Point Patterns

Sergio Rey

Geographic Information Analysis
School of Geographical Sciences and Urban Planning
Arizona State University

Outline

Outline

Central Tendency

Purpose

- Provide a “center point”
- Similar to first moment of a distribution
- “Representative point”

Measures

- Mean Center
- Weighted Mean Center
- Median Center
- Center of Minimum Distance

Example Data

i	x_i	y_i	w_i
1	20	40	10
2	30	60	20
3	34	52	10
4	40	40	20
5	44	42	10
6	48	62	80
7	50	10	10
8	60	50	90
9	90	90	100

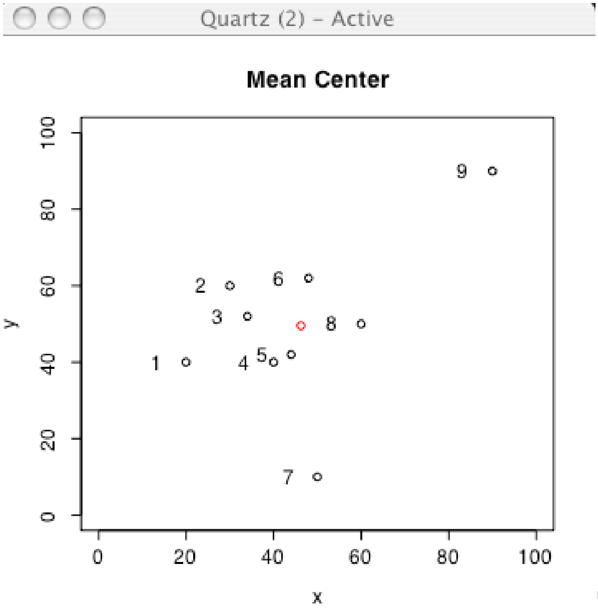
Mean Center

(x_m, y_m)

$$x_m = 1/n \sum_{i=1}^{n} x_i \tag{1}$$

$$y_m = 1/n \sum_{i=1}^{n} y_i \tag{2}$$

Mean Center



Quartz (2) – Active
Mean Center
y
x
0
20
40
60
80
100
0
20
40
60
80
100
1
2
3
4
5
6
7
8
9

Weighted Mean center

(x_m, y_m)

$$x_m = \sum_{i=1}^{n} x_i \frac{w_i}{\sum_{i=1}^{n} w_i} \tag{3}$$

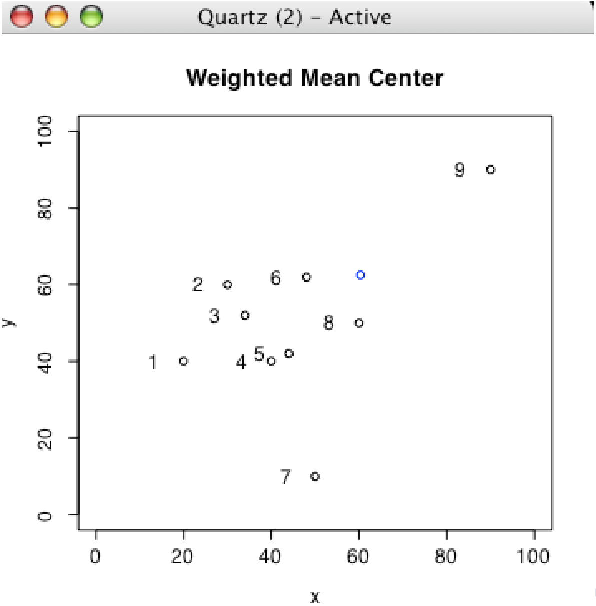
$$y_m = \sum_{i=1}^{n} y_i \frac{w_i}{\sum_{i=1}^{n} w_i} \tag{4}$$

(Corrected)

w_i weight

- Marked point patterns
- Continuous mark
- Not categorical mark

Weighted Mean Center



Median Center

Definition(s)

English Statistics: The intersection of two orthogonal axes, each which has an equal number of points on either side.

American: The center of minimum travel.

English Median Center

Manhattan Median

$$Min\ f(x_m, y_m) = \sum_{i=1}^{n} |x_i - x_m| + |y_i - y_m| \tag{5}$$

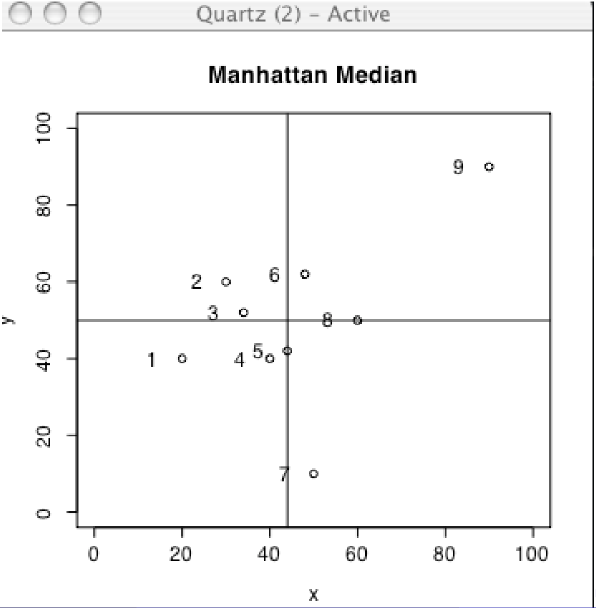
Advantages

- Can be found very quickly
- No calculations are typically required (other than intersection)

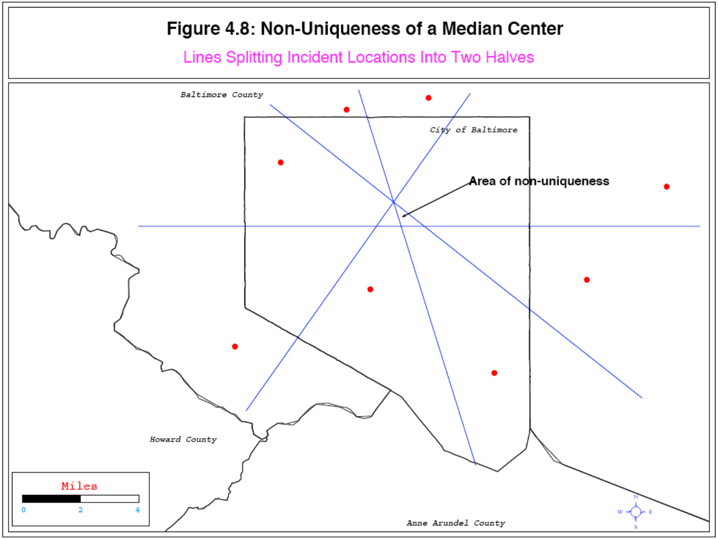
Disadvantage

- Never unique with even *n*
- Always unique with odd *n*
- Not unique under axis rotation

Manhattan Median



Non-Uniqueness



Center of Minimum Travel

Euclidean Median

The location from which the sum of the Euclidean distances to all points in a distribution is a minimum.

Euclidean Median

$$Min\ f(x_m, y_m) = \sum_{i=1}^{n} \sqrt{(x_i - x_m)^2 + (y_i - y_m)^2} \tag{6}$$

Weighted Euclidean Median

$$Min\ f(x_m, y_m) = \sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} \sqrt{(x_i - x_m)^2 + (y_i - y_m)^2} \tag{7}$$

Euclidean Median

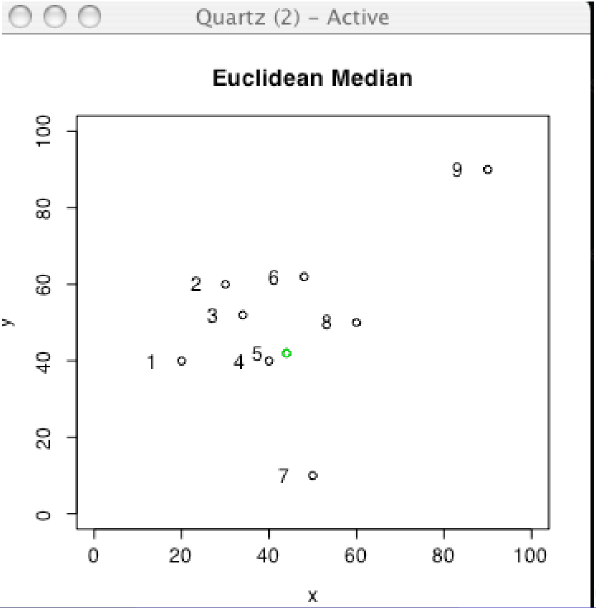
Weber Problem

Find the optimal location for a factory: one that minimizes transport costs between sources of raw materials and delivery to the market.

Solutions

- No closed form solution
- First iterative algorithm: Kuhn and Kuenne (1962)
- Important for more general location allocation problems

Euclidean Median



Outline

Dispersion and Orientation

Measures

- Standard Distance
- Major/minor axes
- Standard Deviational Ellipse

Standard Distance

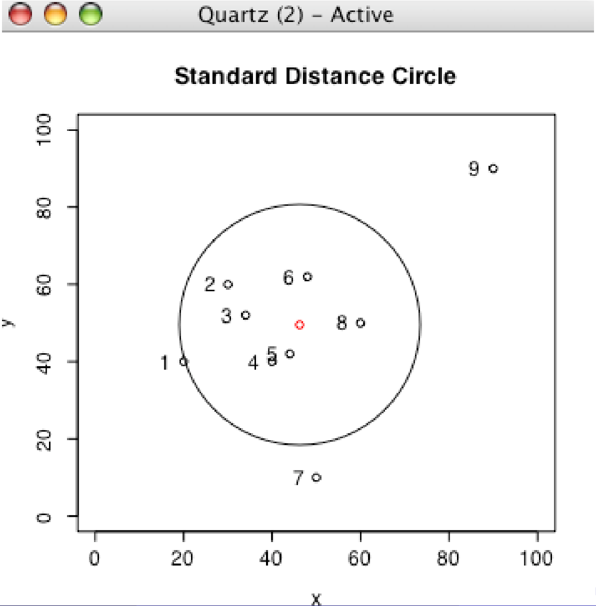
Euclidean Based

$$SD = \sqrt{\frac{\sum_{i=1}^{n}(x_i - x_m)^2}{n} + \frac{\sum_{i=1}^{n}(y_i - y_m)^2}{n}} \tag{8}$$

Uses

- Similar to standard deviation
- Combine with Mean Center for “outlier detection”
- Sensitive to extreme values

Standard Distance Circle



Standard Deviational Ellipse

Relative to Standard Distance

- Measures dispersion
- Sensitive to *shape* of distribution
- Measures dispersion in two dimensions

Components

- Angle of rotation
- Dispersion along major axis
- Dispersion along minor axis

Standard Deviational Ellipse

Major, minor axes

- Major axis defines the direction of maximum spread in the distribution
- Minor axis is orthogonal to major axis
- Minor axis defines the direction of minimum spread

Steps

1. Determine rotation angle of Y-axis
2. Calculate standard deviations for transposed axes
3. Determine length of axes
4. Determine area of the ellipse

Rotation Angle Θ

$$\Theta = ARCTAN\left\{\left(\sum_i (x_i-\bar{x})^2 - \sum_i (y_i-\bar{y})^2\right) + \left[\left(\sum_i (x_i-\bar{x})^2 - \sum_i (y_i-\bar{y})^2\right)^2 + 4\left(\sum_i (x_i-\bar{x})(y_i-\bar{y})\right)^2\right]^{1/2}\right\} / 2\sum_i (x_i-\bar{x})(y_i-\bar{y})$$

Standard Deviations On Transposed Axes

S_x

$$S_x = \sqrt{2\frac{(\sum_{i=1}^{n}(x_i - \bar{x})Cos(\Theta) - \sum_{i=1}^{n}(y_i - \bar{y})Sin(\Theta))^2}{n-2}} \tag{9}$$

S_y

$$S_y = \sqrt{2\frac{(\sum_{i=1}^{n}(x_i - \bar{x})Sin(\Theta) - \sum_{i=1}^{n}(y_i - \bar{y})Cos(\Theta))^2}{n-2}} \tag{10}$$

Ellipse Axes

Lengths

$$L_x = 2S_x \tag{11}$$

$$L_y = 2S_y \tag{12}$$

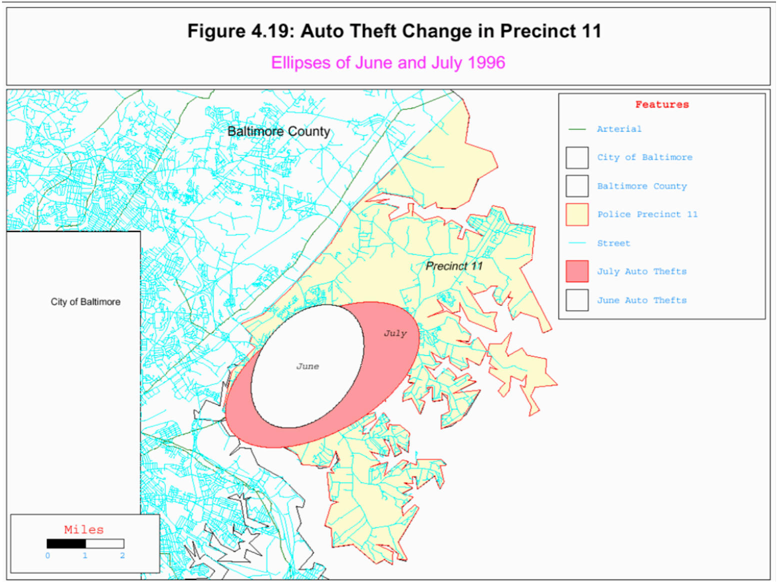
Mid Point

Mean Center of Point Pattern (x_m, y_m)

Area

$$A = \pi S_x S_y \tag{13}$$

Standard Deviational Ellipse



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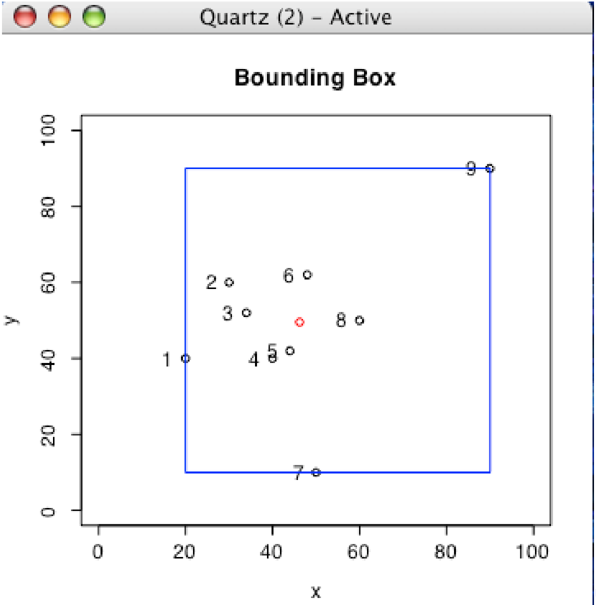
Shape Analysis of Point Patterns

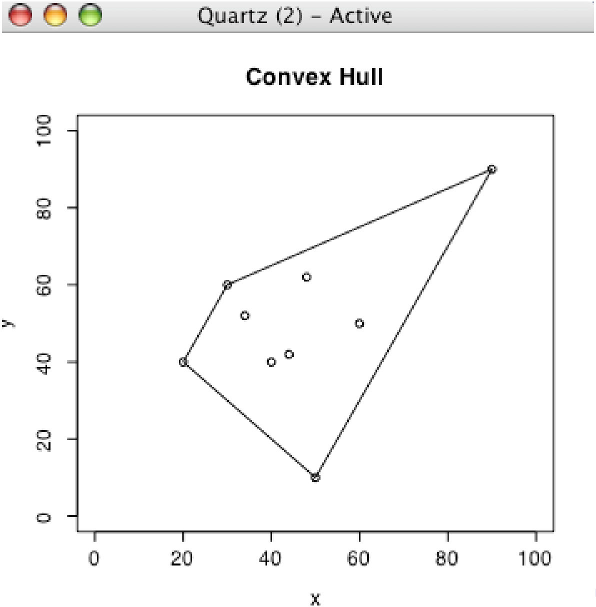
Geometry

- Bounding Box
- Convex Hulls

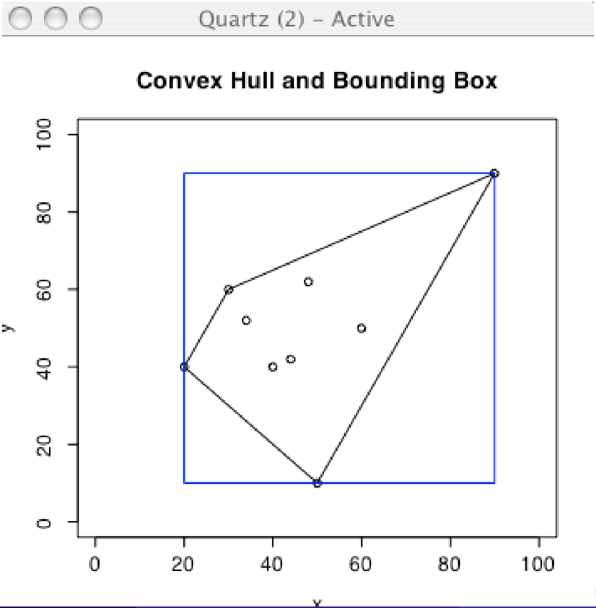
Bounding Box



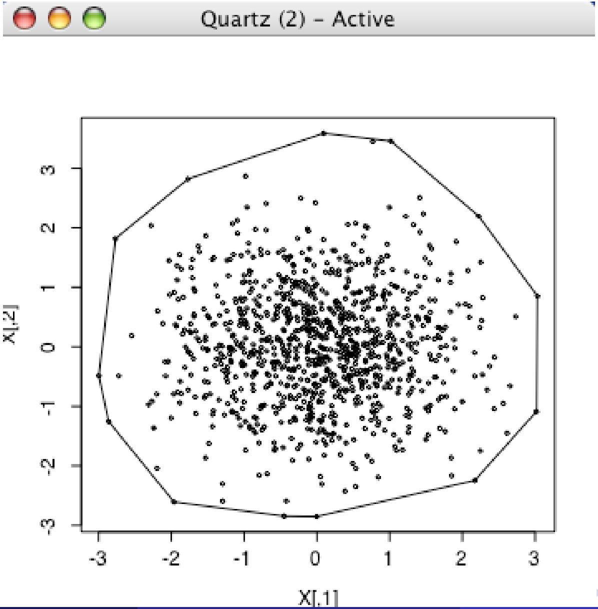
Convex Hull



Convex Hull and Bounding Box



Convex Hull (Large n)



Nested Convex Hulls

Quadrat Counts

Basic Approach

- Impose a tessellation over the area
- Count number of points in each cell
- Compare observed counts against expected counts under the null of CSR

Expected Counts

- Relies on relationship between Poisson-CSR-Binomial
- Treat each cell as independent
- $E[x_i] = \lambda |A_i|$ where λ is the overall area intensity and $|A_i|$ is the area of cell i

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Outline

Quadrat Counts: Test Statistic

χ^2 statistic

- Regular tessellation (Grid with $m \times k$ cells)
- m rows
- k cols
- Equal sized cells

$$\chi^2 = \sum_{i=1}^{m} \sum_{j=1}^{k} (x_{i,j} - E[x_{i,j}])^2 / (\lambda |A_{i,j}|) \tag{14}$$

Under the null of csr our test statistic has a $\chi^2(m \times k - 1)$ distribution

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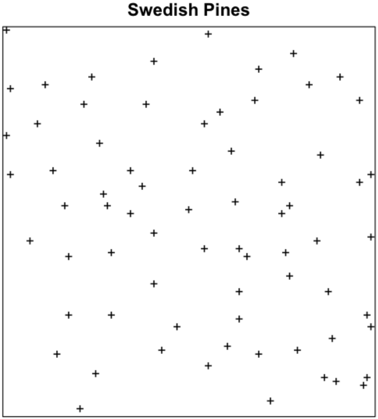

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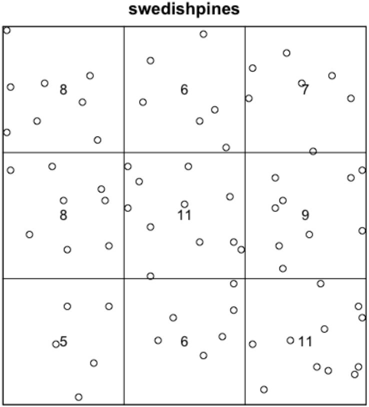

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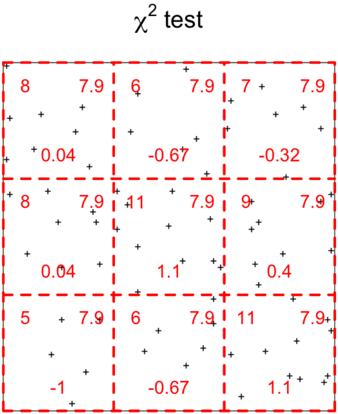

$$\chi^2 = 4.6761,\ df = 8,\ p - value = 0.7916$$

Outline

Quadrat Counts

Issues

- Choice of tessellation
  - how many cells?
  - what cell shape?
  - locations random or fixed?
- Edge effects
- Spatial dependence
  - Independent cell counts
  - Independent locations

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1. Centrography
   - Central Tendency
   - Dispersion and Orientation
   - Geometry
2. Quadrat Counts
   - Test Statistic
   - Issues
3. Monte Carlo Simulation
   - Motivation

Monte Carlo Simulation

Basic Approach

- Specify test statistic
- Calculate test statistic on observed pattern: ψ
- Specify a null hypothesis (H_o)
- Specify an alternate hypothesis (H_1)
- Simulate Empirical Sampling Distribution of $\psi | H_o$
  - Draw *nsim* realizations under the null.
  - Calculate ψ_i where $i = 1, 2, \ldots, nsim$.
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Computational Approximation to Inference

Motivations

- Substitute capital for labor
- Practical when no analytical results are available
- Very flexible

Issues

- Not generalizable beyond data at hand
- Less powerful than exact tests (if available)
- May be computationally expensive

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- Specify a null hypothesis (H_o)
- Specify an alternate hypothesis (H_1)
- Simulate Empirical Sampling Distribution of $\psi | H_o$
  - Draw *nsim* realizations under the null.
  - Calculate ψ_i where $i = 1, 2, \ldots, nsim$.
  - Compare ψ to distribution of ψ_i.

Pseudo Significance Level

p-value

$$p(\chi^2) = \frac{1 + \sum_{i=1}^{nsim} \phi_i}{nsim + 1} \tag{15}$$

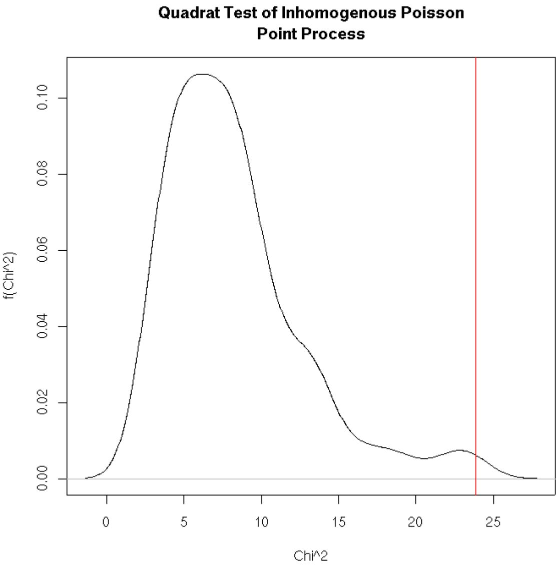
where:

$$\phi_i = \begin{cases} 1 & if\ \psi_i^2 \geq \chi^2, \\ 0 & otherwise \end{cases} \tag{16}$$

Code: ihhpsim.r

```
source("quadcounts.r")
source("ihppsim.r")
pp=ippsim(100)*9+1
ppt=quadcount(pp[,1],pp[,2])
set.seed(100)
nsim=99
source("hppsim.r")
results=matrix(0,nsim+1,1)
for(i in 1:nsim){
    pp=csr(100,1,1,10,10)
    t=quadcount(pp$x,pp$y)
    results[i]=t$chi2
}
results[100]=ppt$chi2
plot(density(results),main="Quadrat Test of Inhomogenous Po
Point Process",xlab="Chi^2",ylab="f(Chi^2)")
abline(v=ppt$chi2,col='red')
```

Empirical Sampling Distribution



Pseudo Significance Level

p-value

$$p(\chi^2) = \frac{1 + \sum_{i=1}^{nsim} \phi_i}{nsim + 1} \quad (17)$$

where:

$$\phi_i = \begin{cases} 1 & if\ \psi_i^2 \geq \chi^2, \\ 0 & otherwise \end{cases} \quad (18)$$

p-value

$$p(\hat{\chi}^2) = \frac{1 + 0}{99 + 1} = 0.01 \quad (19)$$

Pseudo Significance Level

p-value

$$p(\chi^2) = \frac{1 + \sum_{i=1}^{nsim} \phi_i}{nsim + 1} \tag{17}$$

where:

$$\phi_i = \begin{cases} 1 & if\ \psi_i^2 \geq \chi^2, \\ 0 & otherwise \end{cases} \tag{18}$$

p-value

$$p(\hat{\chi}^2) = \frac{1 + 0}{99 + 1} = 0.01 \tag{19}$$