

Shortest Path*

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Algorithm Course: Shanghai Jiao Tong University

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Outline

1 Introduction to Shortest Path

- Definition
- Property
- Application

2 Single Source Shortest Paths

- Problem Statement
- Dijkstra's Algorithm
- Bellman-Ford Algorithm

3 All-Pair Shortest Paths

- Matrix Multiplication
- Floyd-Warshall Algorithm
- Johnson's Algorithm

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Paths in Graphs

Definition

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \rightarrow R$, where $|V| = n$ and $|E| = m$. The **weight** of path $P = v_1 \rightarrow \dots \rightarrow v_k$ is defined to be

$$w(P) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

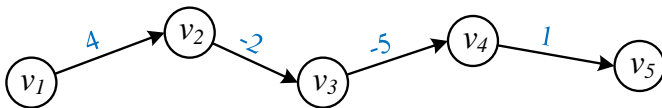
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Example: $w(P) = -2$



Shortest Path

Definition

A *shortest path* from u to v is a path of minimum weight from u to v .
The *shortest path weight* from u to v is defined as

$$d(u, v) = \min\{w(P) \mid P \text{ is a path from } u \text{ to } v\}$$

Note: $d(u, v) = +\infty$ if no path from u to v exists.

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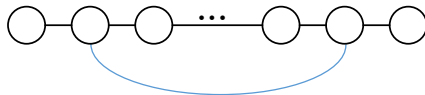
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Optimal Substructure. A subpath of a shortest path is a shortest path.

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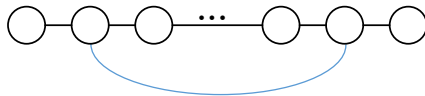
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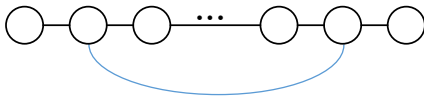


Triangle Inequality. $\forall v_1, v_2, v_3 \in V, d(v_1, v_2) \leq d(v_1, v_3) + d(v_3, v_2)$.

Properties of Shortest Path

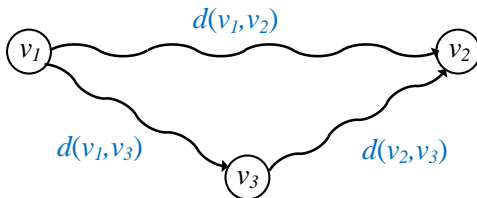
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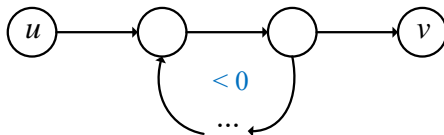
Well-Definedness of Shortest Paths

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

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Example:



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Shortest Path Applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

Ref.: Network Flows: Theory, Algorithms, and Applications, R.K. Ahuja, T.L. Magnanti, and J.B. Orlin, Prentice Hall, 1993

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Single-Source Shortest Paths

Definition (Single-Source Shortest Paths Problem)

From a given source vertex $s \in V$, find the shortest-path weights $d(s, v)$ for all $v \in V$.

- If all edge weights $w(u, v)$ are **nonnegative**, all shortest-path weights must exist.
- If all edge weights $w(u, v)$ can be **negative**, the shortest-path weights may not exist because of negative circle.

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- **Nonnegative** weight \Rightarrow Dijkstra's Algorithm
- **Negative** weight \Rightarrow Bellman-Ford Algorithm

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Dijkstra's Algorithm

IDEA: Greedy

- Maintain a set S of vertices whose shortest-path distances from s are known.
- At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
- Update the distance estimates of vertices adjacent to v .

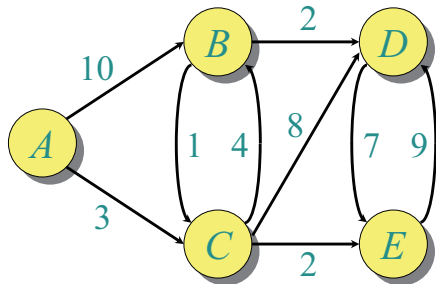
Dijkstra's Algorithm

Algorithm 1: Dijkstra's Algorithm

```
1 foreach  $u \in V$  do
2    $\text{INSERT}(Q, u)$ ;
3 while  $Q \neq \emptyset$  do
4    $u \leftarrow \text{EXTRACT-MIN}(Q)$ ;
5    $S \leftarrow S \cup \{u\}$ ;
6   foreach  $v \in \text{Adj}[u]$  do
7     if  $d[v] > d[u] + w(u, v)$  then
8        $d[v] \leftarrow d[u] + w(u, v)$ ;    /* Relaxation Step */
9      $\text{DECREASE-KEY}(Q, v)$ ;
```

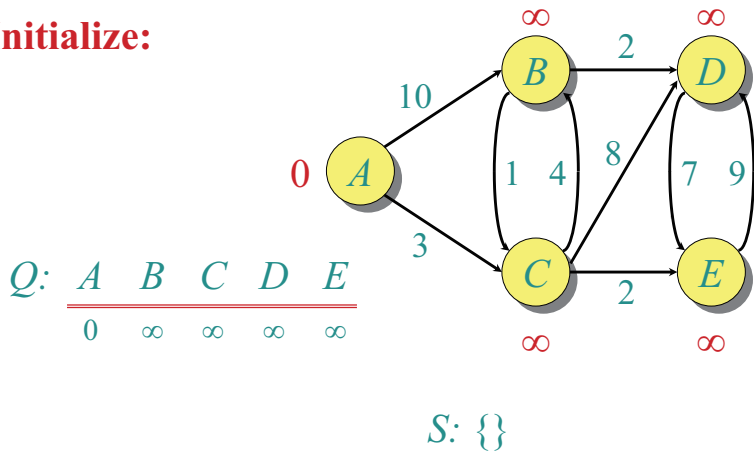
Example of Dijkstra's Algorithm

**Graph with
nonnegative
edge weights:**



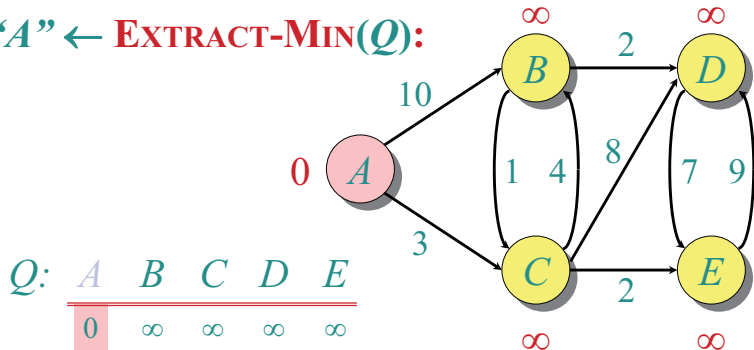
Example of Dijkstra's Algorithm (cont.)

Initialize:



Example of Dijkstra's Algorithm (cont.)

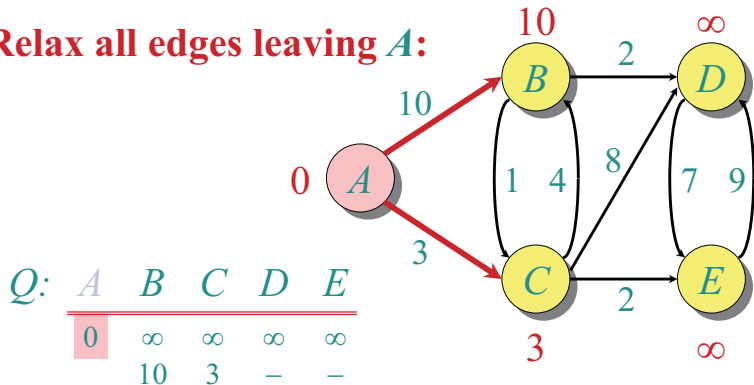
"A" ← EXTRACT-MIN(Q):



$S: \{A\}$

Example of Dijkstra's Algorithm (cont.)

Relax all edges leaving A :



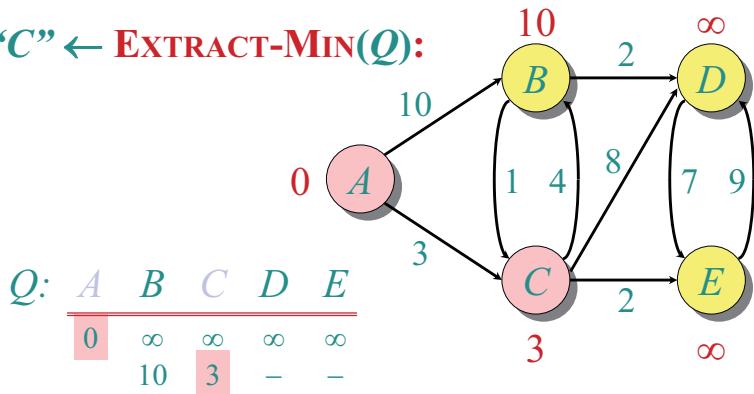
Q :

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	—	—

$S: \{A\}$

Example of Dijkstra's Algorithm (cont.)

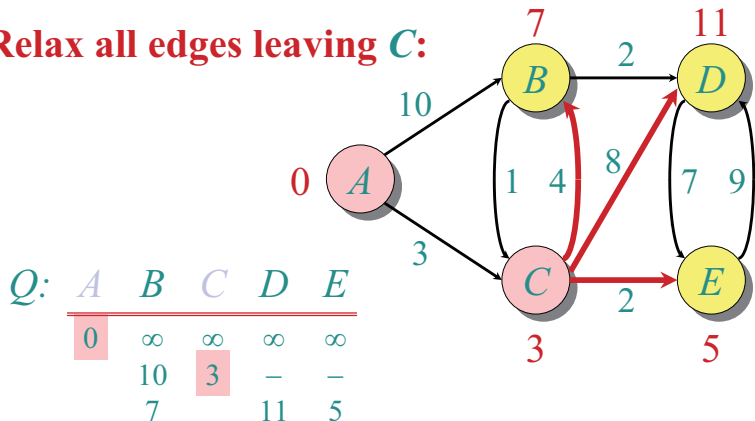
“C” \leftarrow **EXTRACT-MIN**(Q):



S: {A, C}

Example of Dijkstra's Algorithm (cont.)

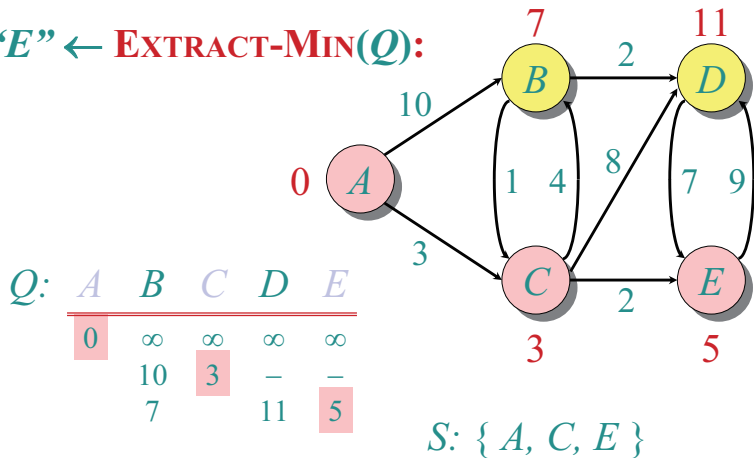
Relax all edges leaving **C**:



$S: \{A, C\}$

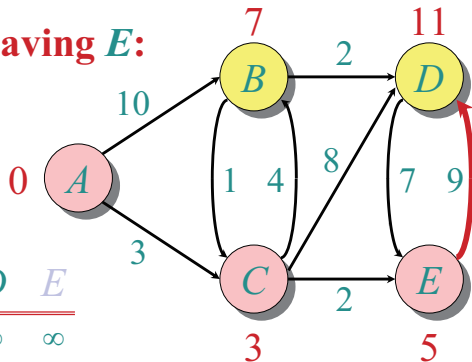
Example of Dijkstra's Algorithm (cont.)

"E" ← EXTRACT-MIN(Q):



Example of Dijkstra's Algorithm (cont.)

Relax all edges leaving E :



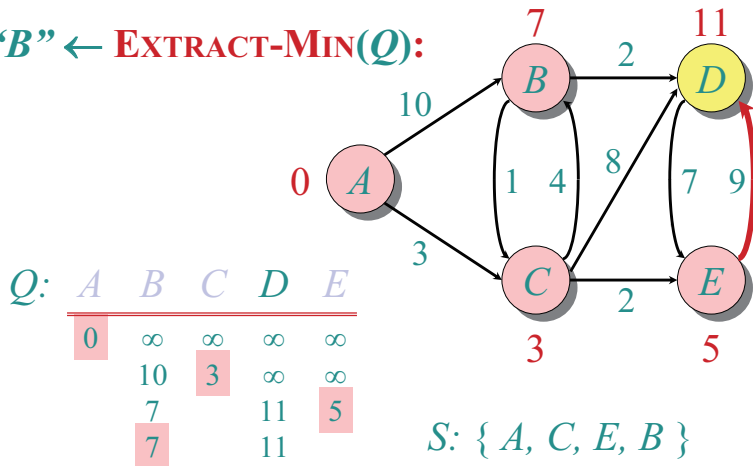
Q :

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	

$S: \{A, C, E\}$

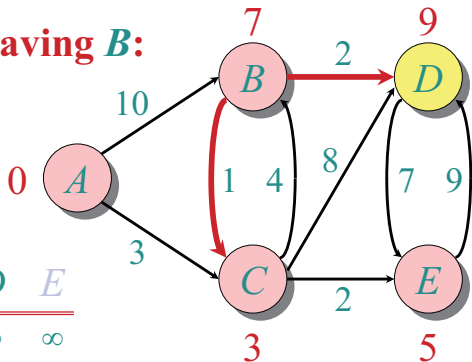
Example of Dijkstra's Algorithm (cont.)

"B" ← EXTRACT-MIN(Q):



Example of Dijkstra's Algorithm (cont.)

Relax all edges leaving B :



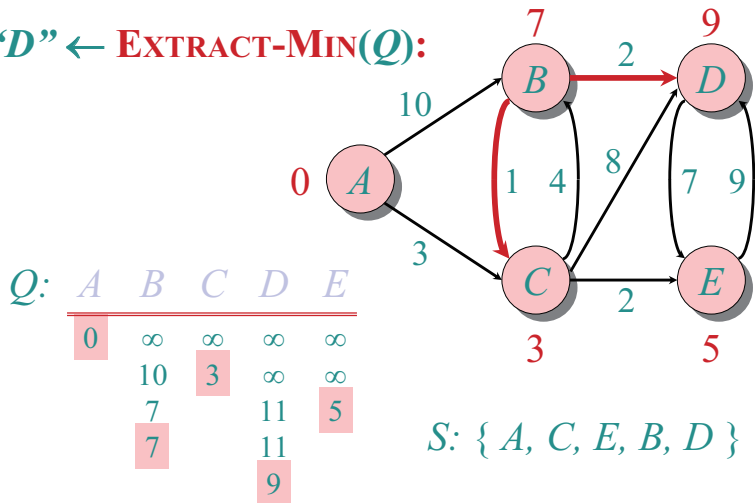
Q :

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

$S: \{A, C, E, B\}$

Example of Dijkstra's Algorithm (cont.)

"D" ← EXTRACT-MIN(Q):



Correctness of Dijkstra's Algorithm

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow +\infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq d(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

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Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow +\infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq d(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Proof. Suppose not. Let v be the first vertex for which $d[v] < d(s, v)$, and let u be the vertex that caused $d[v]$ to change:

$$d[v] = d[u] + w(u, v).$$

Then,

$$\begin{aligned} d[v] &< d(s, v) && \text{supposition} \\ &\leq d(s, u) + d(u, v) && \text{triangle inequality} \\ &\leq d(s, u) + w(u, v) && \text{sh. path} \leq \text{specific path} \\ &\leq d[u] + w(u, v) && v \text{ is first violation} \end{aligned}$$

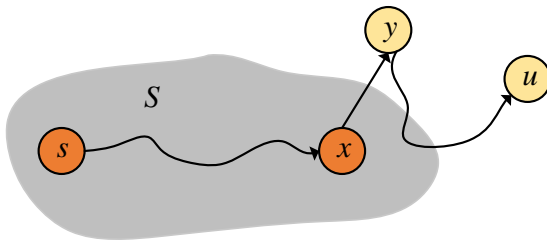
Correctness of Dijkstra's Algorithm (Cont.)

Theorem. Dijkstra's algorithm terminates with $d[v] = d(s, v)$ for all $v \in V$.

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Proof. It suffices to show that $d[v] = d(s, v)$ for every $v \in V$ when v is added to S . Suppose u is the first vertex added to S for which $d[u] \neq d(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u , and let x be its predecessor.

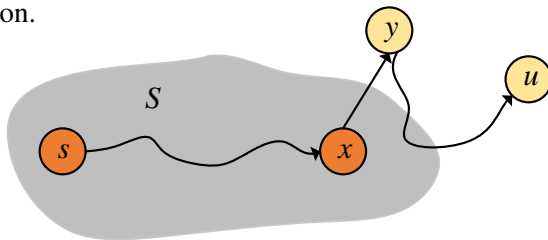


Correctness of Dijkstra's Algorithm (Cont.)

Proof. (cont.) Since u is the first vertex violating the claimed invariant, we have $d[x] = d(s, x)$. Since subpaths of shortest paths are shortest paths, it follows that $d[y]$ was set to $d(s, x) + w(x, y) = d(s, y)$ when (x, y) was relaxed just after x was added to S .

Consequently, we have $d[y] = d(s, y) \leq d(s, u) \leq d[u]$. However, $d[u] \leq d[y]$ by our choice of u in Dijkstra's Algorithm, so $d[y] = d(s, y) = d(s, u) = d[u]$.

Contradiction. □



Analysis of Dijkstra's Algorithm

Algorithm 1: Dijkstra's Algorithm

```
1 for  $u \in V$  do
2    $\text{INSERT}(Q, u);$                                 /*  $|V|$  times */
3 while  $Q \neq \emptyset$  do
4    $u \leftarrow \text{EXTRACT-MIN}(Q);$                     /*  $|V|$  times */
5    $S \leftarrow S \cup \{u\};$ 
6   foreach  $v \in \text{Adj}[u]$  do
7     if  $d[v] > d[u] + w(u, v)$  then
8        $d[v] \leftarrow d[u] + w(u, v);$ 
9        $\text{DECREASE-KEY}(Q, v);$     /*  $\text{degree}(u)$  times */
```

Analysis of Dijkstra's Algorithm (Cont.)

Handshaking Lemma $\Rightarrow O(E)$ implicit DECREASE-KEY.

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Performance:

- Array implementation optimal for dense graphs ($\Theta(n^2)$ edges).
- Binary heap much faster for sparse graphs ($\Theta(n)$ edges).
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but probably not worth implementing.

Implementation	EXTRACT-MIN	INSERT/ DECREASE-KEY	$ V \times \text{EXTRACT-MIN} +$ $(V + E) \times \text{INS/DEC}$
Array	$O(V)$	$O(1)$	$O(V ^2)$
Binary heap	$O(\log V)$	$O(\log V)$	$O((V + E) \log V)$
d -ary heap	$O(\frac{d \log V }{\log d})$	$O(\frac{\log V }{\log d})$	$O\left(\frac{(d V + E) \log V }{\log d}\right)$
Fibonacci heap	$O(\log V)^*$	$O(1)^*$	$O(V \log V + E)$

* Amortized Analysis

Unweighted Graph

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Use FIFO queue instead of priority queue \Rightarrow breadth-first search

Time = $O(n + m)$.

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Correctness:

- The FIFO queue in breadth-first search mimics the priority queue in Dijkstra;
- **Invariant:** v comes after u in queue implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.

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Note: negative weight is allowed.

Shortest Paths

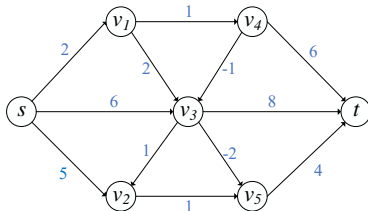
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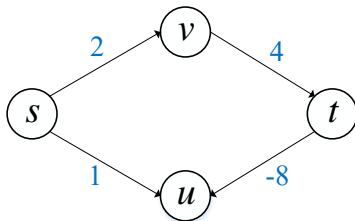
Example: Nodes represent agents in a financial setting and $w(u, v)$ is cost of transaction in which we buy from agent u and sell to v .



Shortest Path: Failed Attempt

Dijkstra:

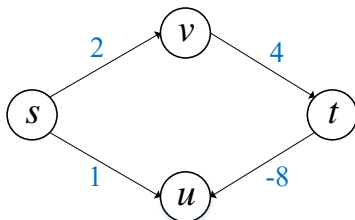
Maybe fail if edge costs are negative.



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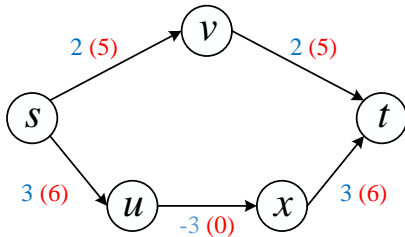
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Maybe fail if edge costs are negative.



Re-weighting:

Adding a constant to every edge weight can fail.



Dynamic Programming

Definition

$OPT(i, v)$ is the length of shortest $s-v$ path P using at most i edges.

Case 1: P uses at most $i - 1$ edges.

$$\triangleright OPT(i, v) = OPT(i - 1, v)$$

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Case 2: P uses exactly i edges

- \triangleright if (u, v) is the last edge, then OPT uses (u, v) , and then selects best s - u path using at most $i - 1$ edges

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$$OPT(i, v) = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{if } v \neq s, i = 0 \\ \min\{OPT(i - 1, v), \min_{(u,v) \in E} \{OPT(i - 1, u) + w(u, v)\}\} & \text{otherwise} \end{cases}$$

Shortest Paths: Implementation

Algorithm 2: Dynamic Programming

```
1 foreach node  $u \in V$  do  
2    $M[0, u] \leftarrow \infty$ ;  
3  $M[0, s] \leftarrow 0$ ;  
4 for  $i = 1$  to  $n$  do  
5   foreach edge  $(u, v) \in E$  do  
6      $M[i, v] \leftarrow \min\{M[i-1, v], M[i-1, u] + w(u, v)\}$ ;
```

Algorithm Analysis: $O(mn)$ time, $O(n^2)$ space

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ as shortest s - v path found so far;
- No need to check edges of the form (u, v) unless $M[u]$ changed.

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Theorem. Throughout the algorithm, $M[v]$ is length of some s - v path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest s - v path using $\leq i$ edges.

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Overall impact.

- Memory: $O(m + n)$;
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

Algorithm 3: Bellman-Ford Algorithm

```
1 foreach node  $u \in V$  do
2    $M[u] \leftarrow \infty$ ;
3    $predecessor[u] \leftarrow \emptyset$ ;
4  $M[s] \leftarrow 0$ ;
5 for  $i = 1$  to  $n - 1$  do
6   foreach node  $u \in V$  do
7     if  $M[u]$  has been updated in previous iteration then
8       foreach edge  $(u, v) \in E$  do
9         if  $M[v] > M[u] + w(u, v)$  then
10           $M[v] \leftarrow M[u] + w(u, v)$ ;
11           $predecessor[v] \leftarrow u$ ;
12   If no  $M[v]$  changed in this iteration, stop.
```


Detecting Negative Cycles

Lemma. If $OPT(n, v) = OPT(n - 1, v)$ for all v , then no negative cycles.

Proof. Bellman-Ford Algorithm. □

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Lemma. If $OPT(n, v) < OPT(n - 1, v)$ for some node u , then (any) shortest path from s to v contains a cycle W . Moreover W has negative cost.

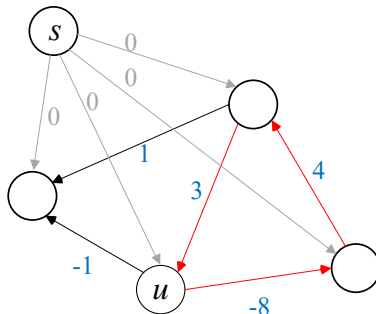
Proof.

- $OPT(n, v) < OPT(n - 1, v) \Rightarrow P$ has exactly n edges;
- By pigeonhole principle, P must contain a directed cycle W ;
- Deleting W yields a s - v path with $< n$ edges $\Rightarrow W$ has negative cost. □

Detecting Negative Cycles

Theorem. Can detect negative cost cycle in $O(mn)$ time.

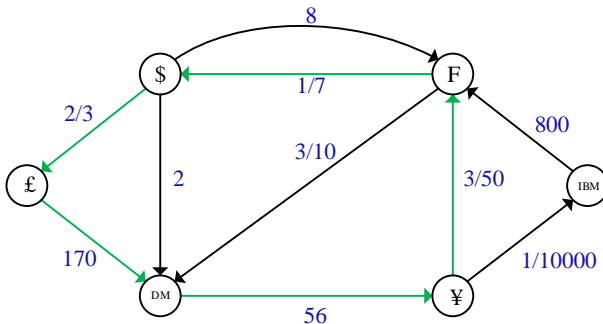
- Add new node s and connect it to all nodes with 0-cost edge.
- Check if $OPT(n, u) = OPT(n - 1, u)$ for all nodes u . if no, then extract cycle from shortest path from s to u .



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



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Definition

Definition (All-Pair Shortest Paths Problem)

Given Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w : E \rightarrow R$, find $n \times n$ matrix of shortest path lengths $d(i, j)$ for all $i, j \in V$.

Definition

Definition (All-Pair Shortest Paths Problem)

Given Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w : E \rightarrow R$, find $n \times n$ matrix of shortest path lengths $d(i, j)$ for all $i, j \in V$.

IDEA #1:

- Run Bellman-Ford once from each vertex.
- Time = $O(n^2m)$.
- Dense graph $\Rightarrow O(n^4)$ time.

Good first try!

Dynamic Programming

Consider the $n \times n$ adjacency matrix $A = (a_{ij})$ of the digraph, and define $d_{ij}^{(m)}$ as the weight of a shortest path from i to j that uses at most m edges.

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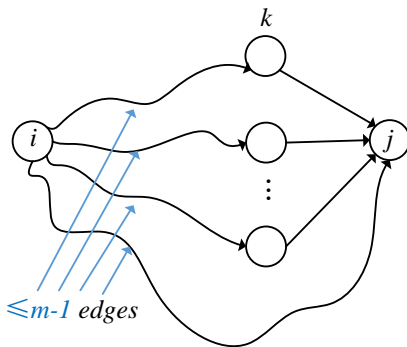
Claim: We have

$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

and for $m = 1, 2, \dots, n-1$

$$d_{ij}^{(m)} = \min_k \{d_{ik}^{(m-1)} + a_{kj}\}$$

Proof of Claim



Note: No negative-weight cycles implies

$$d(i,j) = d_{ij}^{(n-1)} = d_{ij}^{(n)} = d_{ij}^{(n+1)} = \dots$$

Matrix Multiplication

Compute $C = A \times B$, where C , A , and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^n a_{ik} \times b_{kj}$$

Time = $\Theta(n^3)$ using the standard algorithm.

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Thus, $D^{(m)} = D^{(m-1)} \text{ “} \times \text{” } A$.

Identity matrix = $I = D^{(0)} = (d_{ij}^{(0)}) =$

$$\begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

Matrix Multiplication (cont.)

The $(\min, +)$ multiplication is **associative**, and with the real numbers, it forms an algebraic structure called a **closed semiring**.

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yielding $D^{(n-1)} = (d(i, j))$.

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Time = $\Theta(n^4)$. No better than $n \times$ Bellman-Ford.

Improved Matrix Multiplication Algorithm

Repeated squaring: $A^{2k} = A^k \times A^k$.

Compute $A^2, A^4, A^8, \dots, A^{2^{\lceil \log_2(n-1) \rceil}}$ ($O(\log n)$ squarings).

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To detect negative-weight cycles, check the diagonal for negative values in $O(n)$ additional time.

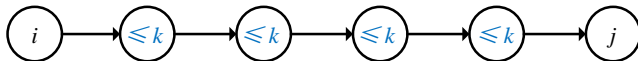
Outline

- 1 Introduction to Shortest Path
 - Definition
 - Property
 - Application
- 2 Single Source Shortest Paths
 - Problem Statement
 - Dijkstra's Algorithm
 - Bellman-Ford Algorithm
- 3 All-Pair Shortest Paths
 - Matrix Multiplication
 - Floyd-Warshall Algorithm
 - Johnson's Algorithm

Floyd-Warshall algorithm

Also dynamic programming, but faster!

Define $c_{ij}^{(k)}$ as the weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \dots, k\}$.

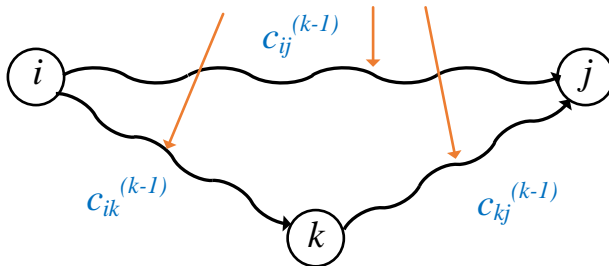


Thus, $d(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

Floyd-Warshall Recurrence

$$c_{ij}^{(k)} = \min_k \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

Intermediate nodes in $\{1, 2, \dots, k-1\}$



Pseudocode for Floyd-Warshall

Algorithm 4: Floyd-Warshall Algorithm

```
1 for  $k \leftarrow 1$  to  $n$  do
2   for  $i \leftarrow 1$  to  $n$  do
3     for  $j \leftarrow 1$  to  $n$  do
4       if  $c_{ij} > c_{ik} + c_{kj}$  then
5          $c_{ij} \leftarrow c_{ik} + c_{kj};$ 
```

Analysis:

- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code and efficient in practice.

Transitive Closure of a Directed Graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$

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Graph Reweighting

Theorem. Given a label $h(v)$ for each $v \in V$, reweight each edge $(u, v) \in E$ by $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$. Then, all paths between the same two vertices are reweighted by the same amount.

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Proof. Let $P = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ be a path in the graph. We have:

$$\begin{aligned}\hat{w}(P) &= \sum_{i=1}^{k-1} \hat{w}(v_i, v_{i+1}) \\ &= \sum_{i=1}^{k-1} (w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})) \\ &= \sum_{i=1}^{k-1} w(v_i, v_{i+1}) + h(v_1) - h(v_k) \\ &= w(P) + h(v_1) - h(v_k)\end{aligned}$$

Johnson's Algorithm

- ① Find a vertex labeling h such that $\hat{w}(u, v) \geq 0$ for all $(u, v) \in E$ by using Bellman-Ford to solve the difference constraints

$$h(v) - h(u) \leq w(u, v)$$

or determine that a negative-weight cycle exists.

▷ Time = $O(mn)$

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- ▷ Time = $O(mn)$
- ② Run Dijkstra's algorithm from each vertex using \hat{w} .
 - ▷ Time = $O(mn + n^2 \log n)$
- ③ Reweight each shortest-path length $\hat{w}(P)$ to produce the shortest-path lengths $w(P)$ of the original graph.
 - ▷ Time = $O(n^2)$

Total time = $O(mn + n^2 \log n)$.