# Homework 3

*Due Date: October 27, 2025*

## 1 - Hermite Interpolation

Given the function

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1, 1],$$

with the function values and first derivatives at the nodes

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$$

### Question 1.1

Construct the cubic Hermite interpolant $H(x)$ and evaluate $H(0.5)$.

*Solution.* The cubic Hermite interpolant on an interval $[x_0, x_1]$ is determined by the shape functions:

$$h_{00}(t) = 2t^3 - 3t^2 + 1, \quad h_{01}(t) = -2t^3 + 3t^2$$

$$h_{10}(t) = t^3 - 2t^2 + t, \quad h_{11}(t) = t^3 - t^2$$

Provided $h = x_1 - x_0$ and $t = \frac{x-x_0}{h} \in [0, 1]$. The cubic Hermite interpolant is given by:

$$H(x) = f(x_0)h_{00}(t) + f(x_1)h_{01}(t) + h\left[f'(x_0)h_{10}(t) + f'(x_1)h_{11}(t)\right]$$

We will construct the Hermite interpolant piecewise on the intervals $[-1, 0]$ and $[0, 1]$.

To start off with, the following table summarizes the function and derivative values at the nodes:

| Node $x_j$ | $f(x_j)$ | $f'(x_j)$ |
|---|---|---|
| -1 | $\frac{1}{26}$ | $\frac{50}{676} = \frac{25}{338}$ |
| 0 | 1 | 0 |
| 1 | $\frac{1}{26}$ | $-\frac{25}{338}$ |

Where $f'(x) = \frac{-50x}{(1+25x^2)^2}$.

On $[-1, 0]$, we have $h = 1$ and $t = x + 1$. So therefore we have:

$$h_{00}(x+1) = 2(x+1)^3 - 3(x+1)^2 + 1 = 2x^3 + 3x^2$$

$$h_{01}(x+1) = -2(x+1)^3 + 3(x+1)^2 = -2x^3 - 3x^2 + 1$$

$$h_{10}(x+1) = (x+1)^3 - 2(x+1)^2 + (x+1) = x^3 + x^2$$
$$h_{11}(x+1) = (x+1)^3 - (x+1)^2 = x^3 + 2x^2 + x$$

Substituting into the interpolant formula, we get:

$$H(x) = \frac{1}{26}(2x^3 + 3x^2) + 1(-2x^3 - 3x^2 + 1) + 1 \cdot \frac{25}{338}(x^3 + x^2) + 0$$

Simplifying, we have:

$$H(x) = -\frac{625}{338}x^3 - \frac{475}{169}x^2 + 1, \quad x \in [-1, 0]$$

On $[0, 1]$, we have $h = 1$ and $t = x$. So therefore our shape functions are as defined initially with $x$ in place of $t$. Substituting into the interpolant formula, we get:

$$H(x) = 1(2x^3 - 3x^2 + 1) + \frac{1}{26}(-2x^3 + 3x^2) + 0 - \frac{25}{338}(x^3 - x^2)$$

Simplifying, we have:

$$H(x) = \frac{625}{338}x^3 - \frac{475}{169}x^2 + 1, \quad x \in [0, 1]$$

So the cubic Hermite interpolant is:

$$H(x) = \begin{cases} -\frac{625}{338}x^3 - \frac{475}{169}x^2 + 1, & x \in [-1, 0] \\ \frac{625}{338}x^3 - \frac{475}{169}x^2 + 1, & x \in [0, 1] \end{cases}$$

Evaluating at $x = 0.5$, we get that:

$$H(0.5) = \frac{625}{338}(0.5)^3 - \frac{475}{169}(0.5)^2 + 1 = \frac{1429}{2704} \approx 0.528479290$$

## Question 1.2

Evaluate the absolution error of $f(0.5) - H(0.5)$, and show that the error agrees the prior estimate of

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2$$

for some $\xi$ between $x$ and $x_j$.

*Solution.* Note that $f(0.5) = \frac{4}{29} \approx 0.1379310345$.

The absolute error with our estimate $H(0.5)$ is:

$$|f(0.5) - H(0.5)| = \left| \frac{4}{29} - \frac{1429}{2704} \right| = \frac{30625}{78416} \approx 0.390548$$

The fourth derivative of $f(x)$ is:

$$f^{(4)}(x) = \frac{15000\,(3125x^4 - 250x^2 + 1)}{(25x^2+1)^5}$$

The maximum of $|f^{(4)}(\xi)|$ for $\xi \in [-1, 1]$ occurs at $x = 0$, yielding:

$$|f^{(4)}(0)| = 15000$$

The product term evaluated at $x = 0.5$ is:

$$\prod_{j=1}^{2} (0.5 - x_j)^2 = (0.5 - 0)^2 (0.5 - 1)^2 = \left(\frac{1}{2}\right)^2 \cdot \left(-\frac{1}{2}\right)^2 = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

We also have that $(2n+2)! = 4! = 24$ for $n = 1$. So putting everything together, we can estimate the error as:

$$|f(0.5) - H(0.5)| \leq \frac{15000}{24} \cdot \frac{1}{16} = \frac{625}{16} \approx 39.0625$$

So therefore our absolute error is well within the established bound.

<u>Extra: Absolute Error with Standard Hermite Interpolant</u>

*Note: Question 1.1 was originally done by hand with the regular Hermite Interpolant resulting in a polynomial of degree 5. This was the error analysis conducted for that version.*

The absolute error with our estimate $P_5(0.5)$ is:

$$|f(0.5) - P_5(0.5)| = \left| \frac{4}{29} - \frac{6341}{10816} \right| = \frac{140625}{313664} \approx 0.4483300602$$

To estimate the error using the provided formula, we first need to compute the sixth derivative of $f(x)$, which works out to:

$$f^{(6)}(x) = \frac{11250000\,(109375x^6 - 21875x^4 + 525x^2 - 1)}{(25x^2+1)^7}$$

The maximum of $|f^{(6)}(\xi)|$ for $\xi \in [-1, 1]$ occurs at $x = 0$, yielding:

$$|f^{(6)}(0)| = 11250000$$

The product term evaluated at $x = 0.5$ is:

$$\prod_{j=0}^{2}(0.5 - x_j)^2 = (0.5 + 1)^2(0.5 - 0)^2(0.5 - 1)^2 = \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(-\frac{1}{2}\right)^2 = \frac{9}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{9}{64}$$

We also have that $(2n + 2)! = 6! = 720$ for $n = 2$.

So putting everything together, we can estimate the error as:

$$|f(0.5) - P_5(0.5)| \leq \frac{11250000}{720} \cdot \frac{9}{64} = \frac{140625}{64} \approx 2197.265625$$

So therefore our absolute error is well within the established bound.

## Question 1.3

Obtain the cubic Hermite interpolant in Newton form $P(x)$., by constructing the Hermite divided-difference table.

*Solution.* The code for Lagrange Interpolation from the previous assignment was modified compute the Hermite Interpolation:

Listing 1: 1.3 Python

```
import numpy as np

def f(x):
    return 1 / (1 + 25 * x**2)

def f_prime(x):
    return -50 * x / (1 + 25 * x**2)**2

def hermite_coefficients(nodes, f=f, f_prime=f_prime):
    # Parameterize nodes for Hermite interpolation
    z = np.concatenate((nodes, nodes))
    sorted_indexes = z.argsort()
    z = z[sorted_indexes]
    num_nodes = len(z)

    # Set up dd table with zeroth dd
    dd_table = np.array([[f(zi) for zi in z]])

    # First divided difference
    f_prime_nodes = np.array([f_prime(xi) for xi in nodes])
    zeros = np.zeros(len(nodes))
    first_dd = np.concatenate((f_prime_nodes, zeros))[sorted_indexes]
    for j in range(num_nodes - 1):
        if j % 2 == 1:
            first_dd[j] = (dd_table[0, j + 1] - dd_table[0, j]) / (z[j + 1] - z[j])
    dd_table = np.vstack([dd_table, first_dd])

    # Remaining Divided Differences
    for i in range(2, num_nodes):
        ith_dd = np.zeros(num_nodes)

        for j in range(num_nodes - i):
            # Calculate ith divided differences
            ith_dd[j] = (dd_table[i - 1, j + 1] - dd_table[i - 1, j]) / (z[j + i] - z[j])

        dd_table = np.vstack([dd_table, ith_dd])
```

```

    coefficients = np.array([dd_table[i, 0] for i in range(dd_table.shape[0])])
    return coefficients, dd_table.T, z

def generate_hermite(nodes, n):
    # Get coefficients
    a = hermite_coefficients(nodes)[0]
    # Start with function and constant term
    equation = f"P_{{{2*n+1}}}(x) = {a[0]}"
    w = []
    # Build (x - xi) terms
    for xi in nodes:
        if abs(xi) <= 1e-14:
            w.append('(x)')
        elif xi < 0:
            w.append(f"(x + {abs(xi)})")
        else:
            w.append(f"(x - {xi})")
    b = ""

    # Build polynomial string
    for i in range(1, len(a)):
        for j in range(i):
            # Multiply (x - xi) terms
            b += '^2' if j%2 ==1 else w[int(j/2)]
        # Multiply (x - xi) product with current coefficient and add term
        if a[i] > 0:
            equation += f" + {a[i]}{b}"
        elif a[i] < 0:
            equation += f" - {abs(a[i])}{b}"
        b = ""

    return equation

if __name__ == '__main__':
    nodes_1 = np.array([-1,0])
    equation_1 = generate_hermite(nodes_1, 1)
    table_1, z_1 = hermite_coefficients(nodes_1)[1:3]

    nodes_2 = np.array([0,1])
    equation_2 = generate_hermite(nodes_1, 1)
    table_2, z_2 = hermite_coefficients(nodes_1)[1:3]

    equations = [equation_1,equation_2]
    tables = [table_1, table_2]
    z = [z_1,z_2]

```

```
    with open("./outputs_3/hermite.txt", 'w') as file:
        for k in range(2):
            file.write(equations[k] +'\n\n')

            file.write('\\begin{center}\n')
            file.write('\\begin{tabular}{|c|c|c|c|c|c|}\n')
            file.write('\\hline\n')
            file.write('& $z_i$ & $f[z_i]$ & 1st dd. & 2nd dd. & 3rd dd. \\\\\n
')
            file.write('\\hline\n')
            for i in range(tables[k].shape[0]):
                file.write(f'$z_{i}$ & ${z[k][i]}$ ')
                for j in range(tables[k].shape[1]):
                    file.write(f'& ${tables[k][i,j]:.4f}$ ')
                file.write('\\\\ \n')
            file.write('\\hline\n')
            file.write('\\end{tabular}\n')
            file.write('\\end{center}\n\n')
```

The divided-difference tables and Hermite Interpolants generated were:

On $[-1, 0]$:

| | $z_i$ | $f[z_i]$ | 1st dd. | 2nd dd. | 3rd dd. |
|---|---|---|---|---|---|
| $z_0$ | $-1$ | $0.0385$ | $0.0740$ | $0.8876$ | $-1.8491$ |
| $z_1$ | $-1$ | $0.0385$ | $0.9615$ | $-0.9615$ | $0.0000$ |
| $z_2$ | $0$ | $1.0000$ | $0.0000$ | $0.0000$ | $0.0000$ |
| $z_3$ | $0$ | $1.0000$ | $0.0000$ | $0.0000$ | $0.0000$ |

$$P_3(x) = 0.038461538461538464 + 0.07396449704142012(x+1) \\ + 0.8875739644970414(x+1)^2 - 1.849112426035503(x+1)^2(x)$$

On $[0, 1]$:

| | $z_i$ | $f[z_i]$ | 1st dd. | 2nd dd. | 3rd dd. |
|---|---|---|---|---|---|
| $z_0$ | $0$ | $1.0000$ | $0.0000$ | $-0.9615$ | $1.8491$ |
| $z_1$ | $0$ | $1.0000$ | $-0.9615$ | $0.8876$ | $0.0000$ |
| $z_2$ | $1$ | $0.0385$ | $-0.0740$ | $0.0000$ | $0.0000$ |
| $z_3$ | $1$ | $0.0385$ | $0.0000$ | $0.0000$ | $0.0000$ |

$$P_3(x) = 1.0 - 0.9615384615384616(x)^2 \\ + 1.849112426035503(x)^2(x-1)$$

## Question 1.4

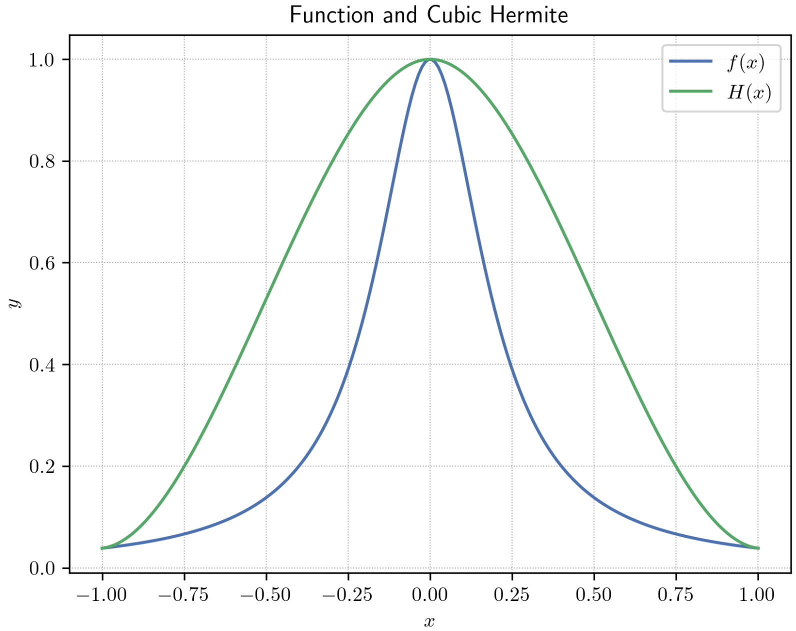
Plot $f(x)$, $H(x)$ and $P(x)$ on $x \in [-1, 1]$.

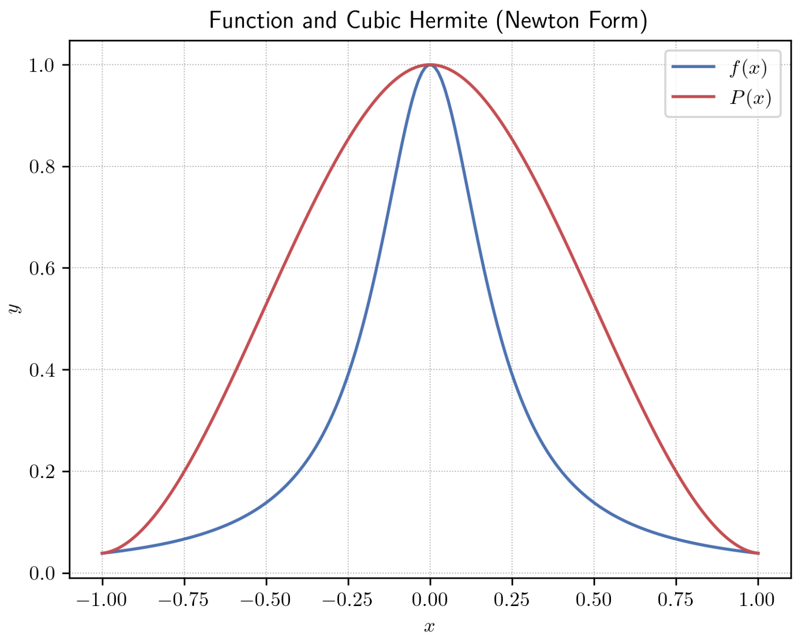
*Solution.* The following code was used to plot the functions:

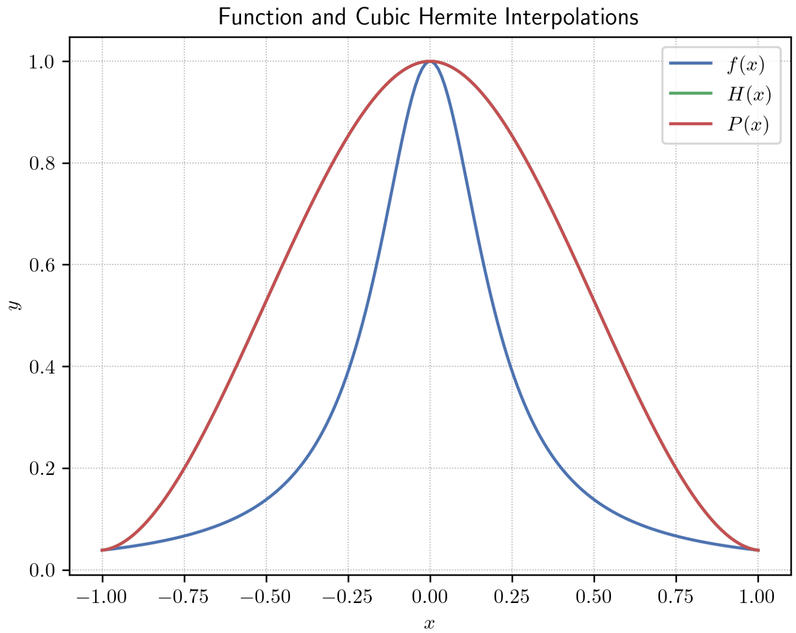
Listing 2: 1.4 Python

```python
import numpy as np
import matplotlib.pyplot as plt

if __name__ == "__main__":
    #<Some matplotlib styling and enable LaTeX>
    x = np.linspace(-1, 1, 200)
    f_y = f(x)

    conditions = [x <= 0, x > 0]
    h_y = np.piecewise(x, conditions, [h_1, h_2])
    p_y = np.concatenate(
        (calculate_hermite(nodes_1, x[:100]), calculate_hermite(nodes_2, x[100:]))
    )

    plt.plot(x, f_y, label="$f(x)$")
    plt.plot(x, h_y, label="$H(x)$")
    plt.xlabel("$x$")
    plt.ylabel("$y$")
    plt.legend()
    plt.title("Function and Cubic Hermite")
    plt.savefig("./outputs_3/hermite_plot_h.png", dpi=300)
    plt.cla()

    # Similar Plotting for P(x) and for all three together
```

The following plots were generated:






We observe that both Hermite interpolants $H(x)$ and $P(x)$ are identical and loosely follow the behavior of $f(x)$ on the interval $[-1, 1]$.

# 2 - Simple Analysis on Finite Difference

Consider the forward difference formula for the first derivative of a smooth function $f(x)$:

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}$$

The total error for this approximation can be expressed as:

$$E(h) = \frac{C_1}{h} + C_2 h$$

with the optimal step size $h$:

$$h_{opt} = \sqrt{\frac{C_1}{C_2}}$$

where $C_1$ and $C_2$ are constants depending on $f(x)$, machine precision $\varepsilon$, and its derivatives.

Let $f(x) = e^x$, and take $x = 1.5$.

## Question 2.1

Compute $f'(1.5)$ using the forward difference formula for a range of step sizes $h = 10^{-k}$, where $k = 1, 2, \ldots, 10$.

*Solution.* The following code was used to compute the forward difference approximations:

Listing 3: 2.1 Python

```python
import numpy as np

def forward_diff(f, x, step=1):
    return (f(x + step) - f(x)) / step

if __name__ == "__main__":
    with open("./outputs_3/forward_diff.txt", "w") as file:
        for k in range(1,11):
            h =10**(-k)
            f_prime = forward_diff(np.exp, 1.5, h)
            file.write(f'\\[D_h^{{({k})}}f(1.5) = {f_prime}\\]\n')
```

The following results were obtained for $D_h^{(k)} f(1.5)$:

$$D_h^{(1)} f(1.5) = 4.713433540570504$$

$$D_h^{(2)} f(1.5) = 4.5041723976187775$$

$$D_h^{(3)} f(1.5) = 4.483930662008362$$

$$D_h^{(4)} f(1.5) = 4.481913162264206$$

$$D_h^{(5)} f(1.5) = 4.4817114789097445$$

$$D_h^{(6)} f(1.5) = 4.48169131139764$$

$$D_h^{(7)} f(1.5) = 4.481689304114411$$

$$D_h^{(8)} f(1.5) = 4.481689064306238$$

$$D_h^{(9)} f(1.5) = 4.481689686031132$$

$$D_h^{(10)} f(1.5) = 4.48169501510165$$

## Question 2.2

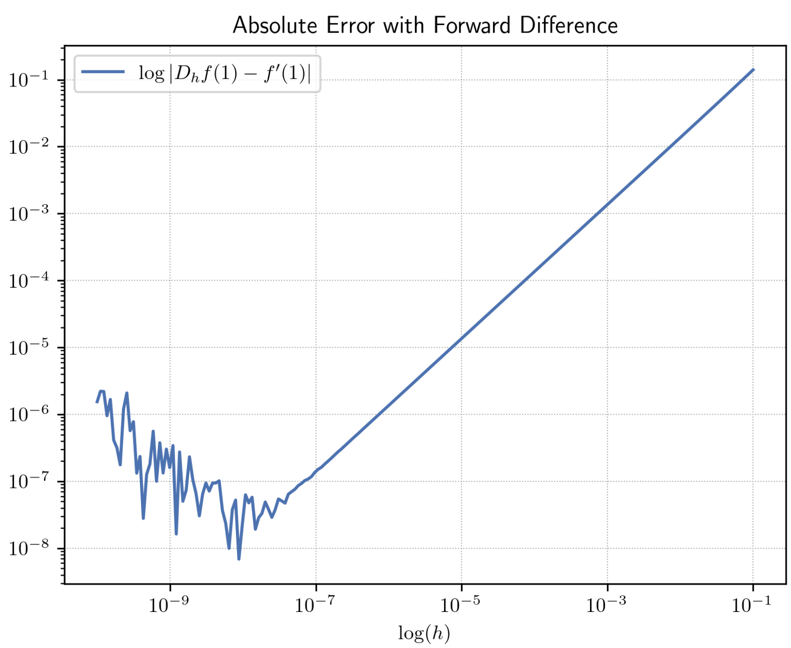
Plot the absolute error $|D_h f(1) - f'(1)|$ verses $h$ on a log-log scale. (Report code)

*Solution.* The following code was used to compute and plot the absolute error:

Listing 4: 2.2 Python

```
import numpy as np
import matplotlib.pyplot as plt

def calc_absolute_error(f, step, eval_x=1, diff_method=forward_diff):
    approx = diff_method(f, eval_x, step)
    return np.abs(approx - f(eval_x))

if __name__ == "__main__":
    # <Some matplotlib styling and enable LaTeX>
    h = np.logspace(-1, -10, 200)
    error = calc_absolute_error(np.exp, h)
    plt.loglog(h, error, label='$\\log|D_hf(1) - f\'(1)|$')
    plt.xlabel('$\\log(h)$')
    plt.title('Absolute Error in Forward Difference Approximation')
    plt.legend()
    plt.savefig('./outputs_3/forward_diff_graph.png', dpi=300)
    plt.show()
```

The resulting plot was:




We observe that machine error starts to occur as $h$ gets smaller than approximately than $10^{-7}$.

## Question 2.3

Identify the $h$ that minimizes the total error and compare it with the predicted $h_{opt}$.

*Solution.* From the plot above, we observe that the minimum error occurs around $h \approx 10^{-8}$.

Machine error is $\varepsilon \sim 10^{-16}$ for double precision floating point numbers. From the notes, we have that $C_1 \sim \varepsilon|f(1)|$ and $C_2 \sim \frac{|f''(1)|}{2}$. This resulted in $h_{opt} \propto \sqrt{\varepsilon} \approx 10^{-8}$, which aligns with our observed minimum error point.

## Question 2.4

Use the one-sided three points scheme to approximate $f'(1.5)$, and repeat question 2.2.

*Solution.* The one-sided three-point finite difference formula for the first derivative is given by:

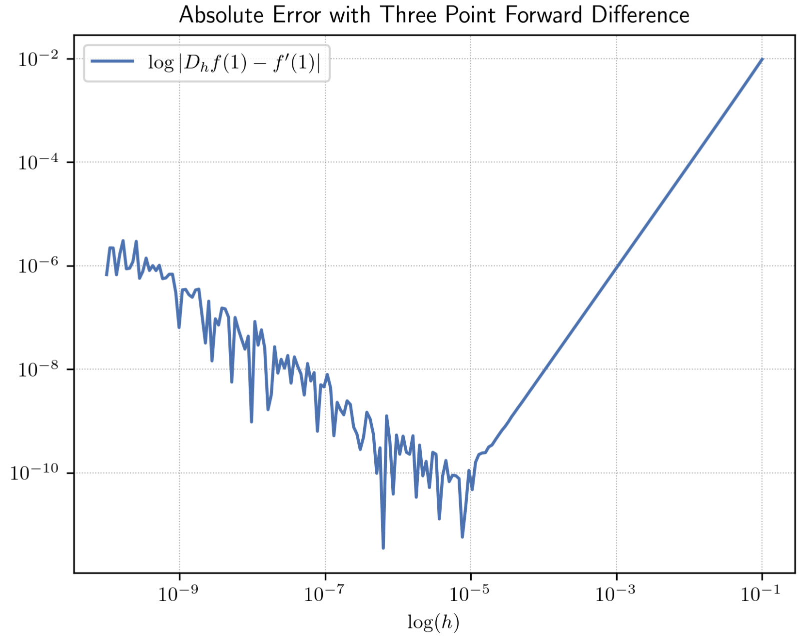
$$D_h f(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

The following code was used to compute and plot the absolute error using the three-point forward difference scheme:

Listing 5: 2.4 Python

```
import numpy as np
import matplotlib.pyplot as plt

def three_point_forward_diff(f, x, step=1):
    return (-3 * f(x) + 4 * f(x + step) - f(x + 2 * step)) / (2 * step)

if __name__ == "__main__":
    # <Some matplotlib styling and enable LaTeX>
    h = np.logspace(-1, -10, 200)
    error = calc_absolute_error(np.exp, h, diff_method=three_point_forward_diff
    )

    plt.loglog(h, error, label='$\\log|D_hf(1) - f\'(1)|$')
    plt.xlabel('$\\log(h)$')
    plt.title('Absolute Error with Three Point Forward Difference')
    plt.legend()
    plt.savefig('./outputs_3/three_point_forward_diff_graph.png', dpi=300)
    plt.show()
```

The resulting plot was:




We observe that machine error starts to occur as $h$ gets smaller than approximately than $10^{-5}$.

# 3 - Solve Burger's Equation by Finite Difference Scheme

Consider the 1D viscous Burgers' equation:

$$u_t + uu_x = vu_{xx}, \quad x \in [0,1],\ t > 0,\ v = 0.2$$

Design, analyze, and implement a numerical method that is stable and achieves at least second-order accuracy in both space and time.

## Question 3.1

Report finite difference scheme used.

*Solution.* We will be using RK4 with 2nd order finite differences for space.

First let us discretize our system. Let $x_0, x_1, ..., x_n$ be a partition of $[0,1]$ with step size $\Delta x = h = \frac{1}{n}$. Let $t_0, t_1, ..., t_m$ be a partition of $[0,T]$ with step size $\Delta t$. Let $u_i$ be the numerical approximation of $u(x_i, t)$.

We can construct our schemes for $u_x, u_{xx}$ from the Taylor expansion of $u$ centered at $x_i$:

$$u(x) = u(x_i) + u_x(x_i)(x - x_i) + \frac{u_{xx}(x_i)}{2}(x - x_i)^2 + \frac{u_{xxx}(x_i)}{6}(x - x_i)^3 + ...$$

Which gives us for $x = x_i - h$ and $x = x_i + h$:

$$u(x_i - h) = u(x_i) - hu_x(x_i) + \frac{h^2}{2}u_{xx}(x_i) - \frac{h^3}{6}u_{xxx}(x_i) + O(h^4)$$

$$u(x_i + h) = u(x_i) + hu_x(x_i) + \frac{h^2}{2}u_{xx}(x_i) + \frac{h^3}{6}u_{xxx}(x_i) + O(h^4)$$

Approximate $u_x$ centrally by:

$$u(x_i + h) - u(x_i - h) = 2hu_x(x_i) + O(h^3)$$

Which gives us the second-order scheme:

$$u_x(x_i) \approx \frac{u(x_i + h) - u(x_i - h)}{2h} + O(h^2)$$

Approximate $u_{xx}$ with the usual second-order central difference:

$$u(x_i + h) - 2u(x_i) + u(x_i - h) = h^2 u_{xx}(x_i) + O(h^4)$$

Which gives us the second-order scheme:

$$u_{xx}(x_i) \approx \frac{u(x_i + h) - 2u(x_i) + u(x_i - h)}{h^2} + O(h^2)$$

Rearranging Burger's equation for $u_t$, we have:

$$u_t = -uu_x + vu_{xx}$$

Substituting in our finite difference approximations, we obtain the ODE:

$$\frac{du_i}{dt} \approx -u_i \cdot \frac{u_{i+1} - u_{i-1}}{2h} + v \cdot \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

Which is $O(h^2) = O(\Delta x^2)$ accurate in space. Denote the right-hand side above as $F(u_i)$.

More compactly, write the system as:

$$\frac{du}{dt} = F(u) + O(\Delta x^2)$$

Where $u = [u_0, u_1, ..., u_n]^T$.

We will now use RK4 to approximate $u^{n+1}$, which is given by:

$$u^{n+1} = u^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Such that:

$$k_1 = F(u^n)$$
$$k_2 = F\left(u^n + \frac{\Delta t}{2}k_1\right)$$
$$k_3 = F\left(u^n + \frac{\Delta t}{2}k_2\right)$$
$$k_4 = F\left(u^n + \Delta t k_3\right)$$

RK4 is $O(\Delta t^4)$ accurate in time, so our overall scheme is $O(\Delta t^4) + O(\Delta x^2)$. We also note that RK4 would have CFL conditions for stability:

$$\Delta t \leq \frac{C_1 \Delta x}{\max|u|}, \quad \Delta t \leq \frac{C_2 \Delta x^2}{v}$$

Where $C_1, C_2$ are constants depending on the scheme. In practice, we can choose $\Delta t$ based on the more restrictive of the two conditions above and pick smaller $C_1, C_2$ to be safe.

## Question 3.2

Use the initial condition as a sine wave $\sin(2\pi x)$. Implement the scheme and plot the numerical solution at $T = 1$ second.

*Solution.* We have Burger's equation:

$$u_t + uu_x = vu_{xx}, \quad x \in [0,1],\ t > 0,\ v = 0.2$$

With periodic initial condition:

$$u(x,0) = \sin(2\pi x)$$

This means that $u(0,t) = u(1,t)$ for all $t \geq 0$. Since our data is periodic, no extra scheme is needed for the boundaries and we can just treat $u$ as a circular array.

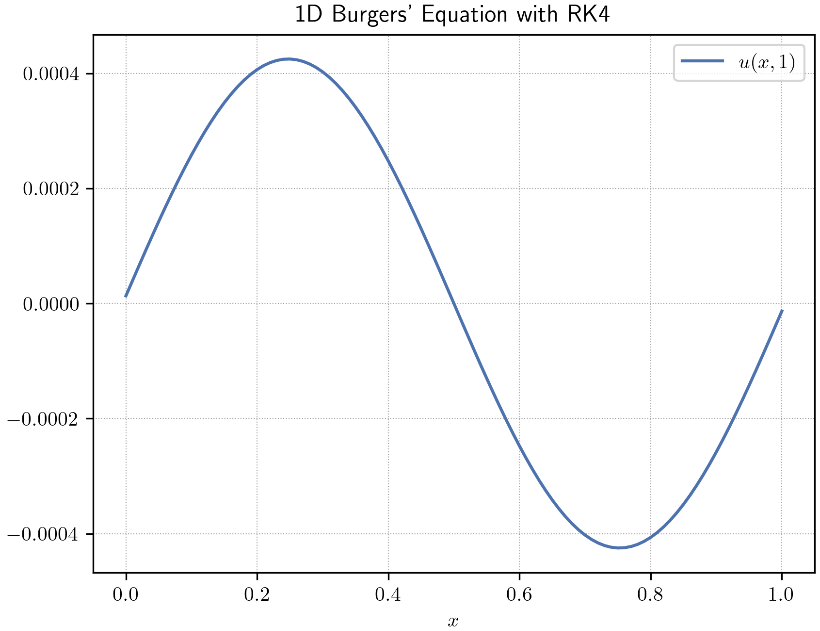
The following code was used to implement the scheme:

Listing 6: 3.2 Python

```python
import numpy as np
import matplotlib.pyplot as plt
from copy import deepcopy

def initial_condition(x):
    return np.sin(2 * np.pi * x)

def rk4_step(u, dt, dx, v):
    def F(u):
        # Compute spatial derivatives using central differences
        u_x = (np.roll(u, -1) - np.roll(u, 1)) / (2 * dx)
        u_xx = (np.roll(u, -1) - 2 * u + np.roll(u, 1)) / (dx**2)
        return -u * u_x + v * u_xx

    k1 = F(u)
    k2 = F(u + 0.5 * dt * k1)
    k3 = F(u + 0.5 * dt * k2)
    k4 = F(u + dt * k3)

    return u + (dt / 6) * (k1 + 2 * k2 + 2 * k3 + k4)

def burgers_rk4(dx, t_final, v=0.2, start_x=0, end_x=1, return_history=False):
    # Take ceil for num_nodes
    num_nodes = int((end_x - start_x) / dx) + 1
    x = np.linspace(start_x, end_x, num_nodes)

    # The CFL condition for diffusion is more restrictive
    # as for initial condition u = sin(2pi x), max|u| = 1
    # Pick C_2 = 0.5 to be safe
    dt = 0.5 * dx**2 / v
```

```
    num_time_steps = int(np.ceil(t_final / dt))
    # Adjust dt to fit exactly into t_final
    dt = t_final / num_time_steps

    u = initial_condition(x)

    # Store history of u for Question 3.4
    if return_history:
        t = 0
        u_history = [deepcopy(u)]
        times = [t]

    for _ in range(num_time_steps):
        u = rk4_step(u, dt, dx, v)
        if return_history:
            t += dt
            u_history.append(deepcopy(u))
            times.append(t)

    if return_history:
        return x, u, np.array(times), np.array(u_history)

    return x, u

if __name__ == "__main__":
    dx = 1e-1
    t_final = 1
    x, u = burgers_rk4(dx, t_final)

    # <Some matplotlib styling and enable LaTeX>

    plt.plot(x, u, label=f'$u(x, {t_final})$')
    plt.title("1D Burgers' Equation with RK4")
    plt.xlabel("$x$")
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4.png', dpi=300)
    plt.show()
```

The resulting plot at $T = 1$ second was:



## Question 3.3

Compute $||u - u_h||_{L^2}$ with a different refined mesh, and show your spatial convergence in a log-log plot.

*Solution.* The discrete $L^2$ norm was used to compute our error:

$$||u - u_h||_{L^2} = \sqrt{\Delta x \sum_{i=1}^{N} (u_i - u_{h,i})^2}$$

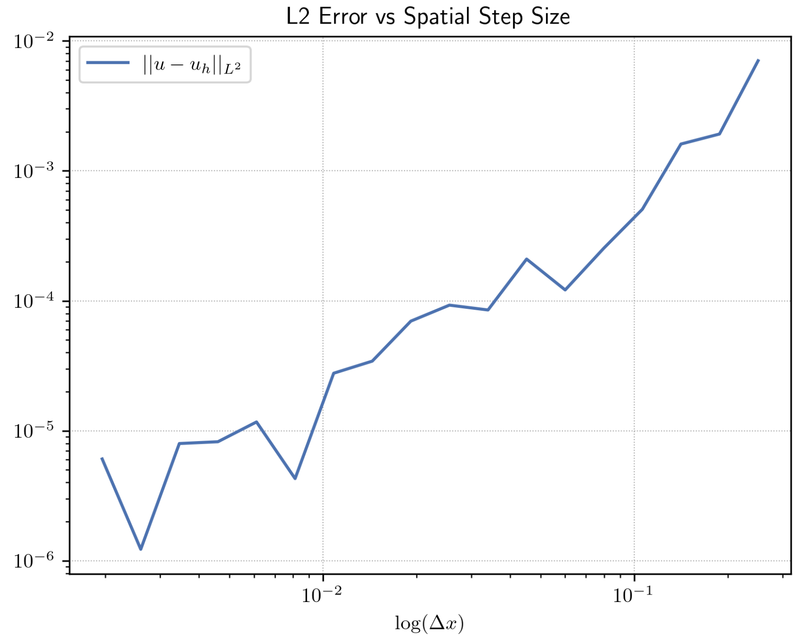
Where $N$ is the number of spatial nodes in the coarser mesh, $u_i$ is the exact solution at node $i$, and $u_{h,i}$ is the numerical solution at node $i$.

The following code was used to compute the $L^2$ error and plot spatial convergence:

Listing 7: 3.3 Python

```python
import numpy as np
import matplotlib.pyplot as plt

def discrete_l2(u_num, u_ref, dx):
    return np.sqrt(np.sum((u_num - u_ref)**2) * dx)

def spacial_convergence(dx_list, t_final, v=0.2):

    # dx_exact will serve as our "exact" solution
    # Use a finer dx than minimum dx in dx_list
    dx_exact = min(dx_list) / 4
    x_exact, u_exact = burgers_rk4(dx_exact, t_final, v)

    errors = []
    for dx in dx_list:
        x_approx, u_approx = burgers_rk4(dx, t_final, v)
        # Interpolate exact solution to the current mesh of x_approx
        u_exact_on_x_approx = np.interp(x_approx, x_exact, u_exact)
        error = discrete_l2(u_approx, u_exact_on_x_approx, dx)
        errors.append(error)
    return dx_list, errors

if __name__ == "__main__":
    # <Some matplotlib styling and enable LaTeX>
    dx_list = np.logspace(-2, -9, 18, base=2)
    dx, error = spacial_convergence(dx_list, t_final)
    plt.loglog(dx, error, label='$||u - u_h||_{L^2}$')
    plt.xlabel('$\\log(\\Delta x)$')
    plt.title('L2 Error vs Spatial Step Size')
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4_error.png', dpi=300)
    plt.show()
```

The plot generated for spatial convergence at $T = 1$ was:




A few notes:

1. $\Delta x = \frac{1}{2}$ has been excluded from the plot as it had favorable error due to the coarse mesh aligning well with the sine wave at $T = 1$.
2. The fluctuations in the error can be explained by additional interpolation error when mapping the exact solution to the coarser mesh and/or point 1 above.

Overall, we observe second-order spatial convergence as $\Delta x$ decreases.

## Question 3.4

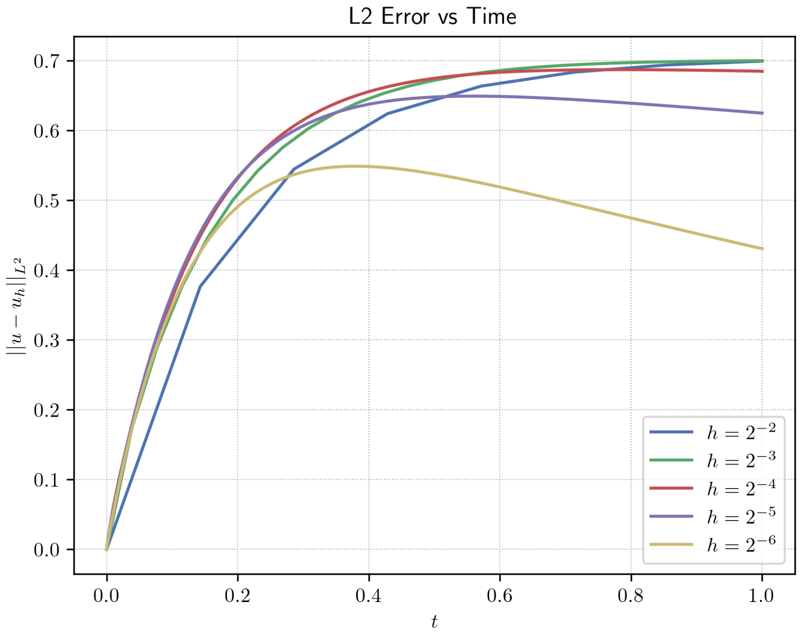
Plot your $||u - u_h||_{L^2}$ vs time.

*Solution.* The following code was used to compute and plot the $L^2$ error vs time:

Listing 8: 3.4 Python

```python
import numpy as np
import matplotlib.pyplot as plt

def norm_vs_time(dx, t_final, x_exact, u_exact_history, v=0.2):
    x_approx, _, times, u_history = burgers_rk4(dx, t_final, v, return_history=True)

    norms = []
    for u_approx, u_exact in zip(u_history, u_exact_history):
        u_exact_on_x_approx = np.interp(x_approx, x_exact, u_exact)
        norm = discrete_l2(u_approx, u_exact_on_x_approx, dx)
        norms.append(norm)
    return times, norms

if __name__ == "__main__":
    # <Some matplotlib styling and enable LaTeX>
    dx_list = np.logspace(-2, -6, 5, base=2)

    dx_exact = min(dx_list) / 4
    x_exact, _, _, u_exact_history = burgers_rk4(dx_exact, t_final, return_history=True)

    for index, dx in enumerate(dx_list):
        time, norms = norm_vs_time(dx, t_final, x_exact, u_exact_history)
        plt.plot(time, norms, label=f'$h = 2^{{{-index-2}}}$')
    plt.xlabel('$t$')
    plt.ylabel(r'$||u - u_h||_{L^2}$')
    plt.title('L2 Error vs Time')
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4_error_vs_time.png', dpi=300)
    plt.show()
```

The plot generated for $L^2$ error vs time was:




We can see that the $L^2$ error generally increases with time and then stabilizes. The increase makes sense as numerical errors accumulate over time steps.

Finer meshes (smaller $h$) consistently yield lower errors throughout the simulation at $T = 1$.

# Appendix

The complete code used for this assignment is provided in the appendix for reference. Files can be accessed directly at this GitHub repository.

Listing 9: question_1.py

```
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return 1 / (1 + 25 * x**2)


def f_prime(x):
    return -50 * x / (1 + 25 * x**2) ** 2


def h_1(x):
    return (-625 / 338) * x**3 - (475 / 169) * x**2 + 1


def h_2(x):
    return (625 / 338) * x**3 - (475 / 169) * x**2 + 1


def hermite_coefficients(nodes, f=f, f_prime=f_prime):
    # Parameterize nodes for Hermite interpolation
    z = np.concatenate((nodes, nodes))
    sorted_indexes = z.argsort()
    z = z[sorted_indexes]
    num_nodes = len(z)

    # Set up dd table with zeroth dd
    dd_table = np.array([[f(zi) for zi in z]])

    # First divided difference
    f_prime_nodes = np.array([f_prime(xi) for xi in nodes])
    zeros = np.zeros(len(nodes))
    first_dd = np.concatenate((f_prime_nodes, zeros))[sorted_indexes]
    for j in range(num_nodes - 1):
        if j % 2 == 1:
            first_dd[j] = (dd_table[0, j + 1] - dd_table[0, j]) / (z[j + 1] - z
[j])
    dd_table = np.vstack([dd_table, first_dd])

    # Remaining Divided Differences
    for i in range(2, num_nodes):
```

```
        ith_dd = np.zeros(num_nodes)

        for j in range(num_nodes - i):
            # Calculate ith divided differences
            ith_dd[j] = (dd_table[i - 1, j + 1] - dd_table[i - 1, j]) / (
                z[j + i] - z[j]
            )

        dd_table = np.vstack([dd_table, ith_dd])

    coefficients = np.array([dd_table[i, 0] for i in range(dd_table.shape[0])])
    return coefficients, dd_table.T, z


def generate_hermite(nodes, n):
    # Get coefficients
    a = hermite_coefficients(nodes)[0]
    # Start with function and constant term
    equation = f"P_{{{2 * n + 1}}}(x) = {a[0]}"
    w = []
    # Build (x - xi) terms
    for xi in nodes:
        if abs(xi) <= 1e-14:
            w.append("(x)")
        elif xi < 0:
            w.append(f"(x + {abs(xi)})")
        else:
            w.append(f"(x - {xi})")
    b = ""

    # Build polynomial string
    for i in range(1, len(a)):
        for j in range(i):
            # Multiply (x - xi) terms
            b += "^2" if j % 2 == 1 else w[int(j / 2)]
        # Multiply (x - xi) product with current coefficient and add term
        if a[i] > 0:
            equation += f" + {a[i]}{b}"
        elif a[i] < 0:
            equation += f" - {abs(a[i])}{b}"
        b = ""

    return equation


def calculate_hermite(nodes, x_coords=[], function=f):
    # Get coefficients
```

```python
    a, z = hermite_coefficients(nodes, function)[:3:2]
    # If no x_coords provided, use nodes as x_coords
    if len(x_coords) == 0:
        x_coords = nodes
    # Start with constant term
    y = a[0]
    # Build (x - zi) terms
    w = [(x_coords - zi) for zi in z]
    # Temporary variable to hold (x - zi) product
    b = 1
    for i in range(1, len(a)):
        for j in range(i):
            # Multiply (x - zi) terms
            b *= w[j]
        # Multiply (x - zi) product with current coefficient
        y += a[i] * b
        b = 1

    return y


if __name__ == "__main__":
    nodes_1 = np.array([-1, 0])
    equation_1 = generate_hermite(nodes_1, 1)
    table_1, z_1 = hermite_coefficients(nodes_1)[1:3]

    nodes_2 = np.array([0, 1])
    equation_2 = generate_hermite(nodes_2, 1)
    table_2, z_2 = hermite_coefficients(nodes_2)[1:3]

    equations = [equation_1, equation_2]
    tables = [table_1, table_2]
    z = [z_1, z_2]

    with open("./outputs_3/hermite.txt", "w") as file:
        for k in range(2):
            file.write(equations[k] + "\n\n")

            file.write("\\begin{center}\n")
            file.write("\\begin{tabular}{|c|c|c|c|c|}\n")
            file.write("\\hline\n")
            file.write("& $z_i$ & $f[z_i]$ & 1st dd. & 2nd dd. & 3rd dd. \\\\\n")
            file.write("\\hline\n")
            for i in range(tables[k].shape[0]):
                file.write(f"$z_{i}$ & ${z[k][i]}$ ")
                for j in range(tables[k].shape[1]):
```

```
                    file.write(f"& ${tables[k][i, j]:.4f}$ ")
                file.write("\\\\ \n")
            file.write("\\hline\n")
            file.write("\\end{tabular}\n")
            file.write("\\end{center}\n\n")

    plt.rcParams["text.usetex"] = True
    plt.rcParams["axes.grid"] = True
    plt.rc("grid", color="#a6a6a6", linestyle="dotted", linewidth=0.5)
    plt.style.use("seaborn-v0_8-deep")
    prop_cycle = plt.rcParams["axes.prop_cycle"]
    default_colors = prop_cycle.by_key()["color"]

    x = np.linspace(-1, 1, 200)
    f_y = f(x)

    conditions = [x <= 0, x > 0]
    h_y = np.piecewise(x, conditions, [h_1, h_2])
    p_y = np.concatenate(
        (calculate_hermite(nodes_1, x[:100]), calculate_hermite(nodes_2, x[100:]))
    )

    plt.plot(x, f_y, label="$f(x)$")
    plt.plot(x, h_y, label="$H(x)$")
    plt.xlabel("$x$")
    plt.ylabel("$y$")
    plt.legend()
    plt.title("Function and Cubic Hermite")
    plt.savefig("./outputs_3/hermite_plot_h.png", dpi=300)
    plt.cla()

    plt.plot(x, f_y, label="$f(x)$")
    plt.plot(x, p_y, label="$P(x)$", color = default_colors[2])
    plt.xlabel("$x$")
    plt.ylabel("$y$")
    plt.legend()
    plt.title("Function and Cubic Hermite (Newton Form)")
    plt.savefig("./outputs_3/hermite_plot_p.png", dpi=300)
    plt.cla()

    plt.plot(x, f_y, label="$f(x)$")
    plt.plot(x, h_y, label="$H(x)$")
    plt.plot(x, p_y, label="$P(x)$")
    plt.xlabel("$x$")
    plt.ylabel("$y$")
    plt.legend()
```

```
    plt.title("Function and Cubic Hermite Interpolations")
    plt.savefig("./outputs_3/hermite_plot.png", dpi=300)
```

Listing 10: question_2.py

```
import numpy as np
import matplotlib.pyplot as plt

def forward_diff(f, x, step=1):
    return (f(x + step) - f(x)) / step

def three_point_forward_diff(f, x, step=1):
    return (-3 * f(x) + 4 * f(x + step) - f(x + 2 * step)) / (2 * step)

def calc_absolute_error(f, step, eval_x=1, diff_method=forward_diff):
    approx = diff_method(f, eval_x, step)
    return np.abs(approx - f(eval_x))

if __name__ == "__main__":
    with open("./outputs_3/forward_diff.txt", "w") as file:
        for k in range(1,11):
            h =10**(-k)
            f_prime = forward_diff(np.exp, 1.5, h)
            file.write(f'\\[D_h^{{({k})}}f(1.5) = {f_prime}\\]\n')

    plt.rcParams["text.usetex"] = True
    plt.rcParams["axes.grid"] = True
    plt.rc("grid", color="#a6a6a6", linestyle="dotted", linewidth=0.5)
    plt.style.use("seaborn-v0_8-deep")

    h = np.logspace(-1, -10, 200)
    error = calc_absolute_error(np.exp, h)

    plt.loglog(h, error, label='$\\log|D_hf(1) - f\'(1)|$')
    plt.xlabel('$\\log(h)$')
    plt.title('Absolute Error with Forward Difference')
    plt.legend()
    plt.savefig('./outputs_3/forward_diff_graph.png', dpi=300)
    plt.show()

    h = np.logspace(-1, -10, 200)
    error = calc_absolute_error(np.exp, h, diff_method=three_point_forward_diff
    )

    plt.loglog(h, error, label='$\\log|D_hf(1) - f\'(1)|$')
    plt.xlabel('$\\log(h)$')
    plt.title('Absolute Error with Three Point Forward Difference')
    plt.legend()
```

```
    plt.savefig('./outputs_3/three_point_forward_diff_graph.png', dpi=300)
    plt.show()
```

Listing 11: question_3.py

```
import numpy as np
import matplotlib.pyplot as plt
from copy import deepcopy

def initial_condition(x):
    return np.sin(2 * np.pi * x)

def rk4_step(u, dt, dx, v):
    def F(u):
        # Compute spatial derivatives using central differences
        u_x = (np.roll(u, -1) - np.roll(u, 1)) / (2 * dx)
        u_xx = (np.roll(u, -1) - 2 * u + np.roll(u, 1)) / (dx**2)
        return -u * u_x + v * u_xx

    k1 = F(u)
    k2 = F(u + 0.5 * dt * k1)
    k3 = F(u + 0.5 * dt * k2)
    k4 = F(u + dt * k3)

    return u + (dt / 6) * (k1 + 2 * k2 + 2 * k3 + k4)

def burgers_rk4(dx, t_final, v=0.2, start_x=0, end_x=1, return_history=False):
    # Take ceil for num_nodes
    num_nodes = int((end_x - start_x) / dx) + 1
    x = np.linspace(start_x, end_x, num_nodes)

    # The CFL condition for diffusion is more restrictive
    # as for initial condition u = sin(2pi x), max|u| = 1
    # Pick C_2 = 0.5 to be safe
    dt = 0.5 * dx**2 / v
    num_time_steps = int(np.ceil(t_final / dt))
    # Adjust dt to fit exactly into t_final
    dt = t_final / num_time_steps

    u = initial_condition(x)

    # Store history of u for Question 3.4
    if return_history:
        t = 0
        u_history = [deepcopy(u)]
        times = [t]

    for _ in range(num_time_steps):
```

```
        u = rk4_step(u, dt, dx, v)
        if return_history:
            t += dt
            u_history.append(deepcopy(u))
            times.append(t)

    if return_history:
        return x, u, np.array(times), np.array(u_history)

    return x, u

def discrete_l2(u_num, u_ref, dx):
    return np.sqrt(np.sum((u_num - u_ref)**2) * dx)

def spacial_convergence(dx_list, t_final, v=0.2):

    # dx_exact will serve as our "exact" solution
    # Use a finer dx than minimum dx in dx_list
    dx_exact = min(dx_list) / 4
    x_exact, u_exact = burgers_rk4(dx_exact, t_final, v)

    errors = []
    for dx in dx_list:
        x_approx, u_approx = burgers_rk4(dx, t_final, v)
        # Interpolate exact solution to the current mesh of x_approx
        u_exact_on_x_approx = np.interp(x_approx, x_exact, u_exact)
        error = discrete_l2(u_approx, u_exact_on_x_approx, dx)
        errors.append(error)
    return dx_list, errors

def norm_vs_time(dx, t_final, x_exact, u_exact_history, v=0.2):
    x_approx, _, times, u_history = burgers_rk4(dx, t_final, v, return_history=True)

    norms = []
    for u_approx, u_exact in zip(u_history, u_exact_history):
        u_exact_on_x_approx = np.interp(x_approx, x_exact, u_exact)
        norm = discrete_l2(u_approx, u_exact_on_x_approx, dx)
        norms.append(norm)
    return times, norms

if __name__ == "__main__":
    dx = 1e-2
    t_final = 1
    x, u = burgers_rk4(dx, t_final)

    plt.rcParams["text.usetex"] = True
```

```python
    plt.rcParams["axes.grid"] = True
    plt.rc("grid", color="#a6a6a6", linestyle="dotted", linewidth=0.5)
    plt.style.use("seaborn-v0_8-deep")

    plt.plot(x, u, label=f'$u(x, {t_final})$')
    plt.title("1D Burgers' Equation with RK4")
    plt.xlabel("$x$")
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4.png', dpi=300)
    plt.cla()

    dx_list = np.logspace(-2, -9, 18, base=2)
    dx, error = spacial_convergence(dx_list, t_final)
    plt.loglog(dx, error, label='$||u - u_h||_{L^2}$')
    plt.xlabel('$\\log(\\Delta x)$')
    plt.title('L2 Error vs Spatial Step Size')
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4_error.png', dpi=300)
    plt.show()

    dx_list = np.logspace(-2, -6, 5, base=2)

    dx_exact = min(dx_list) / 4
    x_exact, _, _, u_exact_history = burgers_rk4(dx_exact, t_final, return_history=True)

    for index, dx in enumerate(dx_list):
        time, norms = norm_vs_time(dx, t_final, x_exact, u_exact_history)
        plt.plot(time, norms, label=f'$h = 2^{{{-index-2}}}$')
    plt.xlabel('$t$')
    plt.ylabel(r'$||u - u_h||_{L^2}$')
    plt.title('L2 Error vs Time')
    plt.legend()
    plt.savefig('./outputs_3/burgers_rk4_error_vs_time.png', dpi=300)
    plt.show()
```