Assignment 2

Due Date: September 22, 2025

1 - Lagrange Interpolation and Error Analysis

Let $f \in C^{n+1}[a,\, b]$, and let P_n be its Lagrange interpolating polynomial at the distinct nodes $x_0, x_1, \ldots, x_n \in [a,\, b]$. The interpolation error is:

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \quad \xi \in (a,\, b)$$

Suppose $f(x) = e^x$ on the interval $[0,\, 1]$, for equally spaced nodes $x_i = \frac{i}{n}$.

Question 1.1

Derive an explicit bound for the maximum error $\max_{x\in[0,1]} |R_n(x)|$.

Proof. To bound $|R_n(x)|$, we start with the error formula:

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right|, \quad \xi \in (0, 1)$$

Note $\frac{1}{(n-1)!}$ is a constant. We can bound $R_n(x)$ by bounding $f^{(n+1)}(\xi)$ and $|\prod_{i=0}^{n}(x - x_i)|$ separately.

Now notice that $f^{(n+1)}(x) = f(x) = e^x$ for all n. Since e^x is strictly increasing on $[0, 1]$, we have that

$$f^{(n+1)}(\xi) = \max_{x\in[0,1]} |f^{(n+1)}(x)| = f(1) = e \tag{1}$$

Next, we need to bound $|\prod_{i=0}^{n}(x - x_i)|$. Let $\omega_n(x) = \prod_{i=0}^{n}(x - x_i)$. Note that $\omega_n(x)$ is a polynomial of degree $n+1$. Since $\omega_n(x)$ is continuous on the compact interval $[0, 1]$, it attains its maximum at some point in $[0, 1]$. We may write that:

$$|\omega_n(x)| \le W_n \tag{2}$$

Where $W_n = \max_{x\in[0,1]} |\omega_n(x)| = |\omega_n(x^*)|$ for some $x^* \in [0, 1]$.

Now, we can use (1) and (2) to bound $|R_n(x)|$ from above:

$$|R_n(x)| \le \frac{e}{(n+1)!} W_n$$

Since $R_n(x)$ is continuous on the compact interval $[0, 1]$, $\sup_{x\in[0,1]} R_n(x) = \max_{x\in[0,1]} R_n(x)$.
Since the supremum is the least upper bound, we have that:

$$\max_{x\in[0,1]} |R_n(x)| \le \frac{e}{(n+1)!} W_n$$

Which gives us an explicit bound for the remainder. □

Question 1.2

Show the asymptotic decay of this error as $n \to \infty$.

Proof. From the previous part, we have that:

$$\max_{x\in[0,1]} |R_n(x)| \leq \frac{e}{(n+1)!} W_n$$

Let us first make a more crude estimate of W_n. Note that for any $x \in [0,1]$ and for each i, we have:

$$|x - x_i| \leq 1$$

Since there are $n+1$ terms in the product, we can write:

$$|\omega_n(x)| = \left|\prod_{i=0}^{n}(x - x_i)\right| \leq 1^{n+1} = 1$$

Therefore, we get the crude bound:

$$W_n = \max_{x\in[0,1]} |\omega_n(x)| \leq 1$$

Plugging this into our error bound, we get:

$$\max_{x\in[0,1]} |R_n(x)| \leq \frac{e}{(n+1)!}$$

Taking a limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} \max_{x\in[0,1]} |R_n(x)| \leq \lim_{n\to\infty} \frac{e}{(n+1)!} = 0$$

Therefore, we have shown that the error decays to 0 as $n \to \infty$. □

2 - Interpolation Programming Exercise

Consider the fuction:

$$f(x) = \frac{1}{1 + 20x}, \quad x \in [-1, 1]$$

Question 2.1

Construct the interpolation polynomial $P_n(x)$ at equidistant nodes for $n = 5$.

Solution. The following methods were used to compute this polynomial:

Listing 1: 2.1 Python

```python
import numpy as np

def f(x):
    return 1 / (1 + 20 * x**2)

def lagrange_coefficients(nodes):
    x = nodes
    num_nodes = len(nodes)
    # Add zeroth divided differences
    dd_table = np.array([[f(xi) for xi in nodes]])

    # Calculate divided difference table
    for i in range(1, num_nodes):
        ith_dd = np.zeros(num_nodes)

        for j in range(num_nodes - i):
            # Calculate ith divided differences
            ith_dd[j] = (dd_table[i - 1, j + 1] - dd_table[i - 1, j]) / (
                x[j + i] - x[j]
            )

        # Append the ith divided difference row to the table
        dd_table = np.vstack([dd_table, ith_dd])

    # Extract coefficients (first column of the table)
    a = np.array([dd_table[i, 0] for i in range(dd_table.shape[0])])
    return a

def generate_lagrange(nodes, degree):
    # Get coefficients
    a = lagrange_coefficients(nodes)
    # Start with function and constant term
    equation = f"P_{{{degree}}}(x) = {a[0]}"
    w = []
    # Build (x - xi) terms
```

```python
    for xi in nodes:
        if abs(xi) <= 1e-14:
            w.append(f'({xi})')
        elif xi < 0:
            w.append(f"(x + {abs(xi)})")
        else:
            w.append(f"(x - {xi})")
    b = ""

    # Build polynomial string
    for i in range(1, len(a)):
        for j in range(i):
            # Multiply (x - xi) terms
            b += w[j]
        # Multiply (x - xi) product with current coefficient and add term
        if a[i] >= 0:
            equation += f" + {a[i]}{b}"
        else:
            equation += f" - {abs(a[i])}{b}"
        b = ""

    return equation

# Sample Execution
if __name__ == "__main__":
    # Generate 6 nodes for 5th degree polynomial
    nodes = np.linspace(-1, 1, 6)
    equation = generate_equation(nodes, 5)
    with open("./plots_2/q2_2/lagrange_equations.txt", "w") as file:
        file.write(equation + "\n")
```

The interpolation polynomial $P_5(x)$ at equidistant nodes for $n = 5$ is given by:

$$
\begin{aligned}
P_5(x) = 0.047619047619047616 \\
&+ 0.18583042973286878(x + 1.0) \\
&+ 1.1227255129694158(x + 1.0)(x + 0.6) \\
&- 2.0647825525874324(x + 1.0)(x + 0.6)(x + 0.19999999999999996) \\
&+ 1.2904890953671473(x + 1.0)(x + 0.6)(x + 0.19999999999999996) \\
&(x - 0.20000000000000018) \\
&- 3.552713678800501e - 15(x + 1.0)(x + 0.6)(x + 0.19999999999999996) \\
&(x - 0.20000000000000018)(x - 0.6000000000000001)
\end{aligned}
$$

Question 2.2

Plot $f(x)$ and $P_n(x)$, and report the behavior of the interpolation error as n increases from 5 to 10 and 20.

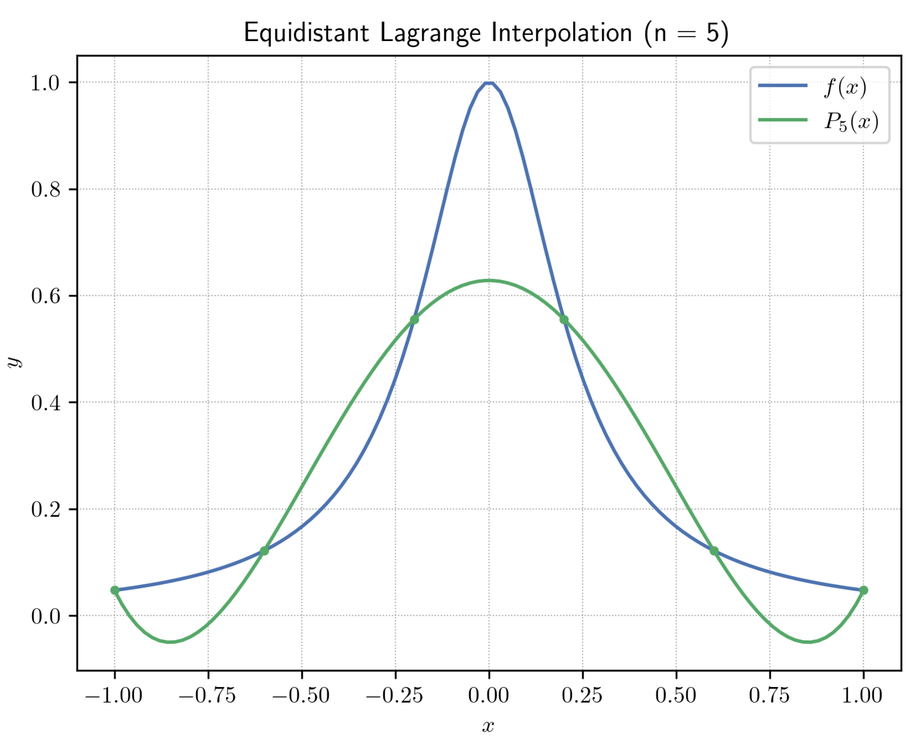
Solution. In addition to the functions defined in the previous part, the following functions were used to help plot $f(x)$ and $P_n(x)$:

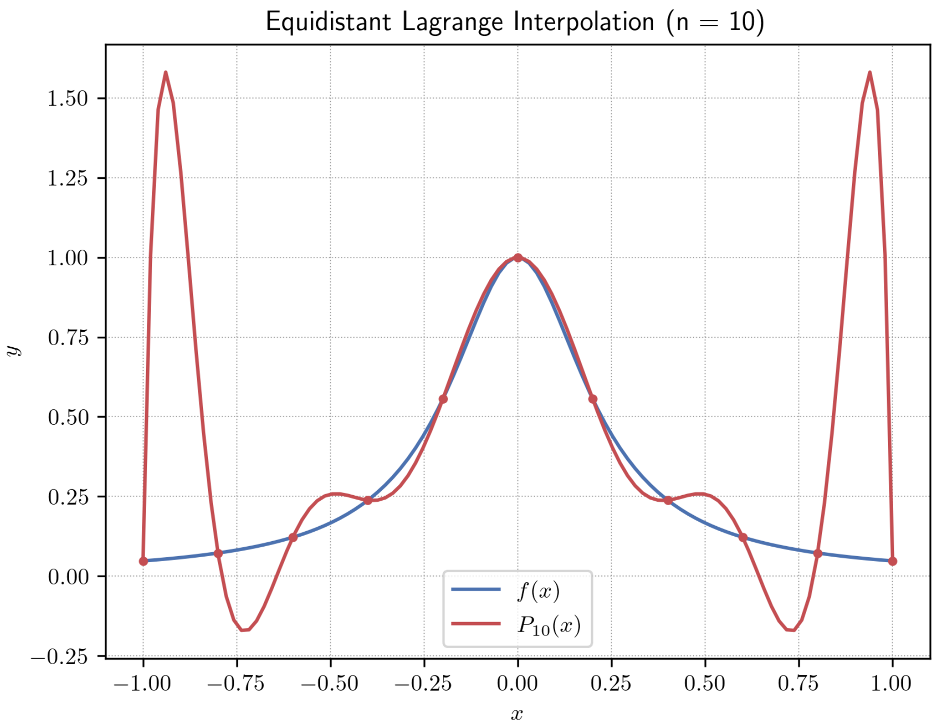
Listing 2: 2.2 Python

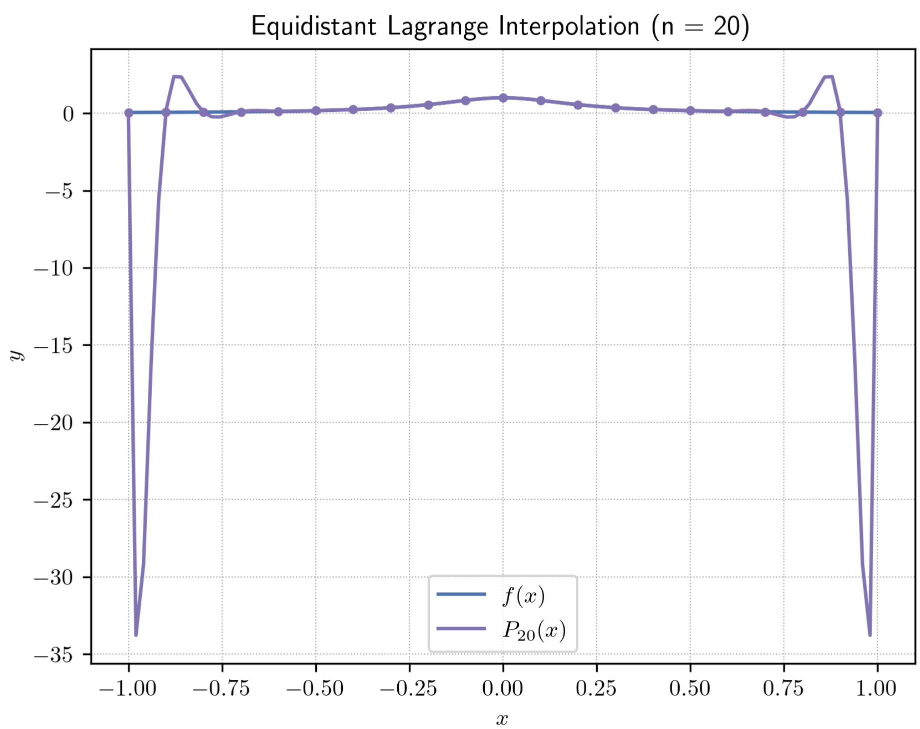
```
import numpy as np
import matplotlib.pyplot as plt
# I created a Figure class for figure management.
# See figure.py at end of this pdf for implementation.
from figure import Figure

def calculate_lagrange(nodes, a, x):
    # Start with constant term
    y = a[0]
    # Build (x - xi) terms
    w = [(x - xi) for xi in nodes]
    # Temporary variable to hold (x - xi) product
    b = 1

    for i in range(1, len(a)):
        for j in range(i):
            # Multiply (x - xi) terms
            b *= w[j]
        # Multiply (x - xi) product with current coefficient
        y += a[i] * b
        b = 1

    return y

def calculate_lagrange_output(nodes, x_coords=[]):
    # Get coefficients
    a = lagrange_coefficients(nodes)
    # If no x_coords provided, use nodes as x_coords
    if len(x_coords) == 0:
        x_coords = nodes
    # Calculate y coordinates for each x coordinate
    y_coordinates = np.array([calculate_lagrange(nodes, a, x) for x in x_coords
    ])

    return y_coordinates

# Sample Execution
if __name__ == "__main__":
```

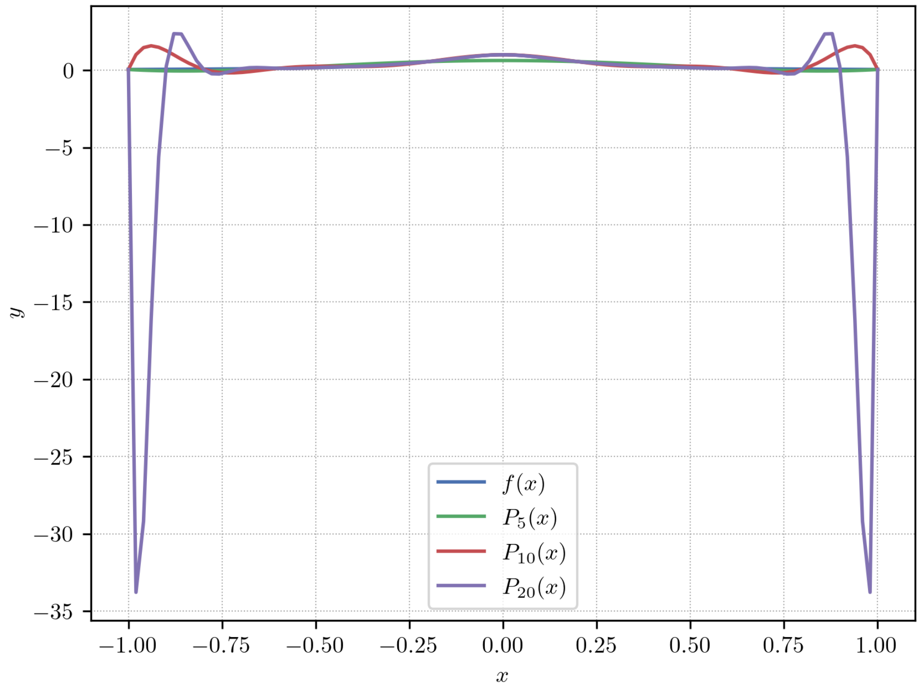
```python
    fig_f.title =  "Equidistant Lagrange Interpolation"
    x = np.linspace(-1, 1, 100)
    f_x = f(x)
    nodes_x = np.linspace(-1, 1, 6)
    nodes_y = calculate_lagrange_output(nodes_x)
    p_5 = calculate_lagrange_output(nodes_x, x)

    # Figure attributes:
    # x_sets, y_sets, labels, colors(indexes from default rotation list),
    # markers, linestyles, title, xlabel, ylabel
    # single or multiple axes can be added during initialization
    fig_f = Figure(x, f_x, r"$f(x)$")
    fig_p5 = Figure(x, p_5, r"$P_5(x)$", 1)
    fig_nodes = Figure(nodes_x, nodes_y, "", 1, ".", "")

    # Returns a new Figure that merges fig_f, fig_p5, and fig_nodes.
    # The title of the new figure is title + " (n = 5)"
    fig = fig_f.copy().merge([fig_p5, fig_nodes], title + " (n = 5)")

    # Returns the matplotlib figure and saves png with 300 dpi
    fig.get_figure("./plots_2/q2_2/p5.png")
    plt.show()
```

The following plots show $f(x)$ and $P_n(x)$ for $n = 5, 10, 20$:




Equidistant Lagrange Interpolation (n = 10)
y
x
$f(x)$
$P_{10}(x)$
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00


Equidistant Lagrange Interpolation (n = 20)
y
x
$f(x)$
$P_{20}(x)$
0
−5
−10
−15
−20
−25
−30
−35
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00



As n increases from 5 to 10 and 20, we see that the interpolation error decreases towards the center of the interval $[-1, 1]$. However, the error increases significantly near the edges of the interval, demonstrating Runge's phenomenon.

Question 2.3

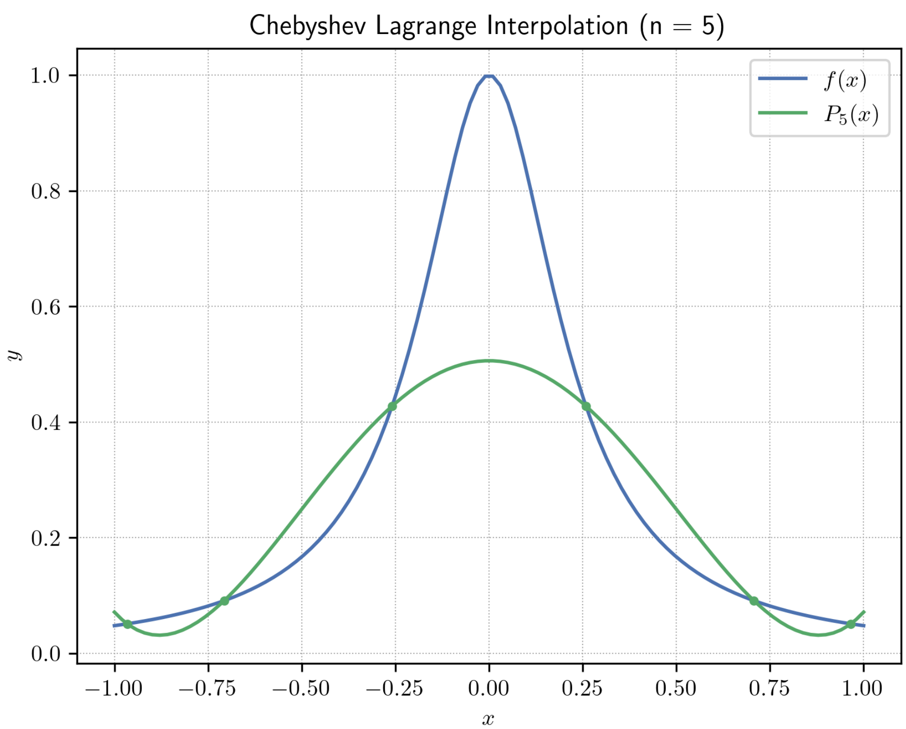
Repeat the interpolation using Chebyshev nodes, and compare the results with the equidistant-node case.

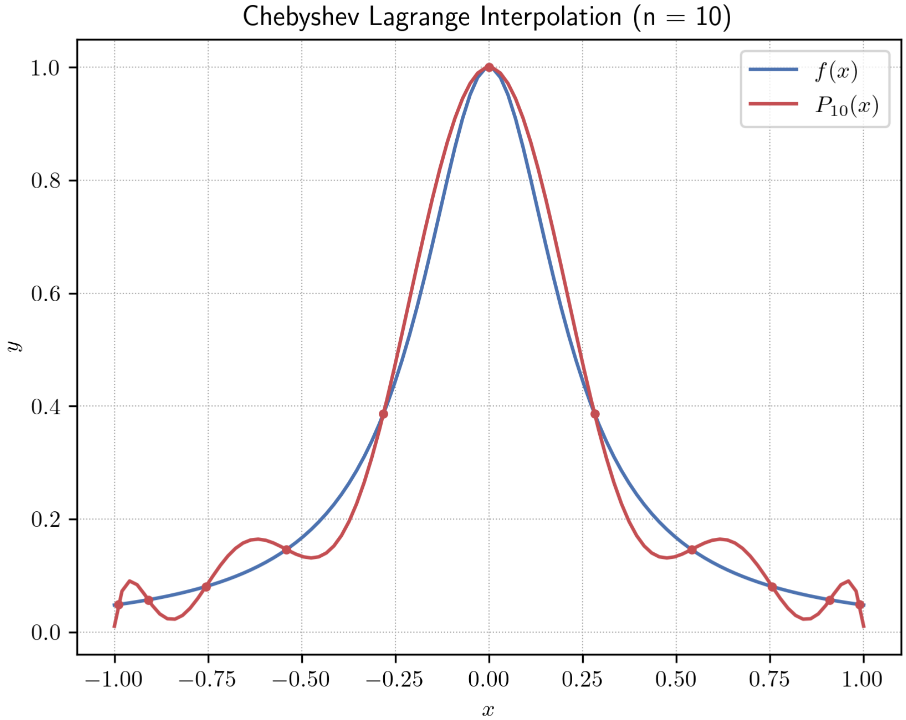
Solution. In addition to the functions defined in the previous parts, the following function was used to generate the Chebyshev nodes:

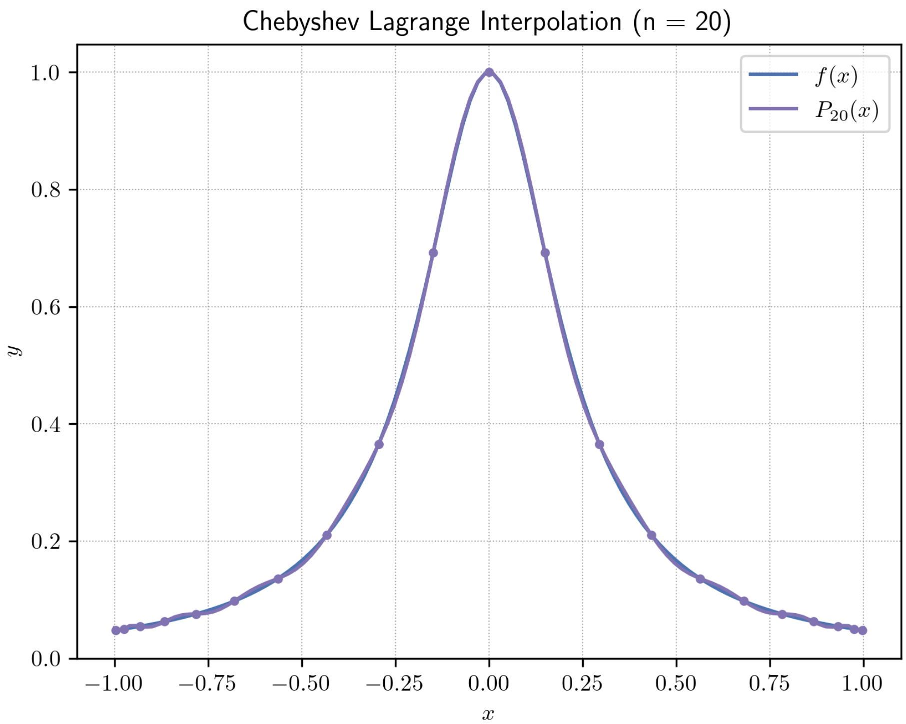
Listing 3: 2.3 Python

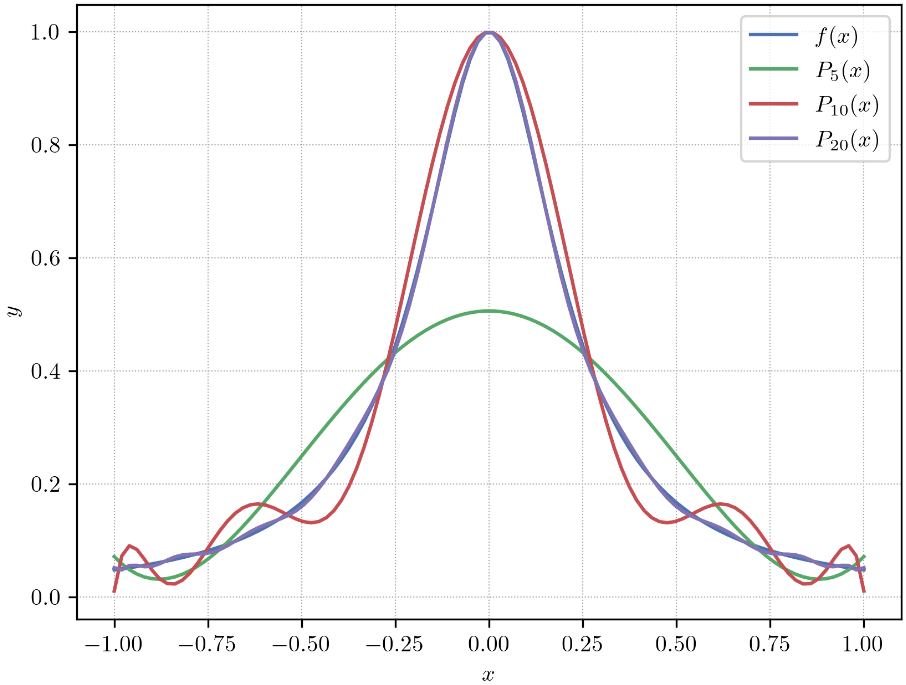
```
import numpy as np

def chebyshev_nodes(n):
    nodes = np.array(
        [(math.cos((2 * k - 1) * math.pi / (2 * n))) for k in range(1, n + 1)]
    )

    return nodes
```

Th following plots show $f(x)$ and $P_n(x)$ for Chebyshev nodes with $n = 5, 10, 20$:




Chebyshev Lagrange Interpolation (n = 10)
$f(x)$
$P_{10}(x)$
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00


Chebyshev Lagrange Interpolation (n = 20)
$f(x)$
$P_{20}(x)$
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00



Comparing these plots to those with equidistant nodes, we see that using Chebyshev nodes significantly reduces the interpolation error across the entire interval $[-1, 1]$. The oscillations near the edges of the interval are much less pronounced, demonstrating that Chebyshev nodes help mitigate Runge's phenomenon and provide a more accurate approximation of $f(x)$.

3 - Chebyshev Polynomials and Their Roots

The Chebyshev polynomials of the first kind is defined by:

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1]$$

Question 3.1

Prove that the numbers

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n$$

are the roots of $T_n(x)$.

Proof. We want to find the values of x such that $T_n(x) = 0$. By the definition of Chebyshev polynomials, we have:

$$T_n(x) = \cos(n \arccos x)$$

Setting $T_n(x) = 0$, we get:

$$T_n(x) = \cos(n \arccos x) = 0$$

The cosine function is zero at odd multiples of $\frac{\pi}{2}$. So $T_n(x)$ can only equal zero when $n \arccos x$ is an odd multiple of $\frac{\pi}{2}$. Therefore, we can write:

$$n \arccos x = \frac{(2k-1)\pi}{2}, \quad k \in \mathbb{Z}$$

Note that for $n = 0$, $T_0(x) = \cos(0 \cdot \arccos(x)) = 1$ which has no roots. So consider $n \geq 1$. Dividing both sides by n, we have:

$$\arccos x = \frac{(2k-1)\pi}{2n}, \quad k \in \mathbb{Z}$$

Taking the cosine of both sides, we get:

$$x = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k \in \mathbb{Z}$$

Therefore, the set of x defined above are the roots of $T_n(x)$ for $n \geq 1$. Note that if we restrict k to the integers $1, 2, \ldots, n$, we exactly get that:

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n$$

are the roots of $T_n(x)$. □

Question 3.2

Show that these roots are distinct and lie in the interval $(-1, 1)$.

Proof. To show that these roots are distinct, we need to show that for $k_1 \neq k_2$, we have $x_{k_1} \neq x_{k_2}$. Assume $k_1, k_2 \in \{1, 2, \ldots, n\}$ such that $k_1 \neq k_2$ and $k_1 < k_2$. Then, assuming for fixed $n \geq 1$:

$$x_{k_1} = \cos\left(\frac{2k_1 - 1}{2n}\pi\right), \quad x_{k_2} = \cos\left(\frac{2k_2 - 1}{2n}\pi\right)$$

Note that $\frac{2k_1-1}{2n}\pi$ and $\frac{2k_2-1}{2n}\pi$ are distinct angles in the interval $(0, \pi)$. Since the cosine function is strictly decreasing on the interval $[0, \pi]$, and since $\frac{2k_1-1}{2n}\pi < \frac{2k_2-1}{2n}\pi$ are distinct, we have:

$$x_{k_1} = \cos\left(\frac{2k_1 - 1}{2n}\pi\right) > \cos\left(\frac{2k_2 - 1}{2n}\pi\right) = x_{k_2}$$

By the law of trichotomy, $x_{k_1} \neq x_{k_2}$. Therefore, the roots x_k are distinct for $k = 1, 2, \ldots, n$.

Next, we need to show that these roots lie in the interval $(-1, 1)$. Note that from above, we have shown that x_k is a strictly decreasing sequence. Therefore, we only need to show that the largest root x_1 is less than 1 and the smallest root x_n is greater than -1.

First, let us show that $x_1 < 1$:

$$x_1 = \cos\left(\frac{2(1) - 1}{2n}\pi\right) = \cos\left(\frac{\pi}{2n}\right)$$

Note that $\frac{\pi}{2n} \in (0, \frac{\pi}{2}]$. In this interval, the range of the cosine function is $[0, 1)$. Therefore, we have:

$$x_1 = \cos\left(\frac{\pi}{2n}\right) < 1 \tag{1}$$

Next, let us show that $x_n > -1$:

$$x_n = \cos\left(\frac{2n - 1}{2n}\pi\right) = \cos\left(\pi - \frac{\pi}{2n}\right)$$

Note that $\pi - \frac{\pi}{2n} \in [\frac{\pi}{2}, \pi)$. In this interval, the range of the cosine function is $(-1, 0]$. Therefore, we have:

$$x_n = \cos\left(\pi - \frac{\pi}{2n}\right) > -1 \tag{2}$$

Combining (1) and (2), we have shown that:

$$-1 < x_n < x_k < x_1 < 1$$

Therefore, x_k are distinct roots that lie in the interval $(-1, 1)$ for $k = 1, 2, \ldots, n$. $\square$

Question 3.3

Write your own code to generate $T_n(x)$ using the recursion formula.

Solution. The following code was used to generate the Chebyshev polynomials::

Listing 4: 3.3 Python

```python
import sympy as smp

def chebyshev_poly(n):
    # Base Cases
    if n == 0:
        return 1
    elif n == 1:
        return smp.symbols("x")
    # Recursive Case
    else:
        return 2 * smp.symbols('x') * chebyshev_poly(n - 1) - chebyshev_poly(n
    - 2)

def generate_chebyshev(n):
    x = smp.symbols('x')
    poly = smp.expand(chebyshev_poly(n))
    function = f"T_{{{n}}}({x})"
    equation = function + " = " + smp.latex(poly)
    return equation

# Sample Execution
if __name__ == "__main__":
with open("./plots_2/q3_3/chebyshev_polynomials.txt", "w") as file:
    for n in range(6):
        equation = generate_chebyshev(n)
        file.write(equation + "\n")
```

The Chebyshev polynomials $T_n(x)$ for $n = 0, 1, \dots, 5$ are given by:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$

4 - Lagrange Interpolation for Nonsmooth Function

Let $f(x) = |x|$ on the interval $[-1, 1]$.

Question 4.1

Construct the interpolation polynomial $P_n(x)$ for equidistant nodes when n is even.

Question 4.2

Show that $P_n(x)$ is an even polynomial.

Question 4.3

Investigate analytically (for small n) how well $P_n(x)$ approximates $f(x)$.

Question 4.4

Discuss why convergence may be slower for nonsmooth functions.