Special Numbers

ITT9131 Konkreetne Matemaatika

Contents

Next section

Next subsection

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Example: splitting a four-element set into two parts

$$\begin{array}{cccc} \{1,2,3\}\cup\{4\} & \{1,2,4\}\cup\{3\} & \{1,3,4\}\cup\{2\} & \{2,3,4\}\cup\{1\} \\ \{1,2\}\cup\{3,4\} & \{1,3\}\cup\{2,4\} & \{1,4\}\cup\{2,3\} & \end{array}$$

Hence $\left\{ 4 \atop 2 \right\} = 7$

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases: (1)

$k = 0$ We can partition a set into no nonempty parts if and only if the set is empty.

That is: $\left\{ n \atop 0 \right\} = [n = 0]$.

$k = 1$ We can partition a set into one nonempty part if and only if the set is nonempty.

That is: $\left\{ n \atop 1 \right\} = [n > 0]$.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases: (2)

$k = n$ If $n > 0$, the only way to partition a set with n elements into n nonempty parts, is to put every element by itself.
That is: $\left\{ {n \atop n} \right\} = 1$. (This also matches the case $n = 0$.)

$k = n-1$ Choosing a partition of a set with n elements into $n-1$ nonempty subsets, is the same as choosing the two elements that go together.
That is: $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases (3)

$k = 2$ Let X be a set with two or more elements.

- Each partition of X into two subsets is identified by two ordered pairs $(A, X \setminus A)$ for $A \subseteq X$.
- There are 2^n such pairs, but $(\emptyset, X)$ and $(X, \emptyset)$ do not satisfy the nonemptiness condition.
- Then $\left\{ n \atop 2 \right\} = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$ for $n \geqslant 2$.

In general, $\left\{ n \atop 2 \right\} = (2^{n-1} - 1)[n \geqslant 1]$

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty groups, then decide which group to add the nth element to.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty groups out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left\{ n \atop k \right\} = k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}, \qquad \text{for } n > 0,$$

that yields the following triangle:

Stirling's triangle for subsets

n	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$	$\left\{ {n \atop 6} \right\}$	$\left\{ {n \atop 7} \right\}$	$\left\{ {n \atop 8} \right\}$	$\left\{ {n \atop 9} \right\}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

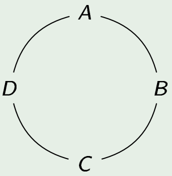
Next subsection

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Circle is a cyclic arrangement



- Circle can be written as $[A, B, C, D]$;
- It means that $[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C]$;
- It is not same as $[A, B, D, C]$ or $[D, C, B, A]$.

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[{n \atop k} \right]$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Example: splitting a four-element set into two circles

$$[1,2,3]\ \ [4] \qquad [1,2,4]\ \ [3] \qquad [1,3,4]\ \ [2] \qquad [2,3,4]\ \ [1]$$

$$[1,3,2]\ \ [4] \qquad [1,4,2]\ \ [3] \qquad [1,4,3]\ \ [2] \qquad [2,4,3]\ \ [1]$$

$$[1,2]\ \ [3,4] \qquad [1,3]\ \ [2,4] \qquad [1,4]\ \ [2,3]$$

Hence $\left[{4 \atop 2} \right] = 11$

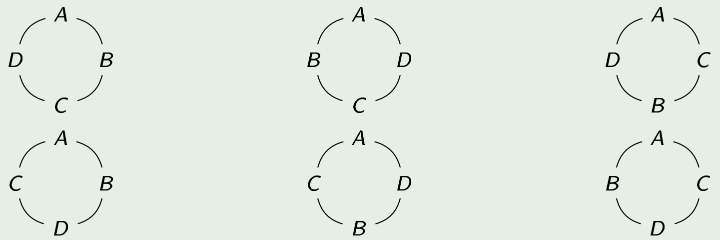
Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[{n \atop k}\right]$, read "n cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Some special cases (1):

$k=1$ To arrange one circle of n objects: choose the order, and forget which element was the first. That is: $\left[{n \atop 1}\right] = n!/n = (n-1)!$.



Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Some special cases (2):

$k = n$ Every circle is the singleton and there is just one partition into circles. That is, $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ for any n:

$$[1] \quad [2] \quad [3] \quad [4]$$

$k = n-1$ The partition into circles consists of $n-2$ singletons and one pair. So $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$, the number of ways to choose a pair:

$$[1,2] \quad [3] \quad [4] \qquad [1,3] \quad [2] \quad [4] \qquad [1,4] \quad [2] \quad [3]$$

$$[2,3] \quad [1] \quad [4] \qquad [2,4] \quad [1] \quad [3] \qquad [3,4] \quad [1] \quad [2]$$

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[n \atop k \right]$, read "n cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty cycles, then decide which element to put the nth one *after*.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty cycles out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left[n \atop k \right] = (n-1) \left[n-1 \atop k \right] + \left[n-1 \atop k-1 \right], \qquad \text{for } n > 0,$$

that yields the following triangle:

Stirling's triangle for circles

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$	$\left[{n \atop 7}\right]$	$\left[{n \atop 8}\right]$	$\left[{n \atop 9}\right]$
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Warmup: A closed formula for $\begin{bmatrix} n \\ 2 \end{bmatrix}$

Theorem

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\,H_{n-1}\,[n \geqslant 1]$$

Warmup: A closed formula for $\begin{bmatrix} n \\ 2 \end{bmatrix}$

Theorem

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1}\,[n \geqslant 1]$$

The formula is true for $n=0$ and $n=1$ ($H_0 = 0$ as an empty sum) so let $n \geqslant 2$.

- For $k = 1, \ldots, n-1$ there are $\binom{n}{k}$ ways of splitting n objects into a group of k and one of $n-k$. Each such way appears once for k, and once for $n-k$.
- To each splitting correspond $\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix} = (k-1)!(n-k-1)!$ pairs of cycles.
- Then

$$\begin{aligned} \begin{bmatrix} n \\ 2 \end{bmatrix} &= \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} (k-1)!(n-k-1)! \\ &= \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \\ &= \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{n} \left(\frac{1}{k} + \frac{1}{n-k} \right) \\ &= (n-1)!H_{n-1} \end{aligned}$$

Next subsection

Basic Stirling number identities, for integer $n \geqslant 0$

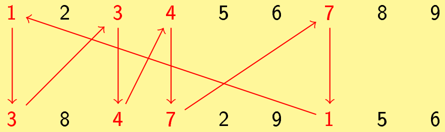
Some identities and inequalities we have already observed:

- $\left\{ {n \atop 0} \right\} = \left[{n \atop 0} \right] = [n = 0]$
- $\left\{ {n \atop 1} \right\} = [n > 0]$ and $\left[{n \atop 1} \right] = (n-1)![n > 0]$
- $\left\{ {n \atop 2} \right\} = (2^{n-1} - 1)[n > 0]$ and $\left[{n \atop 2} \right] = (n-1)! H_{n-1}[n > 0]$
- $\left\{ {n \atop n-1} \right\} = \left[{n \atop n-1} \right] = \binom{n}{2} = \frac{n(n-1)}{2}$
- $\left\{ {n \atop n} \right\} = \left[{n \atop n} \right] = \binom{n}{n} = 1$
- $\left\{ {n \atop k} \right\} = \left[{n \atop k} \right] = \binom{n}{k} = 0$, if $k > n$ or $k < 0$

Basic Stirling number identities (2)

For any integer $n \geqslant 0$, $\sum_{k=0}^{n} \left[{n \atop k} \right] = n!$

Permutations define cyclic arrangement and vice versa,
for example:



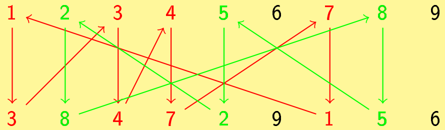
Thus the permutation $\pi = 384729156$ of $\{1,2,3,4,5,6,7,8,9\}$ is equivalent to the circle arrangement

$[1,3,4,7]$ $[2,8,5]$ $[6,9]$

Basic Stirling number identities (2)

For any integer $n \geqslant 0$, $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$

Permutations define cyclic arrangement and vice versa,
for example:



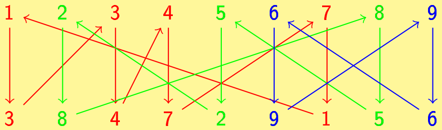
Thus the permutation $\pi = 384729156$ of $\{1,2,3,4,5,6,7,8,9\}$ is equivalent to the circle arrangement

$$[1,3,4,7] \quad [2,8,5] \quad [6,9]$$

Basic Stirling number identities (2)

For any integer $n \geqslant 0$, $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$

Permutations define cyclic arrangement and vice versa,
for example:



Thus the permutation $\pi = 384729156$ of $\{1,2,3,4,5,6,7,8,9\}$ is equivalent to the circle arrangement

$$[1,3,4,7] \quad [2,8,5] \quad [6,9]$$

Basic Stirling number identities (3)

Observation

$$
\begin{aligned}
x^0 &= x^{\underline{0}} \\
x^1 &= x^{\underline{1}} \\
x^2 &= x^{\underline{1}} + x^{\underline{2}} \\
x^3 &= x^{\underline{1}} + 3x^{\underline{2}} + x^{\underline{3}} \\
x^4 &= x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + x^{\underline{4}} \\
&\cdots\cdots
\end{aligned}
$$

n	${n \brace 0}$	${n \brace 1}$	${n \brace 2}$	${n \brace 3}$	${n \brace 4}$	${n \brace 5}$
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

Does the following general formula hold?

$$x^n = \sum_k {n \brace k} x^{\underline{k}}$$

Basic Stirling number identities (3a)

Inductive proof of $x^n = \sum_k \left\{ n \atop k \right\} x^{\underline{k}}$

- Considering that $x^{\underline{k+1}} = x^{\underline{k}}(x-k)$ we obtain that $x \cdot x^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}}$
- Hence

$$\begin{aligned}
x \cdot x^{n-1} = x\sum_k \left\{ n-1 \atop k \right\} x^{\underline{k}} &= \sum_k \left\{ n-1 \atop k \right\} x^{\underline{k+1}} + \sum_k \left\{ n-1 \atop k \right\} kx^{\underline{k}} \\
&= \sum_k \left\{ n-1 \atop k-1 \right\} x^{\underline{k}} + \sum_k \left\{ n-1 \atop k \right\} kx^{\underline{k}} \\
&= \sum_k \left(\left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\} \right) x^{\underline{k}} = \sum_k \left\{ n \atop k \right\} x^{\underline{k}}
\end{aligned}$$

Q.E.D.

Basic Stirling number identities (4)

Observation

$$x^{\overline{0}} = x^0$$

$$x^{\overline{1}} = x^1$$

$$x^{\overline{2}} = x^1 + x^2$$

$$x^{\overline{3}} = 2x^1 + 3x^2 + x^3$$

$$x^{\overline{4}} = 6x^1 + 11x^2 + 6x^3 + x^4$$

$$\dots\dots$$

Generating function for Stirling cycle numbers:

$$x^{\overline{n}} = \sum_k \left[n \atop k \right] x^k, \qquad \text{for } n \geqslant 0$$

Basic Stirling number identities (4a)

Generating function of the Stirling numbers of the first kind

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} z^k = z^{\overline{n}} \quad \forall n \geqslant 0$$

The formula is clearly true for $n = 0$ and $n = 1$.
If it is true for $n - 1$, then:

$$\begin{aligned}
z^{\overline{n}} &= z^{\overline{n-1}}(z+n-1) \\
&= \left(\sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} z^k \right) (z+n-1) \\
&= \sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} z^{k+1} + (n-1) \sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} z^k \\
&= \sum_k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} z^k + (n-1) \sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} z^k \\
&= \sum_k \left((n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) z^k ,
\end{aligned}$$

whence the thesis.

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Proof

As $x^{\underline{k}} = (-1)^k (-x)^{\overline{k}}$, we can rewrite the known equalities as:

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^k (-x)^{\overline{k}} \text{ and } (-1)^n (-x)^{\underline{n}} = \sum_k \left[{n \atop k} \right] x^k$$

But clearly $x^n = (-1)^n (-x)^n$, so by replacing x with $-x$ we get the thesis.

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Corollary

$$\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} = \sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{n-k} = [m = n]$$

Indeed,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} \left(\sum_m \left[{k \atop m} \right] x^m \right) = \sum_m \left(\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} \right) x^m$$

must hold for every x; the other equality is proved similarly.

Stirling's inversion formula (cf. Exercise 6.12)

Statement

Let f and g be two functions defined on $\mathbb{N}$ with values in $\mathbb{C}$.
The following are equivalent:

1. For every $n \geqslant 0$,
$$g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k).$$
2. For every $n \geqslant 0$,
$$f(n) = \sum_k \left[{n \atop k} \right] (-1)^k g(k).$$

Stirling's inversion formula (cf. Exercise 6.12)

Proof

If $g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k)$ for every $n \geqslant 0$, then also for $n \geqslant 0$

$$\begin{aligned}
\sum_k \left[{n \atop k} \right] (-1)^k g(k) &= \sum_k \left[{n \atop k} \right] (-1)^k \sum_m \left\{ {k \atop m} \right\} (-1)^m f(m) \\
&= \sum_{k,m} (-1)^{k+m} f(m) \left[{n \atop k} \right] \left\{ {k \atop m} \right\} \\
&= \sum_{k,m} (-1)^{2n-k-m} f(m) \left[{n \atop k} \right] \left\{ {k \atop m} \right\} \\
&= \sum_m (-1)^{n-m} f(m) \sum_k (-1)^{n-k} \left[{n \atop k} \right] \left\{ {k \atop m} \right\} \\
&= \sum_m (-1)^{n-m} f(m) [m = n] \\
&= f(n).
\end{aligned}$$

The other implication is proved similarly.

Next subsection

Stirling's triangles in tandem

Basic recurrences of Stirling numbers yield for every integers k, n a simple law:

$$\left[{n \atop k}\right] = \left\{{-k \atop -n}\right\} \quad \text{with} \left[{n \atop 0}\right] = \left\{{n \atop 0}\right\} = [n=0] \text{ and } \left[{0 \atop k}\right] = \left\{{0 \atop k}\right\} = [k=0]$$

n	$\left\{{n \atop -5}\right\}$	$\left\{{n \atop -4}\right\}$	$\left\{{n \atop -3}\right\}$	$\left\{{n \atop -2}\right\}$	$\left\{{n \atop -1}\right\}$	$\left\{{n \atop 0}\right\}$	$\left\{{n \atop 1}\right\}$	$\left\{{n \atop 2}\right\}$	$\left\{{n \atop 3}\right\}$	$\left\{{n \atop 4}\right\}$	$\left\{{n \atop 5}\right\}$
-5	1										
-4	10	1									
-3	35	6	1								
-2	50	11	3	1							
-1	24	6	2	1	1						
0	0	0	0	0	0	1					
1	0	0	0	0	0	0	1				
2	0	0	0	0	0	0	1	1			
3	0	0	0	0	0	0	1	3	1		
4	0	0	0	0	0	0	1	7	6	1	
5	0	0	0	0	0	0	1	15	25	10	1

Stirling numbers cheat sheet

- $\left\{\begin{matrix} n \\ 0 \end{matrix}\right\} = \left[\begin{matrix} n \\ 0 \end{matrix}\right] = [n = 0]$
- $\left\{\begin{matrix} n \\ 1 \end{matrix}\right\} = [n > 0]$ and $\left[\begin{matrix} n \\ 1 \end{matrix}\right] = (n-1)![n > 0]$
- $\left\{\begin{matrix} n \\ 2 \end{matrix}\right\} = (2^{n-1} - 1)[n > 0]$ and $\left[\begin{matrix} n \\ 2 \end{matrix}\right] = (n-1)!H_{n-1}[n > 0]$
- $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\} = \left[\begin{matrix} n \\ n-1 \end{matrix}\right] = \binom{n}{2} = \frac{n(n-1)}{2}$
- $\left\{\begin{matrix} n \\ n \end{matrix}\right\} = \left[\begin{matrix} n \\ n \end{matrix}\right] = \binom{n}{n} = 1$
- $\left\{\begin{matrix} n \\ k \end{matrix}\right\} = \left[\begin{matrix} n \\ k \end{matrix}\right] = \binom{n}{k} = 0$, if $k > n$ or $k < 0$
- $\left\{\begin{matrix} n \\ k \end{matrix}\right\} = k\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} + \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}$ and $\left[\begin{matrix} n \\ k \end{matrix}\right] = (n-1)\left[\begin{matrix} n-1 \\ k \end{matrix}\right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix}\right]$
- $\sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} x^{\underline{k}} = x^n$ and $\sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] x^k = x^{\overline{n}}$
- $\sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] = n!$
- $\sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} (-1)^{n-k} x^{\overline{k}} = x^n$ and $\sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] (-1)^{n-k} x^k = x^{\underline{n}}$
- $\sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} \left[\begin{matrix} k \\ m \end{matrix}\right] (-1)^k = \sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] \left\{\begin{matrix} k \\ m \end{matrix}\right\} (-1)^k = [m = n]$

Next section

Fibonacci numbers: Idea

Fibonacci's problem

A pair of baby rabbits is left on an island.

- A baby rabbit becomes adult in one month.
- A pair of adult rabbits produces a pair of baby rabbits each month.

How many pairs of rabbits will be on the island ofter n months?
How many of them will be adult, and how many will be babies?

Leonardo Fibonacci (1175–1235)

Fibonacci numbers: Idea

Fibonacci's problem

A pair of baby rabbits is left on an island.

- A baby rabbit becomes adult in one month.
- A pair of adult rabbits produces a pair of baby rabbits each month.

How many pairs of rabbits will be on the island ofter n months?
How many of them will be adult, and how many will be babies?

Solution (see Exercise 6.6)

- On the first month, the two baby rabbits will have become adults.
- On the second month, the two adult rabbits will have produced a pair of baby rabbits.
- On the third month, the two adult rabbits will have produced *another* pair of baby rabbits, while the other two baby rabbits will have become adults.
- And so on, and so on . . .

Leonardo Fibonacci (1175–1235)

Fibonacci Numbers: Definition

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

Formulae for computing:

- $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$
- $f_n = \frac{\Phi^n - \hat{\Phi}^n}{\sqrt{5}}$ ("Binet form")

The golden ratio

The constant $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ is called golden ratio :

If a line segment a is divided into two sub-segments b and $a-b$ so that $a : b = b : (a-b)$, then

$$\frac{a}{b} = \Phi \text{ and } \frac{b}{a} = -\hat{\Phi}$$

Fibonacci Numbers: Definition

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

Formulae for computing:

- $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$
- $f_n = \frac{\Phi^n - \hat{\Phi}^n}{\sqrt{5}}$ ("Binet form")

The golden ratio

The constant $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ is called golden ratio :

If a line segment a is divided into two sub-segments b and $a - b$ so that $a : b = b : (a - b)$, then

$$\frac{a}{b} = \Phi \text{ and } \frac{b}{a} = -\hat{\Phi}$$

Generating Function for Fibonacci Numbers

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

Generating Function for Fibonacci Numbers

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

$$\left.\begin{array}{llllll} \langle f_0, & f_1, & f_2, & f_3, & f_4, & \cdots\rangle \\ \langle 0, & 1, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \cdots\rangle \end{array}\right\} \leftrightarrow F(x)$$

Generating Function for Fibonacci Numbers

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

$$\left.\begin{array}{llllll} \langle f_0, & f_1, & f_2, & f_3, & f_4, & \cdots \rangle \\ \langle 0, & 1, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \cdots \rangle \end{array}\right\} \leftrightarrow F(x)$$

Applying Addition to some known generating functions:

$$\begin{array}{rlllllllc} & \langle 0, & 1, & 0, & 0, & 0, & \cdots \rangle & \leftrightarrow & x \\ & \langle 0, & f_0, & f_1, & f_2, & f_3, & \cdots \rangle & \leftrightarrow & xF(x) \\ + & \langle 0, & 0, & f_0, & f_1, & f_2, & \cdots \rangle & \leftrightarrow & x^2 F(x) \\ \hline & \langle 0, & 1 + f_0, & f_1 + f_0,, & f_2 + f_1, & f_3 + f_2, & \cdots \rangle & \leftrightarrow & x + xF(x) + x^2 F(x) \end{array}$$

Generating Function for Fibonacci Numbers

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

$$\left.\begin{array}{llllll} \langle f_0, & f_1, & f_2, & f_3, & f_4, & \cdots\rangle \\ \langle 0, & 1, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \cdots\rangle \end{array}\right\} \leftrightarrow F(x)$$

Applying Addition to some known generating functions:

$$\begin{array}{ccccccccc} & \langle 0, & 1, & 0, & 0, & 0, & \cdots\rangle & \leftrightarrow & x \\ & \langle 0, & f_0, & f_1, & f_2, & f_3, & \cdots\rangle & \leftrightarrow & xF(x) \\ + & \langle 0, & 0, & f_0, & f_1, & f_2, & \cdots\rangle & \leftrightarrow & x^2 F(x) \\ \hline & \langle 0, & 1 + f_0, & f_1 + f_0,, & f_2 + f_1, & f_3 + f_2, & \cdots\rangle & \leftrightarrow & x + xF(x) + x^2 F(x) \end{array}$$

Closed form of the generating function: $F(x) = \frac{x}{1-x-x^2}$

Evaluation of Coefficients: Factorization

- We know from the previous lecture that

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

- Let's try to represent a generating function in the form:

$$\begin{aligned} G(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A \sum_{n \geqslant 0} (\alpha x)^n + B \sum_{n \geqslant 0} (\beta x)^n \\ &= \sum_{n \geqslant 0} (A\alpha^n + B\beta^n) x^n \end{aligned}$$

- The task is to find such constants A, B, α, β that

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{A - A\beta x + B - B\alpha x}{(1-\alpha x)(1-\beta x)}$$

Evaluation of Coefficients: Factorization

- We know from the previous lecture that

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

- Let's try to represent a generating function in the form:

$$\begin{aligned} G(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A \sum_{n \geqslant 0} (\alpha x)^n + B \sum_{n \geqslant 0} (\beta x)^n \\ &= \sum_{n \geqslant 0} (A\alpha^n + B\beta^n) x^n \end{aligned}$$

- The task is to find such constants A, B, α, β that

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{A - A\beta x + B - B\alpha x}{(1-\alpha x)(1-\beta x)}$$

Evaluation of Coefficients: Factorization

- We know from the previous lecture that

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

- Let's try to represent a generating function in the form:

$$\begin{aligned} G(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A \sum_{n \geqslant 0} (\alpha x)^n + B \sum_{n \geqslant 0} (\beta x)^n \\ &= \sum_{n \geqslant 0} (A\alpha^n + B\beta^n) x^n \end{aligned}$$

- The task is to find such constants A, B, α, β that

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{A - A\beta x + B - B\alpha x}{(1-\alpha x)(1-\beta x)}$$

Factorization for Fibonacci (2)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

Factorization for Fibonacci (2)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

To factorize $1-x-x^2$

- Solve the equation $w^2-wx-x^2=0$ (i.e. $w=1$ gives the special case $1-x-x^2=0$):

$$w_{1,2}=\frac{x\pm\sqrt{x^2+4x^2}}{2}=\frac{1\pm\sqrt{5}}{2}x$$

- Therefore

$$w^2-wx-x^2=\left(w-\frac{1+\sqrt{5}}{2}x\right)\left(w-\frac{1-\sqrt{5}}{2}x\right)$$

and

$$1-x-x^2=\left(1-\frac{1+\sqrt{5}}{2}x\right)\left(1-\frac{1-\sqrt{5}}{2}x\right)$$

Factorization for Fibonacci (2)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

A general trick

Let $p(x)=\sum_{k=0}^{n} a_k x^k$ be a polynomial over $\mathbb{C}$ of degree n such that $a_0 = p(0) \neq 0$.

- Then all the roots of p have a multiplicative inverse.
- Consider the "reverse" polynomial

$$p_R(x) = \sum_{k=0}^{n} a_k x^{n-k} = x^n p\left(\frac{1}{x}\right)$$

- Then α is a root of p if and only if $1/\alpha$ is a root of p_R, because if $p(x) = a_n(x-\alpha_1)\cdots(x-\alpha_n)$, then $p_R(x) = a_n(1-\alpha_1 x)\cdots(1-\alpha_n x)$.

Factorization for Fibonacci (3)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

Denote $\Phi = \frac{1+\sqrt{5}}{2}$ (golden ratio):

- "phi hat" is

$$\widehat{\Phi} = 1-\Phi = 1-\frac{1+\sqrt{5}}{2} = \frac{2-1-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$$

- and we have

$$1-x-x^2 = (1-\Phi x)\left(1-\widehat{\Phi}x\right)$$

Factorization for Fibonacci (3)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

Denote $\Phi = \frac{1+\sqrt{5}}{2}$ (golden ratio):

- "phi hat" is

$$\widehat{\Phi} = 1-\Phi = 1-\frac{1+\sqrt{5}}{2} = \frac{2-1-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$$

- and we have

$$1-x-x^2 = (1-\Phi x)\left(1-\widehat{\Phi}x\right)$$

Factorization for Fibonacci (4)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\Phi x)(1-\widehat{\Phi}x) & = & 1-x-x^2 \\ (A+B)-(A\widehat{\Phi}+B\Phi)x & = & x \end{cases}$$

To find A and B:

- Solve

$$\begin{cases} A+B=0 \\ A\widehat{\Phi}+B\Phi=-1 \end{cases}$$

- This is $A=1/(\Phi-\widehat{\Phi})$:

$$\begin{aligned} A &= 1/(\Phi-\widehat{\Phi}) \\ &= 1/\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{2}{1+\sqrt{5}-1+\sqrt{5}} = \frac{1}{\sqrt{5}} \end{aligned}$$

Factorization for Fibonacci (4)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\Phi x)(1-\widehat{\Phi}x) & = & 1-x-x^2 \\ (A+B)-(A\widehat{\Phi}+B\Phi)x & = & x \end{cases}$$

To find A and B:

- Solve

$$\begin{cases} A+B=0 \\ A\widehat{\Phi}+B\Phi=-1 \end{cases}$$

- This is $A=1/(\Phi-\widehat{\Phi})$:

$$\begin{aligned} A &= 1/(\Phi-\widehat{\Phi}) \\ &= 1/\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{2}{1+\sqrt{5}-1+\sqrt{5}} = \frac{1}{\sqrt{5}} \end{aligned}$$

Factorization for Fibonacci (5)

To conclude:

- We have $\alpha = \Phi = (1+\sqrt{5})/2$, $\beta = \widehat{\Phi} = (1-\sqrt{5})/2$, $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$
- Generating function

$$\begin{aligned} G(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1-\Phi x} - \frac{1}{1-\widehat{\Phi} x} \right) \end{aligned}$$

- Closed formula for f_n

$$\begin{aligned} f_n &= A\alpha^n + B\beta^n \\ &= \frac{1}{\sqrt{5}} \left(\Phi^n - \widehat{\Phi}^n \right) \end{aligned}$$