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On the Algebraic Theory of Graph Colorings

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ABSTRACT

Some well-known coloring problems of graph theory are generalized as a single algebraic problem about chain-groups. This is transformed into a problem about the finite projective geometries over $GF(2)$. The geometrical problem is solved up to the 5-dimensional case.

1. INTRODUCTION

Let E be any finite set. We define a *chain* on E as a mapping f of E into $GF(2)$. If $x \in E$ then fx is the *coefficient* of x in f. Two or more chains on E are added, to give a new chain on E, by adding the coefficients for each $x \in E$. The chains on E evidently constitute an additive group. Any subgroup of this group is called a *chain-group* on E.

Let N be a chain-group on E. It includes as its zero element the chain on E in which each coefficient is zero. This is the *zero* chain on E, denoted in equations by the symbol 0. It is convenient to postulate the existence of a zero chain even when E is null, and to say that there are then no other chains on E. We refer to the elements of E as the *cells* of N. A cell x of N is *filled* if $fx = 1$ for some chain f of N, and is *empty* otherwise. We say N is a *full* chain-group if it has no empty cells.

A *coloring* of N is a pair $\{f, g\}$ of chains of N such that for each $x \in E$ either $fx = 1$ or $gx = 1$. We say N is *chromatic* if it has a coloring and *achromatic* if it has none. A chromatic chain-group is necessarily full.

In this paper we give a conjectural sufficient condition for N to be chromatic, and we verify the conjecture for all chain-groups having not more than 6 linearly independent chains. Before doing this however we show that some of the best known coloring problems in graph theory can be stated as coloring problems for appropriate chain-groups.

In a more general theory of chain-groups other coefficient domains than $GF(2)$ may be used [3]. The chain-groups of the present paper are then distinguished as "binary."

2. The Cycles and Coboundaries of a Graph

Let G be a graph, $V(G)$ its set of vertices, and $E(G)$ its set of edges. Chains on $V(G)$ and $E(G)$ are called 0-*chains* and 1-*chains* of G, respectively.

Suppose $x \in V(G)$ and $X \in E(G)$. We define $\eta(X, x) \in GF(2)$ as follows. $\eta(X, x) = 0$ if x is not incident with X or if X is a loop whose two ends coincide in x. In the remaining case, in which X is a link having x as one end and some other vertex as the other, $\eta(X, x) = 1$.

The *boundary* ∂f of a 1-chain f of G is a 0-chain of G defined as follows:

$$(\partial f)(x) = \sum_{X \in E(G)} \eta(X, x) f X, \tag{1}$$

for all $x \in V(G)$. If $\partial f = 0$ we say that f is a *cycle* of G.

We deduce from (1) that boundaries satisfy the law

$$\partial(f + g) = \partial f + \partial g. \tag{2}$$

Hence the cycles of G are the chains of a chain-group $\Gamma(G)$ on $E(G)$. We call this the *cycle-group* of G.

The *coboundary* δg of a 0-chain g of G is a 1-chain of G defined as follows.

$$(\delta g)(X) = \sum_{X \in E(G)} \eta(X, x)\, gx, \tag{3}$$

for all $X \in E(G)$. Thus the coefficient of X in δg is obtained by adding the coefficients in g of the two ends of X.

We deduce from (3) that coboundaries satisfy the law

$$\delta(f + g) = \delta f + \delta g. \tag{4}$$

Hence the coboundaries of the 0-chains of the 0-chains of G are the elements of a chain-group $\Delta(G)$ on $E(G)$. We call this the *coboundary-group* of G.

Given two chains f and g on a set E we write

$$(f \cdot g) = \sum_{x \in E} (fx)(gx) \tag{5}$$

We say f and g are *orthogonal* if $(f \cdot g) = 0$.

2.1. *A 1-chain f of G is a cycle if and only if it is orthogonal to every coboundary*

PROOF: Let g be any 0-chain of G. Then

$$\begin{aligned}(f \cdot \delta g) = &\sum_{X \in E(G)} fX \sum_{x \in V(G)} \eta(X, x)\, gx \\ &\sum_{x \in V(G)} gx \sum_{X \in E(G)} \eta(X, x)\, fX \\ = &(g \cdot \partial f).\end{aligned}$$

We deduce that if f is a cycle then $(f \cdot \delta g) = 0$. Conversely, if $(f \cdot \delta g) = 0$ for every 0-chain g, then ∂f must be a zero chain and f must be a cycle.

It follows from the definition of a coboundary that an edge X of G is empty in $\Delta(G)$ if and only if it is a loop.

An edge X of G is called an *isthmus* if the following condition holds: $V(G)$ can be partitioned into two sets U and V such that X is the only edge with one end in U and the other in V. An equivalent statement of this condition runs as follows: there is a chain f of $\Delta(G)$ for which X is the only edge with a non-zero coefficient.

It follows from (2.1) that every isthmus of G is an empty cell of $\Gamma(G)$.

3. Colorings of a Graph

A *4-coloring* of a graph G is a coloring of its vertices in four colors, which need not all be used, in such a way that the two ends of any edge have different colors. Thus if G has a loop it can have no 4-coloring.

As our four colors we take the four 2-vectors $\alpha = (1, 1)$, $\beta = (1, 0)$, $\gamma = (0, 1)$ and $\delta = (0, 0)$, with components in $GF(2)$. Then we can describe a 4-coloring of G as an ordered pair $\{f_1, f_2\}$ of 0-chains of G, the color assigned to a vertex x being (f_1x, f_2x). The condition for a given pair $\{f_1, f_2\}$ of 0-chains to be a 4-coloring is that for each edge X the two ends shall have different coefficients either in f_1 or in f_2, that is X shall have coefficient 1 either in δf_1 or in δf_2.

3.2. *$\{f_1, f_2\}$ is a 4-coloring of G if and only if $\{\delta f_1, \delta f_2\}$ is a coloring of $\Delta(G)$.*

A graph G is called *trivalent* if each vertex is incident with just three edges, loops being counted twice. It is evident that when such a graph has a loop it also has an isthmus. A *Tait coloring* of a trivalent graph is a coloring of the edges in three colors so that each vertex is incident with one edge of each color. We take the three colors to be the vectors α, β, and γ defined above.

3.2 *Let G be a trivalent graph. Then G has a Tait coloring if and only if $\Gamma(G)$ has a coloring.*

PROOF: Suppose first that G has a Tait coloring. We can represent the coloring by an ordered pair $\{f_1, f_2\}$ of 1-chains of G, the color of an edge X being (f_1X, f_2X). Any vertex x of G is incident with just three edges X_α, X_β, and X_γ, these having colors α, β, and γ, respectively. Accordingly just two of these edges have coefficient 1 in f_1, and just two in f_2. It follows that x has coefficient 0 in both ∂f_1 and ∂f_2. We deduce that f_1 and f_2 are cycles of G. Hence, since the color δ does not appear in a Tait coloring, $\{f_1, f_2\}$ is a coloring of $\Gamma(G)$.

Conversely, suppose $\{f_1, f_2\}$ is a coloring of $\Gamma(G)$. To each edge X of G we assign the color (f_1X, f_2X), which is not δ. Let x be any vertex of G and let X_1, X_2, and X_3 be its three incident edges, any loop being written twice. If two of these edges have the same color the third must have color δ, since f_1 and f_2 are cycles of G. We deduce that in fact X_1, X_2, and X_3 are assigned three distinct colors. Hence $\{f_1, f_2\}$ is a Tait coloring of G.

4. Rank and Duality

Let N be a chain-group on a set E. The *rank* $r(N)$ of N is the maximum number of linearly independent chains of N. Any set of $r(N)$ linearly independent chains of N is a *chain-basis* of N. Each chain of N has a unique expression as a linear combination of the members of any given chain-basis. Hence the number of distinct chains of N is $2^{r(N)}$.

Those chains on E which are orthogonal to all the chains of N constitute a chain-group N^* on E called the *dual* chain-group of N. By the theory of homogeneous linear equations we have

4.1. $r(N^*) = n - r(N)$,

where n is the number of cells of N. Two applications of this result show that $r(N^{**}) = r(N)$. Hence N and N^{**} have the same number of chains. But it is clear from the definitions that each chain of N is a chain of N^{**}. We thus have

4.2. $N^{**} = N$.

If G is any graph we have

4.3. $\Gamma(G) = (\Delta(G))^*$,

by 2.1. We can therefore use 4.2 to extend the results of Section 2. Thus a 1-chain f of G is a coboundary if and only if it is orthogonal to every cycle. This implies that every empty cell of $\Gamma(G)$ is an isthmus of G.

5. Minors of Chain-Groups

Let N be a chain-group on a set E and let S be any subset of E.

If f is any chain of N we define its *restriction* to S, denoted by f_S, as that chain on S in which each element of S has the same coefficient as in f. The restrictions to S of the chains of N evidently constitute a chain-group on S. We denote this chain-group by $N \cdot S$ and call it the *reduction* of N to S.

If $\{f, g\}$ is a coloring of N it is clear that $\{f_S, g_S\}$ is a coloring of $N \cdot S$.

5.1. *If $N \cdot S$ is achromatic then N is achromatic.*

Another chain-group on S is constituted by the restrictions to S of those chains of N in which the coefficient of each member of $E - S$ is zero. We denote this chain-group by $N \times S$ and call it the *contraction* of N to S.

5.2. *Suppose $T \subseteq S \subseteq E$. Then*

$$(N \cdot S) \cdot T = N \cdot T,$$
$$(N \times S) \times T = N \times T.$$

These identities are obvious from the definitions.

A *minor* of N is a chain-group of the form $(N \times S) \cdot T$, where $T \subseteq S \subseteq E$. The reductions and contractions of N, including N itself, are minors of N, since $N \cdot S = (N \times E) \cdot S$, $N \times S = (N \times S) \cdot S$, and $N = (N \times E) \cdot E$.

5.3. *Suppose $T \subseteq S \subseteq E$. Then*

$$(N \times S) \cdot T = (N \cdot (E - (S - T))) \times T,$$
$$(N \cdot S) \times T = (N \times (E - (S - T))) \cdot T.$$

PROOF: $(N \times S) \cdot T$ consists of the restrictions to T of those chains of N in which the coefficients of the members of $E - S$ are zero. But $E - S = (E - (S - T)) - T$. Hence $(N \times S) \cdot T$ consists of the restrictions to T of those chains of N in which the coefficients of the members of $(E - (S - T)) - T$ are zero. It is thus identical with $(N \cdot (E - (S - T))) \times T$.

We have now established the first of the required identities. We derive the second by writing $E - (S - T)$ for S.

5.4. *Every minor of a minor of N is a minor of N.*

This is a simple consequence of 5.2 and 5.3.

5.5. *Let S be any subset of E. Then*

$$(N \cdot S)^* = N^* \times S,$$
$$(N \times S)^* = N^* \cdot S.$$

PROOF: Let f be any chain on S. Let f' be the chain on E in which the members of $E - S$ have zero coefficients and the members of S have the same coefficients as in f. Now f is orthogonal to every chain of $N \cdot S$ if and only if f' is orthogonal to every chain of N, that is, if and only if f is a chain of $N^* \times S$.

We have now established the first of the required identities. To derive the second we write N^* for N, take dual chain-groups, and use 4.2.

5.6. *The minors of N^* are the duals of the minors of N.*

This follows from 5.3 and 5.5.

6. Subgraphs and Contractions

Let G be a graph and let S be any subset of $E(G)$. We write $G : S$ for the subgraph of G whose edges are the members of S and which includes all the vertices of G. We also write $G \cdot S$ for the subgraph of G defined by the edges of S and their incident vertices, that is, the graph obtained from $G : S$ by deleting its isolated vertices. We call $G \cdot S$ the *reduction* of G to S.

We construct a graph $G \times S$ as follows. Its edges are the members of S. Its vertices are in 1-1 correspondence with the components of $G : (E(G) - S)$. An edge X is incident with a vertex x in $G \times S$ if and only if X is incident, as an edge of G, with a vertex of the component of $G : (E(G) - S)$ corresponding to x. We can put this briefly by saying that $G \times S$ is derived from G by contracting the components of $G : (E(G) - S)$ to single vertices. We call $G \times S$ the *contraction* of G to S.

6.1. *Let S be any subset of $E(G)$. Then*

$$\begin{aligned}
\Delta(G \cdot S) &= \Delta(G) \cdot S,\\
\Delta(G \times S) &= \Delta(G) \times S,\\
\Gamma(G \cdot S) &= \Gamma(G) \times S,\\
\Gamma(G \times S) &= \Gamma(G) \cdot S.
\end{aligned}$$

The first two of these identities at least are readily verified from the definitions. The other two may be derived from them by taking dual chain-groups and then applying 4.3 and 5.5.

7. Conjectures

We define an *irreducible* chain-group as a full achromatic chain-group which has no full achromatic minor other than itself. Our object in this paper is to classify the irreducible chain-groups of rank ≤ 6.

Let us relate this procedure to what is usually done in the theory of graph-colorings.

The simplest loopless graph with no 4-coloring is the *complete* 5-*graph*. This has just five vertices, each pair of vertices being joined by exactly one edge. *Hadwiger's Conjecture* (in the relevant special case) runs as follows: *if G is loopless and not* 4-*colorable then some reduction of a contraction of G is a complete* 5-*graph* [1].

We describe a chain-group as *graphic* or *cographic* if it can be represented as the coboundary group or cycle group, respectively, of some graph. The minors of a graphic chain-group are graphic and those of a cographic one are cographic, by 6.1. Thus Hadwiger's Conjecture asserts that the only irreducible chain-group which is graphic is the coboundary group of the complete 5-graph.

The writer has not seen an explicitly stated analog of Hadwiger's Conjecture applying to Tait colorings. The best-known example of a trivalent graph having no isthmus and no Tait coloring is the *Petersen graph* [2]. This is constructed from two disjoint pentagons $a_1a_2a_3a_4a_5$ and $b_1b_3b_5b_2b_4$ by making the five joins a_ib_i. Let us make the following conjecture: *the only irreducible chain-group which is cographic is the cycle group of the Petersen graph.* (See 3.2.)

8. Embeddings in a Projective Geometry

In this section we consider sets of points in a finite projective geometry $PG(q, 2)$. We admit the degenerate cases $q = -1$, 0 and 1 of this geometry. We suppose a system of homogeneous coordinates in $PG(q, 2)$ is given and we identify each point with its coordinate vector. The zero $(q+1)$-vector, which represents no point of $PG(q, 2)$ is represented in formulae by the symbol 0. If $q = -1$ there are no points, but the zero vector may be held to exist.

Let N be a chain-group on a set E. Let F be a mapping of E onto a set of points of $PG(q, 2)$. We call F an *embedding* of N in $PG(q, 2)$ if the following condition holds. *Let f be any non-zero chain on E. Then*

$$\sum_{x \in E} (fx)(Fx) = 0$$

if and only if f is a chain of N^*.

If U is a set of points of $PG(q, 2)$ we write $\Sigma(U)$ for the subspace of

$PG(q, 2)$ generated by the points of U. We note that any embedding F of N in $PG(q, 2)$ is also an embedding of N in $\Sigma(FE)$, and in any subspace of $PG(q, 2)$ containing FE. The maximum number of linearly independent points in FE is found by the theory of homogeneous linear equations to be $n - r(N^*)$, where n is the number of cells of N. This is $r(N)$, by 4.1. Thus the dimension of $\Sigma(FE)$ is $r(N) - 1$.

8.1. *A chain-group N on E has an embedding in some $PG(q, 2)$ if and only if it is full.*

PROOF: Suppose first that N is not full. Then there is a chain f of N^* such that just one cell, y say, of N has a non-zero coefficient in f. In any embedding F of N in a $PG(q, 2)$ we would have $Fy = 0$. But this is impossible since no point of $PG(q, 2)$ has a zero coordinate vector.

Suppose next that N is full. Let the chains of N be enumerated as $f_1, f_2, \cdots, f_k$, where $k = 2^{r(N)}$. Let F be a mapping of E onto a set of points of $PG(k - 1, 2)$ such that $Fx = (f_1x, f_2x, \cdots, f_kx)$ for each $x \in E$. Now let f be an arbitrary non-zero chain on E. The points Fx, $x \in E$ satisfy the linear relation

$$\sum_{x \in E} (fx)(Fx) = 0$$

if and only if f is orthogonal to each of the chains f_i, that is, if and only if f is a chain of N^*. The fullness of N ensures that the vectors Fx defined above are all non-zero.

8.2. *Let U be a set of points in $PG(q, 2)$. Let F be a mapping of a finite set E onto U. Then F is an embedding in $PG(q, 2)$ of some chain-group N on E.*

PROOF: The linear relations holding between the points Fx, $x \in E$, define the dual of such a chain-group.

8.3. *Let F be an embedding in $PG(q, 2)$ of a chain-group N on E. Let S be any subset of E. Let H be the mapping of S into $PG(q, 2)$ such that $Hx = Fx$ for each $x \in S$. Then H is an embedding of $N \cdot S$ in $PG(q, 2)$.*

PROOF: The linear relations holding between the points Hx, $x \in S$, correspond to those chains of N^* in which all the members of $E - S$ have zero coefficients. The theorem thus follows from 5.5.

Let U and V be subsets of $PG(q, 2)$ such that $U \subseteq V$. We say U is *closed* in V if no point of $V - U$ is in $\Sigma(U)$.

Suppose U is closed in V and that the dimension of U is u. We can find a subspace Σ of $PG(q, 2)$ whose dimension is $q - u - 1$ and which has no common point with $\Sigma(U)$. Each point P of $V - U$ now determines a subspace $\Sigma(U \cup \{P\})$ of dimension $u + 1$. This subspace intersects Σ in a unique point P'. We refer to the mapping $P \to P'$ of $V - U$ onto a subset of Σ as the *projection of V into Σ from* the closed subset U of V.

8.4. *Let F be an embedding in $PG(q, 2)$ of a chain-group N on E. Let S be any subset of E. Then $N \times S$ is full if and only if FS and $F(E - S)$ are disjoint and $F(E - S)$ is closed in FE.*

PROOF: $N \times S$ is full if and only if there is no chain of $(N \times S)^*$ in which just one cell has a non-zero coefficient, that is, if and only if there is no chain of N^* in which just one cell of S has a non-zero coefficient, by 5.5. Thus $N \times S$ is not full if and only if there exists $y \in S$ such that Fy is a linear combination of points Fx such that $x \in E - S$. Such a point y exists if and only if either FS meets $F(E - S)$ or $F(E - S)$ is not closed in FE.

8.5. *Let F be an embedding in $PG(q, 2)$ of a chain-group N on E. Let U be a closed subset of FE and let H be the projection of FE from U onto a suitable subspace Σ of $PG(q, 2)$. Write $S = E - F^{-1}U$. Let F_1 be the restriction of F to S. Then HF_1 is an embedding of $N \times S$ in Σ.*

PROOF: HF_1 is a mapping of S onto the subset $H(FE - U)$ of Σ. Let f be any non-zero chain on S. Then the linear relation

$$\sum_{x \in S} (fx)(HF_1 x) = 0$$

holds if and only if

$$\sum_{x \in S} (fx)(Fx)$$

is a linear combination of points Fy, $y \in F^{-1}U = E - S$. This happens if and only if f is a chain of $N^* \cdot S$, that is, of $(N \times S)^*$, by 5.5. The theorem follows.

8.6. *Let F be an embedding in $PG(q, 2)$ of a chain-group N on E. Then there is a 1-1 correspondence $f \to \Sigma_f$ between the non-zero chains f of N and the subspaces Σ_f of $\Sigma(FE)$ having dimension $r(N) - 2$, such that Fx is on Σ_f if and only if $fx = 0$, for each $x \in E$.*

PROOF: Let f be any non-zero chain of N.
For each point P of $\Sigma(FE)$ we have

$$P = \sum_{x \in E} (gx)(Fx),$$

for some chain g on E. We write

$$u(f, P) = \sum_{x \in E} (gx)(fx).$$

This definition fixes $u(f, P)$ uniquely. For if also $P = \Sigma(g_1 x)(Fx)$ we have $\Sigma(g_1 x + gx)(Fx) = 0$ and therefore $g_1 + g \in N^*$. Hence $\Sigma(g_1 x)(fx) = \Sigma(gx)(fx)$ by orthogonality.

Consider the set Σ_f of all points P of $\Sigma(FE)$ such that $u(f, P) = 0$. Clearly Fx is a point of Σ_f if and only if $fx = 0$. If P and Q are distinct points of $\Sigma(FE)$ which are either both in Σ_f or both outside Σ_f it is clear that $P + Q$ is a point of Σ_f. Hence Σ_f is a subspace of $\Sigma(FE)$ which meets every line in $\Sigma(FE)$. It is thus a subspace of $\Sigma(FE)$ of dimension $r(N) - 2$, since $u(f, P)$ is not zero for every P.

The number of subspaces of $\Sigma(FE)$ of dimension $r(N) - 2$ is 0 if $r(N) = 0$ and $2^{r(N)} - 1$ if $r(N) > 0$. It is thus equal to the number of non-zero chains of N. Moreover if f and g are distinct non-zero chains of N, then Σ_f and Σ_g differ in their intersections with FE. The theorem follows.

9. BLOCKS

Let k be a positive integer. A set B of points of $PG(q, 2)$ is a *k-block* if its dimension is at least k and it includes at least one point from each subspace of $PG(q, 2)$ of dimension $q - k$.

A k-block B is *minimal* if no proper subset of B is a k-block.

We denote the dimension of any point-set U in $PG(q, 2)$ by dU. Let C be a non-null subset of a k-block B. We define a *tangent* of C in B as any $(q - k)$-space in $PG(q, 2)$ which contains all the points of C but no point of B which is independent of them. We call B a *tan-*

gential k-block if every non-null subset of B, of dimension not exceeding $q - k$, has a tangent in B.

As an example of a k-block we may take any k-space in $PG(q, 2)$. This is trivially both minimal and tangential.

9.1. *Let U be a subset of a subspace Σ of $PG(q, 2)$ such that $dU \geq k$. Then U is a k-block in Σ if and only if it is a k-block in $PG(q, 2)$. Moreover it is a tangential k-block in Σ if and only if it is a tangential k-block in $PG(q, 2)$.*

This result follows from the facts that any $(q - k)$-space in $PG(q, 2)$ intersects Σ in a space of dimension at least $d\Sigma - k$, and that any $(d\Sigma - k)$-space in Σ is the intersection with Σ of a $(q - k)$-space in $PG(q, 2)$.

9.2. *Every tangential k-block is minimal.*

PROOF: Suppose B is a tangential k-block which is not minimal. Then there is a point P of B such that $B - \{P\}$ is a k-block. But each tangent of P, or more precisely $\{P\}$, is a $(q - k)$-space not meeting $B - \{P\}$. This contradiction establishes the theorem.

9.3. *Let B be a k-block in $PG(q, 2)$. Then B is not tangential if and only if it has a closed subset C such that the projection of B from C transforms B into another k-block.*

PROOF: Given a closed subset C of B we consider the projection of B from C into a suitable space Σ. We recall that Σ has dimension $q - dC - 1$ and does not meet $\Sigma(C)$. The projection transforms $B - C$ into a subset B_1 of Σ.

Suppose first that B is not tangential. Then we can choose C so that $dC \leq q - k - 1$ and C has no tangent. Then $d\Sigma \geq k$. If B_1 is not a k-block there is a subspace Σ_1 of Σ, of dimension $q - dC - k - 1$, which is on no point of B_1. The join of Σ_1 and $\Sigma(C)$ is a subspace of $PG(q, 2)$ of dimension $q - k$, and it meets B only in the points of C. But this is contrary to the choice of C. We deduce that B_1 is a k-block.

Conversely suppose B_1 to be a k-block for some C. Then $d\Sigma \geq k$, and $dC \leq q - k - 1$. Let Σ_2 be any $(q - k)$-space in $PG(q, 2)$ that contains C. It meets Σ in a $(d\Sigma - k)$-space and so contains a point of B_1. This implies that Σ_2 contains a point of B not in C. Hence C has no tangent in B, and B is not tangential.

We proceed to relate the theory of 2-blocks to that of full achromatic chain-groups. The basic idea is due to O. Veblen [4].

9.4. *Let F be an embedding in $PG(q, 2)$ of a chain-group N on E. Then N is achromatic if and only if FE is a 2-block.*

PROOF: N is full, by 8.1. Assume it chromatic. If $r(N) \leq 2$ then $d(FE) \leq 1$ and FE is not a 2-block. If $r(N) \geq 3$ we can find two distinct non-zero chains f and g of N such that $\{f, g\}$ is a coloring of N. The intersection of Σ_f and Σ_g is a subspace of $\Sigma(FE)$ of dimension $r(N) - 3 = d(FE) - 2$, and it includes no point of FE, by 8.6. Again we find that FE is not a 2-block.

Conversely suppose FE is not a 2-block. If $r(N) = 0$ then E is null, since N is full, and N has the coloring $\{0, 0\}$. If $r(N) = 1$ then the only non-zero chain f on N yields a coloring $\{f, f\}$, since N is full. If $r(N) = 2$ then any two linearly independent chains of N determine a coloring.

If $r(N) \geq 3$, that is $d(FE) \geq 2$, there is a subspace Σ of $\Sigma(FE)$, of dimension $d(FE) - 2$ which contains no point of FE. We can represent Σ as the intersection of two distinct subspaces of $\Sigma(FE)$ of dimension $d(FE) - 1$. By 8.6 we can write these as Σ_f and Σ_g, where f and g are distinct non-zero chains of N. It then follows from 8.6 that $\{f, g\}$ is a coloring of N.

We have now verified that N is chromatic whenever FE is not a 2-block. This completes the proof.

9.5. *Let F be an embedding in $PG(q, 2)$ of an achromatic chain-group N on E. Then N is irreducible if and only if the following conditions hold:*

(i) *F is a* 1-1 *mapping of E onto FE.*

(ii) *FE is a tangential* 2-*block in $PG(q, 2)$.*

PROOF: Assume first that N is irreducible. If condition (i) does not hold there is a proper subset S of E such that $FS = FE$. Then $N \cdot S$ is achromatic, by 8.3 and 9.4. But this is contrary to assumption.

If condition (i) holds and condition (ii) fails there is a non-null closed subset C of FE such that the projection of FE from C determines a new 2-block B_1, by 9.3. It then follows from 8.5 that N has a full achromatic reduction which is distinct from N itself. This is contrary to assumption.

Assume next that conditions (i) and (ii) are satisfied. If N is not irreducible it has a full achromatic minor $(N \cdot S) \times T$ which is distinct from N itself, by 5.3. Then $FS - FT$ is a closed non-null subset of FS, by 8.3 and 8.4. The projection of FS from $FS - FT$ into a suitable subspace Σ of $PG(q, 2)$ determines a 2-block B, by 8.5 and 9.4.

Now let U be the set of all points of FE in $\Sigma(FS - FT)$. Evidently U is a closed subset of FE containing $FS - FT$. The projection of FE from U into Σ determines a point-set B_1 in Σ which contains the 2-block B. But then B_1 is itself a 2-block, and so the 2-block FE is not tangential.

10. 2-Blocks in Two and Three Dimensions

It follows from 9.5 that the problem of classifying the irreducible chain-groups of rank ≤ 6 is equivalent to that of classifying the tangential 2-blocks of dimension ≤ 5. In the remainder of this paper we are concerned with the latter form of our problem.

We define an *n-stigm* as a set of n points in $n - 2$ dimensions such that each $n - 1$ of them are linearly independent. The n points of an n-stigm thus sum to zero. In particular a 3-stigm consists of two distinct points P and Q and their sum $P + Q$. It is thus a line of the geometry.

An *odd stigm* is an n-stigm for which n is odd and ≥ 3.

10.1. *Every k-block contains an odd stigm.*

Proof: Let B be a k-block in $PG(q, 2)$. We can find a set of $dB + 1 \geq k + 1$ linearly independent points of B. Call this set D. Let Z be the set of all points of $\Sigma(B)$ which are sums of even numbers of points of D. Then Z is a subspace of $\Sigma(B)$ of dimension $dB - 1$. It therefore contains a point Q of B, by 9.1. Then Q and the even number of points of D of which it is the sum constitute an odd stigm contained in B.

10.2. *The odd stigms are the minimal* 1*-blocks.*

Proof: Let S be an odd stigm in $PG(q, 2)$. If possible let Σ be a $(q - 1)$-space in $PG(q, 2)$ which does not meet S. The line joining any two distinct points of S meets Σ, that is, the sum of any two distinct points of S is on Σ. It follows that the sum of any even number (> 0) of points of S is a point of Σ. But each point of S is the sum of the remaining points, even in number. Hence $S \subseteq \Sigma$. From this contradiction we deduce that S is a 1-block.

Since each 1-block contains an odd stigm, by 10.1, and since it is evident that no odd stigm can contain another, the theorem follows.

10.3. *Let B be a k-block in $PG(q, 2)$, and let Σ be a subspace of $\Sigma(B)$ of dimension $dB - k + 1$. Then $B \cap \Sigma$ contains an odd stigm.*

PROOF: Any subspace of Σ of dimension $dB - k$ must contain a point of B, by the definition of a k-block. Hence $B \cap \Sigma$ is a 1-block. The theorem now follows from 10.1.

In $PG(2, 2)$, the Fano plane, the only 2-block is that consisting of all seven points of the space. We refer to it as the *Fano block*. It is trivially tangential.

It can be verified that the irreducible chain-group corresponding to the Fano block is neither graphic nor cographic.

10.4. *The only minimal 2-block which is 3-dimensional is the complement in $PG(3, 2)$ of the 5-stigm. It is tangential.*

PROOF: Let Q be a 5-stigm in $PG(3, 2)$. Since it contains no line its complementary set S is a 2-block. Now each point of $PG(3, 2)$ is a sum of points of Q, and the five points of Q sum to 0. Hence each point of S is a sum of two distinct points of Q, that is, each point of S is on a line which otherwise lies entirely in Q. It follows that the 2-block S is minimal and tangential.

Conversely let S be any minimal 3-dimensional 2-block in $PG(3, 2)$. Then S contains no Fano block, that is, the complementary set Q of S meets each plane of $PG(3, 2)$. Thus Q is a 1-block. Since S meets each line, Q must contain a 5-stigm Q', by 10.1. But the complementary set of Q' in $PG(3, 2)$ is a minimal 2-block, by the result already proved. Hence $Q' = Q$, by the definition of S.

By 10.4 the only 3-dimensional tangential 2-block is the complementary set of the 5-stigm. It can be verified that the points and lines in the 2-block constitute a Desargues configuration. We therefore refer to it as the *Desargues block*. The corresponding irreducible chain-group is the coboundary group of the complete 5-graph. The 10 triangles in this graph correspond to the 10 lines of the Desargues configuration.

11. Four Dimensions

In our investigations of a hypothetical tangential 2-block S in $n \geq 4$ dimensions we use the following notation. P being a fixed point of S we let π denote a tangent of P, an $(n - 2)$-space of $PG(n, 2)$ meeting S only in P. There are just three $(n - 1)$-spaces on π, and we denote them by τ_1, τ_2 τ_3. By 10.3 they contain odd stigms St_1, St_2, and St_3, respectively each containing P, by 10.2. We denote the subspaces of $PG(n, 2)$ generated by these odd stigms by σ_1, σ_2, and σ_3, and the intersections of these with π by λ_1, λ_2, and λ_3, respectively.

We have the following general theorem.

11.1. *Suppose St_1 and St_2 are 5-stigms $PA_1B_1C_1D_1$ and $PA_2B_2C_2D_2$, respectively, and that the planes λ_1 and λ_2 are identical. Then the following propositions hold.*

(i) *The notation can be adjusted so that A_1A_2, B_1B_2, C_1C_2 and D_1D_2 are concurrent in a point Z of τ_3.*

(ii) *If X_i and Y_i are distinct elements of $\{A_i, B_i, C_i, D_i\}$, then each tangent of $\{X_i, Y_i\}$ is on $P + Z$. $(i = 1, 2)$.*

(iii) *$\tau_3 \cap S$ meets the join of σ_1 and σ_2 only in P.*

Proof: Let $m(A, i)$ be the line of intersection of λ_i and the plane $B_iC_iD_i$, and similarly for $m(B, i)$, $m(C, i)$, and $m(D, i)$. $(i = 1, 2.)$ Then for each i the lines $m(X, i)$ are the four distinct lines on λ_i which do not pass through P. Hence we can adjust the notation in St_2 so that $m(X, 1) = m(X, 2)$, $= m(X)$ say. ($X = A$, B, C, or D). We must then have $X_1 + Y_1 = X_2 + Y_2$, where X and Y are any two of the letters A, B, C and D. This implies that

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D_1 + D_2,$$

and so establishes (i).

It is sufficient to prove (ii) for $X_i = A_1$ and $Y_i = B_1$. Any tangent $t(A_1B_1)$ of $\{A_1, B_1\}$ meets $\lambda_1 = \lambda_2$ only in $A_1 + B_1$. It meets σ_2 in a line on $A_1 + B_1$ but not on λ_1. This can only be $\{A_1 + B_1, P + A_2, P+B_2\}$ The plane determined by this line and A_1B_1 cuts τ_3 in the line on $A_1 + B_1$, $P + A_1 + A_2$, and $P + A_1 + B_2$. Hence $t(A_1B_1)$ is on $P + A_1 + A_2 = P + Z$.

To prove (iii) we consider the 3-space α containing A_1, A_2, B_1, B_2, C_1, and C_2. The line PD_1 cuts it only in the point $P + D_1 = A_1 + B_1 + C_1$. But the six lines joining $A_1 + B_1 + C_1$ to A_1, A_2, B_1, B_2, C_1, and C_2 are all distinct. The only other line on α and $A_1 + B_1 + C_1$ is that passing through Z. Hence the tangent of $\{P, D_1\}$ is on Z, and therefore Z is not a point of S. Moreover $P + Z$ is not a point of S, by (ii).

Let β denote the intersection with τ_3 of the join of σ_1 and σ_2. Then β is the third 3-space of the pencil on λ_1 determined by σ_1 and σ_2. Let T be any point of β, not on PZ. The line $T(Z + P)$ meets λ_1 in a point other than P, that is, a point $X_1 + Y_1$, where X_1 and Y_1 are distinct points in $St_1 - \{P\}$. Tangents of $\{X_1, Y_1\}$ must pass through $X_1 + Y_1$, $Z + P$ (by (ii)) and T. Hence T is not a point of S. This completes the proof.

11.2. *In the case $n = 4$ at most one of τ_1, τ_2 and τ_3 contains a 5-stigm in S.*

PROOF: If the theorem fails we may suppose τ_1 and τ_2 to contain 5-stigms St_1 and St_2 of S, respectively. We may then put $\sigma_1 = \tau_1$, $\sigma_2 = \tau_2$, and $\lambda_1 = \lambda_2 = \pi$. But then P is the only point of $S \cap \tau_3$, by 11.1, and this contradicts 10.3.

11.3. *In the case $n = 4$ each of τ_1, τ_2 and τ_3 contains a line of S.*

PROOF: If the theorem fails we may, by 11.2, suppose that St_1 is a 5-stigm and that St_2 and St_3 are lines. Let α be the plane through P determined by St_2 and St_3. It meets τ_1 in a line m through P. Since this line is not on π it must pass through a second point of St_1, say A_1.

Consider a tangent of $\{B_1, C_1\}$. It has a point in common with α, and this point can only be $P + A_1$ since the other six points of α are all in S. Hence the tangent of $\{B_1, C_1\}$ passes through $(B_1 + C_1) + (P + A_1)$, that is D_1. But this is a contradiction.

11.4. *In the case $n = 4$ each 3-space contains a line of S.*

PROOF: If possible let α be a 3-space containing no line of S. Then $S \cap \alpha$ contains a 5-stigm Q, by 10.3. Moreover Q is the whole of $S \cap \alpha$, since each point of $\alpha - Q$ is collinear with two points of Q. If P is chosen to be a point of Q, then α contains a tangent β of P made up of the sums of even numbers of points of $Q - \{P\}$. Applying 11.3 with $\pi = \beta$ we obtain a contradiction.

11.5. *There is no tangential* 4-*dimensional* 2-*block*.

PROOF: Suppose S is such a 2-block. Let P be one of its points. By 11.3 there are three distinct lines of S through P. We denote them by PQ_1R_1, PQ_2R_2, and PQ_3R_3. The three lines are not coplanar since otherwise they would constitute a Fano block, contrary to 9.2. Let α be the 3-space generated by them.

In α we can distinguish four distinct lines PQ_1, PQ_2, PQ_3, and PX through P, where $X = Q_1 + Q_2 + Q_3$, and a plane δ meeting each of these lines only in P. We tabulate the points of these five subspaces, in terms of the four independent points P, Q_1, Q_2, and Q_3, as follows:

$$\begin{aligned}
PQ_1 &= \{P, Q_1, P + Q_1\}.\\
PQ_2 &= \{P, Q_2, P + Q_2\}.\\
PQ_3 &= \{P, Q_3, P + Q_3\}.\\
PX &= \{P, Q_1 + Q_2 + Q_3, P + Q_1 + Q_2 + Q_3\}.\\
\delta &= \{P, Q_1 + Q_2, Q_2 + Q_3, Q_1 + Q_3, P + Q_1 + Q_2,\\
&\qquad\qquad P + Q_2 + Q_3, P + Q_1 + Q_3\}.
\end{aligned}$$

We note that all 15 points of α appear in this table.

Let β and γ be the other two 3-spaces on δ. They contain lines m_β and m_γ of S, respectively, by 11.4.

If J is any point of S not in α, any tangent of J meets α in a line. Since this line does not meet S it can only be on δ. Hence the tangent of J lies either in β or in γ. It therefore meets one of the lines m_β and m_γ. We deduce that $m_\beta - \{P\}$ and $m_\gamma - \{P\}$ are the complete intersections of S with $\beta - \delta$ and $\gamma - \delta$, respectively. It also follows that we can adjust the notation so that neither $Q_1 + Q_2$, $Q_2 + Q_3$ nor $Q_1 + Q_3$ is in S.

Let J and K be points of S in $\beta - \delta$ and $\gamma - \delta$, respectively. Suppose them to be collinear with $X = Q_1 + Q_2 + Q_3$. Consider the 3-space ξ generated by Q_1, Q_2, Q_3, J, and K. Then $\xi \cap \alpha$ is the plane made up of Q_1, Q_2, Q_3, $Q_1 + Q_2$, $Q_2 + Q_3$, $Q_1 + Q_3$, and $Q_1 + Q_2 + Q_3$. This plane meets δ in a line containing no point of S. Hence the intersections of ξ with β and γ are planes meeting S in single points, their intersections with m_β and m_γ, respectively. Accordingly $S \cap \xi$ is the 5-stigm $Q_1Q_2Q_3JK$, unless $X \in S$. But X cannot be a point of S since each of the seven lines on X and α passes through a point of S other than X. We have thus obtained a contradiction of 11.4.

A similar argument, in which Q_i is replaced by $R_i = P + Q_i$, applies if J and K are collinear with $P + X$.

We deduce that neither X nor $P + X$ is a point of S, and that no two points of S outside α are collinear with X or $P + X$.

Suppose m_β and m_γ have a common point, necessarily in δ. We can permute the numerical suffixes to arrange that this point is P or $P + Q_1 + Q_2$. Let the plane determined by m_β and m_γ meet α in the line x. Then x is not on δ, and it is not on either of the points X and $P + X$ by the result just proved. Hence all three points of x must be in S. This is impossible since S contains no Fano block, by 9.2.

In the remaining case m_β and m_γ meet δ in distinct points Y_β and Y_γ, respectively.

Let J and K be the points other than Y_γ on m_γ. The plane Jm_β meets α in a line x_J. Moreover x_J lies entirely in S, since no two points of S outside α can be collinear with X or $P + X$. Hence the only point of Jm_β not in S is $J + Y_\beta$. Similarly the plane Km_β meets α in a line x_K of S, and the only point of Km_β not in S is $K + Y_\beta$.

Let B be any point of $S \cap \alpha$ not on δ, x_J or x_K. (There are at least two such points.) Any tangent of B meets Jm_β and Km_β and so passes through the points $J + Y_\beta$ and $K + Y_\beta$. It therefore passes through $J + K = Y_\gamma$. This is impossible since Y_γ is a point of S distinct from B.

This completes the proof of the theorem.

12. Five Dimensions

In what follows we assume that the tangential 2-block S under discussion is 5-dimensional. Using the notation set out in Section 11 we have the following theorem.

12.2. *Let St_1 and St_2 be 5-stigms. Suppose further that λ_1 and λ_2 are identical. Let α be the 3-space which is the intersection of τ_3 with the join of σ_1 and σ_2. Then $S \cap \tau_3$ is a line not on α.*

PROOF: We adjust the notation for St_1 and St_2 to agree with 11.1, (i). By 11.1, (iii), there is no point of $S - \{P\}$ on α.

Let J and K be distinct points of $S - \{P\}$ on $\tau_3 - \alpha$. Assume that they are not collinear with P. Clearly $J + K$ is a point of α.

Suppose first that $J + K$ is in λ_1. Without loss of generality we can write $J + K = A_1 + B_1$. The set $U = \{C_1, D_2, J\}$ generates a plane μ

which meets σ_1 only in C_1, σ_2 only in D_2 and α only in $C_1 + D_2$. Any tangent θ of U meets the plane $A_1B_1D_1$ in one of the points $A_1 + B_1$, $A_1 + D_1$, $B_1 + D_1$, and $A_1 + B_1 + D_1$. But if $A_1 + D_1 \in \theta$ we have $A_2 + D_2 \in \theta$ and $A_2 \in \theta$, by 11.1, (i). Similarly, if $B_1 + D_1 \in \theta$, then $B_2 + D_2 \in \theta$ and $B_2 \in \theta$. Moreover, if $A_1 + B_1 + D_1$ is in θ, then $P + C_1 \in \theta$ and $P \in \theta$.

From these contradictions we deduce that $A_1 + B_1 \in \theta$. But then $J + K \in \theta$ and $K \in \theta$. This implies that K is a point of the plane C_1D_2J. But in this case we have $C_1 + D_2 = J + K = A_1 + B_1$, which is impossible since D_2 is not on τ_1.

In the remaining case $J + K$ is on $\alpha - \lambda_1 = \alpha - \pi$.

Suppose first that $J + K = Z$. The plane PD_1J meets the join of σ_1 and σ_2 only in the line PD_1. Let θ be any tangent of $\{P, D_1, J\}$. Then θ meets the plane $A_1A_2B_1B_2Z$ in one of the points Z, $A_1 + B_1$ and $A_1 + B_2$. But if $A_1 + B_1 \in \theta$ we have $P + A_1 + B_1 + D_1 \in \theta$, that is, $C_1 \in \theta$. If $A_1 + B_2 \in \theta$, then $A_1 + D_1 + B_2 \in \theta$, $A_2 + B_2 + D_2 \in \theta$, and $P + A_2 + B_2 + D_2 \in \theta$, that is, $C_2 \in \theta$. Moreover, if $Z \in \theta$, then $J + K \in \theta$ and $K \in \theta$. In each case the definition of θ is contradicted.

Suppose next that $J + K = Z + P$. The plane A_1B_1J meets the join of σ_1 and σ_2 only in the line A_1B_1. Let θ be a tangent of $\{A_1, B_1, J\}$. By the argument used for $\{A_1, B_1\}$ in the proof of 11.1, (ii), we find that θ is on $Z + P$. Hence $K \in \theta$, contrary to the definition of θ.

In the remaining case we may suppose without loss of generality that $(J + K)(Z + P)$ meets λ_1 in $A_1 + B_1$. As before the tangent of $\{A_1, B_1, J\}$ is on $Z + P$. It is therefore on $J + K$ and K. Again the definition of a tangent is contradicted.

We deduce that any two points of $S - \{P\}$ on $\tau_3 - \alpha$ are collinear with P. Using 10.3 we deduce that $S \cap \tau_3$ is a line on P which is not on α.

12.2. *Let St_1 and St_2 be 5-stigms. Then λ_1 and λ_2 are distinct.*

PROOF: Assume $\lambda_1 = \lambda_2$. By 12.1 St_3 is a line and $St_3 = S \cap \tau_3$.

Let J be any point of S not on the join of σ_1 and σ_2, and not on St_3. Let J' be a point of St_3 other than P. Then $J + J'$ is on the join of σ_1 and σ_2, and is not on π. We can adjust the notation so that $J + J'$ is on σ_1.

Since J' can be replaced by $P + J'$ we can arrange that $J + J'$ is not a point of St_1. We can therefore adjust the notation in St_1 and St_2 so that $J + J'$ is $P + A_1$.

The plane A_2B_2J' meets the join of σ_1 and σ_2 only in the line A_2B_2.

Let θ be one of its tangents. θ meets λ_1 in the point $A_2 + B_2 = A_1 + B_1$. Its line of intersection with σ_1 can only be that on $A_1 + B_1$, $P + A_1$, and $P + B_1$. Hence $J + J' \in \theta$, and so $J \in \theta$. But this is impossible since J is not in the join of σ_2 and St_3.

We deduce that all the points of S in τ_1 and τ_2 belong to σ_1 and σ_2, respectively. But the 3-space α of 12.1 meets S only in P. Of the three 4-spaces on α one is τ_3, a second contains both σ_1 and σ_2, and the third therefore meets S only in P. But this is contrary to 10.3. The theorem follows.

12.3. *Let St_1, St_2 and St_3 be 5-stigms. Then λ_1, λ_2 and λ_3 have no common line.*

PROOF: Assume that λ_1, λ_2, and λ_3 have a common line m. The λ_i are all distinct, by 12.2, and together they include all the points of the 3-space π.

We denote the points of St_i by A_i, B_i, C_i, D_i, and P. We write the points of m as P, X, and Y. We adjust the notation so that A_1B_1, A_2B_2, and A_3B_3 concur in X while C_1D_1, C_2D_2, and C_3D_3 concur in Y.

There is a plane μ_i on A_i and σ_i made up of the sums of even numbers of the points B_i, C_i, D_i, and P. It meets St_i only in A_i. Moreover each line on A_i and σ_i which meets St_i only in A_i must lie in μ_i. Let x_i be the line of intersection of λ_i and μ_i. Like μ_i it is on Y but not on X. The plane of x_1 and x_2 meets λ_3 in a line y, which is on Y but not on X.

Let α be the space generated by the six points A_i, B_i. Suppose that α is a plane. The tangents of $\{C_1, D_1\}$ meet this plane in its seventh point X, and therefore pass through P. But this is impossible since P is not on the line C_1D_1. We deduce that α is a 3-space.

We denote the point $A_1 + A_2 + A_3$ by Z. By the preceding result Z is not X. If $Z = P$, then XP is the only line on P and α which does not pass through one of the points A_i, B_i. Hence the tangents of $\{C_1, P\}$ are on X and therefore on D_1. But this is impossible since D_1 is not on the line PC_1. We deduce that Z is not P. A similar argument with the letters A and B interchanged shows that $B_1 + B_2 + B_3$ is not P. Hence Z is not Y.

It is clear that Z is a point of π. We can therefore adjust the suffixes so that Z is on λ_3.

Suppose A_1, A_2 and C_3 are collinear. The plane $A_1A_2A_3C_3$ meets the 4-spaces τ_i in single lines, and it meets π in the single point $Z = A_3 + C_3$. Let θ be a tangent of $\{A_1, A_2, A_3, C_3\}$. If $Y \in \theta$, then $D_3 \in \theta$.

But the plane $A_1A_2A_3C_3$ meets τ_3 only in the line $\{A_3, C_3, A_3 + C_3\}$ and so is not on B_3 or D_3. We note also that since Z is not on λ_1 or λ_2 the plane is on no point of $St_1 \cup St_2$ other than A_1 and A_2.

We deduce that Y is not on θ. But the intersection of θ and σ_1 is a line meeting St_1 only in A_1. It therefore lies in μ_1. Hence θ meets x_1, and similarly x_2, in a point other than Y. Accordingly θ meets y in a point $J \neq Y$. Since $J \in \lambda_3 - m$ we must have $J = A_3 + C_3$, $A_3 + D_3$, $B_3 + C_3$, or $B_3 + D_3$. We rule out the last three possibilities since θ cannot pass through B_3, D_3, or P. Hence $A_3 + C_3$ is on y.

We can repeat the preceding arguments with A_3 and B_3 interchanged, and with Z replaced by $A_1 + A_2 + B_3$. This change does not affect y. We deduce that $B_3 + C_3$ is also a point of y. But then y is on $A_3 + C_3 + B_3 + C_3 = X$, which is a contradiction.

We deduce that A_1, A_2, and C_3 are not collinear. The argument can be repeated with A_2 interchanged with B_2, or C_3 with D_3, or both. We conclude that neither A_1A_2 nor A_1B_2 passes through C_3 or D_3.

Now $Z \in \lambda_3 - m$. Hence Z is $A_3 + C_3$, $A_3 + D_3$, $B_3 + C_3$, or $B_3 + D_3$. The first two cases imply that $A_1 + A_2 = C_3$ or D_3. Since $A_3 + B_3 = A_2 + B_2$ the other two imply that $A_1 + B_2 = C_3$ or D_3. In each case the result of the preceding paragraph is contradicted.

12.4. *S has at most* 5 *points on any one plane.*

PROOF: If possible let β be a plane containing more than 5 points of S. Since S contains no Fano block there is just one point Z on β which is not in S. Let γ be a 4-space not on Z. Let S' be the set of all points of γ collinear with Z and a member of S.

The dimension of S' is 4. Moreover S' is a 2-block. For if θ is a 2-space in γ not meeting S', then $Z\theta$ is a 3-space not meeting S.

The 2-block S' is tangential. For let T' be any non-null subset of S' of dimension $d \leq 1$. Let T be the set of all points of S collinear with Z and a point of T'. Then the dimension of T is at most 2, and so T has a tangent φ with respect to S. Now φ is on Z. This is obviously true if the dimension of T exceeds that of T', and in particular if T contains a point of β. But in the remaining case φ must meet β in a point not belonging to S, and this can only be Z. It follows that the intersection of φ with γ is a tangent of T' with respect to S'.

We conclude that S' is a 4-dimensional tangential 2-block. But this is impossible, by 11.5.

12.5. *There is a point of S which is on two distinct lines of S.*

PROOF: Assume that no line of S lies in τ_1 or τ_2. Then St_1 and St_2 are 5-stigms. The planes λ_1 and λ_2 are distinct, by 12.2. We write their line m of intersection as $\{P, X, Y\}$. We write the points of St_i as P, A_i, B_i, C_i, and D_i. ($i = 1, 2$.) We adjust the notation so that A_1B_1 and A_2B_2 meet in X while C_1D_1 and C_2D_2 meet in Y.

The 3-spaces σ_1 and σ_2 have only the line m in common. Hence their join is 5-dimensional. Hence the six points A_1, A_2, X, Y, C_1, and C_2 are linearly independent. We proceed to tabulate the 16 points of $\tau_3 - \pi$ as linear combinations of these six.

In the plane A_1A_2X we have

$$A_1 + A_2,$$
$$A_1 + B_2 = A_1 + A_2 + X.$$

Similarly, in C_1C_2Y we have

$$C_1 + C_2,$$
$$C_1 + D_2 = C_1 + C_2 + Y.$$

Four other points are given by

$$P + A_1 + A_2 = A_1 + A_2 + X + Y,$$
$$P + A_1 + B_2 = A_1 + A_2 + Y,$$
$$P + C_1 + C_2 = C_1 + C_2 + X + Y,$$
$$P + C_1 + D_2 = C_1 + C_2 + X.$$

The remaining eight points are

$$A_1 + C_2,$$
$$A_1 + D_2 = A_1 + C_2 + Y,$$
$$B_1 + C_2 = A_1 + C_2 + X,$$
$$B_1 + D_2 = A_1 + C_2 + X + Y,$$
$$C_1 + A_2,$$
$$C_1 + B_2 = C_1 + A_2 + X,$$
$$D_1 + A_2 = C_1 + A_2 + Y,$$
$$D_1 + B_2 = C_1 + A_2 + X + Y.$$

Let St denote the 5-stigm $\{P, P + A_1 + A_2, P + A_1 + B_2, P + C_1 + C_2, P + C_1 + D_2\}$. The 3-space generated by St contains the line m. Hence it is not St_3, by 12.3. We deduce that St_3, whether it is a line or a 5-stigm, must include a point which is collinear with two other points of S not in τ_3.

We now know that S contains a line. We may take P to be a point on this line, and then take the line to be St_3. If there is a line of S in τ_1 or τ_2 it must meet π in P. The theorem then holds. If there is no such line the preceding argument shows that there is a second line of S meeting St_3 in a point other than P.

12.6. *Suppose St_1 is a 5-stigm, while St_2 and St_3 are lines. Let y denote the line of τ_1 coplanar with St_2 and St_3. Let α be the 3-space determined by λ_1 and y. Then α is a tangent of P. Moreover it is the only tangent of P, other than π, on τ_1.*

PROOF: We denote the points of $St_1 - \{P\}$ by A, B, C, and D, and those of $y - \{P\}$ by U and V.

Assume that α contains a point X of S other than P. We note that y is not on σ_1. If it were it would contain A, B, C, or D, contrary to 12.4. It also follows from 12.4 that X is not U or V. Let the plane Xy meet π in a line m. We adjust the notation in St_1 so that $A + B$ is not on m.

Let θ be a tangent of $\{A, B, X\}$. Considering the plane of St_2 and St_3 we see that θ is on U or V. Hence θ is on some point of m other than P. It therefore meets σ_1 in a plane on A and B. Such a plane must be on a third point of St_1. But this is impossible since the plane ABX meets St_1 only in A and B.

We deduce that α is a tangent of P. Suppose, however, that there is a tangent φ of P, distinct from π and α, on τ_1. Then φ is not one of the three 3-spaces on λ_1 and τ_1. Accordingly φ and σ_1 have a line in common which is on P but not on λ_1. But such a line must be on a second point of St_1, and we have a contradiction.

12.7. *Suppose there is no line of S on τ_1. Then we can arrange, choosing another tangent of P as π if necessary, that all the lines of S on P are in τ_1.*

PROOF: If all such lines are on the same space τ_2 or τ_3 we have only to permute suffixes. We suppose therefore that there is at least one of them in each of τ_2 and τ_3.

But if x_2 and x_3 are lines of S on τ_2 and τ_3, respectively, then the line of τ_1 coplanar with them determines with λ_1 a 3-space α which is a tangent of P, by 12.6. Moreover α is the same for all choices of x_2 and x_3, by 12.6. Hence all the lines of S on P lie in the same 4-space on α. The theorem follows.

13. The Petersen Block

In this section we discuss the case in which no point of S is on three distinct lines of S.

We take P to be the common point of two lines x and y of S, as is permissible by 12.5. We may arrange that both x and y are in τ_3, by 12.7. St_1 and St_2 are then 5-stigms. We use the same notation for them as in 12.5.

We observe that $A_1 + A_2$ is on the lines A_1A_2 and B_1B_2. If it is on x or y then it is on three distinct lines of S, contrary to assumption. Hence neither x nor y is on $A_1 + A_2$ or, similarly, on $A_1 + B_2$, $C_1 + C_2$, or $C_1 + D_2$. The only possibilities for x and y are thus

$$\{P,\ A_1 + C_2,\ B_1 + D_2\},$$
$$\{P,\ A_1 + D_2,\ B_1 + C_2\},$$
$$\{P,\ C_1 + A_2,\ D_1 + B_2\},$$
$$\{P,\ C_1 + B_2,\ D_1 + A_2\}.$$

The first two of these lines are not both in S. For suppose they are. The third line in their plane is XPY. Hence any tangent of $\{A_2, D_1\}$ is on either X or Y. It is therefore also on $B_2 = A_2 + X$ or $C_1 = D_1 + Y$, which is contrary to the definition of a tangent. Similarly the last two lines are not both in S. We can therefore adjust the notation so that

$$x = \{P,\ A_1 + D_2,\ B_1 + C_2\},$$
$$y = \{P,\ C_1 + B_2,\ D_1 + A_2\}.$$

Given three skew lines in a 3-space we say that their nine points constitute a *regulus*. The remaining six points of the 3-space lie on two lines.

The three skew lines A_1B_1X, $P(A_1 + D_2)(B_1 + C_2)$, and C_2D_2Y determine a regulus R_1. The two complementary lines are

$$r_1 = \{P + A_1,\ P + C_2,\ A_1 + C_2\},$$
$$s_1 = \{P + B_1,\ P + D_2,\ B_1 + D_2\}.$$

Another regulus R_2 is obtained from R_1 by interchanging the suffixes 1 and 2. We record its complementary lines as

$$r_2 = \{P + A_2,\ P + C_1,\ A_2 + C_1\},$$
$$s_2 = \{P + B_2,\ P + D_1,\ B_2 + D_1\},$$

We observe that each of the two reguli has all its points, except X and Y, in S.

The two reguli are evident in Figure 1. Here the broken lines represent lines of $PG(5, 2)$ which are not lines of S, and the full lines represent

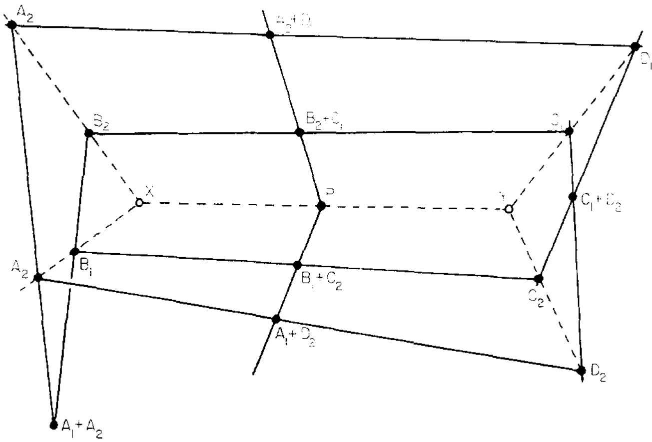

FIGURE 1

lines of S. The assignment of $A_1 + A_2$ and $C_1 + D_2$ to S has however still to be justified.

Let θ be a tangent of $\{A_1, D_1\}$. Then θ is not on X or Y, for otherwise it would be on B_1 or C_1, and these are points of S not on A_1D_1. Hence the line of intersection of θ with the 3-space of R_1 must be r_1 or s_1. The first alternative can be ruled out since it implies that θ is on P. In a similar way we find that θ meets the 3-space of R_2 in r_2.

Moreover we can interchange the suffixes 1 and 2 throughout the preceding argument. We deduce that there is no point of S on any of the four lines r_1, r_2, s_1, s_2.

Let α be the 3-space generated by r_1 and r_2. In the following table we give the point of α collinear with given points of r_1 and r_2.

	$P + A_1$	$P + C_2$	$A_1 + C_2$
$P + A_2$	$A_1 + A_2$	$A_2 + C_2$	$A_1 + B_2 + D_2$
$P + C_1$	$A_1 + C_1$	$C_1 + C_2$	$C_2 + B_1 + D_1$
$A_2 + C_1$	$A_2 + B_1 + D_1$	$C_1 + B_2 + D_2$	$B_1 + B_2 + C_1 + C_2$

Each of the points entered in the third row and the third column is a sum of a point of x or y with a point of S not on the same line x or y. No such point can belong to S since no three lines of S have a common point.

However S is a 2-block and so has at least one point on α. Hence $A_1 + A_2$, $A_2 + C_2$, $A_1 + C_1$, or $C_1 + C_2$ belongs to S. But $A_1 + A_2$ and $C_1 + C_2$ are alike under the symmetry of S as so far determined, as are $A_1 + C_1$ and $A_2 + C_2$. We can adjust the notation so that either $A_1 + A_2$ or $A_1 + C_1$ belongs to S.

We can make a similar investigation of the 3-space β determined by s_1 and s_2. It differs from that of α only in the letter-interchange of A with B and C with D. The conclusion is that $B_1 + B_2$, $B_2 + D_2$, $B_1 + D_1$, or $D_1 + D_2$ belongs to S.

Suppose $A_1 + A_2$, that is $B_1 + B_2$, is not in S. Then from our study of α, with its final adjustment of notation, we have $A_1 + C_1 \in S$. We deduce that $B_1 + D_1$ is not in S, for otherwise P would be on three lines of S, including $\{P, A_1 + C_1, B_1 + D_1\}$. Moreover $D_1 + D_2$, which is $C_1 + C_2$, is not in S, for otherwise C_1 would be on the three distinct lines C_1B_2, C_1C_2, and C_1A_1 of S. We conclude that either $A_1 + A_2$ is in S, or both $A_1 + C_1$ and $B_2 + D_2$ are in S.

Let γ be the 3-space generated by r_1 and s_2. We give a table for the point of γ collinear with given points of r_1 and s_2.

	$P + A_1$	$P + C_2$	$A_1 + C_2$
$P + B_2$	$A_1 + B_2$	$B_2 + C_2$	$A_1 + A_2 + D_2$
$P + D_1$	$A_1 + D_1$	$D_1 + C_2$	$B_1 + C_1 + C_2$
$B_2 + D_1$	$B_1 + C_1 + B_2$	$D_1 + A_2 + D_2$	$B_2 + C_1 + A_1 + D_2$

As before we note that we cannot assign a point entered in the third row or column to S without introducing a third line of S through some point of x or y. But S has some point in γ. Hence $A_1 + B_2$, $B_2 + C_2$, $A_1 + D_1$, or $D_1 + C_2$ is in S.

Assume that $A_1 + C_1$ and $B_2 + D_2$ are both in S. If $A_1 + B_2 \in S$, then A_1 is on the three distinct lines A_1B_2, A_1C_1, and A_1D_2 of S. If $B_2 + C_2 \in S$, then B_2 is on the three distinct lines B_2C_2, B_2D_2, and B_2C_1 of S. If $A_1 + D_1 \in S$, then A_1 is on the three distinct lines A_1D_1, A_1C_1, and A_1D_2 of S. Finally if $D_1 + C_2$, that is $C_1 + D_2$, is in S then C_1 is on the three distinct lines C_1D_2, C_1A_1, and C_1B_2 of S.

We may now suppose that $A_1 + A_2$ is a point of S. If $A_1 + B_2 \in S$, then A_1 is on the three distinct lines A_1A_2, A_1B_2, and A_1D_2 of S. If $A_1 + D_1 \in S$, then A_1 is on the three distinct lines A_1A_2, A_1D_1, and A_1D_2 of S. If $B_2 + C_2$ is a point of S then, since $A_1 + A_2 = B_1 + B_2$, the point B_2 is on the three distinct lines B_2B_1, B_2C_2, and B_2C_1 of S. The only remaining possibility for γ is that $D_1 + C_2$, which is $C_1 + D_2$, is in S.

Counting $A_1 + A_2$ and $C_1 + D_2$ we now have a set S' of 15 points of S, as shown in Figure 1. It can be verified that S' corresponds to an embedding of the cycle-group of the Petersen graph. This can be done by comparing Figure 1 with Figure 2, in which the edges of the Peter-

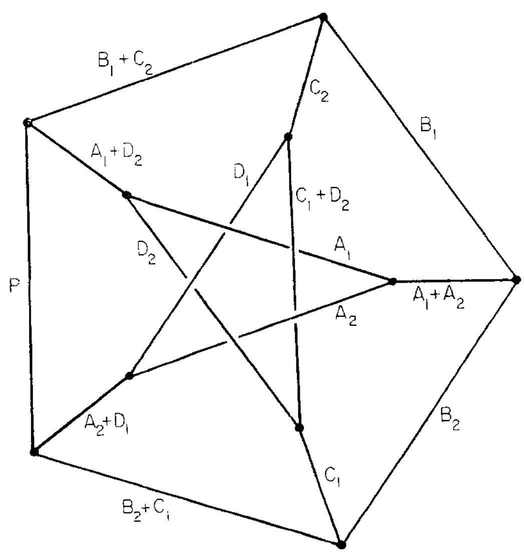

FIGURE 2

sen graph are lettered in correspondence with the points of S'. We note that there are no linear relations between the points of S' other than those given by its 10 lines.

Now the Petersen graph is the standard example of a trivalent graph having no Tait coloring. Hence S' is a 2-block, by 3.2 and 9.4. Accordingly $S' = S$, by 9.2. We refer to S as the *Petersen block.*

It can be verified that the Petersen block is tangential, but we only sketch the proof there. It follows from the fact that the Fano and Desargues blocks correspond only to non-cographic chain-groups. Hence, if G is a Petersen graph, no minor of $\Gamma(G)$ can correspond to a Fano or Desargues block. But if S is not tangential some minor of $\Gamma(G)$ corresponds to a 2-block in 2 or 3 dimensions, by 8.5 and 9.3. and therefore some minor of $\Gamma(G)$ corresponds to a Fano or Desargues block, by 8.3 and 10.4.

We summarize the results of this section in the following theorem.

13.1. *Either S is a Petersen block or it contains three concurrent lines.*

14. Three Concurrent Lines

In the case remaining we can choose P to lie on three concurrent lines $x_i = PA_iB_i$ $(i = 1, 2, 3)$ of S. The three lines are not coplanar since S contains no Fano block. They thus determine a 3-space α. We say they are the *generators* of a *cone* in α. The *axis* of the cone is the line $\{P, Z, Z + P\}$, where $Z = A_1 + A_2 + A_3$. The six points of α not on the axis or a generator of the cone lie on a plane ν through P. We write y_i for the line on ν coplanar with x_j and x_k, where (i, j, k) is a permutation of the sequence (1, 2, 3). The three lines y_i are distinct, and each point of ν lies on one of them.

We observe that any line on α through Z or $Z + P$ must have a common point with a generator. Hence any line on α which does not meet S must lie in ν.

14.1. *The only points of S on α are those of the three generators.*

Proof: No point of S other than P is on any of the lines y_i, by 12.4. If Z or $Z + P$ were in S its tangent would meet α in a line. But this line would meet a generator, and so pass through a second point of S.

14.2. *Suppose P is on a fourth line $x_4 = PA_4B_4$ of S. Then the four lines x_i generate a 4-space τ. Let ξ be the 3-space in τ constituted by the linear combinations of P with the sums of even numbers of the points A_1, A_2, A_3, A_4. Then ξ meets S only in P.*

PROOF: By 14.1 the line x_4 is not on α. Hence the four lines x_i generate a 4-space τ.

If possible let T be a point of $S \cap \xi$ other than P. Then T is not on any of the lines x_i. Let θ be any tangent of $\{A_1, T\}$. It has a common plane φ with τ. There is a 3-space β on φ and τ which does not pass through P. We can adjust the notation so that β meets x_i in A_i. ($i = 1, 2, 3, 4$.)

The four points A_i are not coplanar, since the four lines x_i have no common 3-space. By 14.1 the point T is not in the 3-space generated by any three of the lines x_i. Hence the points T, A_1, A_2, A_3, A_4 constitute a 5-stigm St of S. Each of the three planes of β on A_1T passes through a third point of St. Hence θ passes through a point of S not on A_1T, contrary to its definition as a tangent.

14.3. *There are at most four lines of S on P.*

PROOF: We use the notation of 14.2 and suppose that there is a fifth line $x_5 = PA_5B_5$ of S on P. This line is not on ξ, by 14.2, and not on the 3-space generated by any three of the lines x_1, x_2, x_3, x_4, by 14.1. Hence x_5 is not on τ, and the five lines x_i generate the 5-space.

Let η be the 4-space constituted by the linear combinations of P with the sums of even numbers of the points A_i ($i = 1, 2, 3, 4, 5$). If U is a point of $\eta - \{P\}$ we can adjust the notation, by permuting suffixes, sothat $U \in \xi$. We deduce from 14.2 that η meets S only in P. But this is contrary to 10.3.

14.4. *There are just three lines of S on P. Moreover if π is any tangent of P then each of the three 4-spaces τ_1, τ_2 and τ_3 on π contains just one line of S.*

PROOF: Suppose first that all the lines of S through P belong to τ_3. Then St_1 and St_2 are 5-stigms. The case of two such 5-stigms is analyzed in the proof of 12.5. It is there found that any line through P which is on τ_3 but not on π passes through a point U which is collinear with points $X_1 \in St_1 - \{P\}$ and $X_2 \in St - \{P\}$.

We take the line to be the axis a of our cone in α. Let θ be a tangent of $\{X_1, X_2\}$. Then θ meets α in a line on U, and this line must meet a generator of the cone. But this is contrary to the definition of θ as a tangent.

We deduce that the lines of S on P cannot all be in the same space τ_i.

Suppose there are more than three lines of S on P. Then there are just four such lines, by 14.3. Moreover P has a tangent ξ such that the four lines belong to the same 4-space on ξ, by 14.2. Since this is contrary to the preceding result we deduce that there are only three lines of S on P.

If two of these lines belong to the same space τ_i we can make a new choice of π so as to bring all three into the same space τ_i, by 12.7. But we have seen that this is impossible. The theorem follows.

In view of 14.4 we can adjust the notation so that x_i is in τ_i. $(i = 1, 2, 3)$ It is then clear that y_i is in τ_i and that a is a line in π.

14.5. *τ_i contains just three points D_i, E_i, and F_i of S which are not on x_i. $(i = 1, 2, 3.)$ These three points are non-collinear, and their sum is a point of $y_i - \{P\}$. Their plane does not meet a or x_i.*

PROOF: It is sufficient to prove the theorem for the case $i = 1$.

Let M be the set of all points of S on τ_1 which are not on x_1. Let μ be the space generated by M.

Suppose μ is on P. Then P is the sum of an even number of independent points of M. Hence, by 14.4, there is a 5-stigm of S in τ_1. Hence, by 12.6, we can make a new choice of π so as to bring x_2 and x_3 into the same 4-space τ_i. But this is impossible, by 14.4.

Suppose next that μ is on some point Q of $a - \{P\}$. Then Q is the sum of an even number j of independent points of M. If $j = 2$ there are two points X and Y of M such that XY meets the axis of the cone. But then it is clear that $\{X, Y\}$ can have no tangent in S. We deduce that Q is a sum of four independent points X, Y, Z, T of M. But then μ is 3-dimensional and so meets y_i in a point U. Now U is not in S, by 12.4. Since it is not in π it can only be represented as a sum of an odd number of independent points of M. We may therefore write $X + Y + Z = U$. But then $T + U = Q$, and therefore T is on x_1, contrary to the definition of M.

We deduce that μ does not meet a. Hence μ is at most 2-dimensional.

Assume that μ does not meet y_1. Then we can show that $x_1\mu$ does not meet y_1, except in P. For otherwise we can find points $J \in x_1 - \{P\}$ and $K \varepsilon \mu$ such that $J + K$ is a point of $y_1 - \{P\}$. But then K must be on a, contrary to the preceding result. Hence there exists a 2-space β in π such that of the three 3-spaces on β and τ_1 one is π, a second contains $x_1\mu$, and the third therefore meets S only in P. Moreover we can choose β so that y_1 is in the third 3-space. We can therefore replace π by this 3-space, and so bring x_2 and x_3 into the same 4-space τ_i. But this is impossible, by 14.4.

We deduce that μ includes a point U_1 of $y_1 - \{P\}$. Since U_1 is not in S, by 12.4, it must be the sum of three independent points D_1, E_1 and F_1 of M. The remaining three points of μ are points of π other than P. The theorem follows.

We continue to use the notation of 14.5. We denote the plane $D_iE_iF_i$ by μ_i, and its intersection with π by m_i. ($i = 1, 2, 3.$)

14.6. *The three lines m_i are all distinct.*

PROOF: Suppose the theorem false. Without loss of generality we may suppose $m_1 = m_2$. By the Desargues theorem we can adjust the notation so that D_1D_2, E_1E_2, and F_1F_2 concur in a point Z of τ_3.

Choose $Q \in a - \{P\}$. Write $G_i = D_i + E_i + F_i + Q$ ($i = 1, 2$). Thus $G_1 + G_2 = Z$. Since $D_i + E_i + F_i$ is a point of $y_i - \{P\}$ we have $G_i \in x_i - \{P\}$, $Z \in y_3 - \{P\}$, and $Z + Q \in x_3 - \{P\}$. We note that $Z + Q$ is in S.

Let θ be a tangent of $\{D_1, E_1\}$. We note that D_1E_1 does not meet x_1, x_2, or x_3, by 14.5. The line of intersection of θ with α cannot meet the axis a of our cone. Hence θ is not on Q. Considering the 5-stigm $QD_2E_2F_2G_2$ we find that θ must meet the corresponding 3-space in points $D_1 + E_1 = D_2 + E_2$, $D_2 + T$, and $E_2 + T$, where T is F_2, G_2, or Q. But if $T = F_2$ then θ is on $D_2 + F_2 = D_1 + F_1$ and therefore on F_1. Similarly if $T = G_2$ then θ is on G_1. If $T = Q$ then θ is on $(D_2 + Q) + D_1 = Z + Q$. In each case θ is on a point of S not on D_1E_1, and its definition as a tangent is contradicted.

15. Conclusion

We now investigate the possibilities left over by 14.6. We write

$$U_i = D_i + E_i + F_i$$

($i = 1, 2, 3$). Since $U_i \in y_i - \{P\}$, by 14.5, we have

15.1. $U_1 + U_2 + U_3 = kP$,
where k is an integer, 0 *or* 1.

15.2. *Suppose* $X_i \in \{D_i, E_i, F_i\}$ $(i = 1, 2, 3)$. *Then there is no point* Q *of* $a - \{P\}$ *such that*
$X_1 + X_2 + X_3 = Q.$

PROOF: Assume that such a point Q exists. Suppose first that $X_1 + X_2$ is a point Y_3 of $\{D_3, E_3, F_3\}$. Then any tangent of $\{X_3, Y_3\}$ meets α in a line which intersects x_1, x_2 or x_3, which is absurd. But if $X_1 + X_2$ is a point of x_3. necessarily not P, then X_3 must be a point of y_3, which is contrary to 12.4. We deduce that $X_1 + X_2$ is not a point of S. Similarly $X_2 + X_3$ and $X_1 + X_3$ are not points of S. (See 14.5.) Accordingly the plane $X_1X_2X_3$ meets S only in X_1, X_2, and X_3. But any tangent of $\{X_1, X_2, X_3\}$ must meet α in a line which intersects x_1, x_2, or x_3. This contradiction establishes the theorem.

15.3. *If two of the lines* m_i *are coplanar their plane passes through* P.

PROOF: If the theorem fails we may suppose m_1 and m_2 to be on a plane φ which meets a in a point Q other than P. Let X be a point of φ which is on m_1 but not m_2. Allowing for an interchange of the suffixes 1 and 2 we have four choices for X. If possible we choose X to lie on m_3. Let Y be a point of φ which is on m_2 but not m_1.

We may now write

$$m_1 = \{X, Q + Y, Q + X + Y\},$$
$$m_2 = \{Y, Q + X, Q + X + Y\},$$
$$D_1 = U_1 + X,$$
$$D_2 = U_2 + Y,$$
$$E_1 = U_1 + Q + Y,$$
$$E_2 = U_2 + Q + X,$$
$$F_1 = U_1 + Q + X + Y,$$
$$F_2 = U_2 + Q + X + Y.$$

Suppose m_3 is on the common point $Q + X + Y$ of m_1 and m_2. Then we can put

$$D_3 = U_3 + Q + X + Y,$$
$$D_1 + D_2 + D_3 = Q + kP,$$

by 15.1.

Suppose next that m_3 meets one of the lines m_1 and m_2 but not the other. Then it is on X, by the choice of X. We can now put

$$D_3 = U_3 + X,$$
$$F_1 + D_2 + D_3 = Q + kP.$$

In the remaining case m_3 meets φ only in $X + Y$. It must therefore pass through one of the points P, $P + Q$, $P + X$, and $P + Q + X$. The first two of these are ruled out by 14.5. The other two are equivalent; we can arrange that m_3 is on $P + X$ by a proper choice of X. We can now put

$$D_3 = U_3 + P + X,$$
$$F_1 + D_2 + D_3 = Q + (k + 1)P.$$

In each case we have obtained a contradiction of 15.2. The theorem follows.

15.4. *No two of the lines m_i are coplanar.*

PROOF: If the theorem fails then, by 15.3, we may suppose m_1 and m_2 to have a common plane φ which is on P. We use the same notation for the points of φ, μ_1, and μ_2 as in 15.3, except that Q is replaced by P.

Let θ be a tangent of $\{E_1, F_1\}$. Considering the plane of x_1 and x_2 we see that θ is on either $U_1 + U_2$ or $U_1 + U_2 + P$. But in the first case θ is on

$$\begin{aligned} F_1 + (U_1 + U_2) &= U_1 + P + X + Y + U_1 + U_2 \\ &= U_2 + P + X + Y = F_2. \end{aligned}$$

In the second case θ is on

$$\begin{aligned} E_1 + (P + U_1 + U_2) &= U_1 + P + Y + P + U_1 + U_2 \\ &= U_2 + Y = D_2. \end{aligned}$$

In either case the definition of θ as a tangent is contradicted.

15.5. *The three lines m_i are not mutually skew.*

PROOF: Assume that the three lines are mutually skew. Then they determine a regulus R in π. We write the two complementary lines of R as

$$s = \{X,\ Y,\ X + Y\},$$
$$t = \{Z,\ T,\ Z + T\}.$$

By 14.5 we may suppose that s is the axis a of the cone and that $X + Y = P$.

We may now write the lines m_i as follows.

$$m_1 = \{X + Z,\ Y + T,\ X + Y + Z + T\},$$
$$m_2 = \{X + T,\ Y + Z + T,\ X + Y + Z\},$$
$$m_3 = \{X + Z + T,\ Y + Z,\ X + Y + T\}.$$

We proceed to tabulate some points of S as linear combinations of the six independent points X, Y, Z, T, U_1, and U_2.

$$\begin{aligned}
P &= X + Y,\\
D_1 &= X + Z + U_1,\\
E_1 &= Y + T + U_1,\\
F_1 &= X + Y + Z + T + U_1,\\
D_2 &= X + T + U_2,\\
E_2 &= Y + Z + T + U_2,\\
F_2 &= X + Y + Z + U_2,\\
D_3 &= X + Z + T + U_1 + U_2 + k(X + Y).\\
E_3 &= Y + Z + U_1 + U_2 + k(X + Y),\\
F_3 &= X + Y + T + U_1 + U_2 + k(X + Y).
\end{aligned}$$

We note that

$$D_1 + D_2 + D_3 = X + k(X + Y).$$

But this is X or Y, a point of $a - \{P\}$. This result contradicts 15.2.

From 15.4 and 15.5 we deduce that S can have no three concurrent lines. We may therefore sum up the results of the paper in the following.

THEOREM. *The only tangential 2-blocks of not more than five dimensions are the Fano, Desargues, and Petersen blocks.*

This result suggests the conjecture that a full chain-group N is chromatic if no minor of N corresponds to a Fano, Desargues, or Petersen block.

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