# EXPLORING PARALLEL EFFICIENCY AND SYNERGY FOR MAX-P REGION PROBLEM USING PYTHON

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***Abstract*—Given a set of n areas spatially covering a geographical zone such as a province, forming contiguous regions from homogeneous neighboring areas satisfying a minimum threshold criteria over each region is an interesting NP-hard problem that has applications in various domains such as political science, GIS, etc. In this paper, we focus on a specific case, called Max-p regions problem, in which the main objective is to maximize the number of regions while keeping heterogeneity in each region as small as possible. The solution is broken into two phases. 1) Construction phase: starting from random areas (seeds), the algorithm builds each region by adding neighboring areas to each growing seed until the region reaches out to the threshold. Then, all the unassigned regions called enclaves are assigned to their neighboring regions. 2) Optimization phase: tries to improve the feasible solution from phase one by swapping areas at the border of each region with each neighbors. In this paper, we present a parallel implementation of Max-p problem using Python multiprocessing library. By exploiting a intuitive data structure based on multi-locks, we achieves up 200-fold and 19-fold speeds up over the best sequential algorithm for the construction and optimization phases respectively. Also, the overall end to end running time of our implementation is 35 times faster than this sequential algorithm. We provide extensive experimental results to verify our algorithm.**

***Index Terms*—Clustering, Geospatial, Multiprocessing, Optimization**

## I. Introduction

Clustering geographical areas into homogenous regions have many applications in various domains such as urban development, districting, transport planning, analyzing crime rates etc [1]. The homogeneous region can be defined as a set of contiguous areas with high degree of similarity for a given attribute such as per capita income, population etc. Generally speaking, problems of this kind are NP-hard [2] as the size of geographical areas to be grouped into homogeneous regions increases, the clustering running time increases exponentially. Hence, it is important to utilize the parallel processing capabilities of modern hardware to perform the computation. Furthermore, improving the accuracy of the solution by efficiently taking advantage of parallel synergy is another important aspect of these kinds of problems.

Many researchers and scientists have contributed to the problem of aggregating areas into homogeneous regions. Duque, Anselin, and Rey [2] re-

ferred to it as max-p region problem, Hensen [6] referred to it as clustering under connectivity constraints, Maravalle and Simeone [12] referred to it as regional clustering, Wise [13] referred to it as regionalization. Major challenges in solving this problem are to ensure spatial contiguity of each region, measure homogeneity of each region, explore solution space, and ways to check solution feasibility. Other constraints in solving this problem are the shape of the region, equality of an attribute value across the region, and boundary integrity. Due to these constraints, some different formulations and solution strategies have been contributed by researchers.

Availability of highly disaggregated spatial data and computational resources provides the opportunity for researchers to explore new applications of these clustering models. New opportunities bring new challenges and one of the challenges is ever-increasing data size. DiMOS lab working under the direction of Dr. Prasad has already laid out the roadmap [7], [8] on using GPUs for the parallel processing of geospatial datasets. There are many available algorithms to cluster the areas into homogeneous regions, however, most of them are not scalable. Another challenge is to decide on the number of homogeneous regions to be formed after clustering. Most of the available models require the number of regions as input. Also in order to decide on the best clustering model for a set of given areas, one needs to know about all the available algorithms. Mostly, researchers want to design regions for analysis rather than summarizing and finding the number of regions in the data. In this situation, the researcher may not know the number of regions to be designed, however they may know the condition which will make a region suitable for analysis. This information can be used as criteria to decide the number of regions.

In this paper, we use the clustering model explained in [2] as the max-p problem. The max-p region model is implemented in python, therefore, to have a benchmark for evaluation of our work, we have also used python for this research. Another reason to use python is that lot of researchers are familiar with python and use it on day to day basis for analysis.

The problem aggregates $n$ areas into an unknown maximum number of homogeneous regions, where each region satisfies a minimum threshold value for a given spatial attribute. The problem is formulated as an Integer Linear Programming (ILP) with a two-part objective function. The algorithm has two phases: first, it tries to form as many initial regions as possible (the first part of the objective function) by starting from random areas called seeds and grow them until each region reaches to the threshold. The remaining areas that do not form regions in the first step are called enclaves and are assigned to the regions. In the second phase, the algorithm tries to optimize the objective function among those solutions with the maximum number of regions (the second part of the objective function). This goal is achieved by exchanging areas in the border of two neighboring regions. Unlike other models, this model does not impose any constraint on the shape of the regions, instead, the number of regions ($p$) and shape of the regions are data dependent and are decided by the algorithm at runtime. The max-p model tries to maximize the number of homogeneous regions, based on a predefined minimum threshold value. The degree of aggregation bias is minimized by maximizing the homogeneity within the regions. In this paper, we exploit parallel processing to improve the performance of the max-p model. We also use parallel synergy to improve the accuracy of solutions by trying more solutions in parallel as well as exploiting a fine parallelism model in each solution. Multiple solutions are created simultaneously on multiple processors in parallel and then compared to choose the best feasible solution. Then all the regions of the best feasible solution are processed on multiple processors in parallel to optimize and increase homogeneity within them.

In summary, our key contributions in this work are:

- Parallel implementation of the construction phase of Max-p problem that is up to 200 times faster than the best sequential algorithm for the largest ($56 \times 55$) lattice size.
- Parallelizing the optimization phase of finding

the best feasible solution that is up to 19 times faster than the best sequential algorithm.
- End to end running time of our parallel implementation achieves 35-fold speed up over the best sequential algorithm for the largest $(56 \times 55)$ lattice size.
- Using a parallel data structure based on multiple locks to reduce parallelism overhead.
- By taking advantage of parallel synergy, we often improve the secondary objective function better then sequential algorithm by exploring the exponential search space in parallel.

The rest of the paper is organized as follows. In Section 2, we present literature review. In Section 3, we describe the problem statement. In Section 4, we discuss the algorithm used to solve the problem. In Section 5, we present experimental results and evaluate performance. In Section 6, we summarize the research work.

## II. Literature Review

In this section, we present a brief survey on various methods of clustering areas into set of homogeneous regions. Algorithm used in one of the methods [14], [15] to aggregate areas in to regions has two parts. First part of the algorithm aggregate areas into regions using conventional clustering algorithm. In this part, areas are clustered based on the attributes and not on the locations. In second part of the algorithm, regions are created as subsets of the spatially contiguous areas from the already created clusters. Number of regions created from this method depends on the attributes used to calculate homogeneity within regions [16].

Another method [17], [18] uses $x$ and $y$ coordinates of the centroids of areas as additional attributes in conventional clustering algorithm. Due to these additional attributes nearby areas will be clustered together. Regions will be spatially contiguous, however, contiguity of the areas within the region depends on the weights assigned to $x$ and $y$ coordinates as compared to other attributes [19]. Increase in weights of $x$ and $y$ coordinates will increase the spatial contiguity within the regions, however, it will lead to decrease in weights assigned to other attributes which in turn decreases the homogeneity within the regions. Main challenge of this method is to decide on how to assign weights to the attributes [17], [27], [38].

There are other methods [16], [18], [26], [28], [36] that ensure spatial contiguity based on the information from neighbouring structure. For example, adapted hierarchical clustering algorithms [23], [25], [40] allows to merge clusters only if they share common borders. Another example is graph theory based algorithms [19], [24], [35] which represents areas and neighbouring structures as connected graphs.

The choice of method to aggregate areas into homogeneous regions depends on the problem. For example, if you want the shape of region to be guided by the spatial distribution of variables, than clustering using $x$ and $y$ coordinate method is not appropriate as it generates circular regions. The method purposed in this article ensures contiguity by clustering only those areas into a region which shares a common border.

## III. Problem Statement

The main goal of max-p region problem is to cluster the areas into the maximum number of homogeneous regions such that each region satisfies a predefined minimum threshold value for a given spatially extensive attribute [2]. A spatially extensive attribute refers to a parameter that we want to be greater than a minimum threshold for each cluster. The second goal of this model is to achieve the most homogeneity within each cluster. In the other words, max-p aims to minimize total heterogeneity of all clusters. We will define this term later in this section. The way max-p prioritizes these two goals is that if two clustering configurations satisfy minimum threshold criteria, the model chooses the one with the larger number of regions (maximizing p). In case of a tie, the one with less total heterogeneity is selected.[1]

### A. Preliminary Notation

The following is the formulation of max-p problem [2].

[1]The problem statement and equations 3.1 through 3.4 are taken from [2]

*Area* :
Given n areas $A = \{A_1, A_2, ..., A_n\}$, $(n = |A|)$ and m attributes assigned to each area such that $A_{i,\ y}$ is the y-th attribute of area $A_i$ where $1 \leq y \leq m$. Also, let $l_i$ denote a parameter that we want to be greater than a minimum threshold of area $A_i$.

*Relationship* :
Let $d : A \times A \rightarrow R^+ \cup \{0\}$ be the dissimilarity between areas based on the set of attributes $Y$ such that $d_{ij} \equiv d(A_i,\ A_j)$ satisfies the conditions $d_{ij} \geq 0, d_{ij} = d_{ji}\ and\ d_{ii} = 0$ for $i,\ j = 1,\ 2,\ \ldots,\ n$. Distance functions can be utilized; i.e., $d_{ij}$ can also satify the subadditivity, or triangle inequality, condition: $d_{ij} \leq d_{ik} + d_{kj}$ for $i,\ j,\ k = 1,\ 2,\ \ldots,\ n$.
Let $W = (V,\ E)$ denote the contiguity graph associated with A such that vertices $v_i \in V$ correspond to areas $A_i \in A$ and edges $\{v_i, v_j\} \in E$ if and only if areas $A_i$ and $A_j$ share a common border.

*Feasible Partitions of A* :
Let $P_p = \{R_1,\ R_2,\ ...,\ R_p\}$ denotes a partition of areas $A$ into $p$ regions with $1 \leq p \leq n$ such that:

$$\begin{cases} |R_k| > 0\ for\ k = 1,\ 2, ...,\ p \\ R_k \cap R_{k'} = \theta\ for\ k,\ k' = 1,\ 2, ...,\ p \wedge k \neq k' \\ \cup_{k=1}^{p}\ R_k = A \\ \sum_{A_i \in R_k} l_i \geq threshold > 0 for\ k = 1, 2,\ ....,\ p \end{cases} \tag{1}$$

Let II denotes the set of all feasible partitions of $A$.

We define heterogeneity of a region $R_k$ as Equation 2:

$$h(R_k) = \sum_{ij: A_i, A_j \in R_k, i \leq j} d_{ij} \tag{2}$$

where $R_k \in P_p$. We also define total heterogeneity of partition $P_p$ as the sum of heterogeneity over all of its regions (Equation 3):

$$H(P_p) = \sum_{k=1}^{p} h(R_k) \tag{3}$$

Finally, the max-p problem can be formulated as finding $P_p^* \in II$ such that:

$$\begin{cases} |P_p^*| = max(|P_P^*| : P_p \in II) \\ \nexists P_p \in II : |P_p| = |P_p^*| \quad \text{AND} \quad H(P_p) < H(P_p^*) \end{cases} \tag{4}$$

Let us illustrate this with a basic example. Table 1 shows a regular lattice with $n = 12$ areas from $A_1$ to $A_{12}$. We consider two attributes: 1) y is the average income in an area and l is the number of people in that area where l represents the spatially extensive attribute. The primary objective is to find the maximum number of contiguous regions. We need to group these 12 areas in such a way that each region contains at least 100 people (i.e. threshold = 100). The secondary objective is to find the solution with the lowest heterogeneity based on attribute y. We used Manhattan distance for heterogeneity such that $d_{ij} = |y_i - y_j|$.

As shown in Tables I, there are three solutions for partitioning these areas where Solution 2 and 3 reach to the maximum number of regions. Therefore, Solution 1 is not a feasible solution as there are other solutions with more number of regions. Hence, Solution 2 and Solution 3 are the feasible solutions as we can see that it is not possible to have more than three regions with at least 100 people per region. Also, we can see that Solution 3 has heterogeneity lower than that of Solution 2. Therefore, Solution 3 is the best feasible solution for this example. All the regions of Solution 3 have more than 100 people each. 103 people in $R_{white}$, 115 people in $R_{green}$ and 105 people in $R_{blue}$.

## IV. METHODS

In this section, we present our algorithm to solve the problem of clustering areas into the maximum number of homogeneous regions. The algorithm has two phases, i.e. the construction phase and the optimization phase. The construction phase generates a feasible solution and optimization phase improves the feasible solution by minimizing the heterogeneity within the regions through local swapping of areas between neighbouring regions.[2]

[2]This two phase algorithm is adopted from [2]. We extend the python implementation from [2]

| | | |
|---|---|---|
| $A_1$<br>$y_1 = 3000$,<br>$l_1 = 23$ | $A_2$<br>$y_2 = 3200$,<br>$l_2 = 27$ | $A_3$<br>$y_3 = 3300$,<br>$l_3 = 33$ |
| $A_4$<br>$y_4 = 5200$,<br>$l_4 = 21$ | $A_5$<br>$y_5 = 5000$,<br>$l_5 = 20$ | $A_6$<br>$y_6 = 3100$,<br>$l_6 = 20$ |
| $A_7$<br>$y_7 = 5300$,<br>$l_7 = 24$ | $A_8$<br>$y_8 = 5400$,<br>$l_8 = 25$ | $A_9$<br>$y_9 = 5100$,<br>$l_9 = 25$ |
| $A_{10}$<br>$y_{10} = 7500$,<br>$l_{10} = 35$ | $A_{11}$<br>$y_{11} = 7100$,<br>$l_{11} = 30$ | $A_{12}$<br>$y_{12} = 7300$,<br>$l_{12} = 40$ |

(a) Input: $n = 12$, $m = 2$

| | | |
|---|---|---|
| $A_1$<br>$y_1 = 3000$,<br>$l_1 = 23$ | $A_2$<br>$y_2 = 3200$,<br>$l_2 = 27$ | $A_3$<br>$y_3 = 3300$,<br>$l_3 = 33$ |
| $A_4$<br>$y_4 = 5200$,<br>$l_4 = 21$ | $A_5$<br>$y_5 = 5000$,<br>$l_5 = 20$ | $A_6$<br>$y_6 = 3100$,<br>$l_6 = 20$ |
| $A_7$<br>$y_7 = 5300$,<br>$l_7 = 24$ | $A_8$<br>$y_8 = 5400$,<br>$l_8 = 25$ | $A_9$<br>$y_9 = 5100$,<br>$l_9 = 25$ |
| $A_{10}$<br>$y_{10} = 7500$,<br>$l_{10} = 35$ | $A_{11}$<br>$y_{11} = 7100$,<br>$l_{11} = 30$ | $A_{12}$<br>$y_{12} = 7300$<br>$l_{12} = 40$ |

(b) Solution 1: $p = 2$, $H = 36,500$

| | | |
|---|---|---|
| $A_1$<br>$y_1 = 3000$,<br>$l_1 = 23$ | $A_2$<br>$y_2 = 3200$,<br>$l_2 = 27$ | $A_3$<br>$y_3 = 3300$,<br>$l_3 = 33$ |
| $A_4$<br>$y_4 = 5200$,<br>$l_4 = 21$ | $A_5$<br>$y_5 = 5000$,<br>$l_5 = 20$ | $A_6$<br>$y_6 = 3100$,<br>$l_6 = 20$ |
| $A_7$<br>$y_7 = 5300$,<br>$l_7 = 24$ | $A_8$<br>$y_8 = 5400$,<br>$l_8 = 25$ | $A_9$<br>$y_9 = 5100$,<br>$l_9 = 25$ |
| $A_{10}$<br>$y_{10} = 7500$,<br>$l_{10} = 35$ | $A_{11}$<br>$y_{11} = 7100$,<br>$l_{11} = 30$ | $A_{12}$<br>$y_{12} = 7300$,<br>$l_{12} = 40$ |

(c) Solution 2: $p = 3$, $H = 40,100$

| | | |
|---|---|---|
| $A_1$<br>$y_1 = 3000$,<br>$l_1 = 23$ | $A_2$<br>$y_2 = 3200$,<br>$l_2 = 27$ | $A_3$<br>$y_3 = 3300$,<br>$l_3 = 33$ |
| $A_4$<br>$y_4 = 5200$,<br>$l_4 = 21$ | $A_5$<br>$y_5 = 5000$,<br>$l_5 = 20$ | $A_6$<br>$y_6 = 3100$,<br>$l_6 = 20$ |
| $A_7$<br>$y_7 = 5300$,<br>$l_7 = 24$ | $A_8$<br>$y_8 = 5400$,<br>$l_8 = 25$ | $A_9$<br>$y_9 = 5100$,<br>$l_9 = 25$ |
| $A_{10}$<br>$y_{10} = 7500$,<br>$l_{10} = 35$ | $A_{11}$<br>$y_{11} = 7100$,<br>$l_{11} = 30$ | $A_{12}$<br>$y_{12} = 7300$,<br>$l_{12} = 40$ |

(d) Solution 3: $p = 3$, $H = 3,800$

TABLE I: An example of max-p problem, (a) input areas, (b) a solution with $p = 2$, (c) a feasible solution with $p = 3$, (d) another feasible solution with $p = 3$. Solution 3 is chosen because it has lowest total heterogeneity among partitions with maximum number of regions ($p = 3$).

### A. *Construction Phase*

In this phase, areas are clustered into the maximum number of regions based on a minimum threshold value of the spatially extensive attribute. An initial feasible solution is generated at the end of this phase. We repeat this algorithm multiple times with random seeds (100 times for this experiment) to calculate the set of feasible solutions. We run this algorithm on multiple processors simultaneously for each instance to improve the running time of our algorithm. Once all the processes finish successfully, we compare the output of each process and select the best feasible solution. We use the objective function explained in [2] to decide on which solution is the best out of all the solutions.

Objective function is a minimization function which can be formulated as follows [2].

Parameters:
$i$ = index of areas,
$k$ = index of regions,
$c$ = index of contiguity order (areas are ordered such that areas next to each other share border),
$d_{ij}$ = dissimilarity relationship between areas $i$ and $j$ with $i < j$,
$[H]$ = Total heterogeneity

$$[H]t_{ij} = \begin{cases} 1, \ if \ area \ i \ and \ j \ belong \ to \ same \ region \ k \\ 0, \ otherwise; \end{cases}$$

$$[H]x_i^{kc} = \begin{cases} 1, \ if \ A_i \ is \ assigned \ to \ region \ R_k \ in \ order \ c \\ 0, \ otherwise; \end{cases}$$

Minimize:

$$[H]Z = (-\sum_{k=1}^{n}\sum_{i=1}^{n} x_i^{k0}) * 10^h + \sum_i \sum_{j|j>i} d_{ij}t_{ij}$$

The objective function consists of two parts, one part controls the number of regions and second part controls the total heterogeneity $H(P_p)$. The first part is obtained by adding the number of areas designated as root areas ($X_i^{k0}$), and the second part is obtained by adding the pairwise dissimilarities between areas assigned to the same region. Since

the objective function is formulated as a minimization problem, we multiply the first term by minus one.

These two parts are combined together in such a way that there is an implicit hierarchy where the number of regions comes before the impact of total heterogeneity. We achieve this hierarchy by multiplying the first term by a scaling factor $h = 1 + log(\sum_i \sum_{j|j>i} d_{ij})$. For $p$ regions the objective functions starts at $p * 10^h$. This value increases when we add the total heterogeneity, but $h$ is big enough such that, regardless of the value of heterogeneity, the objective function will never reach $-(p-1) * 10^h$.

The Construction phase has two different parts: region growing and enclave assignment.

During region growing a random area is selected from the set of unassigned areas. Then, this area is added to the region. We check if this newly formed region reaches the minimum threshold value of the spatial regional attribute. We stop growing the new region if the minimum threshold value is met, otherwise, we keep adding unassigned neighbouring areas to the region until it reaches the minimum threshold value. This is repeated until it is not possible to grow any new region that satisfies the minimum threshold value. Areas which are not assigned to any region during growing part are called "enclaves". At the end of growing part, the algorithm generates set of growing regions and the enclaves are randomly placed in an enclave queue.

A partial solution generated from growing subphase will be processed further by an enclave assignment step where we pop the enclave queue and attempt to assign the enclave to a randomly selected neighbouring region. If the selected enclave is not yet contiguous to an existing region, it is returned to the end of the enclave queue. The process continues until the enclave queue is exhausted. The algorithm thus requires that the adjacency graph formed on the areas is connected.

We have tried two different algorithm for construction phase. For the first algorithm, we generate multiple solutions on multiple processes without changing the existing algorithm to generate one solution at a time. For the second algorithm, along

**Algorithm 1** : Find best feasible solution [2]

$A$ : set of areas,
$l$ : spatially extensive attribute of areas,
$W$ : neighborhoods,
$t$ : threshold constraint on attribute $l$ at region level,
$\Psi$ : set of partitions before enclave assignment,
$\epsilon$ : set of enclave areas,
$A^u$ : set of unassigned areas,
$ns$ : Number of solutions

**for** each processor;i = 1 to $ns$ **do in parallel**
  **while** $A^u \neq empty$ **do**
    $A_k$ = select at random, one area from $A^u$
    $A^u = A^u - A_k$
    $R_k = \{A_k\}$, randomly selected area is assigned to region
    building_region = True
    $T = l_k$, value of attribute $l$ in region $R_k$
    **while** building_region = True **do**
      **if** $T \geq t$ **then**
        $\Psi = \Psi \cup \{R_k\}$, add region $R_k$ to partition $\Psi$
      **else**
        $N = neighbors \subset A^u$
        **if** $N \neq empty$ **then**
          $A_i$ = random area from $N$
          $A^u = A^u - A_i$
          $R_k = R_k \cup A_i$
          $T = T + l_i$
        **else**
          $\epsilon \cup \{R_k\}$
          building_region = False
        **end if**
      **end if**
    **end while**
  **end while**
  **if** $\Psi \neq empty$ **then**
    feasible = True
    **while** $\epsilon \neq empty$ **do**
      $\epsilon_i$ = random area from $\epsilon$
      $\eta$ = region $\eta \subset \Psi$, that shares border with $\epsilon_i$
      **if** $\eta \neq empty$ **then**
        $R_k$ = random region $\subset \eta$
        $R_k^new = R_k \cup \epsilon_i$
        $\Psi = \Psi - \{R_k\} \cup R_k^new$, update region $R_k in \Psi$
        $\epsilon = \epsilon - \epsilon_i$
      **else**
        feasible = False
      **end if**
    **end while**
  **else**
    feasible = False
  **end if**
**end for**
$P_{feasible} = \Psi$, at this point all the areas have been assigned to a region
return $P_{feasible}$

**Algorithm 2** : Find best feasible solution [2]

$A$ : set of areas,
$l$ : spatially extensive attribute of areas,
$W$ : neighborhoods,
$t$ : threshold constraint on attribute $l$ at region level,
$T$ : threshold value $\Psi$ : set of partitions before enclave assignment,
$\epsilon$ : set of enclave areas,
$A^u$ : set of unassigned areas,
$ns$ : Number of solutions

**for** each processor;i = 1 to $ns$ **do in parallel**
  **while** $A^u \neq empty$ **do**
    $A_k$ = select at random, one area from $A^u$
    $A^u = A^u - A_k$
    $R_k = \{A_k\}$, randomly selected area is assigned to region
    building_region = True
    $T = l_k$, value of attribute $l$ in region $R_k$
    **while** building_region = True **do**
      **if** $T \geq t$ **then**
        $\Psi = \Psi \cup \{R_k\}$, add region $R_k$ to partition $\Psi$
      **else**
        $N = neighbors \subset A^u$
        **if** $N \neq empty$ **then**
**if** $T - t_{R_k} \geq t_N$ **then**
            $R_k = R_k \cup N$, add all the neighbours to the region
            $A^u = A^u - N$
**else**
            $A = t_N - (T - t_{R_k})$, select only those many areas which will make region reach minimum threshold value
            $A^u = A^u - A$
**end if**
          $T = T + l_i$
        **else**
          $\epsilon \cup \{R_k\}$
          building_region = False
        **end if**
      **end if**
    **end while**
  **end while**
  **if** $\Psi \neq empty$ **then**
    feasible = True
    **while** $\epsilon \neq empty$ **do**
      $\epsilon_i$ = random area from $\epsilon$
      $\eta$ = region $\eta \subset \Psi$, that shares border with $\epsilon_i$
      **if** $\eta \neq empty$ **then**
        $R_k$ = random region $\subset \eta$
        $R_k^new = R_k \cup \epsilon_i$
        $\Psi = \Psi - \{R_k\} \cup R_k^new$, update region $R_k in \Psi$
        $\epsilon = \epsilon - \epsilon_i$
      **else**
        feasible = False
      **end if**
    **end while**
  **else**
    feasible = False
  **end if**
**end for**
$P_{feasible} = \Psi$, at this point all the areas have been assigned to a region
return $P_{feasible}$

with generating multiple solutions on multiple processes, we have also changed the growing part of the algorithm to add multiple neighbours at a time instead of one at a time. Adding multiple areas at a time, helped us to improve the running time of the construction phase of the algorithm.

### *B. Optimization Phase*

In this phase, the best feasible solution from construction phase is optimized by minimizing heterogeneity within the regions. We use multiple processors to perform this task in parallel. Each processor selects a region and all its neighbours which are eligible to move. Neighbour areas which can be removed from the region while the region still satisfying minimum threshold value and contiguity constraint are eligible to move. Each processor acquires two locks, one on the region where an eligible area is being added and one in the region from where an eligible area is being moved. Then we move the area which minimizes the value of objective function. Once the movement is complete, then both the locks are released and another processor that is ready to move the area from one region to another can acquire the lock and make the change. This process is repeated for all the regions in parallel until there is no such neighbour area which when is moved will further minimize the value of objective function.

First, we find the neighbours of all the areas in a region. After that, we check for each neighbour that if we remove it from the current region, whether or not the region can maintain its contiguity and also satisfies the minimum threshold criteria. All the neighbour areas which can be removed from the region while the region still satisfying minimum threshold value and contiguity constraint are eligible to move.

For each eligible area, we try to move it from its current region to the region assigned to a particular processor. After that, we calculate the objective function value for the new solution and compare it with an already available solution to check if it is smaller than the current value. We take the area which gives us the minimum value of objective

**Algorithm 3** : Optimize $P_{feasible}$ to minimize heterogeneity [2]

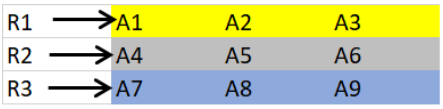
$A$ : set of areas,
$l$ : spatially extensive attribute of areas,
$t$ : threshold constraint on attribute $l$ at region level,
$\Psi$ : set of partitions,
$R$ : set of regions in $\Psi$,
$p$ : number of regions

```
swapping = True
while swapping = True do
    moves = 0
for each processor: i = 1 to p do in parallel
        R_k = random region in Ψ
        N_Bk = neighbors of all the areas of region R_k
        for N_i in N_Bk do
            R_m = region that contains neighbor area N_i
            if l_m ≥ t and doesn't break contiguity then
                N_e = N_e ∪ N_i, set of areas which are eligible for
                movement from region R_m to R_k
            end if
        end for
        if N_e ≠ empty then
            N_ei = Area from N_e that minimizes the objective function
            when moved from region R_m to R_k
            if N_ei ≠ empty then
acquire lock on R_k and R_m
                R_k = R_k ∪ N_ei
                R_m = R_m − N_ei
                move += 1
release lock on R_k and R_m
            end if
        end if
    end for
    if moves = 0 then
        swapping = False
    end if
end while
P_p = Optimized solution with minimum heterogeneity
return P_p
```

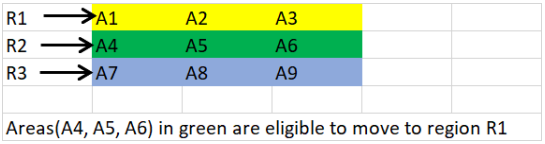
function.

Once we find the area which minimizes the objective function value, then we acquire the lock on both the regions from which area is being moved and also the region to which area is being moved. We acquire the two locks based on the index of the regions. We acquire the lock first on the region with smaller index and then on the region with the larger index. This is done to avoid the race condition. Once the area is moved, then we release the locks on both the regions. Multiple locks are used to reduce the lock overhead.

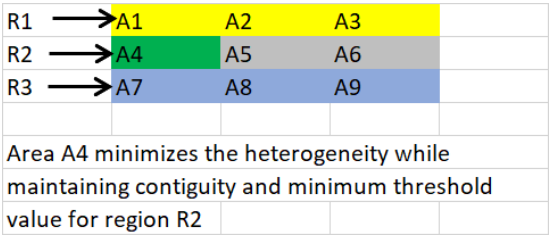
Let us explain the algorithm with the help of an example. Assume we have the feasible solutions($P_p$) with three regions.



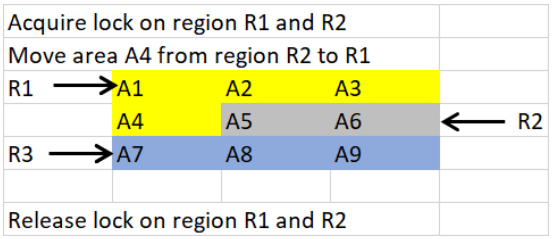

Region $R_1$, $R_2$ and $R_3$ are sent to processors $P_1$, $P_2$, $P_3$. $P_p$ is a shared memory list which can be updated by all the processors. Now on processor $P_1$, first we will find all the neighbour areas of $R_1$, which are eligible to move from their current region. Assume, $N_{R_1}(A_4, A_5, A_6)$ is the list of neighbour areas which are eligible to move from their current region.




Now, we will find the area from $N_{R_1}$, which then moving to $R_1$, will further minimize the objective function. Assume, $A_4$ is the area which minimizes the current value of objective function.




Processor $P_1$ will acquire lock on region $R_1$ and $R_2$ and move the area $A_4$ from region $R_2$ to $R_1$. Once, the movement is complete, $P_1$ will release the lock on region $R_1$ and $R_2$. Now, other processors can acquire the lock and update the list. This is repeated on all the processors concurrently.




3

## V. Experimental Results

## VI. Performance Evaluation

In this section first, we briefly explain the system we used for our experiments and then presents our results.

### A. Experimental Setup

All the experiments employ a compute node that has 16-core Intel Xeon CPUE5-2650 V2 CPU running at a clock speed of 2.60 Ghz with 64GB main memory. All the code for this experiment is written in the Python programming language. The $multiprocessing$ library available in Python is used to connect to multiple processors. Also, we have used a compute node that has 64-core AMD Opteron(TM) Processor 6272 with 64GB main memory for comparison against 16-core Intel.

### B. Experiments and Results

In this section, we evaluate the performance of our algorithm. We have created regular lattices using $numpy$ library available in python. Three different sets of lattices ($20 \times 20$, $33 \times 33$, $55 \times 56$) are created for the evaluation. We considered two areas to be neighbours if and only if they share a line (rook). We have also used three different thresholds for spatially extensive attribute i.e. 25, 100 and 300. We have done several experiments to evaluate the performance parallel max-p region algorithm. All the experiments are done to explore either the parallel efficiency or the parallel synergy or both in the exponential space. Parallel efficiency means to reduce running time of the algorithm by using parallel processing. Parallel synergy means exploring exponential space by using parallel processing and trying to find the best feasible solution which gives us the minimum value of objective function. Our experiments show that we can achieve both efficiency and synergy by using multiple processes. We have divided the results section further into two parts, Evaluating the efficiency and Exploring synergy.

*1) Parallel Efficiency::* We did several experiments to evaluate the performance of our algorithm in terms of running time efficiency.

*Average running time comparison of parallel and sequential max-p region algorithm::* First, we have compared the running time of our algorithm with that of sequential max-p implementation presented in [2]. Table II presents the average running time in seconds for different sets of lattices and different threshold values for both the sequential and the parallel max-p algorithm. All the timings are taken on Intel 16-core system by using 64 processes for the construction phase and for the optimization phase we have used the number of regions as the number of processes. As shown in table II, the speed-up is increasing with lattice size, because smaller lattices form fewer regions. As a result, we are not able to take full advantage of multiprocessing capabilities of the system for small problems. We observed that running time for the optimization phase of a 20x20 lattice for a threshold value of 300 is almost 0, because only one region will be created for $n = 400$ and threshold 300, hence, there will be no optimization phase.

*Average running time comparison of the construction phase of parallel max-p region on 16-core Intel processor to that of 64-core AMD Opteron processor::* We have also compared the running time of the construction phase of parallel max-p region on 16-core Intel processor to that of 64-core AMD Opteron processor. This comparison is done for two lattices 20x20 and 55x56 with a threshold of 100 and 300 by using the different number of processes to find best feasible solution i.e. 8, 16, 32, 64, 128, 256, 512. Figure 1 presents the

[3]These codes can be downloaded the following link: https://github.com/vineysindhu/parallel-max-p-region. Also the code for sequential max-p region is available at the following link: https://github.com/pysal

| Algorithm | Phase | $20 \times 20$ | | | $33 \times 33$ | | | $56 \times 56$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | th=25 | th=100 | th=300 | th=25 | th=100 | th=300 | th=25 | th=100 | th=300 |
| Sequential Max-p | Construction | 10.57 | 56.01 | 154.99 | 64.04 | 395.80 | 1180.36 | 472.89 | 3117.80 | 9814.39 |
| | Optimization | 3.33 | 23.28 | 0.01 | 16.46 | 101.64 | 774.13 | 120.44 | 1573.83 | 6222.92 |
| | Total | 13.90 | 79.29 | 155 | 80.50 | 497.44 | 1954.49 | 593.33 | 4691.63 | 16037.31 |
| Parallel Max-p | Construction | 1.76 | 5.68 | 14.86 | 9.29 | 35.18 | 104.98 | 67.60 | 271.32 | 829.53 |
| | Optimization | 2.43 | 13.25 | 0.16 | 8.61 | 27.04 | 83.36 | 29.05 | 109.86 | 320.85 |
| | Total | 4.19 | 18.93 | 15.02 | 17.90 | 62.22 | 188.34 | 96.65 | 381.18 | 1150.38 |
| | Speedup | 3.31 | 4.19 | 10.31 | 4.50 | 7.99 | 10.38 | 6.14 | 12.30 | 13.94 |
| New Parallel Max-p | Construction | 0.54 | 0.87 | 1.06 | 2.12 | 3.84 | 6.41 | 13.55 | 25.80 | 48.88 |
| | Optimization | 3.61 | 10.85 | 0.17 | 10.64 | 23.98 | 69.91 | 38.24 | 76.70 | 404.20 |
| | Total | 4.15 | 11.72 | 1.23 | 12.76 | 27.82 | 76.32 | 51.79 | 102.50 | 453.08 |
| | Speedup | 3.35 | 6.76 | 126.02 | 6.31 | 17.88 | 25.61 | 11.46 | 45.77 | 35.40 |
| New Sequential Max-p | Construction | 2.21 | 4.31 | 5.95 | 12.80 | 24.39 | 40.94 | 89.39 | 183.14 | 301.93 |
| | Optimization | 3.26 | 15.37 | 0.01 | 13.59 | 93.27 | 412.92 | 94.38 | 754.04 | 3547.81 |
| | Total | 5.47 | 19.68 | 5.96 | 26.39 | 117.66 | 453.86 | 183.77 | 937.18 | 3849.74 |
| | Speedup | 2.54 | 4.03 | 26.01 | 3.05 | 4.23 | 4.31 | 3.23 | 5.01 | 4.17 |

TABLE II: Average running time comparison for three different thresholds(th = Threshold)

charts which show that running time of algorithm decreases with increase in the number of processes up to a certain limit and after that, it starts to increase again. It can be interpreted from the graph that for 16-core Interprocessor threshold to decrease running time is 64 and for 64-core AMD Opteron threshold is 128. Once this threshold is reached, running starts to increase again. Hence, it is clear from the results that we should use 64 processes for 16-core Intel system and 128 processes for 64-core AMD system to get the best running time. We also analysed the CPU usage for each process for the different number of processes. For 16-core Intel system, each process can utilize 100 percent of CPU capacity for up to 16 processes and after that, it decreases to 50 percent for 32 processes. It becomes stable at 17 percent after 128 processes. For 64-core AMD system, each process can utilize 100 percent of CPU capacity for up to 64 processes and starts decreasing after that. It becomes stable at 70 percent after 128 processes.

*Single lock vs multiple locks average running time comparison of the optimization phase::* We started by using single lock on the list of regions in optimization phase. This lead to a lock overhead problem. In order to reduce the lock overhead, we have used multiple locks i.e. one lock for each region. After using multiple locks, we can reduce lock overhead time. Table III presents the running time of the optimization phase for different lattices and different thresholds. It can be seen that after using multiple locks, the running time of the algorithm is reduced.

*Relation between number of areas moved and decrease in objective function value::* We have also analysed the effect of movement of areas from one region to another on objective function value. The decrease in objective function value is directly proportional to the movement of areas from one region to another in each iteration of optimization phase. The more areas move from one region to another, the more the decrease in objective function value. Table IV presents the decrease in objective function value with the decrease in the number of areas moved from one region to another. There are few outliers also, for example in iteration 3 number of areas moved decreased, however, decrease in objective function value increased.

Since we are doing the filtering of the list of areas and regions multiple times in our code, we decided to run a small experiment to explore various methods to filter the list and see which one gives the best performance. We tried the below methods and found that list compression method is the fastest of all the methods to filter the python lists.

Running time using set(a) - set(b) : 2.86102294921875e-06 seconds
Code - list(set(neighbors)-set(enclaves))

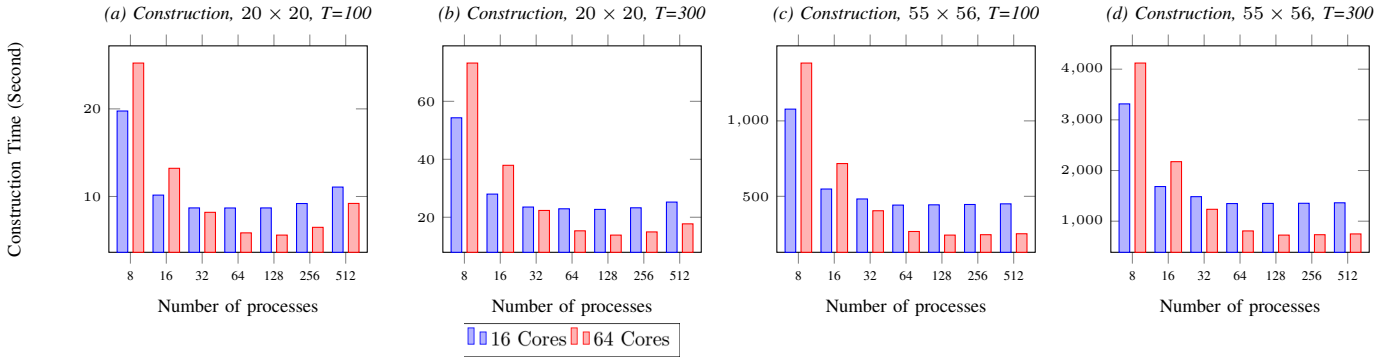


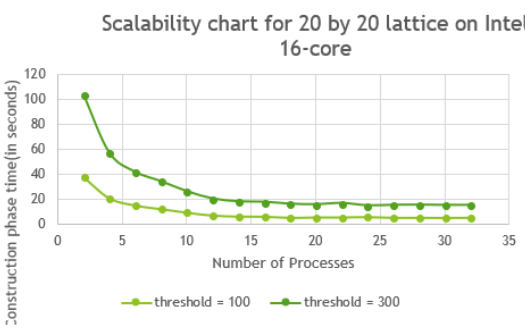

Fig. 1: Average running time of construction phase on 16-core Intel vs 64-core AMD Opteron

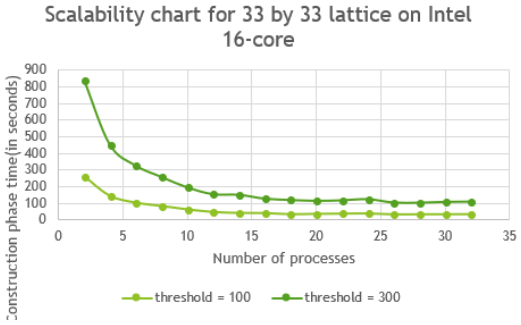
| Lattice | threshold = 100 | | threshold = 300 | |
|---|---|---|---|---|
| | Single Lock | Multiple Locks | Single Lock | Multiple Locks |
| 20x20 (n = 400) | 17.47 | 13.25 | 0.29 | 0.16 |
| 33x33 (n = 1,056) | 41.53 | 27.04 | 133.76 | 83.36 |
| 55x56 (n = 3,080) | 156.47 | 109.86 | 639.63 | 320.85 |

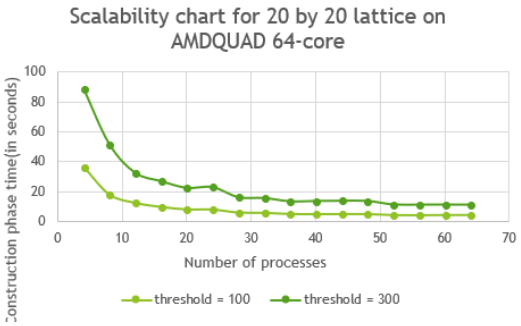
TABLE III: Single lock vs multiple locks average running time for optimization phase in seconds

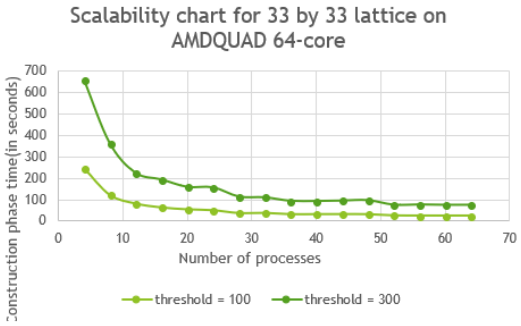
| Iteration | Number of areas moved | Decrease in Objective Function Value | Percentage Decrease |
|---|---|---|---|
| 1 | 12 | 1.59 | 0.026 |
| 2 | 12 | 1.58 | 0.026 |
| 3 | 11 | 1.62 | 0.028 |
| 4 | 11 | 1.14 | 0.020 |
| 5 | 10 | 0.94 | 0.017 |
| 6 | 11 | 1.38 | 0.025 |
| 7 | 10 | 0.63 | 0.012 |
| 8 | 7 | 0.29 | 0.005 |
| 9 | 6 | 0.54 | 0.010 |
| 10 | 4 | 0.15 | 0.003 |
| 11 | 2 | 0.19 | 0.004 |
| 12 | 4 | 0.25 | 0.005 |
| 13 | 4 | 0.09 | 0.002 |
| 14 | 1 | 0.004 | 0.000 |
| 15 | 1 | 0.004 | 0.000 |
| 16 | 0 | 0 | 0.000 |

TABLE IV: Decrease in objective function value with the decrease in areas moved in optimization phase for 20x20 lattice with threshold = 25




Running time using list compression : 2.384185791015625e-06 seconds
Code - [neighbor for neighbor in neighbors if neighbor not in enclaves]

Running time using One of the lists as set : 3.337860107421875e-06 seconds
Code - [neighbor for neighbor in neighbors if neighbor not in set(enclaves)]

Running time using lambda function : 5.0067901611328125e-06 seconds
Code - list(filter(lambda x: x not in set(enclaves), neighbors))

*2) Exploring synergy or accuracy of the parallel max-p region algorithm::* We did the following experiments to explore the synergy of the parallel max-p region algorithm.

*Analysis of top solutions after the construction phase::* First, we analysed the top feasible solutions that we get after construction phase of the algorithm. Table V, VI and VII presents the analysis for 20x20, 33x33 and 55x56 lattices with threshold 25 respectively. All the solutions present in the tables give the maximum number of regions for their respective lattice size. We can see from the table V that our algorithm will select the solution with minimum objective function value after construction phase and tries to optimize it. Hence, Solution 1 will be chosen for optimization. However, when we analysed other top solutions, we found that Solution 5, does not have minimum objective function value after construction phase, but it gives us best results after optimization phase. Similarly, in the table VI, Solution 3 gives the best results after optimization, however, it will not be selected at first by the algorithm. Also, in the table VII, Solution 5 gives the best results.

*Comparison of change in objective function value of sequential and parallel max-p region algorithm::* We have also analysed the change in objective function value during optimization phase and its relation to the number of areas moved from one region to another. Table VIII presents the change in objective function value and the number of areas moved in optimization phase for both sequential max-p and parallel max-p algorithm. As shown in table VIII, the decrease in objective function value is directly

| Solutions | Objective function value after construction phase | Objective function value after optimization phase |
|---|---|---|
| Solution 1 | 63.13 | 54.28 |
| Solution 2 | 63.92 | 54.14 |
| Solution 3 | 63.27 | 54.89 |
| Solution 4 | 63.60 | 55.93 |
| Solution 5 | 63.32 | 51.49 |

TABLE V: Exploring synergy in top solutions for 20x20 lattice with threshold = 25

| Solutions | Objective function value after construction phase | Objective function value after optimization phase |
|---|---|---|
| Solution 1 | 173.94 | 148.38 |
| Solution 2 | 173.72 | 148.36 |
| Solution 3 | 174.49 | 145.25 |
| Solution 4 | 174.39 | 151.35 |

TABLE VI: Exploring synergy in top solutions for 33x33 lattice with threshold = 25

| Solutions | Objective function value after construction phase | Objective function value after optimization phase |
|---|---|---|
| Solution 1 | 498.15 | 410.82 |
| Solution 2 | 493.82 | 411.39 |
| Solution 3 | 497.09 | 413.88 |
| Solution 4 | 497.88 | 415.94 |
| Solution 5 | 497.51 | 408.86 |

TABLE VII: Exploring synergy in top solutions for 55x56 lattice with threshold = 25

proportional to the number of areas moved from one region to another. The parallel max-p algorithm can perform similarly to the sequential max-p algorithm in terms of decrease in the objective function value. However, there are few outliers where sequential max-p algorithm performs much better than the parallel max-p.

*Analysis of objective function value by growing regions in parallel for the construction phase::* We grew regions in parallel to explore synergy in the exponential space. Usually we create 100 solutions and each solution in parallel, however, in each solution, the regions are grown in a single process. Here, we created 20 solutions and within each solution, we grew the regions in parallel. We assigned a total of 60 processes, which means 3 processes to each solution. Each solution will use these processes and try to grow maximum possible regions based on lattice size and threshold. For example, for $20 \times 20$ lattice and threshold of 25, maximum possible regions are 16. These 3 processes will try to grow 16 regions in parallel by seeding from 3 different areas at a time. If any of the regions is not able to reach the minimum threshold value, all the areas of that region will be added to enclaves. Since we did it in parallel, we were able to explore the space in parallel and get the results similar to that we got by generating 100 solutions. For example, we got the maximum number of regions as 13 and objective function value as $51.52$ for $20 \times 20$ lattice with the threshold of 25 by generating 20 solutions in parallel. Hence, by exploring space in parallel, we can get synergy with less number of solutions as compared to exploring it in sequence.

| Algorithm | Phase | $20 \times 20$ | | | $33 \times 33$ | | | $56 \times 56$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | th=25 | th=100 | th=300 | th=25 | th=100 | th=300 | th=25 | th=100 | th=300 |
| Sequential Max-p | Initial | 61.99 | 64.10 | 64.82 | 172.68 | 177.21 | 178.41 | 495.41 | 507.28 | 510.42 |
| | Final | 50.64 | 58.21 | 64.82 | 141.00 | 160.73 | 165.95 | 409.91 | 439.13 | 461.91 |
| | Decrease | 11.35 | 5.89 | 0.00 | 31.68 | 16.48 | 12.46 | 85.50 | 68.15 | 48.51 |
| | Areas Moved | 130 | 79 | 0 | 388 | 245 | 219 | 1069 | 1022 | 745 |
| Parallel Max-p | Initial | 62.05 | 63.78 | 64.82 | 172.71 | 177.40 | 187.34 | 494.98 | 508.21 | 511.75 |
| | Final | 51.15 | 57.41 | 64.82 | 145.25 | 157.44 | 168.55 | 408.84 | 452.52 | 469.81 |
| | Decrease | 10.90 | 6.37 | 0.00 | 26.96 | 19.96 | 9.79 | 86.14 | 55.69 | 41.94 |
| | Areas Moved | 115 | 86 | 0 | 269 | 263 | 151 | 917 | 721 | 702 |
| New Parallel Max-p | Initial | 62.05 | 64.13 | 64.82 | 172.57 | 177.07 | 178.38 | 493.09 | 507.36 | 511.76 |
| | Final | 51.82 | 59.57 | 64.82 | 142.53 | 159.55 | 173.44 | 405.21 | 460.42 | 495.45 |
| | Decrease | 10.23 | 4.56 | 0.00 | 30.04 | 17.52 | 4.94 | 87.88 | 46.94 | 16.31 |
| | Areas Moved | 99 | 79 | 0 | 301 | 226 | 102 | 968 | 651 | 324 |
| New Sequential Max-p | Initial | 63.48 | 64.22 | 64.82 | 172.62 | 177.00 | 179.10 | 492.28 | 507.85 | 512.29 |
| | Final | 50.54 | 58.83 | 64.82 | 143.69 | 151.79 | 173.96 | 405.46 | 441.85 | 483.24 |
| | Decrease | 12.94 | 5.39 | 0.00 | 28.93 | 25.21 | 5.14 | 86.82 | 66.00 | 29.05 |
| | Areas Moved | 147 | 90 | 0 | 357 | 299 | 133 | 1075 | 1050 | 652 |

TABLE VIII: Change in objective function value and number of areas moved in optimization phase for three different thresholds(th = Threshold)

## VII. Conclusion

In this paper, we present a parallel implementation of a constrained clustering problem, where our aim is to cluster areas into a set of the maximum number of homogeneous regions based on a minimum threshold value of a spatially extensive attribute and do so in a computationally efficient manner. We propose a heuristic solution to this problem which can help us achieve our goal and also aide in minimizing aggregation bias. This algorithm can help us to solve many real-life problems. For example, police districting needs headquarters to be allocated in all the territories. Hence areas can be aggregated into regions such that regions are homogeneous in terms of crime types, and each region contains a minimum potential to handle emergency calls. Once the regions are designed, we can decide on the best location for headquarters.

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