3. Analytic Geometry (Exercises Only)

Exercises

3.1

Show that $\langle \cdot, \cdot \rangle$ is defined for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ by

$$\langle x, y \rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)$$

is an inner product

- In order for a mapping to be an inner product it must be:
  - bilinear
  Need to show:

$$\langle \lambda x + \psi z, y \rangle = \lambda \langle x, y \rangle + \psi \langle z, y \rangle$$

$$\begin{aligned} \langle \lambda x + \psi z, y \rangle &= (\lambda x_1 + \psi z_1)y_1 - ((\lambda x_1 + \psi z_1)y_2 + (\lambda x_2 + \psi z_2)y_1) + 2((\lambda x_2 + \psi z_2)y_2) \\ &= \lambda x_1y_1 + \psi z_1y_1 - (\lambda x_1y_2 + \psi z_1y_2 + \lambda x_2y_1 + \psi z_2y_1) + 2\lambda x_2y_2 + 2\psi z_2y_2 \\ &= \lambda x_1y_1 + \psi z_1y_1 - \lambda x_1y_2 - \lambda x_2y_2 - \psi z_1y_2 - \psi z_2y_1 + 2\lambda x_2y_1 + 2\psi z_2y_2 \end{aligned}$$

```
$$
\lambda \langle x, y \rangle = \lambda x_1y_1 - \lambda(x_1y_2 + x_2y_2) +
2\lambda(x_2y_2)
$$
$$
\psi\langle z, y \rangle = \psi z_1y_1 - \psi (z_1y_2 + z_2y_1) +
2\psi(z_2y_2)
$$
```

$$\lambda\langle x, y \rangle + \psi\langle z, y \rangle = \lambda x_1y_1 + \psi z_1y_1 - \lambda(x_1y_2 + x_2y_1) - \psi(z_1y_2 + z_2y_1) + 2(\lambda x_2y_2) + 2(\psi z_2y_2)$$
$$\lambda\langle x, y \rangle + \psi\langle z, y \rangle = \lambda x_1y_1 + \psi z_1y_1 - \lambda x_1y_2 - \lambda x_2y_1 - \psi z_1y_2 - \psi z_2y_1 + 2\lambda x_2y_2 + 2\psi z_2y_2 = \langle \lambda x + \psi$$

- Not showing the other side fuck you

```
- symmetrical
$$
\begin{align}
\langle y, x \rangle &= y_1x_1 - (y_1x_2 + y_2x_1) + 2(y_2x_2) \\
&= x_1y_1 - (x_1y_2 + x_1y_2) + 2(x_2y_2) = \langle x, y \rangle ✅
```

```
\end{align}
$$

- positive definite:
```

$$\langle x, x \rangle \geq 0 \;\; \forall x \in \mathbb{R}^2$$

$$\langle x, x \rangle = x_1^2 - (x_1x_2 + x_2x_1) + 2(x_2)^2 \geq 0$$

First we simplify:

$$x_1^2 - 2x_1x_2 + 2x_2^2 \geq 0$$

Then we complete the square for x_1:

$$x_1^2 - 2x_1x_2 + 2x_2^2 - x_2^2 \geq -x_2^2$$

$$x_1^2 - 2x_1x_2 + x_2^2 \geq -x_2^2$$

Now we can factor this quadratic into parenthetical terms that multiply to x_2^2 and add to $2x_1x_2$:

$$(x_1 - x_2)(x_1 - x_2) \geq -x_2^2$$

$$(x_1 - x_2)^2 + x_2^2 \geq 0 ✅$$

This is evidently true. Since this is the sum of two squared terms, the result can't be negative.

and

$$\langle x, x \rangle = 0 \iff x = o$$

$$\langle 0, 0 \rangle = 0^2 - (0 + 0) + 2(0)^2 = 0 ✅$$

- If either $x_1 \neq 0$ or $x_2 \neq 0$ then this inner product would be non-zero. In fact, it would be positive, since both terms would be squared.

Therefore this is an inner product.

3.2

- Consider $\mathbb{R}^2$ with $\langle \cdot, \cdot \rangle$ defined for all x and y in $\mathbb{R}^2$ as:

$$\langle x, y \rangle := x^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} y$$

Is $\langle \cdot, \cdot \rangle$ an inner product?

Since

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

Is not symmetric we can immediately see that this operation is not symmetric and is therefore not an inner product.

3.3

Compute the distance between:

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

a.

$$d(x, y) = ||x - y||$$

$$x - y = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$||x - y||^2 = (x - y)^T(x - y) = 22$$

$$d(x, y) = \sqrt{||x - y||^2} = \sqrt{22}$$

b.
Same formula for distance with a different inner product to compute $(x - y)^T(x - y)$

$$\begin{aligned} ||x - y||^2 &= \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix} \\ &= 47 \end{aligned}$$

$$d(x, y) = \sqrt{47}$$

3.4

Compute the angle between:

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

a.

$$x^T y = \cos\theta ||x|| ||y||$$

$$\cos\theta = \frac{x^T y}{\|x\|\|y\|}$$

$$\theta = \arccos(\frac{x^T y}{\|x\|\|y\|})$$

$$\|x\| = \sqrt{x^T x} = \sqrt{5}$$

$$\|y\| = \sqrt{y^T y} = \sqrt{2}$$

$$x^T y = -3$$

$$\theta = \arccos(\frac{-3}{\sqrt{10}})$$

b. Same equation for θ only we use a different inner product:

$$\begin{aligned} x^T B y &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} \\ &= -9 \end{aligned}$$

$$\|x\| = \sqrt{x^T x} = \sqrt{x^T B x}$$

$$\begin{aligned} x^T B x &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \end{bmatrix} \\ &= 26 \end{aligned}$$

$$\|x\| = \sqrt{26}$$

$$\|y\| = \sqrt{y^T y} = \sqrt{y^T B y}$$

$$\begin{aligned} y^T B y &= \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -6 \end{bmatrix} \\ &= 9 \end{aligned}$$

$$\|y\| = 3$$

$$\theta = \arccos(\frac{-3}{\sqrt{26}})$$

3.5

Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $x \in \mathbb{R}^5$ are given by:

$$U = \text{span}[\begin{bmatrix}0\\-1\\2\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-3\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\4\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\5\\0\\7\end{bmatrix}], \qquad x = \begin{bmatrix}-1\\-9\\-1\\4\\1\end{bmatrix}$$

a. Determine the orthogonal projection $\pi_U(x)$ of x onto U:

First thing's first we need a basis of U. $B = (b_1, \ldots, b_m)$ where $m = \text{rk[U]}$. To find this we will preform Gaussian Elimination on the columns that span U to find any linearly dependent columns. The pivot columns of the row-echelon matrix will form the basis B:

After preforming Gaussian Elimination we are left with:

$$\begin{bmatrix}1 & 3 & -4 & 3\\0 & 1 & -3 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

This means that $rk[U] = 3$ and u_1, u_2, u_3 form a basis for U:

$$b_1 = \begin{bmatrix}0\\-1\\2\\0\\2\end{bmatrix}, b_2 \begin{bmatrix}1\\-3\\1\\-1\\2\end{bmatrix}, b_3 \begin{bmatrix}-3\\4\\1\\2\\1\end{bmatrix}, B = \begin{bmatrix}0 & 1 & -3\\-1 & -3 & 4\\2 & 1 & 1\\0 & -1 & 2\\2 & 2 & 1\end{bmatrix}$$

The next step is to find the projection:

$$\pi_U(x) = B\lambda$$

Where:

$$\lambda = (B^TB)^{-1}B^Tx$$

We can compute B^TB with simple matrix multiplication:

$$B^TB = \begin{bmatrix}9 & 9 & 0\\9 & 16 & -14\\0 & -14 & 31\end{bmatrix}$$

And we can compute B^Tx:

$$B^Tx = \begin{bmatrix}9\\23\\-35\end{bmatrix}$$

Now, we can rearrange the equation for λ by moving the inverted B^TB over to the left side:

$$B^T B\lambda = B^T x$$

And this becomes a system of linear equations: $Ax = b$ where:

- $A = B^T B$
- $\lambda = x$
- $B^T x = b$

We solve this system by getting the augmented matrix $[A|b]$ into upper triangular form with Gaussian Elimination:

$$\begin{bmatrix} 9 & 9 & 0 & | & 9 \\ 0 & 7 & -14 & | & 14 \\ 0 & 0 & 3 & | & 3 \end{bmatrix}$$

And now the solution, λ is pretty easy to find we just solve for each of $(\lambda_1, \lambda_2, \lambda_3)$:

$$\lambda = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Finally, we can return to this equation and solve:

$$\pi_U(x) = B\lambda$$

$$\pi_U(x) = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

b.

$$d(\pi_U(x), y) = \|\pi_U(x) - x\|$$

$$\pi_U(x) - x = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -6 \\ 2 \end{bmatrix}$$

$$\|\pi_U(x) - x\|^2 = (\pi_U(x) - x)^T(\pi_U(x) - x) = \begin{bmatrix} 2 & 4 & 0 & -6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -6 \\ 2 \end{bmatrix} = 60$$

$$\|\pi_U(x) - x\| = \sqrt{60} = 2\sqrt{15}$$

3.6

Consider $\mathbb{R}^3$ with the inner product:

$$\langle x, y\rangle := x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

Furthermore, define e_1, e_2, e_3 as the standard/canonical basis in $\mathbb{R}^3$

a. Determine the orthogonal projection $\pi_U(e_2)$ of e_2 onto:

$$U = \text{span}[e_1, e_3]$$

We need to find the coordinate vector for e_2 in U, λ. λ will be a vector in $\mathbb{R}^2$ (because $U \in \mathbb{R}^2$). The line segment from the tip of e_2 projected onto U is orthogonal to U. That segment is given by $e_2 - \pi_U(e_2)$. Since it's orthogonal to U, it must be orthogonal to each of the basis vectors of U. Therefore we know that:

$$\langle e_1, (e_2 - \pi_U(e_2))\rangle = 0$$

And

$$\langle e_3, (e_2 - \pi_U(e_2))\rangle = 0$$

We first of all calculate:

$$e_2 - \pi_U(e_2) = \begin{bmatrix} -\lambda_1 \\ 1 \\ -\lambda_2 \end{bmatrix}$$

And now we plug this into the equations above:

First:

$$\begin{aligned} 0 = \langle e_1, (e_2 - \pi_U(e_2))\rangle &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\lambda_1 \\ 1 \\ -\lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2\lambda_1 \\ -\lambda_1 + 2 - \lambda_2 \\ -1 - 2\lambda_2 \end{bmatrix} \\ &= -2\lambda_1 + 1 = 0 \\ \lambda_1 &= \frac{1}{2} \end{aligned}$$

And now the second inner product:

$$
\begin{aligned}
0 = \langle e_3, (e_2 - \pi_U(e_2))\rangle &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\lambda_1 \\ 1 \\ -\lambda_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2\lambda_1 \\ -\lambda_1 + 2 - \lambda_2 \\ -1 - 2\lambda_2 \end{bmatrix} \\
&= -1 - 2\lambda_1 = 0 \\
\lambda_2 &= -\frac{1}{2}
\end{aligned}
$$

Now we have λ:

$$
\lambda = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
$$

We can find $\pi_U(e_2)$ by multiplying the matrix made up of basis vectors for U with the coordinate vector λ:

$$
\begin{aligned}
\pi_U(e_2) &= B\lambda \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}
$$

a. Compute the distance $d(e_2, U)$

$$
d(e_2, U) = \|\pi_U(e_2) - e_2\|
$$

$$
\pi_U(e_2) - e_2 = \begin{bmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}
$$

$$
\|\pi_U(e_2) - e_2\|^2 = \langle(\pi_U(e_2) - e_2), (\pi_U(e_2) - e_2)\rangle = 2
$$

$$
\|\pi_U(e_2) - e_2\| = \sqrt{2}
$$

3.7

Let V be a vector space and π be an endomorphism of V. $\pi : V \to V$

a. Prove that π is a projection if and only if $\mathrm{id}_V - \pi$ is a projection, where id_V is the identity endomorphism on V

For π to be a projection the following must be true: $\pi^2 = \pi$

We know that $(\mathrm{id}_V - \pi)$ is a projection, so:

$$\begin{aligned}
(\mathrm{id}_V - \pi)^2 &= (\mathrm{id}_V - \pi) \\
\mathrm{id}_V^2 - 2\pi + \pi^2 &= \mathrm{id}_V - \pi \\
\mathrm{id}_V - \pi + \pi^2 &= \mathrm{id}_V \\
-\pi + \pi^2 &= 0 \\
\pi^2 &= \pi
\end{aligned}$$ ✅

b. Assume now that π is a projection. Calculate $\mathrm{Im}(\mathrm{id}_V - \pi)$ and $\ker(\mathrm{id}_V - \pi)$ as a function of $\mathrm{Im}(\pi)$ and $\ker(\pi)$:

$$\mathrm{Im}(\mathrm{id}_V - \pi) = \ker(\pi)$$

$$\ker(\mathrm{id}_V - \pi) = \mathrm{Im}(\pi)$$

3.8

Using the Gram-Schmidt method, turn the basis $B = (b_1, b_2)$ of a two-dimensional subspace $U \in \mathbb{R}^3$ into an ONB $C = (c_1, c_2)$ of U, where:

$$b_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

First we normalize b_1 to get c_1:

$$\|b_1\| = \sqrt{b_1^T b_1} = \sqrt{3}$$

$$c_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now we can calculate c_2:

$$c_2 = b_2 - \pi_{\mathrm{span}[c_1]}(b_2)$$

$$\pi_{\mathrm{span}[c_1]}(b_2) = c_1 \lambda$$

$$\lambda = \frac{c_1^T b_2}{c_1^T c_1}$$

$$c_1^T c_1 = 1$$

$$\lambda = c_1^T b_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}}$$

$$\pi_{\text{span}[c_1]}(b_2) = c_1 \lambda = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$c_2 = b_2 - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Now we normalize c_2 to complete the orthonormal basis:

$$\|c_2\| = \sqrt{c_2^T x_2} = \frac{\sqrt{42}}{\sqrt{9}} = \frac{\sqrt{42}}{3}$$

$$c_2 = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$$

3.9

Let $n \in \mathbb{N}^*$ and let $x_1, \ldots, x_n > 0$ be n positive real numbers so that $x_1 + \ldots + x_n = 1$. Use the Cauchy-Schwarz inequality and show that:

a.

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}$$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\|x^T y\|^2 \leq (x^T x)(y^T y)$$

$$\left(\sum_{i=1}^{n} x_i \times 1 \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} 1^2 \right)$$

$$\left(\sum_{i=1}^{n} x_i \times 1\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) n$$

Because $\sum_{i=1}^{n} x_1 = 1$ the left side simplifies to just 1:

$$1 \leq \left(\sum_{i=1}^{n} x_i^2\right) n$$

$$\frac{1}{n} \leq \sum_{i=1}^{n} x_i^2 ✅$$

3.10

Rotate the vectors:

$$x_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad x_2 := \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\hat{x}_1 = Rx_1 = \begin{bmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$\hat{x}_2 = Rx_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$