\Phi)$
- To determine the rank we perform Gaussian Elimination on the matrix and find the number of pivot columns:

1m. $R_2 \leftrightarrow R_1$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

2. $R_2 \leftarrow R_2 - 3R_1, R_3 \leftarrow R_3 - R_1, R_4 \leftarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

3. $R_3 \leftarrow R_3 - 4R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

4. $R_3 \leftarrow \frac{1}{7}R_3$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

5. $R_4 \leftarrow R_4 + R_3$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

6. $R_2 \leftarrow -R_2$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- There are 3 pivot columns therefore $\mathrm{rk}(A_\Phi) = 3$
- Compute the kernel and image of Φ
- The image of A_Φ are the columns associated with the pivot columns of the row-echelon matrix we got in the previous step. In this case, it's just A_Φ:

$$\mathrm{Im}(\Phi) = \mathrm{span}[\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}]$$

- The kernel is the null space of A_Φ
- The kernel only contains the 0 vector since the matrix is full column rank:

$$\ker(\Phi) = \{0\}$$

- What are $\dim(\ker(\Phi))$ and $\dim(\mathrm{Im}(\Phi))$?
- $\dim(\ker(\Phi)) = 0$
- $\dim(\mathrm{Im}(\Phi)) = 3$

2.18

- Let E be a vector space.
- Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$ (i.e, $f \circ g$ is the identity mapping id_E)
- Show that $\ker(f) = \ker(g \circ f)$
    - Automorphisms are *bijective*
    - The identity mapping $f \circ g$ is also an automorphism and therefore *bijective*
    - This means that they are *injective* and *surjective*
    - Them being *injective* means that the null space of either f or g contains only the zero vector because different inputs always map to different outputs. No unique inputs both map to 0.
    - Therefore $\ker(f) = \{0\} = \ker(g \circ f)$
- Show that $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$
    - Again g and $(g \circ f)$ are both automorphisms on E and are *surjective*

    - This means that every possible output is reachable from at least one input. In other words:
    - $\mathrm{Im}(g) = \mathrm{Im}(E) = \mathrm{Im}(g \circ f)$
- Show that $\ker(f) \cap \mathrm{Im}(g) = \{0\}$
- The intersection of $\{0\}$ and E is just $\{0\}$

2.19

- Consider the endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix is:

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- Determine $\ker(\Phi)$ and $\mathrm{Im}(\Phi)$
- To determine both the kernel and image of this linear mapping we first need to perform Gaussian Elimination to get the matrix into row-echelon form:

1. $R_2 \leftarrow R_2 - R_1, R_3 \leftarrow R_3 - R_1$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. $R_2 \leftarrow -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We can go one step further to get this into reduced-row echelon form:

3. $R_1 \leftarrow R_1 - R_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- This is full rank so the $\ker(\Phi)$ contains only the zero vector:

$$\ker(\Phi) = \{0\}$$

- Similarly, the image of Φ spans all of the columns of A_Φ:

$$\mathrm{Im}(\Phi) = \mathrm{span}[\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}]$$

- Determine the transformation matrix $\tilde{A_\Phi}$ with respect to the basis:

$$B = (\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix})$$

- i.e. perform a basis change toward the new basis B

$$\tilde{A_\Phi} = T^{-1}A_\Phi S$$

$$S = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 1 & 0\end{bmatrix}$$

$$T^{-1} = \begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\\1 & 0 & -1\end{bmatrix}$$

$$\tilde{A_\Phi} = \begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}6 & 9 & 1\\--3 & -5 & 0\\-1 & -1 & 0\end{bmatrix}$$

2.20

- Let us consider b_1, b_2, b_1', b_2', 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as:

$$b_1 = \begin{bmatrix}2\\1\end{bmatrix}, b_2 = \begin{bmatrix}-1\\-1\end{bmatrix}, b_1' = \begin{bmatrix}2\\-2\end{bmatrix}, b_2' = \begin{bmatrix}1\\1\end{bmatrix}$$

- Let us define two ordered basis $B = (b_1, b_2)$ and $B' = (b_1', b_2')$ of $\mathbb{R}^2$
- a. Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors:
  - The question itself states that B and B' are both bases or $\mathbb{R}^2$...
  - The two columns of B are linearly independent and so are the two columns of B' therefore they form a basis for $\mathbb{R}^2$
- b. Compute the matrix P_1 that performs a basis change from B' to B
- The vectors b_1' and b_2' written in the new basis must be a linear combination of the basis vectors b_1 and b_2:

$$b_1' = \begin{bmatrix}2\\-2\end{bmatrix} = 4b_1 + 6b_2$$

$$b_2' = \begin{bmatrix}1\\1\end{bmatrix} = 0b_1 + -1b_2$$

- We put these coefficients into a matrix and transpose them to get the transformation matrix P_1:

$$P_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

- We consider c_1, c_2, c_3, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}^3$ as:

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

- and we define $C = (c_1, c_2, c_3)$
  - Show that C is a basis of $\mathbb{R}^3$ e.g. by using determinants:

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

- If the determinant of the matrix is nonzero, then the matrix is invertible and therefore forms a basis for $\mathbb{R}^3$
- We can compute the determinant of C using Sarrus' rule:

$$\begin{aligned} \det(C) &= 1(-1)(-1) + 2(2)(1) + -1(0)(0) - (-1)(-1)(1) - (1)(2)(0) - (2)(0)(-1) \\ &= -1 + 4 - 1 \\ &= 4 \neq 0 ✅ \end{aligned}$$

- The matrix is invertible therefore it forms a basis for $\mathbb{R}^3$
- Let us call $C' = (c'_1, c'_2, c'_3)$ the standard basis for $\mathbb{R}^3$
- Determine the matrix P_2 that performs the basis change from C to C':
- We write the the vectors c_1, c_2, c_3 in terms of the new basis:

$$c_1 = c'_1 + 2c'_2 - c'_3$$

$$c_2 = 0c'_1 - c'_2 + 2c'_3$$

$$c_3 = c'_1 + 0c'_2 - c'_3$$

- Arranging these coefficients into a matrix and transposing it gives us:

$$P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} = C$$

- d. Consider the homomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ such that:

$$\Phi(b_1 + b_2) = c_2 + c_3$$

$$\Phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$$

- Where $B = (b_1, b_2)$ and $C = (c_1, c_2, c_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively
- Determine the transformation matrix A_Φ of Φ with respect to the ordered bases B and C
- By linearity of addition we get:

$$\Phi(b_1 + b_2) = \Phi(b_1) + \Phi(b_2)$$

- and

$$\Phi(b_1 - b_2) = \Phi(b_1) - \Phi(b_2)$$

- So

$$\Phi(b_1 + b_2) + \Phi(b_1 - b_2) = 2\Phi b_1$$

- Plugging in the equations above in terms of C we get:

$$\begin{aligned} 2\Phi(b_1) &= c_2 + c_3 + 2c_1 - c_2 + 3c_3 \\ 2\Phi(b_1) &= 2c_1 + 4c_3 \\ \Phi(b_1) &= c_1 + 2c_3 \end{aligned}$$

- Plugging this back into the first equation:

$$\begin{aligned} \Phi(b_1 - b_2) &= c_2 + c_3 \\ \Phi(b_1) - \Phi(b_2) &= c_2 + c_3 \\ \Phi(b_2) &= -c_1 + c_2 - c_3 \end{aligned}$$

- In summary:

$$\Phi(b_1) = c_1 + 2c_3$$

$$\Phi(b_2) = -c_1 + c_2 - c_3$$

- Putting these coefficients into a matrix and transposing gives us:

$$A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

e. Determine A' the transformation matrix of Φ with respect to the bases B' and C':

$$A' = P_2 A_\Phi P_1 = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$

f.

- Let us consider the vector $x \in \mathbb{R}^2$ whose coordinates in B' are $[2 \quad 3]^T$

- In other words, $x = 2b'_1 + 3b'_2$

  1. Calculate the coordinates of x in B:

  - We can use the projection matrix P_1 to find the coordinates of x in B

$$x_B = P_1 x = \begin{bmatrix}4 & 0 \\ 6 & -1\end{bmatrix} \begin{bmatrix}2 \\ 3\end{bmatrix} = \begin{bmatrix}8 \\ 9\end{bmatrix}$$

```
2. Based on that, compute the coordinates of $\Phi(x)$ expressed in $C$
- We multiply the vector we got from step 1 by $A_\Phi$:
```

$$\Phi(x) = A_\Phi x_B = \begin{bmatrix}1 & -1 \\ 0 & 1 \\ 2 & -1\end{bmatrix} \begin{bmatrix}8 \\ 9\end{bmatrix} = \begin{bmatrix}-1 \\ 9 \\ 7\end{bmatrix}$$

```
3. Then, write $\Phi(x)$ in terms of $c'_1, c'_2, c'_3$:
- We multiply by $P_2$ to go back to basis $C'$:
$$
P_2\Phi(x) = \begin{bmatrix}1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1\end{bmatrix} \begin{bmatrix}-1 \\ 9 \\ 7\end{bmatrix} = \begin{bmatrix}6 \\ -11 \\ 12\end{bmatrix}
$$
4. Use the representation of $x$ in $B'$ and the matrix $A'$ to find this result directly:
```

$$A'x = \begin{bmatrix}0 & 2 \\ -10 & 3 \\ 12 & -4\end{bmatrix} \begin{bmatrix}2 \\ 3\end{bmatrix} = \begin{bmatrix}6 \\ -11 \\ 12\end{bmatrix}$$

- Wow! Cool question.