Menger's Theorem for Infinite Graphs

Dominic Welsh Prize Essay

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Abstract

In the Part B graph theory course we proved a formulation of Menger's theorem with a slick application of the max-flow-min-cut theorem. In this essay, we present a constructive proof of a slight generalisation of that theorem and then move to present the relevant definitions required for a rough introduction to infinite graph theory and a discussion of Aharoni and Berger's proof of a version of Menger's theorem for infinite graphs.

Acknowledgments

This essay inevitably draws greatly from the work of Aharoni and Berger found at arXiv and thanks are also given to Oliver Riordan for his comprehensive lecture notes for the B8.5 Graph Theory course and to Natasha Morrison and Renee Hoekzema for their teaching and feedback in classes for the course.

This essay was written in XƎLaTeX and the source file based upon a dissertation template by Federico Maggi. This template was released under the Creative Commons BY-NC-SA 3.0 license, in accordance with which the source files for the essay are available, under the same license, at my GitHub repository.

Contents

Introduction 1

[2] [1]

Definition 1.1. *A path P in a graph G is a sequence $v_0 v_1 \dots v_n$ of distinct vertices v_i of G such that $v_{i-1} v_i$ is an edge in the graph for each $1 \leq i \leq n$. We say P is an $x - y$ path if $v_0 = x$ and $v_n = y$.*

max-flow-min-cut

The First Proof 2

In this chapter we will recount the way in which Menger's Theorem was presented in the Part B lecture course and the proof as an application of the max-flow-min-cut theorem.

2.1 Measures of Connectedness

Menger's Theorem is an equality between two measures of connectedness within a graph, in this section we build to the definitions of the measures required to state the theorem.

Definition 2.1. *For a graph G, a set $S \subseteq V(G)$* ***separates*** *G if the graph $G - S$ obtained by removing the vertices of S and all incident edges from G is disconnected.*

For example, when G is the cyclic graph C_n any non-empty set $S \subseteq V(G)$ separates G, and in the case where $G = K_3$, then it is clear that any set S with $|S| \geq 2$ will be separating.

Definition 2.2. *A graph G with $|G| > k$ is* ***k-connected*** *if there is no set $S \subseteq V(G)$ with $|S| = k - 1$ such that S separates G.*

Observe that this definition entails that every graph is 0-connected and that a graph is 1-connected if and only if it is a connected graph with at least two vertices. Thus, k-connectedness gives a generalization of the concept of a connected graph, and the maximum k such that a graph is k-connected - or, equivalently, the size of a minimal separating set - provides a natural measure of the connectedness of a graph.

Notation. *We denote by $\kappa(G)$ the maximum k such that G is k-connected*

$\kappa(G)$ gives us a measure of the connectedness of a graph on a global scale. We now extend this to a local measure of how connected two given points in the graph are with relation to removing vertices. Since two adjacent vertices - vertices joined by a single edge - can never be disconnected by removing other vertices we restrict ourselves to non-adjacent vertices and the following definition is as expected.

Definition 2.3. *A set $S \subseteq V(G)$ separates distinct, non-adjacent vertices x and y in G if x and y are in different connected components of $G - S$*

Notation. *We denote by $\kappa_G(x, y)$, or simply $\kappa(x, y)$, the minimum size of a set S separating x and y*

The statement that x and y are in different connected components of $G-S$ is equivalent to saying that there are no paths between x and y in the residual graph $G-S$. In lay terms one might measure the connectedness of two cities by how many different ways there are of travelling between them. For example in Figure 2.1

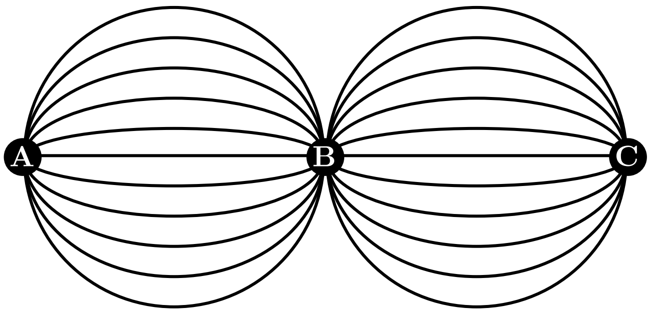

Figure 2.1: Demonstrating levels of connectedness

Definition 2.4. *A pair of $x-y$ paths are **independent** if the only vertices they share are x and y. A set of $x-y$ paths is independent if the paths are pairwise independent.*

As before the idea of connectedness we are seeking to capture is tied in with a dual notion of separation, for if we were to remove the interior (all vertices except the ends x and y) of each of the paths in a maximally independent set of $x-y$ paths then we would disconnect x and y within the graph - there would be no path between them in the remainder, else we could add it to our set of paths contradicting the sets maximality. Since adjacent vertices - those joined by a single edge in the graph - can never be disconnected by removing other vertices from the graph we exclude such vertices from our definition.

Notation. *hi*

Theorem 2.1 (Menger's Theorem)**.** *Let x and y be distinct non-adjacent vertices of a graph G. Then the maximum size of an independent set of x–y paths is $\kappa_G(x, y)$.*

Bibliography

[1] Ron Aharoni *and* Eli Berger. *Menger's Theorem for Infinite Graphs, 2007*

[2] Oliver Riordan. *Lecture Notes for B8.5 Graph Theory, 2015*