Report of MCMC Mixing on Treelike Graphs

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Outline

- Preliminaries of MCMC
- Randomly coloring graphs of bounded treewidth
- Counting independent sets in graphs with bounded bipartite pathwidth

Preliminaries of MCMC

- Irreducible: state space is connected
- Aperiodic: state space is not bipartite
- Irreducible & aperiodic -> converge to a unique stationary distribution
  - Verifying local condition, for all x,y, $\pi(x)p(x,y)=\pi(y)p(y,x)$

- This verifies the correctness of algorithm

Preliminaries of MCMC

- We also want fast mixing in polynomial time
- It is related to the largest eigenvalue of Markov matrix (lazy chain)
- Exponential number of states: impossible to do eigen decomposition

- Idea: verifying local condition to prove global property

Preliminaries of MCMC

- If there is a 'bottleneck' in state graph: hard to converge
- Rapid convergence: no 'bottleneck', hard to prove all partition

- Dual problem: find a multi-commodity flow with low congestion!
- Only need to construct one and verify!

We define the congestion on an edge (σ, σ') with respect to a flow f by

$$\rho_f(\sigma, \sigma') = \frac{1}{q(\sigma, \sigma')} \sum_{\alpha, \beta \in \Omega} \sum_{p:(\sigma,\sigma') \in p \in \mathcal{P}_{\alpha,\beta}} f(p)|p|,$$

and the congestion of f by

$$\rho_f = \max_{(\sigma,\sigma') \in F} \rho_f(\sigma, \sigma').$$

Randomly coloring graphs of bounded treewidth

1.1 Results

Our main result is an algorithm that efficiently samples a $((1+\epsilon)\Delta)$-coloring (almost) uniformly at random if the input graph has logarithmically bounded pathwidth, for any $\epsilon > 0$.[2]

Theorem 1.1. *(Informal) Let $\epsilon > 0$ and G be a graph with maximal degree Δ and pathwidth bounded by $O(\log n)$. There exists a polynomial time algorithm for sampling a $((1+\epsilon)\Delta)$-proper coloring of G (almost) uniformly at random.*

Using the fact that the pathwidth of a graph is at most $O(\log n)$ times its treewidth [35], we have the following corollary.

Corollary 1.2. *(Informal) Let $\epsilon > 0$ and G be a graph with maximal degree Δ and treewidth bounded by a constant. There exists a polynomial time algorithm for sampling a $((1+\epsilon)\Delta)$-proper coloring of G (almost) uniformly at random.*

Randomly coloring graphs of bounded treewidth

- Idea for sampling matchings
  - Construct a mutli-commodity flow
  - Find an injective function from paths going through the edge to state space
- It doesn't work for sampling proper colorings!
  - Unlike matchings, we cannot guarantee the following:
    - part of starting state + part of destination = a proper state
  - Changing one vertex may lead to illegal states, and cases vary!
    - This method allows single-flaw state

Randomly coloring graphs of bounded treewidth

- Allowing single-flaw state
  - Once entering, we can fix it quickly
  - Not too many single-flaw state, so # of flows are polynomial times of sth.
- Could we eliminate single-flaw state?
  - Using a deterministic method to move to proper state?
  - Failed! A counter-example: star graph
  - Reason: deterministic method leads to large congestion!
- So we use randomness to split the flow!

Randomly coloring graphs of bounded treewidth

Let $G = (V, E)$ be a graph with maximal degree Δ and $\epsilon > 0$ such that $(1+\epsilon)\Delta \geq \Delta + 2$. The state space Ω of Markov chain $\mathcal{MC}(G, \epsilon)$ (or simply $\mathcal{MC}$) is the set of all proper and singly-flawed k-colorings of G, for $k = \lceil (1+\epsilon)\Delta \rceil$: $\Omega = \mathcal{C}_p(G, k) \cup \mathcal{C}_{sf}(G, k)$. For simplicity, we henceforth assume that $\epsilon\Delta$, $(1+\epsilon)\Delta$ and $(1+\epsilon)\epsilon^{-1}$ are integers. It is easy to generalize the results to real values thereof. For $\sigma \in \Omega$, the transitions $\sigma \to \sigma'$ of $\mathcal{MC}$ are the following

- Let $\sigma' = \sigma$.
- With probability $1/2$, do nothing (laziness).
- Otherwise, choose a vertex v and color c uniformly at random from V and $[k]$ respectively. Tentatively, set $\sigma'(v) = c$
- If $\sigma' \notin \Omega$, set $\sigma'(v) = \sigma(v)$.

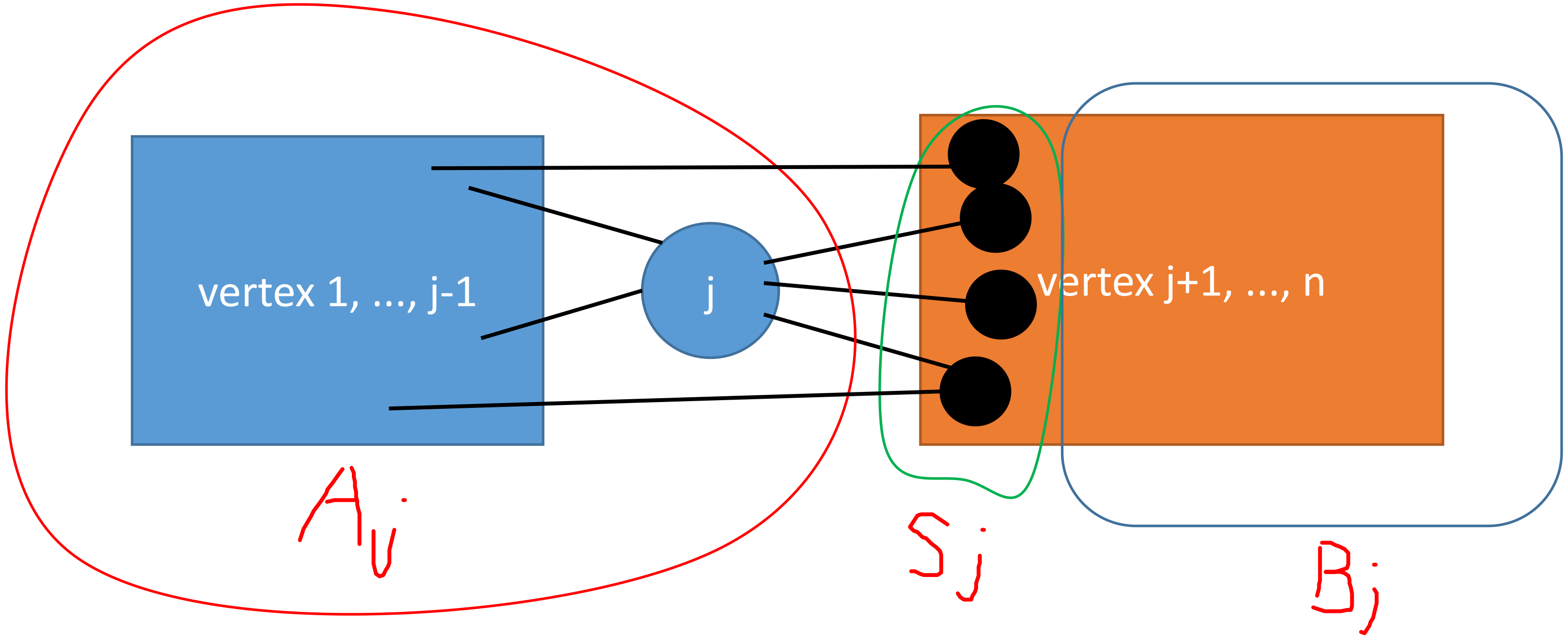
Randomly coloring graphs of bounded treewidth

- Technique: vertex separator, if there are not too many undeterministic vertices, i.e. vertex separator size is O(log n)
- The whole mixing time will be polynomial

- VSN = pathwidth, so pw = O(log n)
- tw = O(1) -> pw = O(log n)

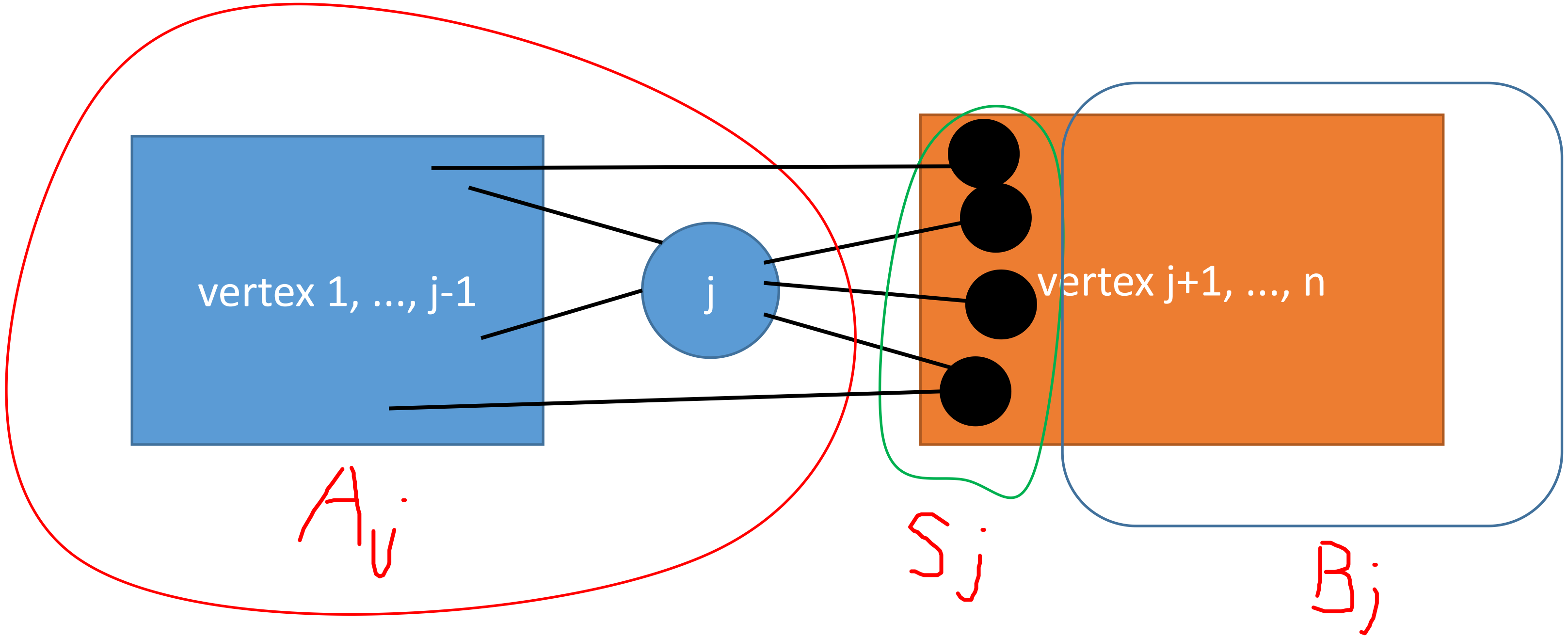
Randomly coloring graphs of bounded treewidth

- Canonical paths: n phases, each phase j with $|S_j|+1$ steps
- Order vertices from 1 to n, using minimal order (i.e., reaching VSN)
- For any a, b $\in \Omega$, phase j will change color of vertex j from a to b, and fix the conflicts



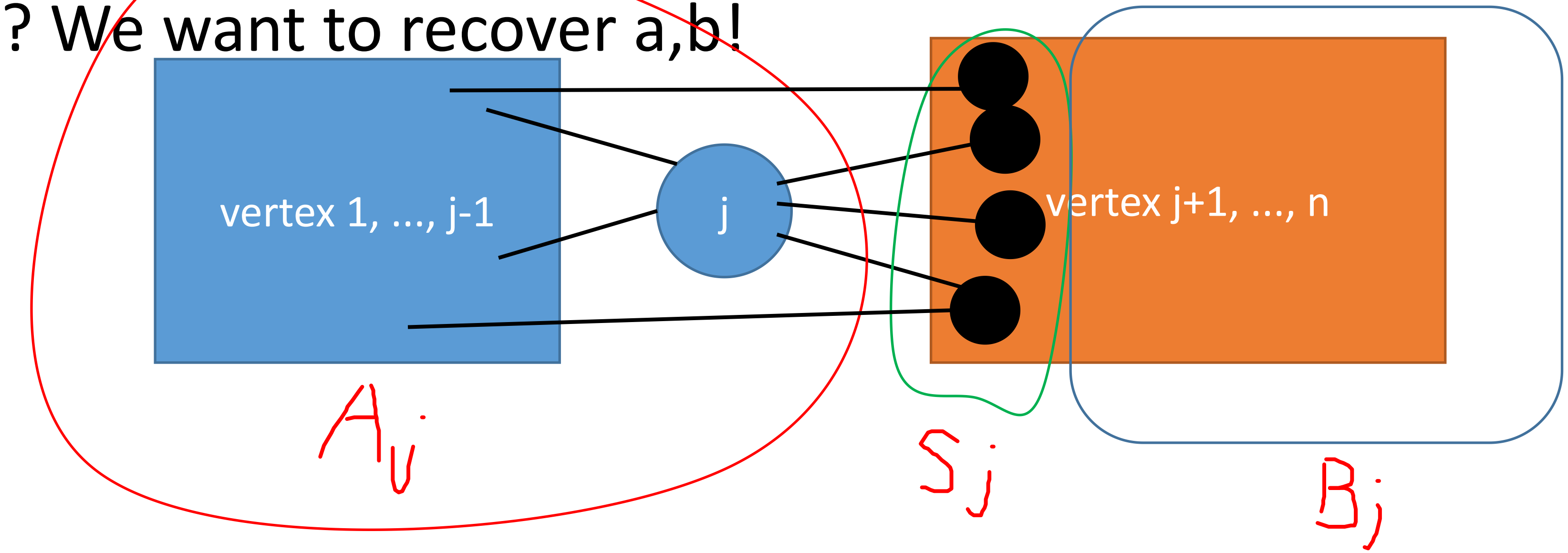
Randomly coloring graphs of bounded treewidth

- First, assume a,b are proper colors
- Phase j step 1: change j to its color in b
- Phase j, the next $|S_j|$ steps: randomly choose one of feasible colors for each vertex in S_j
- VSN is O(log n)



Randomly coloring graphs of bounded treewidth

- Similarly, we want an injective function from (a,b) to something
- In order to bound the number of flows
- From a to b, the current state has same A_j as b, the same B_j as a
- Get a proper coloring X, same A_j as a, same B_j as b!
- How to deal with S_j? We want to recover a,b!



Randomly coloring graphs of bounded treewidth

- Y is $[(1+\epsilon)\epsilon^{-1}]^{|S_j|}$, allowing us to pinpoint $\beta(v)$ for every $v \in S_j$.
- Finally, Z is simply all possible colorings of the vertices of $QS(j,t)$ under α.

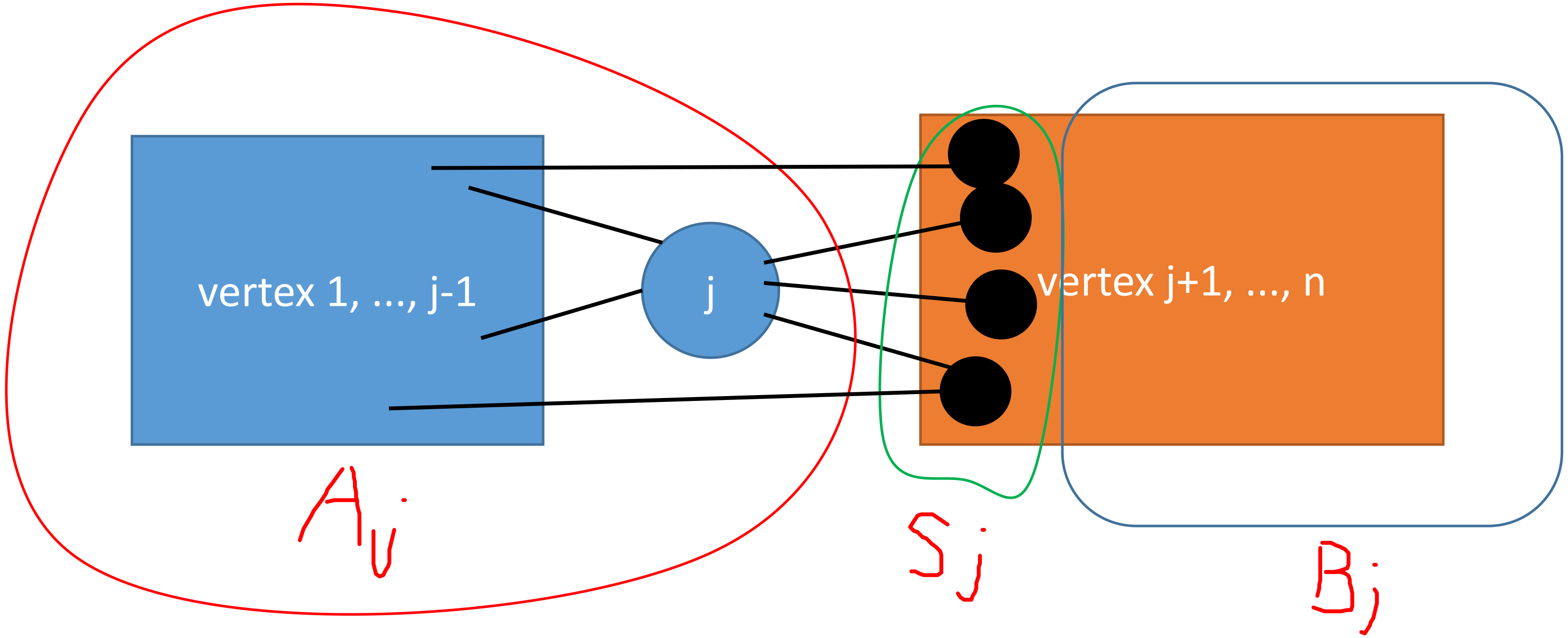
tell the random choice of S_j by b

tell the random choice of (part of) S_j (needs randomness) by a

Clearly x, y, z, j and t allow us to recover α and β. The size of the co-domain is at most

$$|\mathcal{C}_p| \cdot ((1+\epsilon)\epsilon^{-1})^{|S_j|} \cdot k^{|QS(j,t)|}.$$

t tells which step we are in phase j



Why Y is this way?

- Y is $\left[(1+\epsilon)\epsilon^{-1}\right]^{|S_j|}$, allowing us to pinpoint $\beta(v)$ for every $v \in S_j$.
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In canonical path, we 'split' the flow evenly, but not randomly.
Evenly is enough!
For its previous color, we project to a feasible color evenly
We only use ε△ feasible colors (even if we can use more for some vertex), so we can recover the info in (1+ε)/ε size set!
Because in flow size, we only split ε△ times for each 'unknown' state!

Randomly coloring graphs of bounded treewidth

- Each flow will split in the undeterministic vertices, so

Claim 3.3. *The flow routed from α to β through any $t = (\sigma, \sigma') \in F$ in any phase $j \in [n]$ is at most*

$$f_{j,t,\alpha,\beta} \le \frac{\pi(\alpha)\pi(\beta)}{(\epsilon\Delta)^{|QS(j,t)|}}.$$

Randomly coloring graphs of bounded treewidth

- Combining together, we'll get a polynomial mixing time
- What does undeterministic give us? (my view)
  - Each flow becomes smaller
  - But there are many more flows
  - Leading to $|S_j|$ in the exponential part, so we have to bound it in O(log n)

- Another question: it seems that k=$\triangle$+1 also works
  - Since we have single-flaw states, MC is no longer reducible

Randomly coloring graphs of bounded treewidth

- The last problem: flows may start / end at a single-flaw state
- There are not too many, each proper state may have kn times more neighbors, # of flows multiply at most $4k^2n^2$
- mixing time is still polynomial

Corollary 2.2. *For any $G = (V, E)$ such that $|V| = n$ and $k \geq \Delta + 2$,*

$$|\mathcal{C}_{sf}(G, k)| \leq kn|\mathcal{C}_p(G, k)|.$$

Proof. Immediate from the surjectivity of the function g in Lemma 2.1. □

Corollary 2.3. *For any $G = (V, E)$ such that $|V| = n$ and $k \geq \Delta + 2$, there exists a function*

$$g' : \mathcal{C}_{sf}(G, k) \to \mathcal{C}_p(G, k),$$

for which each element in the co-domain has at most kn pre-images in the domain.

Proof. For every $\sigma' \in \mathcal{C}_{sf}$, arbitrarily select one pre-image (σ, v, c) w.r.t. g, and set σ as the image for σ' under g'. □

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