模型算法简介

控制方程

连续性方程在 σ 坐标变换下的形式

$$\frac{\partial \zeta}{\partial t}+\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_\sigma}{\partial \sigma}=0 \tag{1}$$

x, y, z 方向动量方程在 σ 坐标变换下的形式

$$\begin{aligned}\frac{\partial q_x}{\partial t}+\frac{\partial (q_x u)}{\partial x}+\frac{\partial (q_x v)}{\partial y}+\frac{\partial (q_x \tilde{w})}{\partial \sigma}&=-gD\frac{\partial \zeta}{\partial x}-\frac{D}{\rho_0}\frac{\partial p_n}{\partial x}+fq_x\\&+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial q_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial q_x}{\partial y}\right)+\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{\nu_t}{D}\frac{\partial q_x}{\partial \sigma}\right)+f_{IBx}\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial q_y}{\partial t}+\frac{\partial (q_y u)}{\partial x}+\frac{\partial (q_y v)}{\partial y}+\frac{\partial (q_y \tilde{w})}{\partial \sigma}&=-gD\frac{\partial \zeta}{\partial y}-\frac{D}{\rho_0}\frac{\partial p_n}{\partial y}+fq_y\\&+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial q_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial q_y}{\partial y}\right)+\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{\nu_t}{D}\frac{\partial q_y}{\partial \sigma}\right)+f_{IBy}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial q_z}{\partial t}+\frac{\partial (q_z u)}{\partial x}+\frac{\partial (q_z v)}{\partial y}+\frac{\partial (q_z \tilde{w})}{\partial \sigma}&=-gD\frac{\partial \zeta}{\partial \sigma}\\&+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial q_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial q_z}{\partial y}\right)+\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{\nu_t}{D}\frac{\partial q_z}{\partial \sigma}\right)+f_{IBz}\end{aligned} \tag{4}$$

其中，在计算域中使用流量变量代替速度变量计算

$$q_x=Du,\ q_y=Dv,\ q_z=Dw,\ D=h+\zeta,\ \sigma=\frac{z-\zeta}{h+\zeta}=\frac{z-\zeta}{D}$$

在 σ 坐标变换下，垂向速度通过如下形式计算

$$q_\sigma=\frac{q_z}{D}-\frac{q_x}{D}\left(\sigma\frac{\partial D}{\partial x}+\frac{\partial \zeta}{\partial x}\right)-\frac{q_y}{D}\left(\sigma\frac{\partial D}{\partial y}+\frac{\partial \zeta}{\partial y}\right)-\left(\sigma\frac{\partial D}{\partial t}+\frac{\partial \zeta}{\partial t}\right) \tag{5}$$

数值离散方法

连续性方程使用半隐式时间离散，如下所示

$$\begin{aligned}&\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}+\theta\sum_{k=1}^{KBM}\left(\frac{\partial q_x}{\partial x}\right)_{i,k}^{n+1}\Delta\sigma_k+(1-\theta)\sum_{k=1}^{KBM}\left(\frac{\partial q_x}{\partial x}\right)_{i,k}^{n}\Delta\sigma_k\\&\qquad+\theta\sum_{k=1}^{KBM}\left(\frac{\partial q_y}{\partial y}\right)_{i,k}^{n+1}\Delta\sigma_k+(1-\theta)\sum_{k=1}^{KBM}\left(\frac{\partial q_y}{\partial y}\right)_{i,k}^{n}\Delta\sigma_k=0\end{aligned} \tag{6}$$

x, y, z 方向动量方程采用如下格式离散

$$
\begin{aligned}
\frac{q_{xi}^{\ n+1}-q_{xi}^{\ n}}{\Delta t}=Fq_{xi}^{\ n}-gD\theta\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)_i-gD(1-\theta)\left(\frac{\partial\zeta^{n}}{\partial x}\right)_i \\
-\frac{D}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial x}\right)_i+\left[\frac{\partial}{D\partial\sigma}\left(\frac{\nu_{tv}}{D}\frac{\partial q_x^{\ n+1}}{\partial\sigma}\right)\right]_i+f_{IBxi}^{\ \ *}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\frac{q_{yi}^{\ n+1}-q_{yi}^{\ n}}{\Delta t}=Fq_{yi}^{\ n}-gD\theta\left(\frac{\partial\zeta^{n+1}}{\partial y}\right)_i-gD(1-\theta)\left(\frac{\partial\zeta^{n}}{\partial y}\right)_i \\
-\frac{D}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial y}\right)_i+\left[\frac{\partial}{D\partial\sigma}\left(\frac{\nu_{tv}}{D}\frac{\partial q_y^{\ n+1}}{\partial\sigma}\right)\right]_i+f_{IByi}^{\ \ *}
\end{aligned}
\tag{8}
$$

$$
\frac{q_{zi}^{\ n+1}-q_{zi}^{\ n}}{\Delta t}=Fq_{zi}^{\ n}-\frac{D}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial\sigma}\right)_i+\left[\frac{\partial}{D\partial\sigma}\left(\frac{\nu_{tv}}{D}\frac{\partial q_z^{\ n+1}}{\partial\sigma}\right)\right]_i+f_{IBzi}^{\ \ *}
\tag{9}
$$

连续性方程以矩阵形式简化可以得到

$$
\zeta_i^{n+1}+\mathbf{Z}_{1i}\left(\frac{\partial\mathbf{Q}_x^{n+1}}{\partial x}\right)_i+\mathbf{Z}_{1i}\left(\frac{\partial\mathbf{Q}_y^{n+1}}{\partial y}\right)_i=\zeta_i^n-\mathbf{Z}_{2i}\left(\frac{\partial\mathbf{Q}_x^{n}}{\partial x}\right)_i-\mathbf{Z}_{2i}\left(\frac{\partial\mathbf{Q}_y^{n}}{\partial y}\right)_i
\tag{10}
$$

离散动量方程采用分步投影法求解，水平动量方程（以 x 方向为例，y 方向相同）与垂向动量方程修正如下

$$
\begin{aligned}
&\frac{1}{\Delta t}\left[\left(\Delta\sigma_k-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k+1}}-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k-1}}\right)q_{xi,k}^{\ n+1}-\left(\Delta\sigma-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k+1}}\right)q_{xi,k+1}^{\ n+1}-\left(\Delta\sigma-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k-1}}\right)q_{xi,k-1}^{\ n+1}\right] \\
&=\frac{\Delta\sigma_k}{\Delta t}q_{xi,k}^{\ n}+\Delta\sigma_k\left[Fq_{xi,k}^{\ n}-gD\theta\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)_i-gD(1-\theta)\left(\frac{\partial\zeta^{n}}{\partial x}\right)_i-\frac{D}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial x}\right)_{i,k}+f_{IBxi,k}^{\ \ *}\right]
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{1}{\Delta t}\left[\left(\Delta\sigma_k-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k+1}}-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k-1}}\right)q_{zi,k}^{\ n+1}-\left(\Delta\sigma-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k+1}}\right)q_{zi,k+1}^{\ n+1}-\left(\Delta\sigma-\frac{\nu_{tv}\Delta t}{D^2\Delta\sigma_{k-1}}\right)q_{zi,k-1}^{\ n+1}\right] \\
&=\frac{\Delta\sigma_k}{\Delta t}q_{zi,k}^{\ n}+\Delta\sigma_k\left[Fq_{zi,k}^{\ n}-\frac{1}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial\sigma}\right)_{i,k}+f_{IBzi,k}^{\ \ *}\right]
\end{aligned}
\tag{12}
$$

整理并以矩阵形式简化可以得到

$$
\begin{aligned}
\mathbf{A}_{ix}^n q_{xi,k}^{\ n+1}&=\Delta\sigma_k q_{xi,k}^{\ n} \\
&+\Delta t\Delta\sigma_k\left[Fq_{xi,k}^{\ n}-gD\theta\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)_i-gD(1-\theta)\left(\frac{\partial\zeta^{n}}{\partial x}\right)_i-\frac{D}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial x}\right)_{i,k}+f_{IBxi,k}^{\ \ *}\right]
\end{aligned}
\tag{13}
$$

$$
\mathbf{A}_{iz}^n q_{zi,k}^{\ n+1}=\Delta\sigma_k q_{zi,k}^{\ n}+\Delta t\Delta\sigma_k\left[Fq_{zi,k}^{\ n}-\frac{1}{\rho_0}\left(\frac{\partial p^{n+1}}{\partial\sigma}\right)_{i,k}+f_{IBzi,k}^{\ \ *}\right]
\tag{14}
$$

使用矢量与矩阵形式表示求解变量，水平求解方程（以 x 方向为例，y 方向相同）以如下形式表示

$$
\mathbf{A}_{ix}^n\mathbf{Q}_{xi}^{(1)}=\mathbf{G}_{xi}^n-\mathbf{B}_{1i}^n\left(\frac{\partial\zeta^n}{\partial x}\right)_i
\tag{15}
$$

$$\mathbf{A}_{ix}^{n}\left(\mathbf{Q}_{xi}^{(2)}-\mathbf{Q}_{xi}^{(1)}\right)=-\mathbf{B}_{\mathbf{2}i}^{\ \ n}\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)_{i} \tag{16}$$

$$\mathbf{A}_{ix}^{n}\left(\mathbf{Q}_{xi}^{*}-\mathbf{Q}_{xi}^{(2)}\right)=\mathbf{M}_{i}^{n}f_{IBxi}^{\ \ *} \tag{17}$$

$$\mathbf{A}_{ix}^{n}\left(\mathbf{Q}_{xi}^{n+1}-\mathbf{Q}_{xi}^{*}\right)=-\mathbf{C}_{i}^{n}\left(\frac{\partial p^{n+1}}{\partial\sigma}\right)_{i} \tag{18}$$

垂向求解方程以如下形式表示

$$\mathbf{A}_{iz}^{n}\mathbf{Q}_{zi}^{(1)}=\mathbf{G}_{zi}^{n} \tag{19}$$

$$\mathbf{A}_{iz}^{n}\left(\mathbf{Q}_{zi}^{*}-\mathbf{Q}_{zi}^{(1)}\right)=\mathbf{M}_{i}^{n}f_{IBzi}^{\ \ *} \tag{20}$$

$$\mathbf{A}_{iz}^{n}\left(\mathbf{Q}_{zi}^{n+1}-\mathbf{Q}_{zi}^{*}\right)=-\mathbf{C}_{i}^{n}\left(\frac{\partial p^{n+1}}{\partial\sigma}\right)_{i} \tag{21}$$

水位与动压求解

把离散动量方程 (17) 代入连续性方程 (10) 可以得到（y 方向动量方程离散形式同 x 方向一致，故省略）

$$\zeta_{i}^{n+1}-\mathbf{Z}_{1i}\left\{\frac{\partial}{\partial x}\left[\mathbf{A}^{n-1}\mathbf{B}^{n}\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)\right]\right\}_{i}-\mathbf{Z}_{1i}\left\{\frac{\partial}{\partial y}\left[\mathbf{A}^{n-1}\mathbf{B}^{n}\left(\frac{\partial\zeta^{n+1}}{\partial y}\right)\right]\right\}_{i}=\mathbf{BB}_{i} \tag{22}$$

以系数矩阵形式简化可得

$$\mathbf{AP}_{i}\zeta_{i}^{n+1}-\sum_{s=1}^{NS}\mathbf{AP}_{is}\zeta_{is}^{n+1}=\left\langle\mathbf{BB}_{i}\right\rangle \tag{23}$$

其中，系数矩阵形式为

$$\mathbf{AP}_{i}=\Delta S_{i}+\sum_{s=1}^{NS}\mathbf{AP}_{is},\ \mathbf{AP}_{is}=\frac{\mathbf{Z}_{\mathbf{1}i}\mathbf{A}_{i}^{n-1}\mathbf{B}_{i}^{n}y_{\eta is}}{J_{is}\Delta\zeta_{is}}\cos\alpha_{is}\Delta l_{is}-\frac{\mathbf{Z}_{\mathbf{1}i}\mathbf{A}_{i}^{n-1}\mathbf{B}_{i}^{n}y_{\eta is}}{J_{is}\Delta\zeta_{is}}\sin\alpha_{is}\Delta l_{is}$$

同理，把离散动量方程 (18) 与 (21) 代入以 q_z 形式（即方程 (5) 的形式）改写连续性方程 (1) 可以得到（y 方向动量方程离散形式仍然同 x 方向一致，故省略）

$$\mathbf{AP}_{i,k}p_{ni,k}^{n+1}-\mathbf{AP}_{\mathbf{1}i,k}p_{ni,k-1}^{n+1}-\mathbf{AP}_{\mathbf{2}i,k}p_{ni,k+1}^{n+1}-\sum_{s=1}^{NS}\mathbf{AP}_{\mathbf{3}i,k}\zeta_{is}^{n+1}=\left\langle\mathbf{BP}_{i,k}\right\rangle \tag{24}$$

其中，系数矩阵形式为

$$\mathbf{AP}_{i,k}=\mathbf{AP}_{\mathbf{1}i,k}+\mathbf{AP}_{\mathbf{2}i,k}+\sum_{s=1}^{NS}\mathbf{AP}_{\mathbf{3}i,k}$$

$$\mathbf{AP}_{\mathbf{1}}=\frac{\Delta t\Delta S_{i}}{D\rho_{0}\Delta\sigma_{k-1/2}},\ \mathbf{AP}_{\mathbf{2}}=\frac{\Delta t\Delta S_{i}}{D\rho_{0}\Delta\sigma_{k+1/2}},\ \mathbf{AP}_{\mathbf{3}}=\left\langle\frac{D\Delta l_{is}}{\rho_{0}J_{is}\Delta\zeta_{is}}\left(\cos\alpha_{is}y_{\eta}-\sin\alpha_{is}x_{\eta}\right)\right\rangle^{f}$$

浸没边界法 （Immersed Boundary Method, IBM）

基于分布投影法，虚拟边界体积力可通过如下方程求解

$$f_{IBxi,k}^{\ *}=\begin{cases}\dfrac{q_{IBxi,k}^{\ *}-q_{xi,k}^{(2)}}{\Delta t} & \text{in the ghost cells}\\ 0 & \text{in the other cells}\end{cases}\tag{25}$$

$$f_{IBzi,k}^{\ *}=\begin{cases}\dfrac{q_{IBxi,k}^{\ *}-q_{xi,k}^{(1)}}{\Delta t} & \text{in the ghost cells}\\ 0 & \text{in the other cells}\end{cases}\tag{26}$$

即在虚拟边界上，体积力的形式为

$$f_{IBxi,k}^{\ *}=\frac{q_{IBxi,k}^{\ *}}{\Delta t}-\left\{\mathbf{A}^{-1}\left[\mathbf{G}q_{xi,k}^{\ n}-\mathbf{B}_{\mathbf{1}i}^{\ n}\left(\frac{\partial\zeta^{n}}{\partial x}\right)_i-\mathbf{B}_{\mathbf{2}i}^{\ n}\left(\frac{\partial\zeta^{n+1}}{\partial x}\right)_i\right]\cdot\frac{1}{\Delta t}\right\}_{i,k}\tag{27}$$

$$f_{IBzi,k}^{\ *}=\frac{q_{IBzi,k}^{\ *}}{\Delta t}-\left(\mathbf{A}^{-1}\mathbf{G}q_{zi,k}^{\ n}\cdot\frac{1}{\Delta t}\right)_{i,k}\tag{28}$$

式中虚拟边界速度 q_{IB} 在每一时间步更新满足固体边界条件，而虚拟边界位置与网格单元位置通常不重合，因此采用插值方法计算实际网格单元变量 q_{IB}。设置虚拟边界真实速度变量 q_{BP} 与虚拟边界镜像点速度变量 q_{IP}，采用如下方法计算

$$q_{IP}^{\ *}=\phi\left(q_{IP,1}^{\ *},q_{IP,2}^{\ *},q_{IP,3}^{\ *}\ \ldots\ q_{IP,n}^{\ *}\right)\tag{29}$$

其中，ϕ 为任意插值函数，$q_{\mathrm{IP,n}}$ 为所需要的插值点。以距离权重线性插值方法为例

$$q_{IP}^{\ *}=\sum_{k=1}^{n}\omega_k q_{IP,k}\tag{30}$$

$$\omega_k=\frac{1}{\left(d_k^{\ 2}+\varepsilon\right)}\sum_{m=1}^{n}\frac{1}{\left(d_m^{\ 2}+\varepsilon\right)}\tag{31}$$

实际网格单元变量 q_{IB} 可由下式计算得到

$$q_{IB}^{\ *}=\frac{1}{2}\left(q_{BP}^{\ *}+q_{IP}^{\ *}\right)=\frac{1}{2}\left(q_{BP}^{\ *}+\sum_{k=1}^{n}\frac{q_{IP,k}}{\left(d_k^{\ 2}+\varepsilon\right)}\sum_{m=1}^{n}\frac{1}{\left(d_m^{\ 2}+\varepsilon\right)}\right)\tag{32}$$

在虚拟边界区域，求解水位与动压时，对应的边界面系数矩阵取值为 0，保证该处界面速度通量为 0。

运动边界更新

虚拟边界均采用刚性边界，运动边界更新遵循刚体动力学模型。任意刚体运动可以分解为平移与旋转，对应变量速度 **v**，位置 **x**，角速度 ω 与对应旋转状态的四元数 **q**。

平移状态更新过程如下

$$\mathbf{v}^{n+1}=\mathbf{v}^{n}+\Delta t\cdot\sum_{i=1}^{m}\mathbf{F}_i\left(\mathbf{x},\mathbf{v}\right)^{n}\tag{33}$$

$$\mathbf{x}^{n+1} = \mathbf{x}^{n} + \Delta t \cdot \mathbf{v}^{n} \tag{34}$$

旋转状态更新过程如下（刚体每一个离散顶点的运动即相对质心的旋转运动）

$$\boldsymbol{\tau}^{n} = \sum_{i=1}^{m} \left(\mathbf{R}^{n} \mathbf{r}_i^{\,n} \right) \times \mathbf{F}_i \tag{35}$$

$$\boldsymbol{\omega}^{n+1} = \boldsymbol{\omega}^{n} + \Delta t \left(\mathbf{R}^{n} \mathbf{I_0} \mathbf{R}^{nT} \right)^{-1} \boldsymbol{\tau}^{n} \tag{36}$$

$$\mathbf{q}^{n+1} = \mathbf{q}^{n} + \mathbf{q}^{*} \times \mathbf{q}^{n} \tag{37}$$

其中 $\mathbf{q}^{*} = \left[0 \;\; \boldsymbol{\omega}^{n+1} \cdot \Delta t / 2 \right]$，对于任意四元数 $\mathbf{q} = \left[s \;\; \mathbf{v}(x, y, z) \right]$，旋转矩阵具体形式如下所示

$$\mathbf{R} = \begin{bmatrix} s^2 + x^2 - y^2 - z^2 & 2(xy - sz) & 2(xz + sy) \\ 2(xy + sz) & s^2 - x^2 + y^2 - z^2 & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & s^2 - x^2 - y^2 + z^2 \end{bmatrix} \tag{38}$$

在与真实物理边界发生碰撞时，采用冲量法修正速度，如下所示

$$\mathbf{v}_{\mathbf{n},i}^{\;n+1} = -\mu_n \left(\mathbf{v}_i^{\,n} \cdot \mathbf{n}_i \right) \mathbf{n}_i \tag{39}$$

$$\mathbf{v}_{\boldsymbol{\tau},i}^{\;n+1} = -\alpha \left[\mathbf{v}_i^{\,n} - \left(\mathbf{v}_i^{\,n} \cdot \mathbf{n}_i \right) \mathbf{n}_i \right] \tag{40}$$

对于每一个碰撞点，速度位置与冲量通过以下方法计算

$$\mathbf{x}_i^{\,n} = \mathbf{x}^{n} + \mathbf{R}\mathbf{r}_i \tag{41}$$

$$\mathbf{v}_i^{\,n} = \mathbf{v}^{n} + \boldsymbol{\omega} \times \mathbf{R}\mathbf{r}_i \tag{42}$$

$$\mathbf{J}^{n+1} = \left[\mathbf{m}^{-1} - \left(\mathbf{R}\mathbf{r}_i \right)^{*} \mathbf{I}^{-1} \left(\mathbf{R}\mathbf{r}_i \right)^{*} \right]^{-1} \cdot \left(\mathbf{v}_{\mathbf{n},i}^{\;n+1} + \mathbf{v}_{\tau,i}^{\;n+1} - \mathbf{v}_i^{\,n} \right) \tag{43}$$

最终，修正速度与角速度为

$$\mathbf{v}^{n+1} = \mathbf{v}^{n} + \mathbf{m}^{-1} \mathbf{J}^{n+1} \tag{44}$$

$$\boldsymbol{\omega}^{n+1} = \boldsymbol{\omega}^{n} + \mathbf{I}^{-1} \left(\mathbf{R}\mathbf{r}_i \times \mathbf{J}^{n+1} \right) \tag{45}$$

至此完成整个时间步的计算。