Special Numbers

ITT9131 Konkreetne Matemaatika

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Fibonacci numbers: Idea

Fibonacci's problem

A pair of baby rabbits is left on an island.

- A baby rabbit becomes adult in one month.
- A pair of adult rabbits produces a pair of baby rabbits each month.

How many pairs of rabbits will be on the island ofter n months?
How many of them will be adult, and how many will be babies?

Leonardo Fibonacci (1175–1235)

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Solution (see Exercise 6.6)

- On the first month, the two baby rabbits will have become adults.
- On the second month, the two adult rabbits will have produced a pair of baby rabbits.
- On the third month, the two adult rabbits will have produced *another* pair of baby rabbits, while the other two baby rabbits will have become adults.
- And so on, and so on . . .

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Solution (see Exercise 6.6)

month	0	1	2	3	4	5	6	7	8	9	10
baby	1	0	1	1	2	3	5	8	13	21	34
adult	0	1	1	2	3	5	8	13	21	34	55
total	1	1	2	3	5	8	13	21	34	55	89

That is: at month n, there are f_{n+1} pair of rabbits, of which f_n pairs of adults, and f_{n-1} pairs of babies.
(Note: this seems to suggest $f_{-1} = 1 \ldots$)

Leonardo Fibonacci (1175–1235)

Fibonacci Numbers: Main formulas

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

Formulae for computing:

- $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$.
- $f_n = \frac{1}{\sqrt{5}}\left(\Phi^n - \hat{\Phi}^n\right)$ ("Binet form")

  where $\Phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the golden ratio.

Generating function

$$\sum_{n\geqslant 0} f_n z^n = \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi z} - \frac{1}{1-\hat{\Phi} z}\right) \quad \forall z \in \mathbb{C} : |z| < \Phi^{-1},$$

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$$\sum_{n \geqslant 0} f_n z^n = \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \Phi z} - \frac{1}{1 - \hat{\Phi} z} \right) \quad \forall z \in \mathbb{C} : |z| < \Phi^{-1},$$

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Some Fibonacci Identities

Cassini's Identity $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for all $n > 0$

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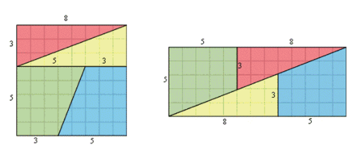
The Chessboard Paradox



Ref: https://en.chessbase.com/post/a-mathematical-cheboard-paradox

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Divisors f_n and f_{n+1} are relatively prime and f_k divides f_{nk}:

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$$\gcd(f_n, f_m) = f_{\gcd(n,m)}$$

Matrix Calculus If A is the 2×2 matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}, \qquad \text{for } n > 0.$$

Observe that this is equivalent to Cassini's identity.

Some Fibonacci Identities (2)

Fibonacci Numbers and Pascal's Triangle: $f_{n+1} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j}$

n	f_n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$
0	**0**	1								
1	**1**	1	1							
2	**1**	1	2	1						
3	**2**	1	3	3	1					
4	**3**	1	4	6	4	1				
5	**5**	1	5	10	10	5	1			
6	**8**	1	6	15	20	15	6	1		
7	**13**	1	7	21	35	35	21	7	1	
8	**21**	1	8	28	56	70	56	28	8	1

Some Fibonacci Identities (3)

Continued fractions

The continued fraction composed entirely of 1s equals the ratio of successive Fibonacci numbers:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{}{a_{n-2} + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} = \frac{f_{n+1}}{f_n},$$

where $a_1 = a_2 = \cdots = a_n = 1$.

For example

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{f_5}{f_4} = \frac{5}{3} = 1.(6)$$

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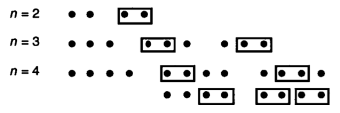
Some applications of Fibonacci numbers

1. Let S_n denote the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain consecutive elements. For example, when $n = 3$ the allowable subsets are $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}$. Therefore, $S_3 = 5$. In general, $S_n = f_{n+2}$ for $n \geqslant 1$.
2. Draw n dots in a line. If each domino can cover exactly two such dots, in how many ways can (non-overlapping) dominoes be placed? For example



   Thus $D_n = f_{n+1}$ for $n > 0$.
3. Compositions: Let T_n be the number of ordered compositions of the positive integer n into summands that are odd. For example, $4 = 1 + 3 = 3 + 1 = 1 + 1 + 1 + 1$ and $5 = 5 = 1 + 1 + 3 = 1 + 3 + 1 = 3 + 1 + 1 = = 1 + 1 + 1 + 1 + 1$. Therefore, $T_4 = 3$ and $T_5 = 5$. In general, $T_n = f_n$ for $n > 0$.
4. Compositions: Let B_n be the number of ordered compositions of the positive integer n into summands that are either 1 or 2. For example, $3 = 1 + 2 = 2 + 1 = 1 + 1 + 1$ and $4 = 2 + 2 = 1 + 1 + 2 = 1 + 2 + 1 = = 2 + 1 + 1 = 1 + 1 + 1 + 1.$. Therefore, $B_3 = 3$ and $B_4 = 5$. In general, $B_n = f_{n+1}$ for $n > 0$.

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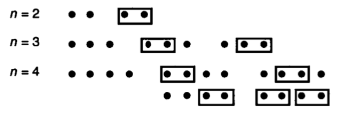

   The number of possible placements D_n of dominoes with n dots, consider the rightmost dot in any such placement P. If this dot is not covered by a domino, then P minus the last dot determines a solution counted by D_{n-1}. If the last dot is covered by a domino, then the last two dots in P are covered by this domino. Removing this rightmost domino then gives a solution counted by D_{n-2}.Taking into account these two possibilities $D_n=D_{n-1}+D_{n-2}$ for $n\geqslant 3$ with $D_1=1$, $D_2=2$. Thus $D_n=f_{n+1}$ for $n>0$.
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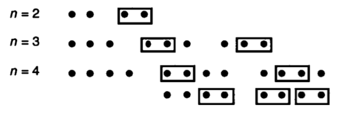

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   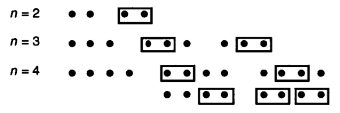

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Observation

$$\lim_{n \to \infty} \widehat{\Phi}^{n} = 0$$

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For example:

$$f_{10} = \left\lfloor \frac{\Phi^{10}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = \left\lfloor 55.00364\ldots + \frac{1}{2} \right\rfloor = \lfloor 55.50364\ldots \rfloor = 55$$

$$f_{11} = \left\lfloor \frac{\Phi^{11}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = \left\lfloor 88.99775\ldots + \frac{1}{2} \right\rfloor = \lfloor 89.49775\ldots \rfloor = 89$$

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- $f_n \to \frac{\Phi^n}{\sqrt{5}}$ as $n \to \infty$
- $f_n = \left\lfloor \frac{\Phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor$
- $\frac{f_n}{f_{n-1}} \to \Phi$ as $n \to \infty$

For example:

$$\frac{f_{11}}{f_{10}} = \frac{89}{55} \approx 1.61818182 \approx \Phi = 1.61803\ldots$$

Fibonacci numbers with negative index: Idea

Question

What can f_n be when n is a negative integer?

We want the basic properties to be satisfied for every $n \in \mathbb{Z}$:

- Defining formula:

$$f_n = f_{n-1} + f_{n-2}\,.$$

- Expression by golden ratio:

$$f_n = \frac{1}{\sqrt{5}}\left(\Phi^n - \hat{\Phi}^n\right).$$

- Matrix form:

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Consequently, Cassini's identity too.)

Note: For $n = 0$, the above suggest $f_{-1} = 1$ ...

Fibonacci numbers with negative index: Formula

Theorem

For every $n \geqslant 1$,

$$f_{-n} = (-1)^{n-1} f_n$$

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Proof: As $(1 - \Phi z) \cdot (1 - \hat{\Phi} z) = 1 - z - z^2$, it is $\Phi^{-1} = -\hat{\Phi} = 0.618\ldots$
Then for every $n \geqslant 1$,

$$\begin{aligned} f_{-n} &= \frac{1}{\sqrt{5}} \left(\Phi^{-n} - \hat{\Phi}^{-n} \right) \\ &= \frac{1}{\sqrt{5}} \left((-\hat{\Phi})^n - (-\Phi)^n \right) \\ &= \frac{(-1)^{n+1}}{\sqrt{5}} \left(\Phi^n - \hat{\Phi}^n \right) \\ &= (-1)^{n-1} f_n \,, \end{aligned}$$

Q.E.D.

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$$\begin{aligned}
f_{-n} &= \frac{1}{\sqrt{5}}\left(\Phi^{-n} - \hat{\Phi}^{-n}\right) \\
&= \frac{1}{\sqrt{5}}\left((-\hat{\Phi})^{n} - (-\Phi)^{n}\right) \\
&= \frac{(-1)^{n+1}}{\sqrt{5}}\left(\Phi^{n} - \hat{\Phi}^{n}\right) \\
&= (-1)^{n-1} f_n\,,
\end{aligned}$$

Q.E.D.

Another proof is by induction with the defining relation in the form $f_{n-2} = f_n - f_{n-1}$, with initial conditions $f_1 = 1$, $f_0 = 0$.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_k f_{n+1} + f_{k-1} f_n$$

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Why generalization?

Because for $k = 1 - n$ we get

$$f_1 = (-1)^{n-2} f_{n-1} f_{n+1} + (-1)^{n-1} f_n^2 \,,$$

which is Cassini's identity multiplied by $(-1)^n$.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_k f_{n+1} + f_{k-1} f_n$$

Proof: For every $n \in \mathbb{Z}$ let $P(n)$ be the following proposition:

$$\forall k \in \mathbb{Z}. f_{n+k} = f_k f_{n+1} + f_{k-1} f_n .$$

- For $n = 0$ we get $f_k = f_k \cdot 1 + 0$.
  For $n = 1$ we get $f_{k+1} = f_k \cdot 1 + f_{k-1} \cdot 1$.
- If $n \geqslant 2$ and $P(n-1)$ and $P(n-2)$ hold, then:

$$\begin{aligned} f_{n+k} &= f_{n-1+k} + f_{n-2+k} \\ &= f_k f_n + f_{k-1} f_{n-1} + f_k f_{n-1} + f_{k-1} f_{n-2} \\ &= f_k f_{n+1} + f_{k-1} f_n . \end{aligned}$$

- If $n < 0$ and $P(n+1)$ and $P(n+2)$ hold, then

$$\begin{aligned} f_{n+k} &= f_{n+2-k} - f_{n+1-k} \\ &= f_k f_{n+3} + f_{k-1} f_{n+2} - f_k f_{n+2} - f_{k-1} f_{n+1} \\ &= f_k f_{n+1} + f_{k-1} f_n . \end{aligned}$$

A note on generating functions for bi-infinite sequences

Question

Can we define f_n for every $n \in \mathbb{Z}$ via a single power series which depends from both positive and negative powers of the variable?
(We can renounce such $G(z)$ to be defined in $z = 0$.)

A note on generating functions for bi-infinite sequences

Question

Can we define f_n for every $n \in \mathbb{Z}$ via a single power series which depends from both positive and negative powers of the variable?
(We can renounce such $G(z)$ to be defined in $z = 0$.)

Answer: Yes, but it would not be practical!

A generalization of Laurent's theorem goes as follows:
Let f be an analytic function defined in an annulus $A = \{z \in \mathbb{C} \mid r < |z| < R\}$.
Then there exists a bi-infinite sequence $\langle a_n \rangle_{n \in \mathbb{Z}}$ such that:

1. the series $\sum_{n \geqslant 0} a_n z^n$ has convergence radius $\geqslant R$;
2. the series $\sum_{n \geqslant 1} a_{-n} z^n$ has convergence radius $\geqslant 1/r$;
3. for every $z \in A$ it is $\sum_{n \in \mathbb{Z}} a_n z^n = f(z)$.

We could set $r = 0$, but the power series $\sum_{n \geqslant 1} a_{-n} z^n$ would then need to have infinite convergence radius! (i.e., $\lim_{n \to \infty} \sqrt[n]{|a_{-n}|} = 0$.) However, $\lim_{n \to \infty} \sqrt[n]{|f_{-n}|} = \Phi$.
Also, the intersection of two annuli can be empty: making controls on feasibility of operations much more difficult to check. (Not so for "disks with a hole in zero".)

Fibonacci numbers cheat sheet

- Recurrence:

$$\begin{aligned} f_0 &= 0\,;\ \ f_1 = 1\,; \\ f_n &= f_{n-1} + f_{n-2} \quad \forall n \in \mathbb{Z}\,. \end{aligned}$$

- Binet form:

$$f_n = \frac{1}{\sqrt{5}} \left(\Phi^n - \hat{\Phi}^n \right) \ \forall n \in \mathbb{Z}\,.$$

- Generating function:

$$\sum_{n \geqslant 0} f_n z^n = \frac{z}{1 - z - z^2} \ \forall z \in \mathbb{C},\ |z| < \frac{1}{\Phi}\,.$$

- Matrix form:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \ \forall n \in \mathbb{Z}\,.$$

- Generalized Cassini's identity:

$$f_{n+k} = f_k f_{n+1} + f_{k-1} f_n \ \forall n, k \in \mathbb{Z}\,.$$

- Greatest common divisor:

$$\gcd(f_m, f_n) = f_{\gcd(m,n)} \ \forall m, n \in \mathbb{Z}\,.$$

Next section

Harmonic numbers

Definition

The harmonic numbers are given by the formula

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{for } n \geqslant 0, \text{ with } H_0 = 0$$

- H_n is the discrete analogue of the natural logarithm.
- The first twelve harmonic numbers are shown in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$	$\frac{83711}{27720}$

Harmonic numbers

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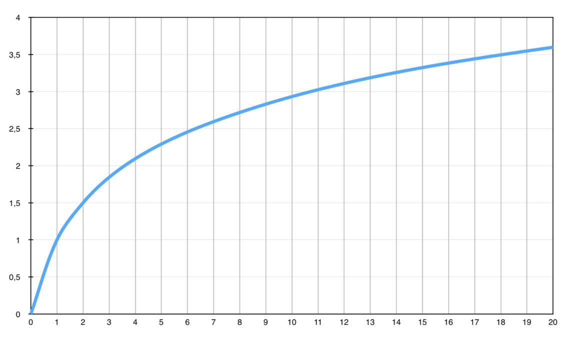

Harmonic numbers

Properties:

- Harmonic and Stirling cyclic numbers: $H_n = \frac{1}{n!}\left[{n+1 \atop 2}\right]$ for all $n \geqslant 1$;
- $\sum_{k=1}^{n} H_k = (n+1)(H_{n+1} - 1)$ for all $n \geqslant 1$;
- $\sum_{k=1}^{n} kH_k = \binom{n+1}{2}\left(H_{n+1} - \frac{1}{2}\right)$ for all $n \geqslant 1$;
- $\sum_{k=1}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right)$ for every $n \geqslant 1$;
- $\lim_{n\to\infty} H_n = \infty$;
- $H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4}$ where $\gamma \approx 0.57721\,56649\,01533$ denotes *Euler's constant*.

Approximation

- $H_{10} \approx 2.92896\,82578\,96$
- $H_{1000000} \approx 14.39272\,67228\,65723\,63138\,11275$

Harmonic numbers

Properties:

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Harmonic numbers

Generating function:

$$\frac{1}{1-z}\ln\frac{1}{1-z} = z + \frac{3}{2}z^2 + \frac{11}{6}z^3 + \frac{25}{12}z^4 + \cdots = \sum_{n\geqslant 0} H_n z^n$$

Indeed, $\frac{1}{1-z} = \sum_{n\geqslant 0} z^n$, $\ln\frac{1}{1-z} = \sum_{n\geqslant 0}\frac{z^n}{n}$, and

$$H_n = \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n} 1^{n-k}\frac{1}{k}$$

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A general remark

If $G(z)$ is the generating function of the sequence $\langle g_0, g_1, g_2, \ldots\rangle$, then $G(z)/(1-z)$ is the generating function of the sequence of the *partial sums* of the original sequence:

$$\text{if } G(z) = \sum_{n\geqslant 0} g_n z^n \text{ then } \frac{G(z)}{1-z} = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} g_k\right) z^n$$

Harmonic numbers and binomial coefficients

Theorem

$$\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$$

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Take $v(x) = \binom{x}{m+1}$: then

$$\Delta v(x) = \binom{x+1}{m+1} - \binom{x}{m+1} = \frac{x^{\underline{m}}}{m!} \cdot \frac{x+1-(x-m)}{m+1} = \binom{x}{m}$$

We can then sum by parts with $u(x) = H_x$ and get:

$$\begin{aligned} \sum \binom{x}{m} H_x \delta x &= \binom{x}{m+1} H_x - \sum \binom{x+1}{m+1} x^{\underline{-1}} \delta x \\ &= \binom{x}{m+1} \left(H_x - \frac{1}{m+1} \right) + C \end{aligned}$$

Then $\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{x}{m+1} \left(H_x - \frac{1}{m+1} \right) \Big|_{x=0}^{x=n+1} = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$, as desired.

Harmonic numbers and binomial coefficients

Theorem

$$\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$$

Corollary

For $m = 0$ we get:

$$\sum_{k=0}^{n} H_k = (n+1)(H_{n+1} - 1) = (n+1)H_n - n$$

For $m = 1$ we get:

$$\sum_{k=0}^{n} kH_k = \frac{n(n+1)}{2} \left(H_{n+1} - \frac{1}{2} \right) = \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4}$$

Harmonic numbers of higher order

Definition

For $n \geqslant 1$ and $m \geqslant 2$ integer, the nth harmonic number of order m is

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$$

As with the "first order" harmonic numbers, we put $H_0^{(m)} = 0$ as an empty sum.

For $m \geqslant 2$ the quantities

$$H_\infty^{(m)} = \lim_{n \to \infty} H_n^{(m)}$$

exist finite: they are the values of the *Riemann zeta function* $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ for $s = m$.

Euler's γ constant

Euler's approximation of harmonic numbers

For every $n \geqslant 1$ the following equality holds:

$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

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$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

For $k \geqslant 2$ we can write:

$$\ln \frac{k}{k-1} = \ln \frac{1}{1 - \frac{1}{k}} = \sum_{m \geqslant 1} \frac{1}{m \cdot k^m}$$

As $\ln(a/b) = \ln a - \ln b$ and $\ln 1 = 0$, by summing for k from 2 to n we get:

$$\ln n = \sum_{k=2}^{n} \sum_{m \geqslant 1} \frac{1}{m \cdot k^m} = \sum_{m \geqslant 1} \sum_{k=2}^{n} \frac{1}{m \cdot k^m} = H_n - 1 + \sum_{m \geqslant 2} \left(H_n^{(m)} - 1 \right)$$

Euler's γ constant

Euler's approximation of harmonic numbers

For every $n \geqslant 1$ the following equality holds:

$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

For $m \geqslant 2$, $H_n^{(m)}$ converges from below to $\zeta(m)$.
It turns out that $\zeta(s) - 1 \sim 2^{-s}$, therefore the series $\sum_{m \geqslant 2} \frac{1}{m} (\zeta(m) - 1)$ converges.
The quantity

$$\gamma = 1 - \sum_{m \geqslant 2} \frac{1}{m} (\zeta(m) - 1)$$

is called *Euler's constant*. The following approximation holds:

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o\left(\frac{1}{n^3}\right)$$

Next section

Next subsection

Eulerian Numbers

Don't mix up with Euler numbers!

$$E = \langle 1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, \ldots \rangle \leftrightarrow \frac{1}{\sinh x} = \frac{2}{e^x + e^{-x}} = \sum_n \frac{E_n}{n!} x^n$$

Eulerian Numbers

Definition

Let $\pi = (\pi_1, \pi_2, \ldots \pi_n)$ be a permutation of $\{1, 2, \ldots, n\}$. An ascent of the permutation π is any index i $(1 \leqslant i < n)$ such that $\pi_i < \pi_{i+1}$. The Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k ascents.

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Examples

- The permutation $\pi = (\pi_1, \pi_2, \ldots \pi_n) = (1, 2, 3, 4)$ has three ascents since $1 < 2 < 3 < 4$ and it is the only permutation in $S_4 = \{1, 2, 3, 4\}$ with three ascents; this is $\left\langle {4 \atop 3} \right\rangle = 1$
- There are $\left\langle {4 \atop 1} \right\rangle = 11$ permutations in S_4 with one ascent: $(1,4,3,2), (2,1,4,3), (2,4,3,1), (3,1,4,2), (3,2,1,4), (3,2,4,1), (3,4,2,1),$ $(4,1,3,2), (4,2,1,3), (4,2,3,1)$, and $(4,3,1,2)$.

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n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Eulerian Numbers

Some identities:

- $\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$ for all $n \geqslant 1$;

Symmetry: $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$ for all $n \geqslant 1$;

Recurrency: $\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$ for all $n \geqslant 2$;

- $\sum\limits_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n!$ for all $n \geqslant 2$;

Worpitzky's identity: $x^n = \sum\limits_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$ for all $n \geqslant 2$;

- $\left\langle {n \atop k} \right\rangle = \sum\limits_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$ for all $n \geqslant 1$;

Stirling numbers: $\left\{ {n \atop k} \right\} = \frac{1}{m!} \sum\limits_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{k}{n-m}$ for all$n \geqslant m$ and $n \geqslant 1$;

Generating f-n: $\frac{1-x}{e^{(x-1)t}-x} = \sum\limits_{n,m} \left\langle {n \atop m} \right\rangle x^m \frac{t^n}{n!}$.

Next subsection

Bernoulli numbers: History

Jakob Bernoulli (1654-1705) worked on the functions:

$$S_m(n) = 0^m + 1^m + \ldots + (n-1)^m = \sum_{k=0}^{n-1} k^m = \sum_0^n x^m \delta x$$

Plotting an expansion with respect to n yields:

$$\begin{array}{lllllll}
S_0(n) & = & n & & & & \\
S_1(n) & = & \frac{1}{2}n^2 & -\frac{1}{2}n & & & \\
S_2(n) & = & \frac{1}{3}n^3 & -\frac{1}{2}n^2 & +\frac{1}{6}n & & \\
S_3(n) & = & \frac{1}{4}n^4 & -\frac{1}{2}n^3 & +\frac{1}{4}n^2 & & \\
S_4(n) & = & \frac{1}{5}n^5 & -\frac{1}{2}n^4 & +\frac{1}{3}n^3 & -\frac{1}{30}n & \\
S_5(n) & = & \frac{1}{6}n^6 & -\frac{1}{2}n^5 & +\frac{5}{12}n^4 & -\frac{1}{12}n^2 & \\
S_6(n) & = & \frac{1}{7}n^7 & -\frac{1}{2}n^6 & +\frac{1}{2}n^5 & -\frac{1}{6}n^3 & +\frac{1}{42}n \\
S_7(n) & = & \frac{1}{8}n^8 & -\frac{1}{2}n^7 & +\frac{7}{12}n^6 & -\frac{7}{24}n^4 & +\frac{1}{12}n^2 \\
S_8(n) & = & \frac{1}{9}n^9 & -\frac{1}{2}n^8 & +\frac{2}{3}n^7 & -\frac{7}{15}n^5 & +\frac{2}{9}n^3 & -\frac{1}{30}n \\
S_9(n) & = & \frac{1}{10}n^{10} & -\frac{1}{2}n^9 & +\frac{3}{4}n^8 & -\frac{7}{10}n^6 & +\frac{1}{2}n^4 & -\frac{3}{20}n^2
\end{array}$$

Bernoulli numbers: History

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Bernoulli observed the following regularities:

- The leading coefficient of S_m is always $\frac{1}{m+1} = \frac{1}{m+1}\binom{m+1}{0}$.
- The coefficient of n^m in S_m is always $-\frac{1}{2} = -\frac{1}{2} \cdot \frac{1}{m+1} \cdot \binom{m+1}{1}$.
- The coefficient of n^{m-1} in S_m is always $\frac{m}{12} = \frac{1}{6} \cdot \frac{1}{m+1} \cdot \binom{m+1}{2}$.
- The coefficient of n^{m-2} in S_m is always 0.
- The coefficient of n^{m-3} in S_m is always $-\frac{m(m-1)(m-2)}{720} = -\frac{1}{30} \cdot \frac{1}{m+1} \cdot \binom{m+1}{4}$.
- The coefficient of n^{m-4} in S_m is always 0.
- The coefficient of n^{m-5} in S_m is always $\frac{1}{42} \cdot \frac{1}{m+1} \cdot \binom{m+1}{6}$.
- And so on, and so on ...

Bernoulli numbers

Definition

The kth Bernoulli number is the unique value B_k such that, for every $m \geqslant 0$,

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

Bernoulli numbers are also defined by the recurrence:

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = [m=0]$$

Observe that the above is simply $S_m(1)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Bernoulli numbers and the Riemann zeta function

Theorem

For every $n \geqslant 1$,

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}$$

In particular,

$$\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$