1 Introduction

A *graph* G consists of a set of *vertices* $V(G)$ and a set of *edges* $E(G)$

Each edge is associated with one or two vertices, its *ends*

An edge is a *loop* if the ends are the same, a *link* if the ends are different

Graphs are assumed to be *finite* - both $V(G)$ and $E(G)$ are finite sets

A *path* is a sequence $(a_0, A_1, a_1, A_2, a_2, \ldots, A_n, a_n)$ of vertices a_i and edges A_j such that:

1. If $1 \leq i \leq n$ the ends of A_i are a_{i-1} and a_i
2. If $1 \leq i \leq n$ then $a_{i-1} = a_i$ iff A_i is a loop

If all terms in path are distinct, then the path is *simple*

If all terms are distinct except that $a_0 = a_n$, then the path is *circular*

$x, y \in V(G)$ are *connected* if there is a path from x to y in G

Connectedness is an equivalence relation on $V(G)$ so that if $V(G) \neq \varnothing$ then it can be partitioned into disjoint non-null subsets $V_1, \ldots, V_k$ such that two vertices of G are connected iff they are in the same V_i

The $G[V_i]$ are the *connected components* of G, they are edge- and vertex-disjoint and cover G

The number of components of G is denoted $p_0(G)$

A graph is *connected* iff $p_0(G) = 0$ or 1, the first case arising if G is the empty graph

A connected graph with no circular path is a tree

$\alpha_0(G) = |G|$ and $\alpha_1(G) = e(G)$

Let Q_n be a finite set of $n > 0$ elements

$f : V(G) \to Q_n$ is an *n-colouring* of G if each edge xy of G has $f(x) \neq f(y)$

The number of n-colourings of G wrt Q_n is denoted by $P(G, n)$

If $V(G) = 0$ then we say $P(G, n) = 1$, also $P(G, n) = 0$ if G contains a loop

When G is loopless, $P(G, n)$ is a polynomial in n of degree $|G|$

For planar graphs, $P(G, n)$ is called the *chromatic polynomial* of G

$$P(G, n) = \sum_S (-1)^{e(S)} n^{p_0(S)} \qquad \text{(summing over spanning subgraphs } S \text{ of } G\text{)}$$

An *orientation* of G distinguishes one end of each edge A as positive, $p(A)$, and one as negative, $q(A)$

If A is a loop, then $p(A) = q(A)$ otherwise $p(A) \neq q(A)$

If $a \in V(G)$ and $A \in E(G)$ then $\eta(A, a) = 0$ if A is a loop or a is not an end of A.
Otherwise, $\eta(A, a) = 1$ or -1 depending as whether a is the positive or negative end of A

A mapping f of $V(G)$ or $E(G)$ into Q_n is a *0-chain* or *1-chain* respectively *on G over Q_n*

If $V(G) = \varnothing$ then there is just one 0-chain on G over Q_n

If $E(G) = \varnothing$ then there is just one 1-chain on G over Q_n

If h is a 0-chain on G over Q_n its *coboundary*, δh is the 1-chain on G over Q_n satisfying

$$(\delta h)(A) = \sum_a \eta(A, a)h(a) \tag{2}$$

for each $A \in E(G)$, equivalently

$$(\delta h)(A) = h(p(A)) - h(q(A)) \tag{2a}$$

If g is a 1-chain, its *boundary* δg is the 0-chain satisfying

$$(\delta g)(a) = \sum_A \eta(A, a)g(A) \tag{3}$$

for each $a \in V(G)$

We call g a *1-cycle* on G over Q_n if $\delta g \equiv \mathbf{0}$

2 Colour-coboundaries and colour-cycles

A *colour-coboundary* or *colour-cycle* on G over Q_n is a 1-chain g on G over Q_n which is a coboundary or a 1-cycle respectivly and which satisfies $g(A) \neq 0$ for each $A \in E(G)$

The number of colour-coboundaries of G over Q_n is denoted $\theta(G, n)$

The number of colour cycles on G over Q_n is denoted $\phi(G, n)$

$\theta(G, n)$ and $\phi(G, n)$ are independent of orientation

If $e(G) = 0$ then we say $\theta(G, n) = \phi(G, n) = 1$

By (2a), the colour-coboundaries on G over Q_n are the coboundaries of the n-colourings of G

Also, $\delta h_1 = \delta h_2$ for 0-chains h_1, h_2 iff $h_1(a) - h_2(a)$ is constant in each component of G, for all $A \in E(G)$:

$$h_1(p(A)) - h_1(q(A)) = h_2(p(A)) - h_2(q(A)) \iff h_1(p(A)) - h_2(p(A)) = h_1(q(A)) - h_2(q(A))$$