Journal of Combinatorial Theory, Series B 102 (2012) 521–529

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The weak 3-flow conjecture and the weak circular flow conjecture

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ARTICLE INFO

Article history:
Received 12 November 2010
Available online 13 October 2011

Keywords:
Orientations modulo k
Star decomposition
3-Flow conjecture

ABSTRACT

We show that, for each natural number $k > 1$, every graph (possibly with multiple edges but with no loops) of edge-connectivity at least $2k^2 + k$ has an orientation with any prescribed outdegrees modulo k provided the prescribed outdegrees satisfy the obvious necessary conditions. For $k = 3$ the edge-connectivity 8 suffices. This implies the weak 3-flow conjecture proposed in 1988 by Jaeger (a natural weakening of Tutte's 3-flow conjecture which is still open) and also a weakened version of the more general circular flow conjecture proposed by Jaeger in 1982. It also implies the tree-decomposition conjecture proposed in 2006 by Bárat and Thomassen when restricted to stars. Finally, it is the currently strongest partial result on the $(2+\epsilon)$-flow conjecture by Goddyn and Seymour.

1. Introduction

An orientation of a multigraph G is a *Tutte-orientation* if, for each vertex x, the indegree of x is congruent to the outdegree modulo 3. Following [1] we say that G *admits all generalized Tutte-orientations* if, for every prescribed integer function $p(x)$ defined on $V(G)$ such that the sum is congruent to $|E(G)|$ modulo 3, it is possible to direct all edges such that each vertex x has outdegree $p(x)$ modulo 3. Tutte's 3-flow conjecture says that every 4-edge-connected graph has a Tutte-orientation. Jaeger [6] suggested the weaker version where 4 is replaced by a larger number, and he called this the *weak 3-flow conjecture*.

Jaeger [5,6] generalized the 3-flow conjecture to the following conjecture which he called the *the circular flow conjecture*:

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0095-8956/$ – see front matter
doi:10.1016/j.jctb.2011.09.003

If k is an odd natural number, and G is a $(2k-2)$-edge-connected multigraph, then G has an orientation such that each vertex has the same indegree and outdegree modulo k.

This conjecture would not be true when k is an even number (because a vertex of odd degree cannot be balanced modulo an even number) also not in the weak version where we replace the edge-connectivity $2k-2$ by a larger function of k. However, the weak version becomes true for generalized orientations provided we add the obvious additional condition on the prescribed outdegrees, as Theorem 2 below shows.

Also, we prove that every 8-edge-connected multigraph admits all generalized Tutte-orientations. In particular, it has a Tutte orientation, and, if it has no multiple edges and size divisible by 3, its edges can be decomposed into claws, as conjectured in [1].

2. Notation and terminology

The terminology and notation are essentially the same as [2,3,7]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. If G is a multigraph, and S is a set of vertices, then the subgraph $G(S)$ induced by S has vertex set S and contains precisely those edges in G which join two vertices of S. If v is a vertex in a graph G, then the *degree* of v is denoted $d(v,G)$ (or just $d(v)$). More generally, if A is a set of vertices in G, then the *degree* of A, which is denoted $d(A,G)$ (or just $d(A)$) is the number of edges with precisely one end in A.

If xy, xz are edges and y, z are distinct, then the deletion of the edges xy, xz and addition of the edge yz is called a *lifting at* x.

3. The set function $t(A)$

Our main result is proved by induction. The main idea is to introduce a set function $t(A,G)$ (or just $t(A)$) which will make the induction work. Let k be a natural number, $k \geqslant 2$. Let G be a multigraph, and let z_0 be a vertex in G. Each edge incident with z_0 is directed, and all other edges are undirected. For each vertex x in G, let $p(x)$ be an integer. Assume that $p(z_0) = d^+(z_0)$. We wish to direct all undirected edges of G such that, for each vertex x distinct from z_0, the outdegree $d^+(x)$ is congruent to $p(x)$ modulo k. If A is a vertex set not containing z_0, we define $p(A)$ as the sum (reduced modulo k) of $p(x)$ (taken over all x in A) minus the number of edges in $G(A)$, the subgraph induced by A. The motivation for this definition is that the number of edges leaving A and directed away from A must be congruent to $p(A)$ modulo k.

Let A be a vertex set not containing z_0. We now define $t(A)$. We first consider the case where A is a vertex x of degree at least k such that x is not adjacent to z_0. Recall that we wish x to have outdegree $p(x)$ modulo k. This can be achieved as follows: We first select some (say m) edges incident with x such that $d(x) - m$ is even. We direct all those m edges in the same direction. (That is, either all are directed away from x or all are directed towards x.) Then we direct half of the remaining edges towards x and the other half away from x. If m is chosen such that it is minimum, then the number m is the *type* of x and is denoted $t(x)$. If $m > 0$ and the m edges are directed towards (respectively away from) x, we call x *negative* (respectively *positive*). Clearly, $t(x)$ has the same parity as $d(x)$, and $t(x)$ is one of the integers $0, 1, \dots, k$. For example, if $k = 3$, $d(x) = 5$, $p(x) = 0$, then we can direct all edges incident with x towards x. That is, we can have $m = 5$. On the other hand, we can also have $m = 1$, so $t(x) = 1$ in this case, and x is positive. If $k = 3, d(x) = 5, p(x) = 1$, then $t(x) = 3$, and x is both positive and negative. If x is joined to z_0 by many edges, then it may not be possible to orient the undirected edges incident with x in this way, but we still use the above definition of $t(x)$ (ignoring the prescribed orientations of the edges between z_0 and x). If A is a vertex set not containing z_0, then we define $t(A)$ and the sign of A in the same way (by treating A like a single vertex).

We can also give an explicit expression of $t(A)$ but we shall not use it in the proof. If $d(A)$ is even, and $p(A) = d(A)/2$, then $t(A) = 0$. If A is positive, then $t(A) \equiv 2p(A) - d(A) \pmod k$. If A is negative, then $t(A) \equiv d(A) - 2p(A) \pmod k$.

4. Orientations modulo 3: The weak 3-flow conjecture

Before we address orientations modulo 3, we note that orientations modulo 2 are easy: For every connected multigraph G and for every prescribed integer function $p(x)$ defined on $V(G)$ such that the sum is congruent to $|E(G)|$ modulo 2, it is possible to direct all edges such that each vertex x has outdegree $p(x)$ modulo 2. Modifying an argument, which I believe is due to Anton Kotzig, we first orient the edges at random. If x is a vertex of wrong outdegree, then there is another vertex y of wrong outdegree. Then take a path (in the undirected sense) from x to y and change all its directions. That decreases the number of vertices of wrong outdegree by 2, and we repeat the argument.

We now consider orientations modulo 3. We consider a multigraph G having a vertex z_0 of degree at most 11 such that all edges incident with z_0 are directed. All other edges are undirected but we want to give all those edges a direction such that each vertex x gets a prescribed outdegree $p(x)$ modulo 3. Assume that $p(z_0) = d^+(z_0)$. Consider the case where $G - z_0$ is connected, and the vertices of $G - z_0$ can be divided into two sets A, B such that there is precisely one edge e between A and B. Suppose also that $p(A)$ is congruent to the number of edges from A to z_0 minus 2. Then it is not possible to direct the undirected edge of G leaving A. We call e a *problematic bridge* in $G - z_0$.

Theorem 1. *Assume that the multigraph G satisfies the following*:

(i) $|V(G)| \geqslant 3$.
(ii) $1 \leqslant d(z_0) \leqslant 11$.
(iii) *For each nonempty vertex set A not containing z_0 and satisfying $|V(G) \setminus A| > 1$, we have $d(A) \geqslant 6 + t(A)$.*
(iv) *The sum of $p(x)$ taken over all vertices of G (including z_0) is congruent to the number of edges of G modulo* 3.

Then all edges not incident with z_0 can be directed such that, for each vertex x distinct from z_0, $d^+(x) \equiv p(x) \pmod 3$ unless $G - z_0$ has a problematic bridge.

Proof. The proof is by induction on the number of edges of G. Assume (reductio ad absurdum) that G is a smallest counterexample. By conditions (ii), (iii), G and $G - z_0$ are connected.

Claim 1. *For any two vertices x, y distinct from z_0, there is at most one edge joining x, y.*

Proof. Suppose (reductio ad absurdum) that there are q edges joining x, y where $q \geqslant 2$. If $|V(G)| > 3$, then we contract x, y into a vertex z, and we use induction with $p(z) = p(x) + p(y) - q$. Then we direct the edges joining x, y such that x gets the desired outdegree $p(x)$ modulo 3. Then y automatically gets the desired outdegree $p(y)$ modulo 3. □

Claim 2. $|V(G)| > 4$.

Proof. Assume first that $|V(G)| = 3$. We have previously noted that $G - z_0$ is connected, so there is at least one edge between x, y. By Claim 1, there is precisely one edge between x, y. As this edge is not a problematic bridge of $G - z_0$, it is possible to orient this edge.

Assume next that $|V(G)| = 4$. By Claim 1, each vertex in $G - z_0$ has degree at most 2 in that graph. Then each vertex in $G - z_0$ is joined to z_0 by at least 4 edges. But this contradicts the condition that z_0 has degree at most 11. □

Claim 3. *If A is a vertex set not containing z_0 such that $|A| > 1$, and $|V(G) \setminus A| > 1$, then $d(A) \geqslant 12$ unless A consists of two neighboring vertices such that one has degree* 6 *and the other has degree* 6 *or* 7.

Proof. Assume $d(A) < 12$. Then we first contract A and use induction. In particular, all edges from A to $V(G) \setminus A$ receive a direction. Then we contract instead $V(G) \setminus A$ into a single vertex which we think of as a new z_0, and then again we use induction (if possible) to also direct the edges in $G(A)$. We may assume that this induction is not possible. Then $G(A)$ has a bridge e which becomes problematic when we have directed the edges not in $G(A)$. We claim that $|A| = 2$. Suppose therefore (reductio ad absurdum) that $|A| > 2$. Then one of the two components of $G(A) - e$ has a vertex set A' with more than one vertex. By condition (iii), there are at least 6 edges leaving the other component of $G(A)$. Hence there are at most 7 edges leaving A'. We now repeat the argument above with A' instead of A. This time we obtain a contradiction because $G(A')$ cannot have a bridge. This proves the claim that $|A| = 2$. As $d(A) < 12$ and both vertices of A have degree at least 6, the two vertices of A are joined by at least one edge (and hence precisely one edge by Claim 1), and the two vertex degrees of A are $6, 6$ or $6, 7$, respectively. □

Claim 4. *Every vertex x distinct from z_0 has at least two neighbors, and, $d(x) = 6 + t(x)$. In particular, the degree of x is one of* $6, 7, 8, 9$.

Proof. As noted earlier, $t(x)$, $d(x)$ have the same parity. Suppose (reductio ad absurdum) that either x has at most one neighbor or $d(x) \geqslant 6 + t(x) + 2$ (or both).

Assume that x has been chosen such that it has maximum degree under this condition.

Consider first the case where x has only one neighbor z. The number of edges from x to z is at least 6, by condition (iii). The number of edges from z to vertices distinct from x is at least 6 by applying condition (iii) to $\{x, z\}$ (if $z \neq z_0$) or $V(G) \setminus \{x, z\}$ (if $z = z_0$). So $d(z) \geqslant 12$.

If $z = z_0$, this contradicts condition (ii), and if $z \neq z_0$ it contradicts the maximality of the degree of x.

Consider next the case where x has at least two neighbors. Let y, z be distinct neighbors of x. (One of y, z may equal z_0.) Now delete an edge from x to y, and an edge from x to z, and add an edge from y to z. We apply induction to the resulting graph G' (with $p(x) - 1$ instead of $p(x)$). To show that this is possible, we first observe that G' satisfies condition (iii) when A is a singleton (by the assumption on the degree of x). Condition (iii) is also satisfied if A is not a singleton because of Claim 3. (Claim 3 implies that $d(A) \geqslant 10$ before the lifting, and hence $d(A) \geqslant 8$ after the lifting. This is sufficient to satisfy (iii), because $t(A) \leqslant 3$, and $t(A)$, $d(A)$ have the same parity.)

Finally, we claim that $G' - z_0$ cannot have a problematic bridge. For otherwise, $V(G) \setminus \{z_0\}$ can be divided into sets A, B such that G' has only one edge between A, B, and hence G has at most 3 edges between A, B. We choose the notation such that G has at most 5 edges from z_0 to A. By Claim 3, A is a singleton u. As u has degree at least 6, there are in G at least 3 edges from z_0 to u and hence at most 8 edges from z_0 to B and hence at most 11 edges leaving B in G. But, Claims 2, 3 imply that there are at least 12 edges leaving B.

This contradiction to the minimality of G proves Claim 4. □

Claim 5. *For every vertex x distinct from z_0, the number of edges joining x, z_0 is less than $d(x)/2 = 3 + t(x)/2$.*

Proof. Let m be the number of edges joining x, z_0. The number of edges from $\{x, z_0\}$ to $V(G) \setminus \{x, z_0\}$ is $d(z_0) + d(x) - 2m \leqslant 11 + d(x) - 2m$. By Claims 2, 3 that number of edges is at least 12. Hence $2m \leqslant d(x) - 1$. □

Claim 6. *If x is a vertex distinct from z_0, then $t(x) > 0$.*

Proof. If $t(x) = 0$, then $d(x) = 6$, and we lift successively the edges incident with x. To see that this is possible, we first use Claims 5, 1 to conclude that no edge-multiplicity of an edge incident with x is greater than the sum of the other edge-multiplicities of edges incident with x. We then lift arbitrarily until some edge-multiplicity of an edge incident with x equals the sum of the other edge-multiplicities. Then there is only one way of performing the remaining liftings. We then apply induction to the resulting graph G'. To see that this is possible it suffices to verify condition (iii) and show that $G' - z_0$ has no problematic bridge. To verify (iii), consider a vertex set A' in G' which does not contain z_0 and also does not contain all vertices of $G' - z_0$. In G we consider the sets A', $A' \cup \{x\}$. We let A denote the one having the smallest number of edges leaving the set. Then

$$d(A, G) \leqslant d\big(A', G'\big) + 2,$$

and hence

$$d\big(A', G'\big) \geqslant 8,$$

by Claim 3. As $d(A', G'), t(A', G')$ have the same parity, G' satisfies condition (iii).

Finally, we claim that $G' - z_0$ cannot have a problematic bridge. For otherwise, $V(G') \setminus \{z_0\}$ can be divided into sets A, B such that G' has only one edge between A, B. We choose the notation such that G' has at most 5 edges from z_0 to A. By the argument in the preceding paragraph (where we verified condition (iii)), A is a singleton u. As u has degree at least 6, there are in G' at least (and hence precisely) 5 edges from z_0 to u and hence at most 6 edges from z_0 to B. Hence there are at most 7 edges leaving B in G'. But, when we verified condition (iii) above, we proved that there are at least 8 edges leaving B in G'.

This contradiction proves Claim 6. □

Claims 6, 4 imply that the degree of x is one of $7, 8, 9$, and hence the last assertion in Claim 3 cannot occur.

Consider now a vertex x distinct from z_0 such that the number m say, of edges between x and z_0 is smallest possible (possibly $m = 0$).

Assume that x is of type i where $1 \leqslant i \leqslant 3$. Then x has degree $6 + i$, by Claim 4. By Claims 1 and 5, the vertex x has $q > 3 + i/2 > 3$ neighbors distinct from z_0. The minimality of m then implies that

$$m \leqslant d(z_0)/(q + 1) \leqslant 11/4,$$

and hence $m \leqslant 2$. This implies that $q \geqslant 5$ because $d(x) \geqslant 7$. The minimality of m now implies that $m \leqslant 1$ because $d(z_0) < 12$. Hence $q \geqslant 6$.

Assume without loss of generality that x is negative. Now x has either i positive neighbors which are pairwise distinct and also distinct from z_0, or else x has $6 - i$ negative neighbors which are pairwise distinct and also distinct from z_0.

In the former case we select i distinct positive neighbors of x and direct (and delete) the edge from each of these towards x. For each such neighbor y, we replace $p(y)$ by $p(y) - 1$. Then also $t(y)$ is reduced by 1. In the latter case we select $6 - i$ distinct negative neighbors of x and direct (and delete) the edge from each of these away from x. For each such neighbor y of x, $p(y)$ is unchanged but $t(y)$ is reduced by 1. In either case $d(y), t(y)$ are reduced by 1. We lift the remaining edges incident with x. It is possible to lift the remaining edges incident with x because $m \leqslant 1$, that is, x is not incident with any multiple edge.

We apply induction to the resulting graph G' where the prescribed outdegrees $p(y)$ have been modified accordingly.

To see that induction is possible, we need only verify the condition (iii) and show that $G' - z_0$ has no problematic bridge. To prove (iii), consider a vertex set A' in G' which does not contain z_0 and does not contain all vertices in $G' - z_0$. If A' is a singleton in G', then it is possible that $d(A', G') = d(A', G) - 1$. But then also the type of A' is reduced by one when going from G to G'. On the other hand, if A' is not a singleton, then we define A either as A' or $A' \cup \{x\}$, and then

$$d(A, G) \leqslant d\big(A', G'\big) + d(x, G)/2,$$

and hence

$$d(A, G) \leqslant d\big(A', G'\big) + 4.$$

Claim 3 implies that

$$d(A, G) \geqslant 12$$

(because the last assertion in Claim 3 cannot occur, as earlier noted). Hence

$$d(A', G') \geqslant 8,$$

which implies that condition (iii) is satisfied in G' because the degree of A' has the same parity as $t(A')$.

Finally, we claim that $G' - z_0$ cannot have a problematic bridge. For otherwise, $V(G') \setminus \{z_0\}$ can be divided into sets A, B such that G' has only one edge between A, B. None of A, B can be a singleton u because then (in G') all edges incident with u, except one, are also incident with z_0. Then (in G) all edges incident with u, except possibly two, are also incident with z_0. But, this contradicts Claim 5.

By Claim 3,

$$24 \leqslant d(A, G) + d(B, G) \leqslant d(z_0, G) + d(x, G) + 2 \leqslant 11 + 9 + 2,$$

a contradiction.

This completes the proof of Theorem 1. $\square$

5. Orientations modulo k: The weak circular flow conjecture

We now extend Theorem 1 to arbitrary k. The proof is the same, except that there is some calculation involving k. On the other hand, there is no condition on problematic bridges. Let $G, z_0, p(x)$ be as in Section 3.

Theorem 2. *Assume that k is a natural number, $k \geqslant 4$, and that the multigraph G satisfies the following*:

(i) $|V(G)| \geqslant 3$.
(ii) $d(z_0) \leqslant 3k^2 + 6k - 13$.
(iii) *For each nonempty vertex set A not containing z_0 and satisfying $|V(G) \setminus A| > 1$, we have $d(A) \geqslant 2k^2 + t(A)$.*
(iv) *The sum of $p(x)$ taken over all vertices of G (including z_0) is congruent to the number of edges of G modulo k.*

Then all edges not incident with z_0 can be directed such that, for each vertex x, $d^+(x) \equiv p(x) \pmod{k}$.

Proof. The proof is by induction on the number of edges of G. Assume (reductio ad absurdum) that G is a smallest counterexample.

Claim 7. *For any two vertices x, y distinct from z_0, there are at most $k-2$ edges joining x, y.*

Proof. Suppose (reductio ad absurdum) that there are q edges, where $q \geqslant k-1$, joining x, y. If $|V(G)| > 3$, then we contract x, y into a vertex z and use induction with $p(z) = p(x) + p(y) - q$. Then we direct the edges joining x, y such that x gets the desired outdegree $p(x)$ modulo k. Then y automatically gets the desired outdegree $p(y)$ modulo k. $\square$

Claim 8. $|V(G)| > 3$.

Proof. If $|V(G)| = 3$, then (ii), (iii) imply that there are at least $k-1$ edges joining the two vertices distinct from z_0 because one of them is joined to z_0 by at most $d(z_0)/2$ edges and hence joined to the other vertex by at least $2k^2 - d(z_0)/2$ edges. But this contradicts Claim 7. $\square$

Claim 9. *If A is a vertex set not containing z_0 such that $|A| > 1$, and $|V(G) \setminus A| > 1$, then $d(A) \geqslant 3k^2 + 6k - 12$.*

Proof. If $d(A) \leqslant 3k^2 + 6k - 13$, then we first contract A and use induction. In particular, all edges from A to $V(G) \setminus A$ receive a direction. Then we contract instead $V(G) \setminus A$ into a single vertex which we think of as a new z_0, and then again we use induction to also direct the edges in $G(A)$. $\square$

Claim 10. *Every vertex x distinct from z_0 has at least two neighbors, and, $d(x) = 2k^2 + t(x)$.*

Proof. As noted earlier, $t(x), d(x)$ have the same parity. Suppose (reductio ad absurdum) that either x has at most one neighbor or

$$d(x) \geqslant 2k^2 + t(x) + 2$$

(or both). Assume that x has been chosen such that it has maximum degree under this condition.

Consider first the case where x has only one neighbor z. The number of edges from z to vertices distinct from x is at least $2k^2$ by applying condition (iii) to $\{x, z\}$ (if $z \neq z_0$) or $V(G) \setminus \{x, z\}$ (if $z = z_0$). So $d(z) \geqslant 4k^2 + t(x) + 2$.

If $z = z_0$, this contradicts condition (ii), and if $z \neq z_0$ it contradicts the maximality of the degree of x.

Consider next the case where x has at least two neighbors. Let y, z be distinct neighbors of x. (One of y, z may equal z_0.) Now delete an edge from x to y, and an edge from x to z, and add an edge from y to z. We apply induction to the resulting graph. This is possible because condition (iii) is satisfied when A is a singleton (by the assumption on the degree of x), and (iii) is also satisfied if A is not a singleton because of Claim 9. (Claim 9 implies that $d(A) \geqslant 3k^2 + 1$ before the lifting, and hence $d(A) \geqslant 3k^2 - 1 \geqslant 2k^2 + t(A)$ after the lifting.) This contradiction to the minimality of G proves Claim 10. □

Claim 11. *For every vertex x distinct from z_0, the number of edges joining x, z_0 is less than $d(x)/2 = k^2 + t(x)/2$.*

Proof. Let m be the number of edges joining x, z_0. The number of edges from $\{x, z_0\}$ to $V(G) \setminus \{x, z_0\}$ is $d(z_0) + d(x) - 2m \leqslant 3k^2 + 6k - 13 + d(x) - 2m$. By Claim 9 that number of edges is $\geqslant 3k^2 + 6k - 12$. Hence $2m \leqslant d(x) - 1$. □

Claim 12. *If x is a vertex distinct from z_0, then $t(x) > 0$.*

Proof. If $t(x) = 0$, then we lift successively the edges incident with x. To see that this is possible, we first use Claims 11, 10, 7 to conclude that no edge-multiplicity of an edge incident with x is greater than the sum of the other edge-multiplicities of edges incident with x. We then lift arbitrarily until some edge-multiplicity of an edge incident with x equals the sum of the other edge-multiplicities. Then there is only one way of performing the remaining liftings. We then apply induction to the resulting graph G'. To see that this is possible it suffices to verify condition (iii). Consider therefore a vertex set A' in G' not containing z_0. In G we consider the sets A', $A' \cup \{x\}$. We let A denote the one having the smallest number of edges leaving the set. Then

$$d(A, G) \leqslant d\big(A', G'\big) + d(x, G)/2.$$

By Claims 9, 10, G' satisfies condition (iii). This contradiction proves Claim 12. □

Consider now a vertex x distinct from z_0 such that the number m say, of edges between x and z_0 is smallest possible (possibly $m = 0$).

Assume that x is of type i where $1 \leqslant i \leqslant k$. Then x has degree $2k^2 + i$, by Claim 10. By Claims 7 and 11, the vertex x has at least $q = (2k^2 + i)/2(k - 2)$ neighbors distinct from z_0. The minimality of m then implies that

$$m \leqslant d(z_0)/(q + 1) \leqslant 3k.$$

Combining this with Claims 7, 10, we conclude that x has at least $q' = (2k^2 + i - 3k)/(k - 2)$ neighbors distinct from z_0. Clearly, $q' \geqslant 2k$. Assume without loss of generality that x is negative. Then either x has i positive neighbors which are pairwise distinct and also distinct from z_0, or else x has $2k - i$ negative neighbors which are pairwise distinct and also distinct from z_0.

In the former case we select i distinct positive neighbors of x and direct an edge from each of these towards x. In the latter case we select $2k-i$ distinct negative neighbors of x and direct an edge from each of these away from x. For each such neighbor y of x we delete the directed edge between x and y. We lift the remaining edges incident with x. It is possible to lift the remaining edges incident with x because each edge-multiplicity of edges incident with x is $\leqslant 3k$, and after deletion of the i or $2k-i$ edges incident with x there are still $2k^2-2k+2$ edges incident with x.

We apply induction to the resulting graph G' where the prescribed outdegrees $p(y)$ have been modified accordingly.

To see that induction is possible, we need only consider the condition (iii). If A' is a singleton in G', then it is possible that $d(A',G')=d(A',G)-1$. But then also the type of A' is reduced by one when going from G to G'. On the other hand, if A' is not a singleton, then we define A either as A' or $A'\cup\{x\}$, and then

$$d(A,G)\leqslant d\big(A',G'\big)+d(x,G)/2.$$

So, if A' does not satisfy condition (iii) in G', then we get a contradiction to Claim 9.

This completes the proof of Theorem 2. □

6. Applications to the 3-flow conjecture, the circular flow conjecture, and decomposition into trees

The following consequence of Theorem 1 is a strengthening of the weak 3-flow conjecture.

Theorem 3. *Every* 8*-edge-connected multigraph admits all generalized Tutte-orientations.*

This follows from Theorem 1 because $6+t(A)$ is one of $6,7,8,9$ but $d(A),t(A)$ have the same parity. (Before we apply Theorem 1, we subdivide an edge by inserting a vertex z_0 of degree 2. We direct the two edges incident with z_0 so that z_0 gets indegree 1 and outdegree 1.)

Jaeger [5,6] generalized the 3-flow conjecture to the following which he called the *the circular flow conjecture*:

If k is an odd natural number, and G is a $(2k-2)$-edge-connected multigraph, then G has an orientation such that each vertex has the same indegree and outdegree modulo k. Note that for $k=3$ this reduces to the 3-flow conjecture which is still open.

We now get the following weakened version.

Theorem 4. *If k is an odd natural number, and G is a $(2k^2+k)$-edge-connected multigraph, then G has an orientation such that each vertex has the same indegree and outdegree modulo k.*

The following conjecture was made in [1].

Conjecture 1. *For each tree T, there exists a natural number k_T such that the following holds: If G is a k_T-edge-connected graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.*

In this conjecture it is important that G has no multiple edges. The conjecture has been verified first for the path of length 4 in [8], and then for the path of length 3 in [9]. In a forthcoming paper [10] I verify it for each path whose length is a power of 2. Now we can verify it for all stars by putting $p(x)=0$ in Theorems 1, 2.

Theorem 5. *Let k be any natural number. If G is a $(2k^2+k)$-edge-connected graph whose size (number of edges) is divisible by k, then the edge set of G can be decomposed into stars with k edges each.*

A star with 3 edges is called a *claw*.

Theorem 6. *If G is an* 8*-edge-connected graph whose size is divisible by* 3*, then G has a claw-decomposition.*

So, we have general decomposition results for families of sparse graphs, although the general decomposition problem is *NP*-complete [4]. Further applications of Theorems 5, 6 to decomposition will be explored in a forthcoming paper.

For the sake of completeness we also state Theorem 1 as a flow result.

Theorem 7. *If G is an* 8*-edge-connected multigraph, then each edge can be oriented and assigned a weight* 1 *or* 2 *such that, for each vertex, the weighted outdegree equals the weighted indegree.*

If we divide the vertex set into two sets A, B then the number of edges directed from A to B is at most twice the number of edges directed from B to A.

Paul Seymour pointed out that the former statement follows by a slight modification of the argument by Anton Kotzig mentioned just before Theorem 1: By Theorem 1, each edge can be oriented and assigned a weight 1 or 2 such that, for each vertex, the weighted outdegree equals the weighted indegree when these numbers are reduced modulo 3. (In fact we do not need the weight 2 at all.) Let P be the set of vertices whose weighted outdegree is strictly greater than the weighted indegree. Let Q be the set of vertices whose weighted outdegree is strictly smaller than the weighted indegree. Assume that the orientation and edge-weighting is chosen such that the sum of weighted outdegrees in P is minimum.

If P is empty, then also Q is empty, and we are done. If P is nonempty, then it is easy to see that there is a directed path from P to Q. Now we reverse all the edge directions, and we interchange between weights 1, 2 in this path. The new orientation has a smaller sum of outdegrees in P, a contradiction.

It is now a challenge to decrease the edge-connectivity in Theorem 7 from 8 to 4 in order obtain Tutte's 3-flow conjecture. Another challenge is to increase the edge-connectivity in order to obtain the so-called $(2+\epsilon)$-*flow conjecture* by Goddyn and Seymour. If that conjecture is true, then the factor 2 in the last statement of Theorem 7 can be replaced by any real number $1+\epsilon > 1$ provided the multigraph is $f(\epsilon)$-edge-connected, where $f(\epsilon)$ is a natural number depending on ϵ only.

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