Numerical Analysis Homework 2

Due day: Sep 22, 2025

Please prepare your solutions in latex and submit the compiled PDF. A easy tool for latex writing is using "overleaf.com" website. Here is the link to a less than 30 mins self tutorial about latex writing:

https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes

Code and pictures should be included using the latex template as follows (Submission of separate code files is not required.):

Include the following packages in latex head:

```
\usepackage{listings}
\usepackage{graphicx}
```

Inside the *document* section, you can display your code by writing:
<u>For python code:</u>

```
\begin{lstlisting}[caption=Python code]
for x in range(5)
   print(x)
\end{lstlisting}
```

<u>for Matlab code:</u>

```
\begin{lstlisting}[caption=Matlab code]
for x = 0:4
    disp(x)
end
\end{lstlisting}
```

<u>To include figure:</u>

```
\begin{figure}[h]
    \centering
    \includegraphics[width=0.8\textwidth]{figure_name.png}
    \caption{Figure caption}
\end{figure}
```

Question 1. Lagrange Interpolation and Error Analysis

Let $f \in C^{n+1}[a,b]$, and let P_n be its Lagrange interpolating polynomial through distinct nodes $x_0, \ldots, x_n \in [a,b]$. The interpolation error is:

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \qquad \xi \in (a,b).$$

Suppose $f(x) = e^x$ on $[0,1]$. For equally spaced nodes $x_i = i/n$:

1. Derive an explicit bound for the maximum error $\max_{x \in [0,1]} |R_n(x)|$.
2. Show the asymptotic decay of this error as $n \to \infty$.

Question 2. Interpolation Programming Exercise

Consider the function:

$$f(x) = \frac{1}{1 + 20x^2}, \quad x \in [-1,1].$$

1. Construct the interpolation polynomial $P_n(x)$ at equidistant nodes for $n = 5$.
2. Plot $f(x)$ and $P_n(x)$, and report the behavior of the interpolation error as n increases from 5 to 10 and 20.
3. Repeat the interpolation using Chebyshev nodes, and compare the results with the equidistant-node case.

You may use any programming language of your choice.

Question 3. Chebyshev Polynomials and Their Roots

The Chebyshev polynomial of the first kind is defined by:

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1,1].$$

1. Prove that the numbers
   $$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n,$$
   are roots of $T_n(x)$.
2. Show that these n roots are distinct and lie in the interval $(-1,1)$.
3. Write your own code (without calling built-in Chebyshev routines) to generate $T_n(x)$ using the recursion formula. Hint, define a function such as $T_n = \mathit{chebyshev}(n)$, to calculate T_n, you call $T_n = 2x * \mathit{chebyshev}(n-1) - \mathit{chebyshev}(n-2)$.
4. Plot the first six polynomials $T_0(x), T_1(x), \ldots, T_5(x)$ on the interval $[-1,1]$.

Question 4. Lagrange interpolation for nonsmooth function

Let $f(x) = |x|$ on $[-1,1]$.

1. Construct the Lagrange interpolation polynomial $P_n(x)$ for equidistant nodes when n is even.
2. Show that $P_n(x)$ is an even polynomial.
3. Investigate analytically (for small n) how well $P_n(x)$ approximates $f(x)$.
4. Discuss why convergence may be slower for nonsmooth functions.