Assignment 2

Due Date: September 22, 2025

Question 1 - Lagrange Interpolation and Error Analysis

Let $f \in C^{n+1}[a,\ b]$, and let P_n be its Lagrange interpolating polynomial at the distinct nodes $x_0, x_1, \ldots, x_n \in [a,\ b]$. The interpolation error is:

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \quad \xi \in (a,\ b)$$

Suppose $f(x) = e^x$ on the interval $[0,\ 1]$, for equally spaced nodes $x_i = \frac{i}{n}$.

Question 1.1

Derive an explicit bound for the maximum error $\max_{x \in [0,1]} |R_n(x)|$.

Proof. To bound $|R_n(x)|$, we start with the error formula:

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right|, \quad \xi \in (0, 1)$$

Note $\frac{1}{(n-1)!}$ is a constant. We can bound $R_n(x)$ by bounding $f^{(n+1)}(\xi)$ and $\left|\prod_{i=0}^{n}(x - x_i)\right|$ separately.

Now notice that $f^{(n+1)}(x) = e^x$ for all n. Since e^x is strictly increasing on $[0, 1]$, we have that

$$f^{(n+1)}(\xi) = \max_{x \in [0,1]} |f^{(n+1)}(x)| = f(1) = e \tag{1}$$

Next, we need to bound $\left|\prod_{i=0}^{n}(x - x_i)\right|$. Let $\omega_n(x) = \prod_{i=0}^{n}(x - x_i)$. Note that $\omega_n(x)$ is a polynomial of degree $n + 1$. Since $\omega_n(x)$ is continuous on the compact interval $[0, 1]$, it attains its maximum at some point in $[0, 1]$. We may write that:

$$|\omega_n(x)| \leq W_n \tag{2}$$

Where $W_n = \max_{x \in [0,1]} |\omega_n(x)| = |\omega_n(x^*)|$ for some $x^* \in [0, 1]$.

Now, we can use (1) and (2) to bound $|R_n(x)|$ from above:

$$|R_n(x)| \leq \frac{e}{(n+1)!} W_n$$

Since $R_n(x)$ is continuous on the compact interval $[0, 1]$, $\sup_{x \in [0,1]} R_n(x) = \max_{x \in [0,1]} R_n(x)$. Since the supremum is the least upper bound, we have that:

$$\max_{x \in [0,1]} |R_n(x)| \leq \frac{e}{(n+1)!} W_n$$

Which gives us an explicit bound for the remainder. $\square$

Question 1.2

Show the asymptotic decay of this error as $n \to \infty$.