Numerical Analysis Homework 3

Due day: Oct 27, 2025

Question 1. Hermite interpolation Given the function

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1, 1].$$

with the function values and $1st$ derivatives at nodes:

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1.$$

1. Construct the cubic Hermite interpolant $H(x)$ and evaluate $H(0.5)$.

2. Evaluate the absolution error of $f(0.5) - H(0.5)$, and show that the error agrees the prior error estimate of

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2, \quad \text{for some } \xi \text{ between } x \text{ and } \{x_j\}.$$

3. Obtain the cubic Hermite interpolant in Newton form $P(x)$, by constructing the Hermite divided-difference table.

4. Plot $f(x)$, $H(x)$ and $P(x)$ on [-1,1].

Question 2. Simple analysis on finite difference

Consider the forward difference approximation for the first derivative of a smooth function $f(x)$:

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

We showed in class that the total error in the derivative approximation can be written as:

$$E(h) = \frac{C_1}{h} + C_2 h,$$

with the optimal choice of step size h:

$$h_{\text{opt}} = \sqrt{\frac{C_1}{C_2}},$$

where C_1 and C_2 are constants depending on $f(x)$, machine precision ε, and its derivatives.

Let $f(x) = e^x$, and take $x = 1.5$.

1. Compute $f'(1.5)$ using the forward difference formula for a range of step sizes $h = 10^{-k}$, where $k = 1, 2, \ldots, 10$.

2. Plot the absolute error $|D_h f(1) - f'(1)|$ versus h on a log-log scale. Code needs to be reported in latex.

3. Identify the h that minimizes the total error and compare it with the predicted h_{opt}.

4. Use the one-sided three points scheme to approximate $f'(1.5)$, and repeat the sub-question #2.

Question 3. Solve nonlinear burger's equation by Finite difference scheme
Consider the 1D viscous Burgers equation

$$u_t + uu_x = \nu u_{xx}, \quad x \in [0,1], t > 0, \nu = 0.2.$$

Your task is to design, analyze, and implement a numerical method that is stable and achieves at least second-order accuracy in space

1. Report your finite difference scheme used.
2. Use the initial condition as a since wave $sin(2\pi x)$. Write your code to implement your scheme, and plot the numerical solution at $T = 1s$.
3. Compute your $||u - u_h||_{L_2}$ with different refined mesh, and show your spatial convergence in log-log plot.
4. Plot your $||u - u_h||_{L_2}$ vs time.

Solutions.

1) Error growth with degree.

From the remainder formula,

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \qquad \xi \in (0, 1).$$

For $f(x) = e^x$, we have $f^{(n+1)}(t) = e^t \leq e$ on $[0, 1]$. Hence

$$|R_n(x)| \leq \frac{e}{(n+1)!} \Big| \prod_{i=0}^{n} (x - x_i) \Big|.$$

Taking the maximum over $x \in [0, 1]$ gives

$$\max_{x \in [0,1]} |R_n(x)| \leq \frac{e}{(n+1)!} \max_{x \in [0,1]} \Big| \prod_{i=0}^{n} (x - x_i) \Big|.$$

On $[0, 1]$, each factor satisfies $|x - x_i| \leq 1$, so

$$0 \leq \max_{x \in [0,1]} \Big| \prod_{i=0}^{n} (x - x_i) \Big| \leq 1.$$

Therefore

$$\max_{x \in [0,1]} |R_n(x)| \leq \frac{e}{(n+1)!} \xrightarrow[n \to \infty]{} 0.$$

1) x_k are roots of T_n.

Let $\theta_k = \frac{2k-1}{2n} \pi \in (0, \pi)$. Then $\arccos(\cos \theta_k) = \theta_k$, so

$$T_n(x_k) = \cos(n \arccos x_k) = \cos(n\theta_k) = \cos\left(\frac{2k-1}{2} \pi\right) = 0$$

Hence each x_k is a root. 2) Distinct and in $(\ -1, 1\ )$.

The θ_k are strictly increasing in $(\ 0, \pi\ )$; since cos is strictly decreasing on $[0, \pi]$, the values $x_k = \cos \theta_k$ are distinct. Because $\theta_k \in (0, \pi)$, we have $x_k \in (-1, 1)$. Thus all n roots lie in $(-1, 1)$ and are distinct. Because T_n has degree n, these are all the roots.