Tutte's Flow Conjectures

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Abstract

For $G = (V, E)$ a directed multigraph and H an abelian group, a map $f : E \to H$ is an H-*flow* if $f(e) \neq 0$ for all $e \in E$ and f obeys Kirchhoff's law: for all $v \in V$

$$\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$$

If $H = \mathbb{Z}$, the group of integers under addition, and k is a positive integer such that $-k < f(e) < k$ for every edge e, we say that f is a k-flow.

W. T. Tutte [5, 6, 7] made a series of conjectures asserting certain weak conditions under which a multigraph G has a 3-, 4- or 5-flow. We will present the statements of these alongside a number of results about group-valued and k-flows including sufficient conditions for low k-flows and the duality between flow and colouring problems. As an example of this we shall demonstrate that the theorem of Grötzsch [2] that *every triangle free planar graph is 3-colourable* is a dual to the 3-flow conjecture with an added assumption of planarity. Finally we will give an exposition of Thomassen's [4] recent proof of a weakening of the 3-flow conjecture and discuss subsequent work.

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Contents

Introduction 1

Seymour's 6-Flow Theorem 2

Lemma 2.1. *For $k > 2$, let $G = (V, E)$ be a bridgeless graph with no nowhere-zero k-flow such that G has $|V| + |E|$ minimal. Then G is simple, cubic and 3-connected.*

Suppose there were a loop e in G, let $G' = G - e$. Any bridge in G' would also be a bridge in G, so G' must be bridgeless. Therefore, by the minimality of G, there must be a nowhere-zero k-flow f on G'. But then we can extend f to a nowhere-zero k-flow on G by setting $f(e) = l$ for some $0 < l < k$, contradicting that G has no such flow. Thus G contains no loops. Now suppose there are vertices $v, w \in V$ with multiple edges $e_1, e_2, \dots, e_n$ between them and let $G'' = G/e_1$. Again any bridge in G'' would be a bridge in G so that G'' is bridgeless and by the minimality of G has a nowhere-zero k-flow f'. We will construct a flow f on G from f'. For all $e \in E \setminus \{e_1, e_2\}$, define $f(e) = f'(e)$. Since $k > 2$, we can either increase or decrease $f'(e_2)$ by one such that $|f'(e_2)|$ remains strictly between 0 and k. If we may increase $f'(e_2)$ by one then define $f(e_2) = f'(e_2) + 1$ and

$$f(e_1) = \begin{cases} 1 & \text{if } e_1 \text{ is oriented in the opposite direction to } e_2 \\ -1 & \text{if } e_1 \text{ is oriented in the same direction as } e_2 \end{cases}$$

Otherwise, define $f(e_2) = f'(e_2) - 1$ and set

$$f(e_1) = \begin{cases} 1 & \text{if } e_1 \text{ is oriented in the opposite direction to } e_2 \\ -1 & \text{if } e_1 \text{ is oriented in the same direction as } e_2 \end{cases}$$

Thus (with respect to f') f fixes the flow through all vertices except v and w and our construction ensures that at v and w any changes to the flow into the vertex is balanced by an equal and opposite change to the flow out of the vertex and vice versa. This ensures that f obeys Kirchhoff's Law, as f' does. In addition, $0 < |f(e)| < k$ for all $e \in E$, so f is a nowhere-zero k-flow on G. This contradicts that G has no such flow and so we have shown that G must be simple.

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