

4. Matrix Decompositions (Exercises Only)

Exercises

4.1

Compute the determinant using Laplace Expansion (using the first row) and the Sarrus Rule for:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

Starting out with the Laplace expansion method, we find the minors:

$$A_{11} = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

Now we find the determinants of the minors:

$$\det(A_{11}) = 16 - 12 = 4$$

$$\det(A_{12}) = 8 - 0 = 8$$

$$\det(A_{13}) = 4 - 0 = 4$$

And now the cofactors:

$$C_{11} = -1^2(4) = 4$$

$$C_{12} = -1^3(8) = -8$$

$$C_{13} = -1^4(4) = 4$$

And now we can compute the determinant:

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(4) + 3(-8) + 5(4) \\ &= 4 - 24 + 20 \\ &= 0 \end{aligned}$$

Now we can compute the determinant using the Sarrus Rule:

$$\begin{aligned}
\det(A) &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\
&= 1(4)(4) + 2(2)(5) + 0(3)(6) - 0(4)(5) - 1(2)(6) - 2(3)(4) \\
&= 16 + 20 - 12 - 24 \\
&= 36 - 36 \\
&= 0 \quad \checkmark
\end{aligned}$$

4.2

Compute the following determinant efficiently:

$$\begin{vmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{vmatrix}$$

Since it doesn't specify a method I think I will get it in upper triangular form using Gaussian Elimination, then take the product of the diagonal elements to find the determinant.

After performing Gaussian Elimination we get:

$$\begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

We didn't perform any row swaps or scaling of rows so we don't need to worry about that. Now we take the product of the diagonal elements to find the determinant:

$$\det(A) = 2(-1)(1)(1)(-3) = 6$$

4.3

Compute the eigenspaces of:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

Starting with: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$:

We start out by finding the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 \\ 1 & 1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = 1$$

- Note that the algebraic multiplicity of the eigenvalue is 2
- This means that the geometric multiplicity must either 1 or 2.

We can find the eigenvectors:

$$\begin{bmatrix} 1 - \lambda_1 & 0 \\ 1 & 1 - \lambda_1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x = 0$$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, the eigenspace is:

$$E_1 = \text{span} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$$

Now $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$:

4.4:

$$\begin{bmatrix} -\lambda & -1 & 1 & 1 \\ 0 & -\lambda & -1 & 3 - \lambda \\ 0 & 1 & -2 - \lambda & 2\lambda \\ 1 & -1 & 1 & -\lambda \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & -1 - \lambda & 0 & 1 \\ 0 & -\lambda & -1 - \lambda & 3 - \lambda \\ 0 & 1 & -1 - \lambda & 2\lambda \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$$

4.5

a.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonalizability:


- Check for eigenvectors that span \mathbb{R}^n

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0$$

$$\lambda = 1$$

Find eigenvectors:

$$\begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} x = 0$$

- Any vector in \mathbb{R}^2 is an eigenvector of this matrix and therefore there are two linearly independent vectors in the null space of $(A - \lambda I)$ meaning this is diagonalizable 

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (1)(1) - 0 = 1 \quad \text{✓}$$

- You may have immediately been able to see these results by recognizing that this matrix is I_2

b.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span \mathbb{R}^n

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(1-\lambda) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 1$$

Find eigenvectors:

- First λ_1 :

$$\begin{bmatrix} 1-0 & 0 \\ 0 & 0-0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = 0$$


$$x_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Now λ_2 :

$$\begin{bmatrix} 1-1 & 0 \\ 0 & 0-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- x_1 and x_2 are linearly independent and form a basis for \mathbb{R}^2 therefore this matrix is diagonalizable 

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = 0 - 0 = 0 \text{ } \times$$

c.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span \mathbb{R}^n

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(1 - \lambda) = 0$$

$$\lambda = 1$$

Find eigenvectors:

$$\begin{bmatrix} 1 - 1 & 1 \\ 0 & 1 - 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

$$x := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The eigenvectors do not form a basis for \mathbb{R}^2 therefore this is not diagonalizable 

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (1)(1) - 0 = 1 \text{ } \checkmark$$

d.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span \mathbb{R}^n

$$\det(A - \lambda I) = \begin{vmatrix} 0 - \lambda & 1 \\ 0 & 0 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0$$

Find eigenvectors:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

$$x := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The eigenvectors do not form a basis for \mathbb{R}^2 therefore this is not diagonalizable ✖

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (0)(0) - (1)(0) = 0 \text{ ✖}$$

4.6

Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

a.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 3 & 0 \\ 1 & 4 - \lambda & 3 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

- We can compute this with a Laplace expansion. It will be easiest to perform Laplace expansion in the third row because there are two zeros, so two of the terms will disappear:

$$\det(A - \lambda I) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$\det(A - \lambda I) = 0C_{31} + 0C_{32} + (1 - \lambda)C_{33}$$

$$\det(A - \lambda I) = (1 - \lambda)C_{33}$$

- To find the C_{33} we first find the determinant of the minor A_{33}

$$A_{33} = \begin{bmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{bmatrix}$$

$$\det(A_{33}) = (2 - \lambda)(4 - \lambda) - 3$$

$$\det(A_{33}) = 8 - 2\lambda - 4\lambda + \lambda^2 - 3$$

$$\det(A_{33}) = \lambda^2 - 6\lambda + 5$$

$$\det(A_{33}) = (\lambda - 5)(\lambda - 1)$$

This gives us the cofactor C_{33} :

$$C_{33} = (\lambda - 5)(\lambda - 1)$$

Now we can return to the characteristic polynomial of $(A - \lambda I)$

$$\det(A - \lambda I) = (1 - \lambda)(\lambda - 5)(\lambda - 1) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 5$$

- note that λ_1 has an algebraic multiplicity of 2
- Now we can compute the eigenvectors of A :
- $\lambda_1 = 1$

$$\begin{bmatrix} 2 - 1 & 3 & 0 \\ 1 & 4 - 1 & 3 \\ 0 & 0 & 1 - 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

- $\lambda_2 = 5$

$$\begin{bmatrix} 2 - 5 & 3 & 0 \\ 1 & 4 - 5 & 3 \\ 0 & 0 & 1 - 5 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 3 \\ 0 & 0 & -4 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore the eigenspaces of A are:

$$E_1 = \text{span} \left[\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right]$$

$$E_5 = \text{span} \left[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right]$$

A is *not* diagonalizable because its eigenvectors do not form a basis for \mathbb{R}^3

b.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

- Doing a Laplace expansion in the first row:

$$\det(A - \lambda I) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\det(A - \lambda I) = (1 - \lambda)C_{11} + (1)C_{12} + (0)C_{13} + (0)C_{14}$$

$$\det(A - \lambda I) = (1 - \lambda)C_{11} + C_{12}$$

- We start by finding the cofactor C_{11} . The first step in that is finding the determinant of the minor A_{11} :

$$A_{11} = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

- Since it's in row echelon form the determinant is just the product of the diagonal elements:

$$\det(A_{11}) = -\lambda^3$$

$$C_{11} = -\lambda^3$$

- Now we can find the second cofactor C_{12} :

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

- The determinant of A_{12} is evidently zero so the entire C_{12} term disappears:

$$\det(A - \lambda I) = (1 - \lambda)(-\lambda^3)$$

$$\lambda_1 = 1 \quad \lambda_2 = 0$$

- Now we find the eigenvectors:
- $\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\lambda_1 = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0$$

- Writing this out as a system of linear equations:

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

- therefore:

$$x = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- Expressing this vector as a linear combination:

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Therefore:

$$x_2 := \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad x_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad x_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- The eigenspaces of A are:

$$E_1 = \text{span} \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]$$

$$E_0 = \text{span} \left[\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

- Putting all of the eigenvectors for A into a matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- We can see that the columns of this matrix are linearly independent. Therefore the eigenvectors form a basis for \mathbb{R}^4 and A is diagonalizable.

4.7

a.

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -8 & 4 - \lambda \end{vmatrix}$$

- This matrix has no real eigenvalues

b.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = 0$$

$$1. R_2 \leftarrow R_2 - R_1$$

$$\begin{vmatrix} 1 - \lambda & 1 & 1 \\ -\lambda & -\lambda & 0 \\ 1 & 1 & 1 - \lambda \end{vmatrix}$$

$$2. R_3 \leftarrow R_3 - R_1$$

$$\begin{vmatrix} 1 - \lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{vmatrix}$$

- Now we perform a Laplace expansion along the first row:

$$\det = (1 - \lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ \lambda & -\lambda \end{vmatrix} + \begin{vmatrix} \lambda & -\lambda \\ \lambda & 0 \end{vmatrix}$$

$$\det = (1 - \lambda)(\lambda^2) - (-\lambda^2) + \lambda^2$$

$$\det = \lambda^2 - \lambda^3 + 2\lambda^2$$

$$= -\lambda^3 + 3\lambda^2$$

$$= -\lambda^2(\lambda - 3)$$

$$\lambda_1 = 3 \quad \lambda_1 = 0$$

- Now we find the eigenvectors:

- $\lambda_1 = 3$

$$\begin{bmatrix} 1 - 3 & 1 & 1 \\ 1 & 1 - 3 & 1 \\ 1 & 1 & 1 - 3 \end{bmatrix} x = 0$$


$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $\lambda_1 = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad x_3 := \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- The matrix is diagonalizable because its eigenvectors form a basis for \mathbb{R}^3 
- The diagonal form of the matrix is:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The basis with respect to which the matrix is diagonal is:

$$B := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

c.

d.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix} = 0$$

- Performing a Laplace expansion along the first row:

$$\det(A - \lambda I) = (5 - \lambda) \underbrace{\begin{vmatrix} 4 & 2 \\ -6 & -4 - \lambda \end{vmatrix}}_{A_{11}} + 6 \underbrace{\begin{vmatrix} -1 & 2 \\ 3 & -4 - \lambda \end{vmatrix}}_{A_{12}} - 6 \underbrace{\begin{vmatrix} -1 & 4 - \lambda \\ 3 & -6 \end{vmatrix}}_{A_{13}}$$

Computing the determinants of the minors:

- A_{11}

$$\det(A_{11}) = \begin{vmatrix} 4 & 2 \\ -6 & -4 - \lambda \end{vmatrix} = (4 - \lambda)(-4 - \lambda) + 12$$

$$\det(A_{11}) = \lambda^2 - 4$$

- A_{12}

$$\det(A_{11}) = \begin{vmatrix} -1 & 2 \\ 3 & -4 - \lambda \end{vmatrix} = \lambda - 2$$

- A_{13}

$$\det(A_{11}) = \begin{vmatrix} -1 & 4 - \lambda \\ 3 & -6 \end{vmatrix} = 3(\lambda - 2)$$

Plugging those determinants back in:

$$\begin{aligned} \det(A - \lambda I) &= (5 - \lambda)(\lambda^2 - 4) + 6(\lambda - 2) - 6[3(\lambda - 2)] \\ &= (5 - \lambda)(\lambda - 2)(\lambda + 2) + 6(\lambda - 2) - 6[3(\lambda - 2)] \end{aligned}$$

Factoring out a $(\lambda - 2)$:

$$\begin{aligned} \det(A - \lambda I) &= (\lambda - 2) [(5 - \lambda)(\lambda + 2) + 6 - 18] \\ &= (\lambda - 2) [5\lambda + 10 - \lambda^2 - 2\lambda + 6 - 16] \\ &= (\lambda - 2) (3\lambda + 10 - \lambda^2 - 12) \\ &= (\lambda - 2) (-\lambda^2 + 3\lambda - 2) \end{aligned}$$

Factoring out a -1 from the right polynomial (we can just ignore it since $\det(A - \lambda I) = 0$ and $\frac{0}{-1} = 0$):

$$\begin{aligned} \det(A - \lambda I) &= (\lambda - 2)(\lambda^2 - 3\lambda + 2) \\ &= (\lambda - 2)(\lambda - 2)(\lambda - 1) \end{aligned}$$

Therefore the eigenvalues are:

$$\lambda_1 = 2 \quad \lambda_2 = 1$$

- Now we find the eigenvectors:
- $\lambda_1 = 2$

$$\begin{bmatrix} 5 - 2 & -6 & -6 \\ -1 & 4 - 2 & 2 \\ 3 & -6 & -4 - 2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} x = 0$$

$$1. R_1 \leftarrow \frac{1}{3}R_1, R_3 \leftarrow \frac{1}{3}R_3:$$

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

$$1. R_2 \leftarrow R_2 + R_1, R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 - 2x_3 = 0$$

$$x_1 = 2x_2 + 2x_3$$

$$x = \begin{bmatrix} 0x_1 + 2x_2 + 2x_3 \\ 0x_1 + x_2 + 0x_3 \\ 0x_1 + 0x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad x_2 := \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\begin{bmatrix} 5-1 & -6 & -6 \\ -1 & 4-1 & 2 \\ 3 & -6 & -4-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} x = 0$$

$$4x_1 - 6x_2 - 6x_3 = 0$$

$$-x_1 + 3x_2 + 2x_3 = 0$$

$$3x_3 - 6x_2 - 5x_3$$

From equation 2:


$$x_1 = 3x_2 + 2x_3$$

Plugging this into equation 1:

$$\begin{aligned}
4(3x_2 + 2x_3) - 6x_2 - 6x_3 &= 0 \\
12x_2 + 8x_3 - 6x_2 - 6x_3 &= 0 \\
6x_2 + 2x_3 &= 0 \\
x_3 &= -3x_2
\end{aligned}$$

Plugging this back into the equation for x_1 :

$$\begin{aligned}
x_1 &= 3x_2 + 2(-3x_2) \\
x_1 &= -3x_2 \\
x &= \begin{bmatrix} 0x_1 - 3x_2 + 0x_3 \\ 0x_1 + x_2 + 0x_3 \\ 0x_1 - 3x_2 + 0x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} \\
x_3 &:= \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}
\end{aligned}$$

- The matrix is diagonalizable because its eigenvectors form a basis for \mathbb{R}^3 
- The diagonal form of the matrix is:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The basis with respect to which the matrix is diagonal is:

$$B := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$$

4.8

Find the SVD:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

- The first step in finding the SVD is finding $A^T A$:

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

- Now we eigendecompose $A^T A$.
- We start by finding the characteristic polynomial:

$$\det(A^T A - \lambda I) = \begin{vmatrix} 13 - \lambda & 12 & 2 \\ 12 & 13 - \lambda & -2 \\ 2 & -2 & 8 - \lambda \end{vmatrix} = 0$$

- Performing a Laplace expansion along the first row:

$$\det(A^T A - \lambda I) = (13 - \lambda) \underbrace{\begin{vmatrix} 13 - \lambda & -2 \\ -2 & 8 - \lambda \end{vmatrix}}_{A_{11}} - 12 \underbrace{\begin{vmatrix} 12 & -2 \\ 2 & 8 - \lambda \end{vmatrix}}_{A_{12}} + 2 \underbrace{\begin{vmatrix} 12 & 13 - \lambda \\ 2 & -2 \end{vmatrix}}_{A_{13}}$$

- Starting with the first minor A_{11} :

$$\begin{aligned} \det(A_{11}) &= (13 - \lambda)(8 - \lambda) - 4 \\ &= 104 - 13\lambda - 8\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 21\lambda + 100 \end{aligned}$$

- Now A_{12}

$$\begin{aligned} \det(A_{12}) &= 96 - 12\lambda + 4 \\ &= 100 - 12\lambda \end{aligned}$$

- Now A_{13}

$$\begin{aligned} \det(A_{13}) &= -24 - 2(13 - \lambda) \\ &= 24 - 26 + 2\lambda \\ &= 2\lambda - 50 \end{aligned}$$

- Plugging back into the equation for $\det(A^T A - \lambda I)$:

$$\begin{aligned} \det(A^T A - \lambda I) &= (13 - \lambda)(\lambda^2 - 21\lambda + 100) - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ &= 13\lambda^2 - 273\lambda + 1300 - \lambda^3 + 21\lambda^2 - 100\lambda - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ &= -\lambda^3 + 13\lambda^2 + 21\lambda^2 - 100\lambda - 273\lambda + 1300 - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ &= -\lambda^3 + 34\lambda^2 - 373\lambda + 1300 - 1200 + 144\lambda + 4\lambda - 100 \\ &= \lambda^3 + 34\lambda^2 - 255\lambda \\ &= -\lambda(\lambda^2 - 34\lambda - 255) \\ &= -\lambda(\lambda - 25)(\lambda - 9) \end{aligned}$$

- This gives us the eigenvalues of $A^T A$:

$$\lambda_1 = 25 \quad \lambda_2 = 9 \quad \lambda_3 = 0$$

- Now we find the eigenvectors of $A^T A$ using the eigenvalues:

- $\lambda_1 = 25$

$$\begin{bmatrix} 13 - 25 & 12 & 2 \\ 12 & 13 - 25 & -2 \\ 2 & -2 & 8 - 25 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

$$1. R_2 \leftarrow R_2 + R_1:$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & -17 \end{bmatrix}$$

$$2. R_3 \leftrightarrow R_2$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 2 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1. R_1 \leftarrow -\frac{1}{12} R_1$$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 2 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix}$$

$$4. R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & -\frac{100}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

$$5. R_2 \leftarrow -\frac{6}{100} R_2$$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 - x_2 - \frac{1}{6}x_3 = 0$$

$$x_3 = 0$$

- We see that:

$$x_1 - x_2 - \frac{1}{6}(0) = 0$$

$$x_1 = x_2$$

Therefore our first eigenvector is:

$$x_1 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- $\lambda_2 = 9$

$$\begin{bmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_1 \leftarrow \frac{1}{4}R_1$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

2. $R_2 \leftarrow R_2 - 12R_1, R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & -32 & -8 \\ 0 & -8 & -2 \end{bmatrix}$$

3. $R_2 \leftarrow -\frac{1}{32}R_2$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & -8 & -2 \end{bmatrix}$$

4. $R_3 \leftarrow R_3 + 8R_2$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 + 3x_2 + \frac{1}{2}x_3 = 0$$

$$x_2 + \frac{1}{4}x_3 = 0$$

$$x_2 = -\frac{1}{4}x_3$$

$$x_1 = -3x_2 - \frac{1}{2}x_3$$

$$x_1 = -3\left(-\frac{1}{4}x_3\right) - \frac{1}{2}x_3$$

$$x_1 = \frac{1}{4}x_3$$

- Our second eigenvector is:

$$x_1 := \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- $\lambda_3 = 0$

$$\begin{bmatrix} 13-0 & 12 & 2 \\ 12 & 13-0 & -2 \\ 2 & -2 & 8-0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & -2 & 8 \\ 13 & 12 & 2 \\ 12 & 13 & -2 \end{bmatrix}$$

$$2. R_1 \leftarrow \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 13 & 12 & 2 \\ 12 & 13 & -2 \end{bmatrix}$$

$$3. R_2 \leftarrow R_2 - 13R_1, R_3 \leftarrow R_3 - 12R_1$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 25 & -50 \\ 0 & 25 & -50 \end{bmatrix}$$

$$4. R_3 \leftarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 25 & -50 \\ 0 & 0 & 0 \end{bmatrix}$$

$$5. R_2 \leftarrow \frac{1}{25}R_2$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 - x_2 + 4x_3 = 0$$

$$x_2 - 2x_3 = 0$$

$$x_2 = 2x_3$$

$$x_1 = x_2 - 4x_3$$

$$x_1 = -2x_3$$

- Our third eigenvector is:

$$x_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

- Now that we have the eigenvectors and eigenvalues for $A^T A$ we can construct two of the three matrices in the SVD: Σ and V
- Σ is a diagonal matrix of the same shape as A with the square root of the eigenvalues along the diagonal put in descending order. These values are the singular values:

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

- V is an orthogonal matrix of right singular vectors. All we need to do to construct V is normalize the eigenvectors of $A^T A$. The eigenvectors are already orthogonal (because $A^T A$ is symmetric), so we just need to normalize them.

$$v_1 = \frac{x_1}{\|x_1\|}$$

$$\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{2}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$v_2 = \frac{x_2}{\|x_2\|}$$

$$\|x_2\| = \sqrt{x_2^T x_2} = \frac{\sqrt{18}}{4}$$

$$v_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$v_3 = \frac{x_3}{\|x_3\|}$$

$$\|x_3\| = \sqrt{x_3^T x_3} = 3$$

$$v_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

- Putting these together gives us the matrix V :

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{bmatrix}$$

- The only matrix left to find is the matrix of left singular vectors U .

$$u_1 = \sigma_1 A v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \sigma_2 A v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- Which finally gives us U :

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- In summary the SVD of A is:

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

4.9

Find the singular value decomposition of:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- The first step in finding the SVD is finding $A^T A$:

$$A^T A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

- Now we eigendecompose $A^T A$.
- We start by finding the characteristic polynomial:

$$\begin{aligned}
 \det(A^T A - \lambda I) &= \begin{vmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{vmatrix} = 0 \\
 &= (5 - \lambda)(5 - \lambda) - 9 \\
 &= 25 - 10\lambda + \lambda^2 - 9 \\
 &= \lambda^2 - 10\lambda + 16 \\
 0 &= (\lambda - 8)(\lambda - 2)
 \end{aligned}$$

- This gives us the eigenvalues of $A^T A$:

$$\lambda_1 = 8 \quad \lambda_2 = 2$$

- Now we find the eigenvectors of $A^T A$ using the eigenvalues:
- $\lambda_1 = 8$

$$\begin{aligned}
 \begin{bmatrix} 5 - 8 & 3 \\ 3 & 5 - 8 \end{bmatrix} x &= 0 \\
 \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} x &= 0
 \end{aligned}$$

- We can get this matrix into row echelon form with Gaussian Elimination:

$$1. R_2 \leftarrow R_2 - R_1$$

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} x = 0$$

$$2. R_1 \leftarrow -\frac{1}{3} R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x = 0$$

- Now writing this as a system of linear equations:

$$x_1 = x_2$$

$$x_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\lambda_2 = 2$

$$\begin{aligned}
 \begin{bmatrix} 5 - 2 & 3 \\ 3 & 5 - 2 \end{bmatrix} x &= 0 \\
 \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} x &= 0
 \end{aligned}$$

- We can get this matrix into row echelon form with Gaussian Elimination:

$$1. R_2 \leftarrow R_2 - R_1$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} x = 0$$

$$2. R_1 \leftarrow \frac{1}{3} R_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

- Now writing this as a system of linear equations:

$$x_1 = -x_2$$

$$x_2 := \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Now that we have the eigenvectors and eigenvalues for $A^T A$ we can construct two of the three matrices in the SVD: Σ and V
- Σ is a diagonal matrix of the same shape as A with the square root of the eigenvalues along the diagonal put in descending order. These values are the singular values:

$$\Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

- V is an orthogonal matrix of right singular vectors. All we need to do to construct V is normalize the eigenvectors of $A^T A$. The eigenvectors are already orthogonal (because $A^T A$ is symmetric), so we just need to normalize them.

$$v_1 = \frac{x_1}{\|x_1\|}$$

$$\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{2}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$v_2 = \frac{x_2}{\|x_2\|}$$

$$\|x_2\| = \sqrt{x_2^T x_2} = \sqrt{2}$$

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Putting these together gives us the matrix V :

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- The only matrix left to find is the matrix of left singular vectors U .

$$u_1 = \sigma_1 A v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \sigma_2 A v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Which finally gives us U :

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In summary the SVD of A is:

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

4.10

Find the best rank-1 approximation of:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

- We can use the SVD we got in question 4.8. To get a rank 1 approximation we use the formula:

$$A_1 = \sigma_1 u_1 v_1^T = 5 \left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right) = 5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2.5 & 2.5 & 0 \\ 2.5 & 2.5 & 0 \end{bmatrix}$$

4.11

Show that for any $A \in \mathbb{R}^{m \times n}$, the matrices $A^T A$ and $A A^T$ possess the same nonzero eigenvalues.

- Using the singular value decomposition:

$$A = U\Sigma V^T$$

- Where U and V are orthogonal matrices and Σ is a diagonal matrix whose diagonal entries are the square root of the eigenvalues of A

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T$$

- Because U is orthogonal:

$$A^T A = V\Sigma^T \Sigma V^T$$

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T$$

- Because V is orthogonal

$$AA^T = U\Sigma\Sigma^T U^T$$

- In both equations we end up with $\Sigma^T \Sigma$ and $\Sigma \Sigma^T$. Because Σ is diagonal $\Sigma^T \Sigma = \Sigma \Sigma^T = D$, where D is a diagonal matrix of the eigenvalues of $A^T A$ and AA^T

4.12

Show that for $x \neq 0$:

$$\max_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$

- Where σ_1 is the largest singular value of $A \in \mathbb{R}^{m \times n}$
- if x is a unit vector, $\|x\|_2 = 1$ so:

$$\|A\|_2 = \max_x \|Ax\|_2$$

If we break A down into its **singular value decomposition**:

$$\|A\|_2 = \max_x \|U\Sigma V^T x\|$$

- Because U is orthogonal, it preserves norms and angles:

$$\|A\|_2 = \max_x \|\Sigma V^T x\|$$

- V^T is also orthogonal, so it preserves norms. We now make a substitution:

$$z = V^T x$$

- because V^T is orthogonal (and therefore preserves lengths) and x is length 1:

$$\|z\|_2 = 1$$

So now our maximization problem becomes:

$$\|A\|_2 = \max_z \|\Sigma z\|$$

The question now is, which z , given the constraint that z must be unit length, maximizes the Euclidean norm of the vector Σz ?

- Recall that Σ is a diagonal matrix of singular values sorted in descending order:

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

This means that we can write out the equation for the Euclidean norm $\|\Sigma z\|_2$ in terms of the components of z and Σ :

$$\|\Sigma z\|_2 = \sqrt{\sigma_1^2 z_1^2 + \dots + \sigma_n^2 z_n^2}$$

Since z is unit length:

$$\sum_{i=1}^n z_i^2 = 1$$

Therefore to maximize $\|\Sigma z\|_2$ we "concentrate" all of the weight possible for z into the element corresponding to the largest singular value, σ_1 :

$$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore, all of the other terms in $\sqrt{\sigma_1^2 z_1^2 + \dots + \sigma_n^2 z_n^2}$ disappear and we are left with:

$$\|A\|_2 = \sqrt{\sigma_1^2} = \sigma_1$$