4. Matrix Decompositions (Exercises Only)

Exercises

4.1

Compute the determinant using Laplace Expansion (using the first row) and the Sarrus Rule for:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

Starting out with the Laplace expansion method, we find the minors:

$$A_{11} = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

Now we find the determinants of the minors:

$$\det(A_{11}) = 16 - 12 = 4$$

$$\det(A_{12}) = 8 - 0 = 8$$

$$\det(A_{13}) = 4 - 0 = 4$$

And now the cofactors:

$$C_{11} = -1^2(4) = 4$$

$$C_{12} = -1^3(8) = -8$$

$$C_{13} = -1^4(4) = 4$$

And now we can compute the determinant:

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(4) + 3(-8) + 5(4) \\ &= 4 - 24 + 20 \\ &= 0 \end{aligned}$$

Now we can compute the determinant using the Sarrus Rule:

$$
\begin{aligned}
\det(A) &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\
&= 1(4)(4) + 2(2)(5) + 0(3)(6) - 0(4)(5) - 1(2)(6) - 2(3)(4) \\
&= 16 + 20 - 12 - 24 \\
&= 36 - 36 \\
&= 0 ✅
\end{aligned}
$$

4.2

Compute the following determinant efficiently:

$$
\begin{vmatrix}
2 & 0 & 1 & 2 & 0 \\
2 & -1 & 0 & 1 & 1 \\
0 & 1 & 2 & 1 & 2 \\
-2 & 0 & 2 & -1 & 2 \\
2 & 0 & 0 & 1 & 1
\end{vmatrix}
$$

Since it doesn't specify a method I think I will get it in upper triangular form using Gaussian Elimination, then take the product of the diagonal elements to find the determinant.

After performing Gaussian Elimination we get:

$$
\begin{bmatrix}
2 & 0 & 1 & 2 & 0 \\
0 & -1 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & -7 \\
0 & 0 & 0 & 0 & -3
\end{bmatrix}
$$

We didn't perform any row swaps or scaling of rows so we don't need to worry about that. Now we take the product of the diagonal elements to find the determinant:

$$
\det(A) = 2(-1)(1)(1)(-3) = 6
$$

4.3

Compute the eigenspaces of:

$$
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}
$$

Starting with: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$:

We start out by finding the characteristic polynomial:

$$
\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0
$$

$$
(1-\lambda)(1-\lambda) = 0
$$

$$
\lambda_1 = 1
$$

- Note that the algebraic multiplicity of the eigenvalue is 2
- This means that the geometric multiplicity must either 1 or 2.

We can find the eigenvectors:

$$\begin{bmatrix} 1-\lambda_1 & 0 \\ 1 & 1-\lambda_1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x = 0$$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, the eigenspace is:

$$E_1 = \text{span}[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]$$

Now $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$:

4.4:

$$\begin{bmatrix} -\lambda & -1 & 1 & 1 \\ 0 & -\lambda & -1 & 3-\lambda \\ 0 & 1 & -2-\lambda & 2\lambda \\ 1 & -1 & 1 & -\lambda \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & -1-\lambda & 0 & 1 \\ 0 & -\lambda & -1-\lambda & 3-\lambda \\ 0 & 1 & -1-\lambda & 2\lambda \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$$

4.5

a.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span $\mathbb{R}^n$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda = 1$$

Find eigenvectors:

$$\begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} x = 0$$

- *Any* vector in $\mathbb{R}^2$ is an eigenvector of this matrix and therefore there are two linearly independent vectors in the null space of $(A - \lambda I)$ meaning this is diagonalizable ✅

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (1)(1) - 0 = 1 ✅$$

- You may have immediately been able to see these results by recognizing that this matrix if I_2

b.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span $\mathbb{R}^n$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(1-\lambda) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 1$$

Find eigenvectors:

- First λ_1:

$$\begin{bmatrix} 1-0 & 0 \\ 0 & 0-0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Now λ_2:

$$\begin{bmatrix} 1-1 & 0 \\ 0 & 0-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- x_1 and x_2 are linearly independent and form a basis for $\mathbb{R}^2$ therefore this matrix is diagonalizable ✅

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = 0 - 0 = 0 ❌$$

c.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span $\mathbb{R}^n$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) = 0$$

$$\lambda = 1$$

Find eigenvectors:

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

$$x := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The eigenvectors do not form a basis for $\mathbb{R}^2$ therefore this is not diagonalizable ❌

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (1)(1) - 0 = 1 ✅$$

d.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Diagonalizability:

- Check for eigenvectors that span $\mathbb{R}^n$

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0$$

Find eigenvectors:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

$$x := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The eigenvectors do not form a basis for $\mathbb{R}^2$ therefore this is not diagonalizable ❌

Invertibility:

- Check that the determinant is nonzero:

$$\det(A) = (0)(0) - (1)(0) = 0 ❌$$

4.6

Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

a.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

- We can compute this with a Laplace expansion. It will be easiest to preform Laplace expansion in the third row because there are two zeros, so two of the terms will disappear:

$$\det(A - \lambda I) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$\det(A - \lambda I) = 0C_{31} + 0C_{32} + (1-\lambda)C_{33}$$

$$\det(A - \lambda I) = (1-\lambda)C_{33}$$

- To find the C_{33} we first find the determinant of the minor A_{33}

$$A_{33} = \begin{bmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A_{33}) &= (2-\lambda)(4-\lambda) - 3 \\ \det(A_{33}) &= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 \\ \det(A_{33}) &= \lambda^2 - 6\lambda + 5 \\ \det(A_{33}) &= (\lambda-5)(\lambda-1) \end{aligned}$$

This gives us the cofactor C_{33}:

$$C_{33} = (\lambda-5)(\lambda-1)$$

Now we can return to the characteristic polynomial of $(A - \lambda I)$

$$\det(A - \lambda I) = (1-\lambda)(\lambda-5)(\lambda-1) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 5$$

- note that λ_1 has an algebraic multiplicity of 2
- Now we can compute the eigenvectors of A:
- $\lambda_1 = 1$

$$\begin{bmatrix} 2-1 & 3 & 0 \\ 1 & 4-1 & 3 \\ 0 & 0 & 1-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

- $\lambda_2 = 5$

$$\begin{bmatrix} 2-5 & 3 & 0 \\ 1 & 4-5 & 3 \\ 0 & 0 & 1-5 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 3 \\ 0 & 0 & -4 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Therefore the eigenspaces of A are:

$$E_1 = \text{span}[\begin{bmatrix}-3\\1\\0\end{bmatrix}]$$

$$E_5 = \text{span}[\begin{bmatrix}1\\1\\0\end{bmatrix}]$$

A is *not* diagonalizable because its eigenvectors do not form a basis for $\mathbb{R}^3$

b.

$$A = \begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix}1-\lambda & 1 & 0 & 0\\0 & -\lambda & 0 & 0\\0 & 0 & -\lambda & 0\\0 & 0 & 0 & -\lambda\end{vmatrix} = 0$$

- Doing a Laplace expansion in the first row:

$$\det(A - \lambda I) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$
$$\det(A - \lambda I) = (1-\lambda)C_{11} + (1)C_{12} + (0)C_{13} + (0)C_{14}$$
$$\det(A - \lambda I) = (1-\lambda)C_{11} + C_{12}$$

- We start by finding the cofactor C_{11}. The first step in that is finding the determinant of the minor A_{11}:

$$A_{11} = \begin{bmatrix}-\lambda & 0 & 0\\0 & -\lambda & 0\\0 & 0 & -\lambda\end{bmatrix}$$

- Since it's in row echelon form the determinant is just the product of the diagonal elements:

$$\det(A_{11}) = -\lambda^3$$

$$C_{11} = -\lambda^3$$

- Now we can find the second cofactor C_{12}:

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

- The determinant of A_{12} is evidently zero so the entire C_{12} term disappears:

$$\det(A - \lambda I) = (1-\lambda)(-\lambda^3)$$

$$\lambda_1 = 1 \quad \lambda_2 = 0$$

- Now we find the eigenvectors:
- $\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\lambda_1 = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0$$

- Writing this out as a system of linear equations:

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

- therefore:

$$x = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- Expressing this vector as a linear combination:

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Therefore:

$$x_2 := \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- The eigenspaces of A are:

$$E_1 = \text{span}[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}]$$

$$E_0 = \text{span}[\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}]$$

- Putting all of the eigenvectors for A into a matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- We can see that the columns of this matrix are linearly independent. Therefore the eigenvectors form a basis for $\mathbb{R}^4$ and A *is* diagonalizable.

4.7

a.

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -8 & 4 - \lambda \end{vmatrix}$$

- This matrix has no real eigenvalues

b.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

1. $R_2 \leftarrow R_2 - R_1$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ -\lambda & -\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

2. $R_3 \leftarrow R_3 - R_1$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{vmatrix}$$

- Now we preform a Laplace expansion along the first row:

$$\det = (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ \lambda & -\lambda \end{vmatrix} + \begin{vmatrix} \lambda & -\lambda \\ \lambda & 0 \end{vmatrix}$$

$$\det = (1-\lambda)(\lambda^2) - (-\lambda^2) + \lambda^2$$

$$\begin{aligned} \det &= \lambda^2 - \lambda^3 + 2\lambda^2 \\ &= -\lambda^3 + 3\lambda^2 \\ &= -\lambda^2(\lambda - 3) \end{aligned}$$

$$\lambda_1 = 3 \quad \lambda_1 = 0$$

- Now we find the eigenvectors:
- $\lambda_1 = 3$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} x = 0$$

$$x_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $\lambda_1 = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x = 0$$

$$x_2 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad x_3 := \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- The matrix is diagonalizable because its eigenvectors form a basis for $\mathbb{R}^3$ ✅
- The diagonal form of the matrix is:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The basis with respect to which the matrix is diagonal is:

$$B := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

c.

d.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

- First we find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix} = 0$$

- Performing a Laplace expansion along the first row:

$$\det(A - \lambda I) = (5-\lambda)\underbrace{\begin{vmatrix} 4 & 2 \\ -6 & -4-\lambda \end{vmatrix}}_{A_{11}} + 6\underbrace{\begin{vmatrix} -1 & 2 \\ 3 & -4-\lambda \end{vmatrix}}_{A_{12}} - 6\underbrace{\begin{vmatrix} -1 & 4-\lambda \\ 3 & -6 \end{vmatrix}}_{A_{13}}$$

Computing the determinants of the minors:

- A_{11}

$$\det(A_{11}) = \begin{vmatrix} 4 & 2 \\ -6 & -4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda)+12$$

$$\det(A_{11}) = \lambda^2 - 4$$

- A_{12}

$$\det(A_{11}) = \begin{vmatrix} -1 & 2 \\ 3 & -4-\lambda \end{vmatrix} = \lambda - 2$$

- A_{13}

$$\det(A_{11}) = \begin{vmatrix} -1 & 4-\lambda \\ 3 & -6 \end{vmatrix} = 3(\lambda - 2)$$

Plugging those determinants back in:

$$\begin{aligned} \det(A-\lambda I) &= (5-\lambda)(\lambda^2-4)+6(\lambda-2)-6[3(\lambda-2)] \\ &= (5-\lambda)(\lambda-2)(\lambda+2)+6(\lambda-2)-6[3(\lambda-2)] \end{aligned}$$

Factoring out a $(\lambda - 2)$:

$$\begin{aligned} \det(A-\lambda I) &= (\lambda-2)\left[(5-\lambda)(\lambda+2)+6-18\right] \\ &= (\lambda-2)[5\lambda+10-\lambda^2-2\lambda+6-16] \\ &= (\lambda-2)(3\lambda+10-\lambda^2-12) \\ &= (\lambda-2)(-\lambda^2+3\lambda-2) \end{aligned}$$

Factoring out a -1 from the right polynomial (we can just ignore it since $\det(A-\lambda I)=0$ and $\dfrac{0}{-1}=0$):

$$\begin{aligned} \det(A-\lambda I) &= (\lambda-2)(\lambda^2-3\lambda+2) \\ &= (\lambda-2)(\lambda-2)(\lambda-1) \end{aligned}$$

Therefore the eigenvalues are:

$$\lambda_1 = 2 \quad \lambda_2 = 1$$

- Now we find the eigenvectors:
- $\lambda_1 = 2$

$$\begin{bmatrix} 5-2 & -6 & -6 \\ -1 & 4-2 & 2 \\ 3 & -6 & -4-2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} x = 0$$

1. $R_1 \leftarrow \frac{1}{3} R_1$, $R_3 \leftarrow \frac{1}{3} R_3$:

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

1. $R_2 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 - 2x_3 = 0$$

$$x_1 = 2x_2 + 2x_3$$

$$x = \begin{bmatrix} 0x_1 + 2x_2 + 2x_3 \\ 0x_1 + x_2 + 0x_3 \\ 0x_1 + 0x_2 + x3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad x_2 := \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1$

$$\begin{bmatrix} 5-1 & -6 & -6 \\ -1 & 4-1 & 2 \\ 3 & -6 & -4-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} x = 0$$

$$4x_1 - 6x_2 - 6x_3 = 0$$

$$-x_1 + 3x_2 + 2x_3 = 0$$

$$3x_3 - 6x_2 - 5x_3$$

From equation 2:

$$x_1 = 3x_2 + 2x_3$$

Plugging this into equation 1:

$$\begin{aligned}4(3x_2 + 2x_3) - 6x_2 - 6x_3 &= 0\\12x_2 + 8x_3 - 6x_2 - 6x_3 &= 0\\6x_2 + 2x_3 &= 0\\x_3 &= -3x_2\end{aligned}$$

Plugging this back into the equation for x_1:

$$\begin{aligned}x_1 &= 3x_2 + 2(-3x_2)\\x_1 &= -3x_2\end{aligned}$$

$$x = \begin{bmatrix}0x_1 - 3x_2 + 0x_3\\0x_1 + x_2 + 0x_3\\0x_1 - 3x_2 + 0x_3\end{bmatrix} = x_2\begin{bmatrix}-3\\1\\-3\end{bmatrix}$$

$$x_3 := \begin{bmatrix}-3\\1\\-3\end{bmatrix}$$

- The matrix is diagonalizable because its eigenvectors form a basis for $\mathbb{R}^3$ ✅
- The diagonal form of the matrix is:

$$D = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$$

- The basis with respect to which the matrix is diagonal is:

$$B := \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}-3\\1\\-3\end{bmatrix}$$

4.8

Find the SVD:

$$A = \begin{bmatrix}3 & 2 & 2\\2 & 3 & -2\end{bmatrix}$$

- The first step in finding the SVD is finding A^TA:

$$A^TA = \begin{bmatrix}3 & 2\\2 & 3\\2 & -2\end{bmatrix}\begin{bmatrix}3 & 2 & 2\\2 & 3 & -2\end{bmatrix} = \begin{bmatrix}13 & 12 & 2\\12 & 13 & -2\\2 & -2 & 8\end{bmatrix}$$

- Now we eigendecompose A^TA.
- We start by finding the characteristic polynomial:

$$\det(A^TA - \lambda I) = \begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = 0$$

- Performing a Laplace expansion along the first row:

$$\det(A^TA - \lambda I) = (13-\lambda)\underbrace{\begin{vmatrix} 13-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix}}_{A_{11}} - 12\underbrace{\begin{vmatrix} 12 & -2 \\ 2 & 8-\lambda \end{vmatrix}}_{A_{12}} + 2\underbrace{\begin{vmatrix} 12 & 13-\lambda \\ 2 & -2 \end{vmatrix}}_{A_{13}}$$

- Starting with the first minor A_{11}:

$$\begin{aligned} \det(A_{11}) &= (13-\lambda)(8-\lambda) - 4 \\ &= 104 - 13\lambda - 8\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 21\lambda + 100 \end{aligned}$$

- Now A_{12}

$$\begin{aligned} \det(A_{12}) &= 96 - 12\lambda + 4 \\ &= 100 - 12\lambda \end{aligned}$$

- Now A_{13}

$$\begin{aligned} \det(A_{13}) &= -24 - 2(13-\lambda) \\ &= 24 - 26 + 2\lambda \\ &= 2\lambda - 50 \end{aligned}$$

- Plugging back into the equation for $\det(A^TA - \lambda I)$:

$$\begin{aligned} \det(A^TA - \lambda I) &= (13-\lambda)(\lambda^2 - 21\lambda + 100) - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ &= 13\lambda^2 - 273\lambda + 1300 - \lambda^3 + 21\lambda^2 - 100\lambda - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ \\ &= -\lambda^3 + 13\lambda^2 + 21\lambda^2 - 100\lambda - 273\lambda + 1300 - 12(100 - 12\lambda) + 2(2\lambda - 50) \\ &= -\lambda^3 + 34\lambda^2 - 373\lambda + 1300 - 1200 - 144\lambda + 4\lambda - 100 \\ &= \lambda^3 + 34\lambda^2 - 255\lambda \\ &= -\lambda(\lambda^2 - 34\lambda - 255) \\ &= -\lambda(\lambda - 25)(\lambda - 9) \end{aligned}$$

- This gives us the eigenvalues of A^TA:

$$\lambda_1 = 25 \qquad \lambda_2 = 9 \qquad \lambda_3 = 0$$

- Now we find the eigenvectors of A^TA using the eigenvalues:
- $\lambda_1 = 25$

$$\begin{bmatrix} 13-25 & 12 & 2 \\ 12 & 13-25 & -2 \\ 2 & -2 & 8-25 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_2 \leftarrow R_2 + R_1$:

$$\begin{bmatrix} -12 & 12 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & -17 \end{bmatrix}$$

2. $R_3 \leftrightarrow R_2$

$$\begin{bmatrix} -12 & 12 & 2 \\ 2 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix}$$

1. $R_1 \leftarrow -\frac{1}{12} R_1$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 2 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix}$$

4. $R_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & -\frac{100}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

5. $R_2 \leftarrow -\frac{6}{100} R_2$

$$\begin{bmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 - x_2 - \frac{1}{6} x_3 = 0$$

$$x_3 = 0$$

- We see that:

$$x_1 - x_2 - \frac{1}{6}(0) = 0$$

$$x_1 = x_2$$

Therefore our first eigenvector is:

$$x_1 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- $\lambda_2 = 9$

$$\begin{bmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_1 \leftarrow \frac{1}{4} R_1$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

2. $R_2 \leftarrow R_2 - 12R_1,\ R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & -32 & -8 \\ 0 & -8 & -2 \end{bmatrix}$$

3. $R_2 \leftarrow -\frac{1}{32} R_2$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & -8 & -2 \end{bmatrix}$$

4. $R_3 \leftarrow R_3 + 8R_2$

$$\begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 + 3x_2 + \frac{1}{2}x_3 = 0$$

$$x_2 + \frac{1}{4}x_3 = 0$$

$$x_2 = -\frac{1}{4}x_3$$

$$x_1 = -3x_2 - \frac{1}{2}d_3$$

$$x_1 = -3(-\frac{1}{4}x_3) - \frac{1}{2}x_3$$

$$x_1 = \frac{1}{4}x_3$$

- Our second eigenvector is:

$$x_1 := \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- $\lambda_3 = 0$

$$\begin{bmatrix} 13-0 & 12 & 2 \\ 12 & 13-0 & -2 \\ 2 & -2 & 8-0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} x = 0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & -2 & 8 \\ 13 & 12 & 2 \\ 12 & 13 & -2 \end{bmatrix}$$

2. $R_1 \leftarrow \frac{1}{2}R_1$

$$\begin{bmatrix} 1 & -1 & 4 \\ 13 & 12 & 2 \\ 12 & 13 & -2 \end{bmatrix}$$

3. $R_2 \leftarrow R_2 - 13R_1$, $R_3 \leftarrow R_3 - 12R_1$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 25 & -50 \\ 0 & 25 & -50 \end{bmatrix}$$

4. $R_3 \leftarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 25 & -50 \\ 0 & 0 & 0 \end{bmatrix}$$

5. $R_2 \leftarrow \frac{1}{25}R_2$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now writing this as a system of linear equations:

$$x_1 - x_2 + 4x_3 = 0$$

$$x_2 - 2x_3 = 0$$

$$x_2 = 2x_3$$

$$x_1 = x_2 - 4x_3$$

$$x_1 = -2x_3$$

- Our third eigenvector is:

$$x_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

- Now that we have the eigenvectors and eigenvalues for $A^T A$ we can construct two of the three matrices in the SVD: Σ and V
- Σ is a diagonal matrix of the same shape as A with the square root of the eigenvalues along the diagonal put in descending order. These values are the singular values:

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

- V is an orthogonal matrix of right singular vectors. All we need to do to construct V is normalize the eigenvectors of A^TA. The eigenvectors are already orthogonal (because A^TA is symmetric), so we just need to normalize them.

$$v_1 = \frac{x_1}{\|x_1\|}$$

$$\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{2}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$v_2 = \frac{x_2}{\|x_2\|}$$

$$\|x_2\| = \sqrt{x_2^T x_2} = \frac{\sqrt{18}}{4}$$

$$v_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$v_3 = \frac{x_3}{\|x_3\|}$$

$$\|x_3\| = \sqrt{x_3^T x_3} = 3$$

$$v_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

- Putting these together gives us the matrix V:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{bmatrix}$$

- The only matrix left to find is the matrix of left singular vectors U.

$$u_1 = \sigma_1 A v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \sigma_2 A v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- Which finally gives us U:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- In summary the SVD of A is:

$$A = U\Sigma V^T$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

4.9

Find the singular value decomposition of:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- The first step in finding the SVD is finding A^TA:

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

- Now we eigendecompose A^TA.
- We start by finding the characteristic polynomial:

$$\begin{aligned}\det(A^TA-\lambda I) &= \begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0\\ &=(5-\lambda)(5-\lambda)-9\\ &=25-10\lambda+\lambda^2-9\\ &=\lambda^2-10\lambda+16\\ 0&=(\lambda-8)(\lambda-2)\end{aligned}$$

- This gives us the eigenvalues of A^TA:

$$\lambda_1=8 \quad \lambda_2=2$$

- Now we find the eigenvectors of A^TA using the eigenvalues:
- $\lambda_1=8$

$$\begin{bmatrix}5-8 & 3\\ 3 & 5-8\end{bmatrix}x=0$$

$$\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}x=0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix}-3 & 3\\ 0 & 0\end{bmatrix}x=0$$

2. $R_1 \leftarrow -\frac{1}{3}R_1$

$$\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}x=0$$

- Now writing this as a system of linear equations:

$$x_1=x_2$$

$$x_1 := \begin{bmatrix}1\\ 1\end{bmatrix}$$

- $\lambda_2=2$

$$\begin{bmatrix}5-2 & 3\\ 3 & 5-2\end{bmatrix}x=0$$

$$\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}x=0$$

- We can get this matrix into row echelon form with Gaussian Elimination:

1. $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} x = 0$$

2. $R_1 \leftarrow \dfrac{1}{3} R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

- Now writing this as a system of linear equations:

$$x_1 = -x_2$$

$$x_2 := \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Now that we have the eigenvectors and eigenvalues for $A^T A$ we can construct two of the three matrices in the SVD: Σ and V
-
    - Σ is a diagonal matrix of the same shape as A with the square root of the eigenvalues along the diagonal put in descending order. These values are the singular values:

$$\Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

- V is an orthogonal matrix of right singular vectors. All we need to do to construct V is normalize the eigenvectors of $A^T A$. The eigenvectors are already orthogonal (because $A^T A$ is symmetric), so we just need to normalize them.

$$v_1 = \frac{x_1}{\|x_1\|}$$

$$\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{2}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$v_2 = \frac{x_2}{\|x_2\|}$$

$$\|x_2\| = \sqrt{x_2^T x_2} = \sqrt{2}$$

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Putting these together gives us the matrix V:

$$V = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

- The only matrix left to find is the matrix of left singular vectors U.

$$u_1 = \sigma_1 A v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \sigma_2 A v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Which finally gives us U:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \$\$ - InsummarytheSVDof\$A\$is:$$

A = U \Sigma V^T

A = \begin{bmatrix}
1 & 0 \
0 & 1 \
\end{bmatrix}
\begin{bmatrix}\sqrt{8} & 0 \ 0 & \sqrt{2} \end{bmatrix}
\begin{bmatrix}
\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \
-\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}
\end{bmatrix}

4.10 Find the best rank-1 approximation of:

A =
\begin{bmatrix}
3 & 2 & 2 \
2 & 3 & -2
\end{bmatrix}

$$-WecanusetheSVDwegotinquestion4.8.Togetarank1approximationweusetheformula:$$

A_1 = \sigma_1u_1v_1^T = 5(\begin{bmatrix}\dfrac{1}{\sqrt{2}} \ \dfrac{1}{\sqrt{2}}\end{bmatrix} \begin{bmatrix}\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0\end{bmatrix}) = 5 \begin{bmatrix}\dfrac{1}{2} & \dfrac{1}{2} & 0 \ \dfrac{1}{2} & \dfrac{1}{2} & 0 \\end{bmatrix} = \begin{bmatrix}2.5 & 2.5 & 0 \ 2.5 & 2.5& 0 \\end{bmatrix}

4.11 Show that for any $A \in \mathbb{R}^{m \times n}$, the matrices A^TA and AA^T possess the

A = U \Sigma V^T

−Where$U$and$V$areorthogonalmatricesand$Σ$isadiagonalmatrixwhosediagonalentriesarethesqu

A^TA = (U\Sigma V^T)^T(U\Sigma V^T) =V\Sigma^TU^TU\Sigma V^T

−Because$U$isorthogonal :

A^TA = V\Sigma^T\Sigma V^T

AA^T = (U \Sigma V^T)(U \Sigma V^T)^T = U \Sigma V^T V\Sigma^TU^T

−Because$V$isorthogonal

AA^T = U\Sigma \Sigma^T U^T

- In both equations we end up with $\Sigma^T \Sigma$ and $\Sigma \Sigma^T$. Because Σ is diagona

\max_x \dfrac{|Ax|_2}{|x|_2} = \sigma_1

−Where$σ_1$isthelargestsingularvalueof$A ∈ ℝ^{m×n}$ − if$x$isaunitvector, $‖x‖_2 = 1$so :

|A|_2 = \max_x|Ax|_2

Ifwebreak$A$downintoits[[SingularValueDecomposition|singularvaluedecomposition]] :

|A|_2 = \max_x |U \Sigma V^Tx|

−Because$U$isorthogonal, itpreservesnormsandangles :

|A|_2 = \max_x |\Sigma V^Tx|

−V^Tisalsoorthogonal, soitpreservesnorms. Wenowmakeasubstitution :

z = V^Tx

−becauseV^Tisorthogonal(andthereforepreserveslengths)and$x$islength1 :

|z|_2 = 1

Sonowourmaximizationproblembecomes :

|A|_2 = \max_z|\Sigma z|

Thequestionnowis, which$z$, giventheconstraintthat$z$mustbeunitlength, maximizestheEuclideann

\Sigma =
\begin{bmatrix}
\sigma_1 & ... & 0 \
... & ... & ... \

0 & ... & \sigma_n \
\end{bmatrix}

$Thismeansthatwecanwriteouttheequation for the Euclidean norm\$\|\Sigma z\|_2\$intermsofthecomponentsof$

|\Sigma z|_2 = \sqrt{\sigma_1^2z_1^2 + ... \sigma_n^2z_n^2}

$$Since\$z\$isunitlength:$$

\sum_{i = 0}^n z_i^2 = 1

$Therefore to maximize\$\|\Sigma z\|_2\$we''concentrate''alloftheweightpossible for\$z\$intotheelementcorrespo$

z = \begin{bmatrix}
1 \ 0 \ \vdots \ 0
\end{bmatrix}

$$Therefore, alloftheothertermsin\$\sqrt{\sigma_1^2z_1^2 + \ldots \sigma_n^2z_n^2}\$disappear and we are left with:$$

|A|_2 = \sqrt{\sigma_1^2} = \sigma_1