Point Pattern Analysis: Processes

SERGIO REY

GPH 483/598
Geographic Information Analysis
School of Geographical Sciences
Arizona State University
Spring 2010

Outline

1. Properties of Point Processes
   - First Order Properties
   - Second Order Property
2. Point Processes
   - Complete Spatial Randomness
   - Homogeneous Poisson Process
   - Inhomogeneous Poisson Process (IPP)

First Order Properties: Spatial Analysis

Mean value of the process in space

- Variation in mean value of the process in space
- Global, large scale spatial trend

First Order Property of Point Patterns, Intensity: λ

- Intensity: λ = number of events expected per unit area
- Estimation of λ
- Spatial variation of λ, $\lambda(s)$, s is a location

$$\lambda(s) = \lim_{ds \to 0} \left\{ \frac{E(Y(ds))}{ds} \right\} \tag{1}$$

Second Order Properties: Spatial Analysis

Spatial Correlation Structure

- Deviations in values from process mean
- Local or small scale effects

Second Order Property of Point Patterns

- Relationship between number of events in pairs of areas
- Second order intensity $\gamma(s_i, s_j)$

$$\gamma(s_i, s_j) = \lim_{ds_i \to 0, ds_j \to 0} \left\{ \frac{E(Y(ds_i)Y(ds_j))}{ds_i ds_j} \right\} \tag{2}$$

Spatial Stationarity

First Order Stationarity

$$\lambda(s) = \lambda \forall s \in A \quad (3)$$

$$E(Y(A)) = \lambda \times A \quad (4)$$

Second Order Stationarity

$$\gamma(s_i, s_j) = \gamma(s_i - s_j) = \gamma(h) \quad (5)$$

- h is the vector difference between locations s_i and s_j
- h encompasses direction and distance (relative location)
- Second order intensity only depends on h for second order stationarity

Spatial Isotropy and Stationarity

Isotropic Process

- When a stationary process is invariant to rotation about the origin.
- Relationship between two events depend only on the distance separating their locations and not on their orientation to each other.
- Depends only on distance, not direction

Usefulness

- Two pairs of events from a stationary process separated by same distance and relative direction should have same "relatedness"
- Two pairs of events from a stationary *and* isotropic process separated by the same distance (irrespective of direction) should have the same "relatedness"
- Both allow for replication and the ability to carry out estimation of the underlying DGP.

Complete Spatial Randomness

CSR

- Standard of Reference
- Uniform: each location has equal probability
- Independent: location of points independent
- Homogeneous Planar Poisson Point Process

Poisson Point Process

Intensity

- number of points in region A : $N(A)$
- intensity: $\lambda = N/|A|$
- implies: $\lambda|A|$ points randomly scattered in a region with area $|A|$
- e.g., 10×1 (points per km^2)

Poisson Distribution

$N(A) \sim Poi(\lambda|A|)$

Poisson Distribution

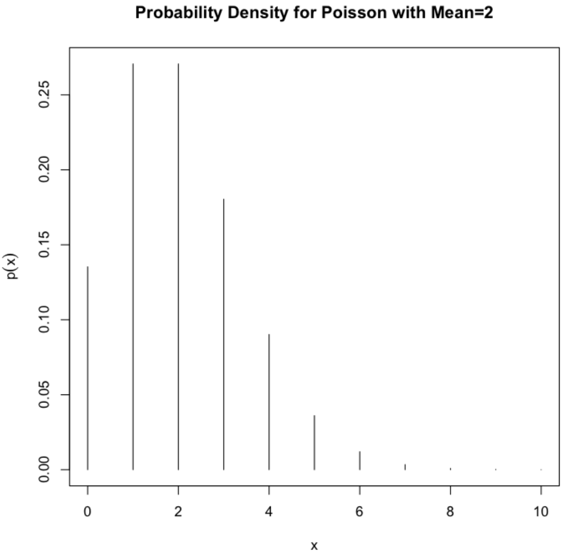
Single Parameter Distribution: $\lambda|A|$

- Generally, λ is the number of events in some well defined *interval*
  - Time: phone calls to operator in one hour
  - Time: accidents at an intersection per week
  - Space: trees in a quadrat
- Let x be a Poisson random variable
  - $E[x] = V[x] = \lambda|A|$

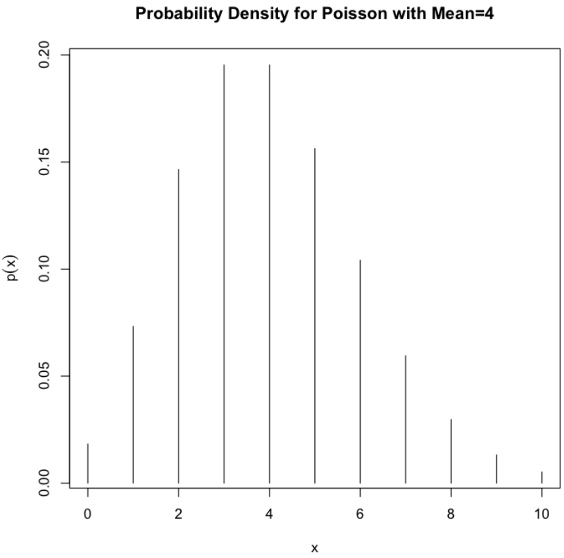
Poisson Distribution

$$P(x) = \frac{e^{-\lambda|A|}(\lambda|A|)^x}{x!} \tag{6}$$

Poisson Distribution $\lambda = 2$



Poisson Distribution $\lambda = 4$



In Space

Single Parameter

$$P[N(A) = x] = e^{-\lambda|A|}(\lambda|A|)^x/x! \quad (7)$$

Spatial Example

CSR with $\lambda = 5/km^2$

- Region = Circle
  - area = $|A| = \pi r^2$
  - $r = 0.1$ km then area ≈ 0.03 km^2
- Probability of Zero Points in Circle

$$
\begin{aligned}
P[N(A) = 0] &= e^{-\lambda|A|}(\lambda|A|)^x/x! && (8)\\
&\approx e^{-5\times 0.03}(5 \times 0.03)^0/0! && (9)\\
&\approx e^{-5\times 0.03} && (10)\\
&\approx 0.86 && (11)
\end{aligned}
$$

Complete Spatial Randomness (CSR)

Homogeneous spatial Poisson point process

1. The number of events occurring within a finite region A is a random variable following a Poisson distribution with mean $\lambda|A|$, with $|A|$ denoting area of A.
2. Given the total number of events N occurring within an area A, the locations of the N events represent an independent random sample of N locations where each location is equally likely to be chosen as an event.

- Criterion 2 is the general concept of CSR (uniform (random)) distribution in A.
- Criterion 1 pertains to the intensity λ.

Homogeneous Poisson process

Implications

1. The number of events in nonoverlapping regions in A are statistically independent.
2. For any region $R \subset A$:

$$\lim_{|R| \to 0} \frac{Pr[exactly\ one\ event\ in\ R]}{|R|} = \lambda > 0 \tag{12}$$

3. $$\lim_{|R| \to 0} \frac{Pr[more\ than\ one\ event\ in\ R]}{|R|} = 0 \tag{13}$$

Homogeneous Poisson process

Implications

- λ is the intensity of the spatial point pattern.
- For a Poisson random variable, Y:

$$E[Y] = \lambda = V[Y] \tag{14}$$

- Provides the motivation for some quadrat tests of CSR hypothesis.
  - If Y_R is the count in quadrat R
  - If $\widehat{E[Y]} < \widehat{V[Y]}$: overdispersion = spatial clustering
  - If $\widehat{E[Y]} > \widehat{V[Y]}$: underdispersion = spatial uniformity

Simulating CSR

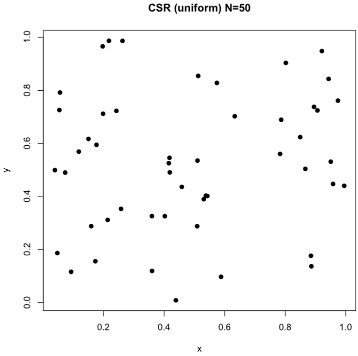
N – *conditioned*

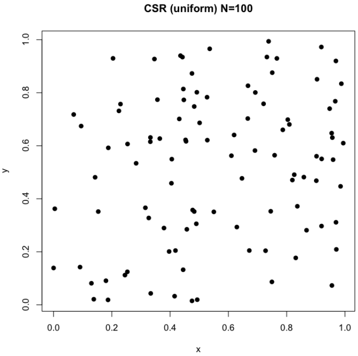
- CSR= uniform distribution
- random uniform draws for x and y point coordinates
- N fixed

λ – *conditioned*

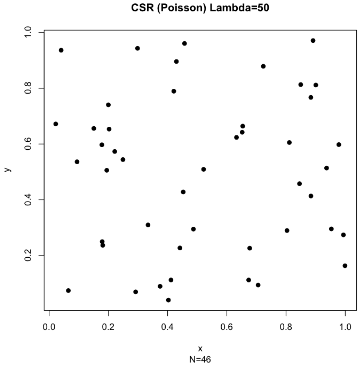
- CSR= Poisson distribution
- λ and $|A|$ given
- $N(A)$ random

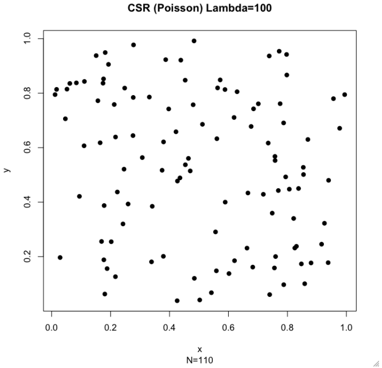
CSR Uniform


CSR (uniform) N=50
y
x
0.0
0.2
0.4
0.6
0.8
1.0


CSR (uniform) N=100
y
x
0.0
0.2
0.4
0.6
0.8
1.0

CSR Poisson


CSR (Poisson) Lambda=50
y
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
x
N=46


CSR (Poisson) Lambda=100
y
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
x
N=110

Limitations of CSR

Stationary Poisson Process

- homogeneous
- translation invariant

Rare in practice

very few (any?) actual processes are CSR

Strawman

- purely a benchmark
- null hypothesis

Inhomogeneous Poisson Process (IPP)

Criteria

1. The number of events occurring within a finite region A is a random variable following a Poisson Distribution with mean $\int_A \lambda(s)ds$.
2. Given the total number of events N occurring within A, the N events represent an independent sample of N locations, with the probability of sampling a particular point s proportional to $\lambda(s)$.

Spatially Variable Intensity $\lambda(s)$

- Useful for *constant risk hypothesis*
- Underlying population at risk is spatially clustered
- Want to control for that since with individual constant risk apparent clusters would be generated.
- Compare pattern against constant risk, not CSR.

Inhomogeneous Poisson Process

Implications

- Apparent clusters can occur solely due to heterogeneities in the intensity function $\lambda(s)$.
- Individual event locations still remain independent of one another.
- Process is not stationary due to intensity heterogeneity

HPP vs. IPP

HPP is a special case of IPP with a constant intensity

CSR vs. Constant Risk Hypotheses

CSR

- Intensity is spatially constant
- Population at risk assumed spatially uniform
- Useful null hypothesis if these conditions are met

Constant Risk Hypothesis

- Population density variable
- Individual risk constant
- Expected number of events should vary with population density
- Clusters due to deviation from CSR
- Clusters due to deviation from CSR and Constant Risk