Greedy Algorithms*

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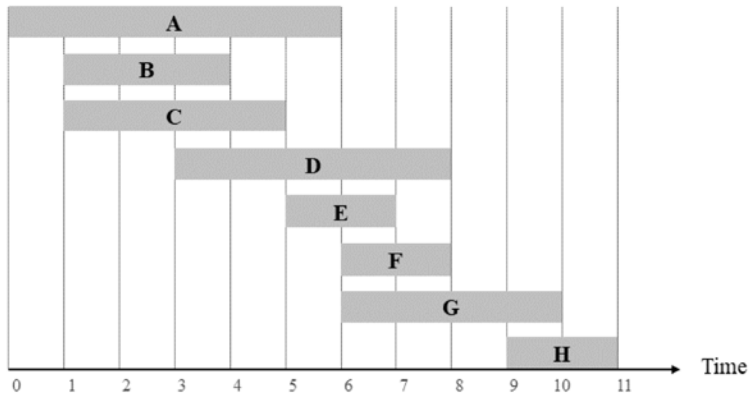
Outline

1. Basic Methodology
   - Interval Scheduling
   - Interval Partitioning
   - Scheduling to Minimize Lateness
2. More Examples
   - Optimal Caching
   - Coin Changing

Interval Scheduling: An Introductory Example

- Job j starts at s_j and finishes at f_j.
- Two jobs are compatible if they don't overlap.

Goal: find maximum subset of mutually compatible jobs.



Greedy Strategy

Optimization Problem: Given a problem Π with domain $\mathbf{X}$, choose a subset or determine a sequence according to some maximization or minimization objective. (Each $X \in \mathbf{X}$ is an instance of Π)

General Template: Consider each item $x_i \in X$ of problem Π (in some order), make choice that looks best at the moment.

Note: it makes a *locally optimal* choice in hope that this choice will lead to a *globally optimal* solution.

Interval Scheduling Problem: Consider jobs in some natural order. Take each job provided to judge its compatibility with the ones already taken.

Attempts

[Earliest start time] Consider jobs in ascending order of s_j.

Counter Example:

[Earliest finish time] Consider jobs in ascending order of f_j.

[Shortest interval] Consider jobs in ascending order of $f_j - s_j$.

Counter Example:

[Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Counter Example:

Greedy Interval Scheduling Algorithm

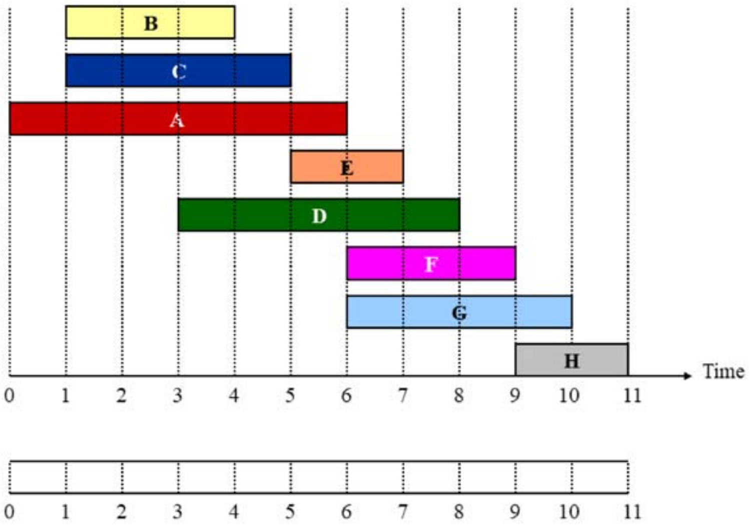
Algorithm 1: Greedy Interval Scheduling

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$;
$A \leftarrow \emptyset$ ;                    // set of jobs selected
for $j = 1$ to $n$ do
    if job $j$ is compatible with $A$ then
        $A \leftarrow A \cup \{j\}$;
return $A$;
```

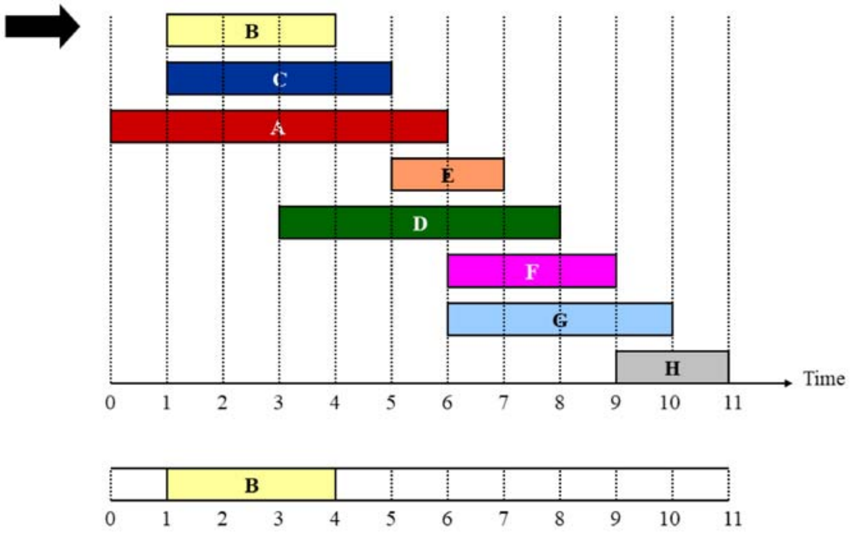
Implementation: $O(n \log n)$.

- After each iteration, set job j^* that was added last to A.
- Job j is compatible with A if $s_j \ge f_{j^*}$.

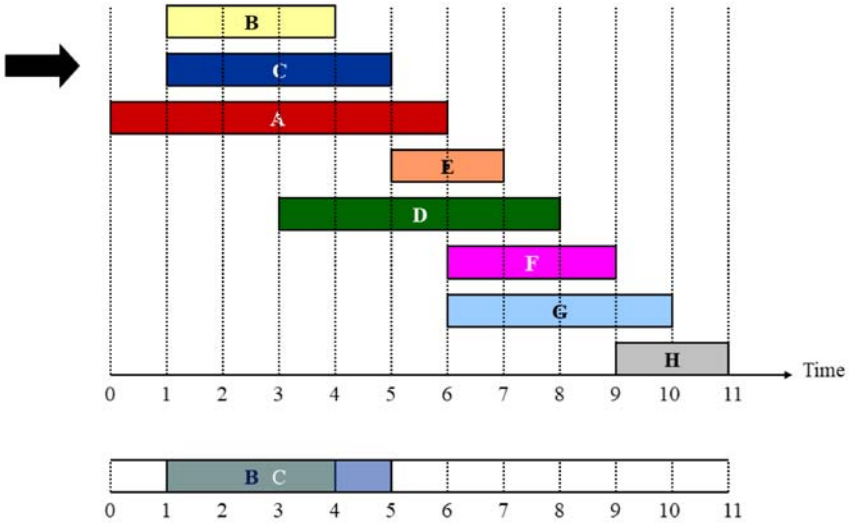
Demo



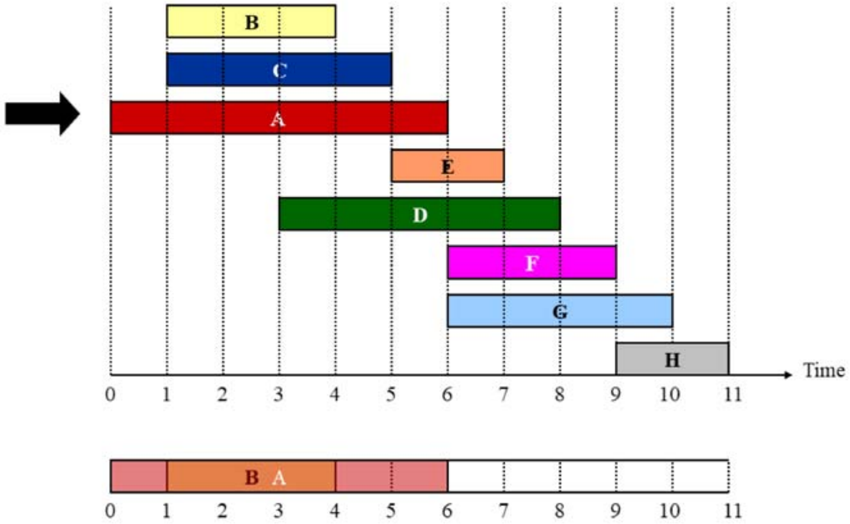
Demo



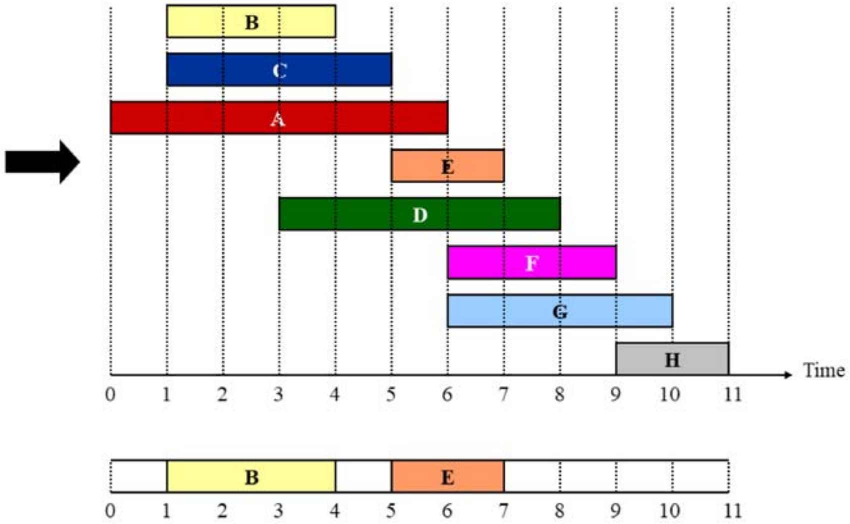
Demo



Demo



Demo



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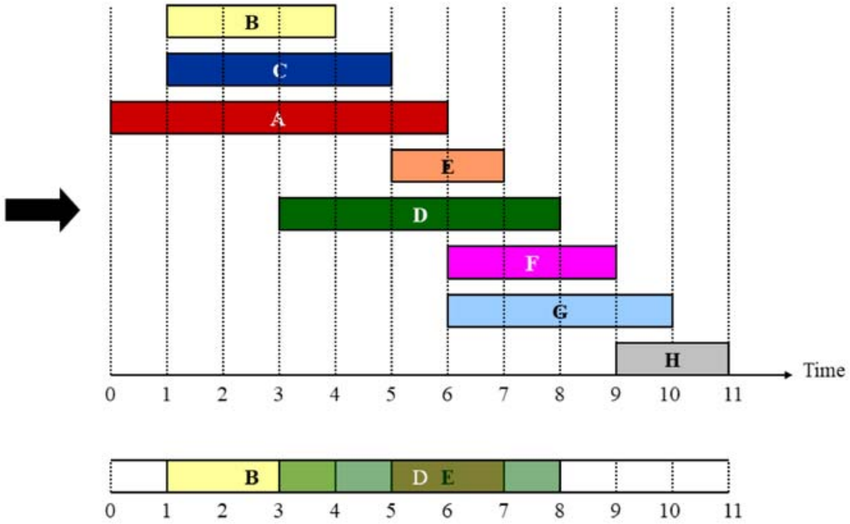

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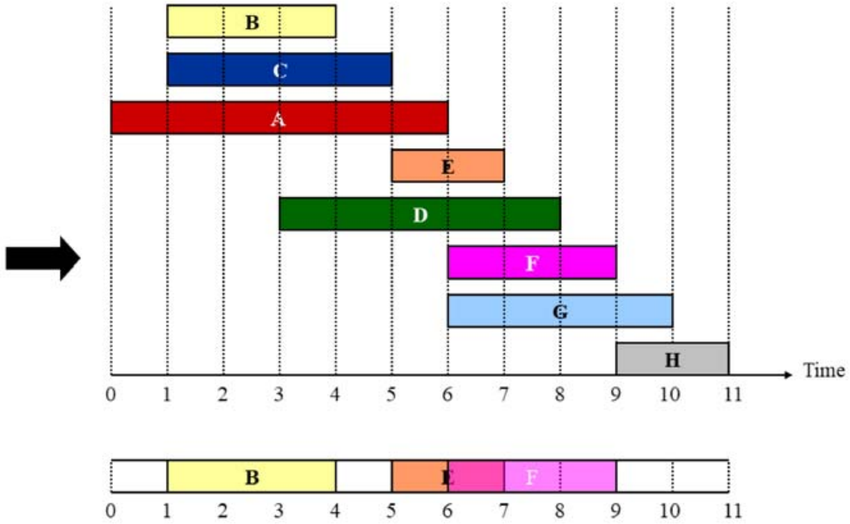
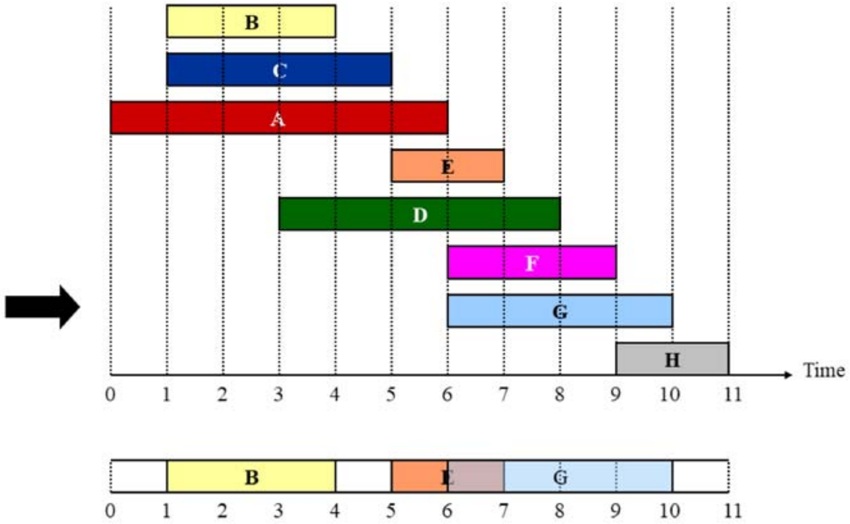
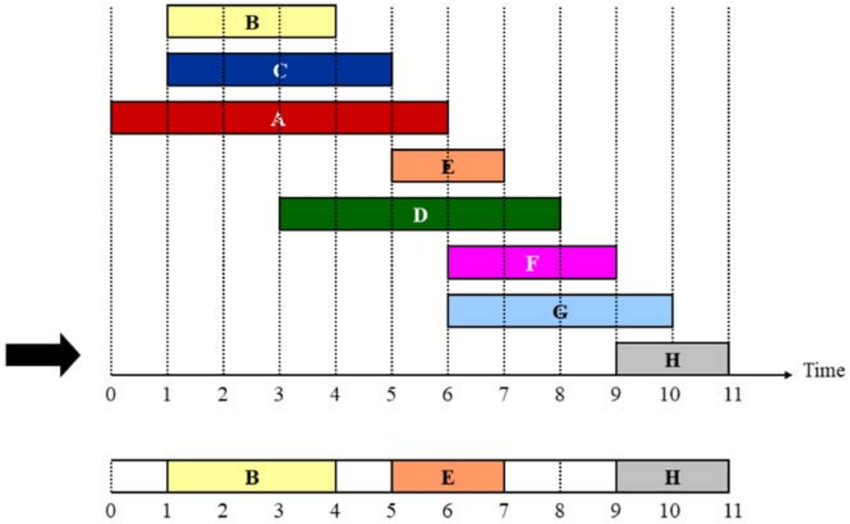


Demo



Demo



Notation

Greedy Solution: $\{B, E, H\}$

Optimal Solutions: (not necessarily unique)

$\{A, F, H\}, \{B, E, H\}, \{B, F, H\}, \{C, E, H\}, \{C, F, H\}$

Feasible Solutions: (can work, but may not be the best)

$\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}, \{H\};$
$\{A, F\}, \{A, G\}, \{A, H\}, \{B, E\}, \{B, F\}, \{B, G\}, \{B, H\},$
$\{C, E\}, \{C, F\}, \{C, G\}, \{C, H\}, \{D, H\}, \{E, H\}, \{F, H\};$
$\{A, F, H\}, \{B, E, H\}, \{B, F, H\}, \{C, E, H\}, \{C, F, H\}.$

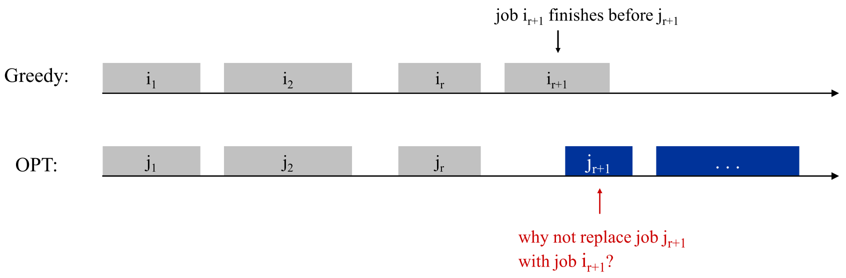
Correctness Analysis

Theorem. Greedy Interval Scheduling algorithm is optimal.

Proof. (by contradiction) Assume greedy is not optimal.

Let $i_1, i_2, \cdots, i_k$ denote set of jobs selected by greedy.

Let $j_1, j_2, \cdots, j_m$ denote set of jobs in an optimal solution with $i_1 = j_1$, $i_2 = j_2$, $\cdots$, $i_r = j_r$ for the largest possible value of r.



job i_{r+1} finishes before j_{r+1}

why not replace job j_{r+1} with job i_{r+1}?

Correctness Analysis

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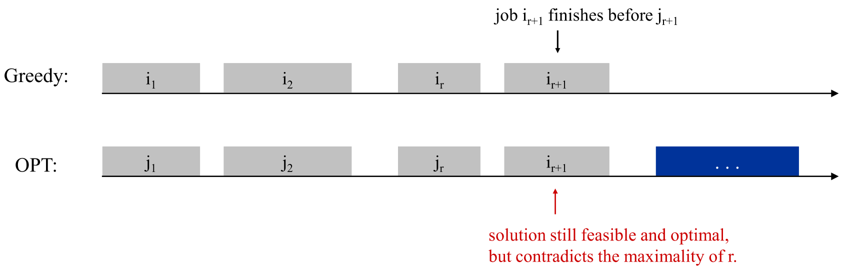
Let $j_1, j_2, \cdots, j_m$ denote set of jobs in an optimal solution with $i_1 = j_1$, $i_2 = j_2$, $\cdots$, $i_r = j_r$ for the largest possible value of r.



job i_{r+1} finishes before j_{r+1}

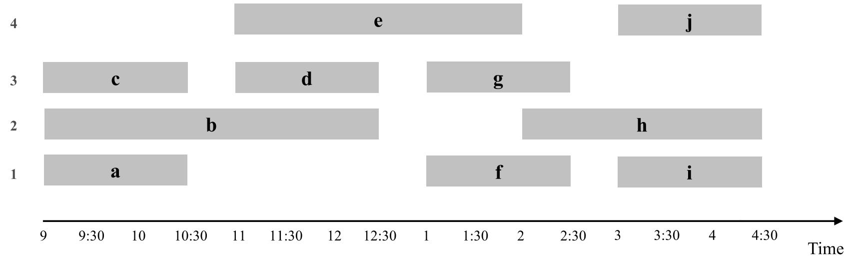
solution still feasible and optimal, but contradicts the maximality of r.

Interval Partitioning

Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses 4 classrooms to schedule 10 lectures.

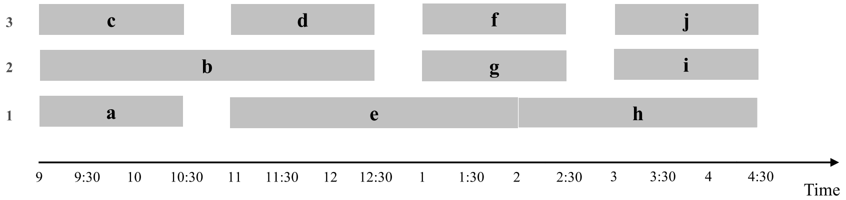


Interval Partitioning

Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses only 3.

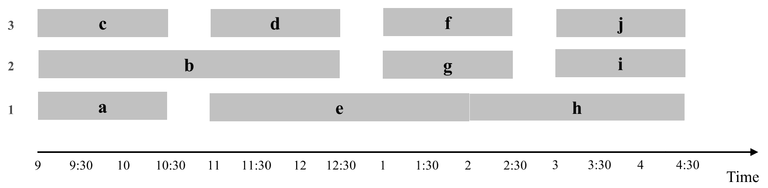


Lower Bound on Optimal Solution

Definition: The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Example: Depth of schedule $= 3 \Rightarrow$ The schedule is optimal.



Question. Does there always exist a schedule equal to depth of intervals?

Greedy Interval Partitioning Algorithm

Algorithm 2: Interval Partitioning Greedy Algorithm

```
Sort intervals by starting time so that $s_1 \leq s_2 \leq ... \leq s_n$;
$d \leftarrow 0$;            // number of allocated classrooms
for $j = 1$ to $n$ do
    if lecture $j$ is compatible with some classroom $k$ then
        schedule lecture $j$ in classroom $k$;
    else
        allocate a new classroom $d + 1$;
        schedule lecture $j$ in classroom $d + 1$;
        $d \leftarrow d + 1$;
return $A$;
```

Implementation: $O(n \log n)$.

- For classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Correctness Proof

Key Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Proof. Let $d =$ number of classrooms that the algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all $d - 1$ other classrooms. (These d jobs each end after s_j.)

Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j. Thus, we have d lectures overlapping at time $s_j + \varepsilon$. ☐

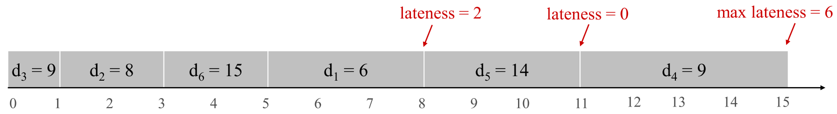
Key observation $\Rightarrow$ all schedules use $\geq d$ classrooms.

Scheduling to Minimize Lateness

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $l_j = \max\{0, f_j - d_j\}$.

Goal: schedule all jobs to minimize maximum lateness $L = \max l_j$.

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Attempt: Consider jobs in ascending order by some strategy

[Shortest processing time first] Sort by processing time t_j.

counterexample

	1	2
t_j	1	10
d_j	100	10

[Earliest deadline first] Sort by deadline d_j.

[Smallest slack] Sort by slack $d_i - t_i$.

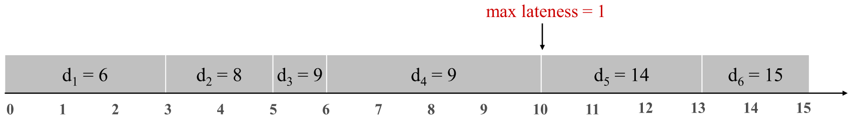
counterexample

	1	2
t_j	1	10
d_j	2	10

A Greedy Algorithm: Earliest Deadline First

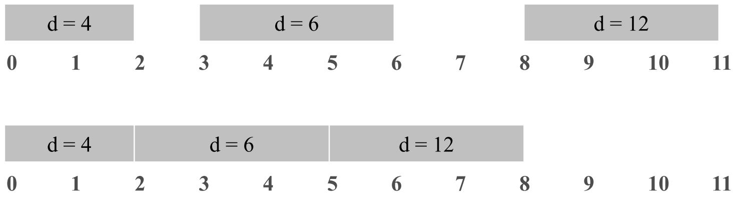
```
Algorithm 3: Greedy Minimizing Lateness
Sort $n$ jobs by deadline so that $d_1 \leq d_2 \leq ... \leq d_n$;
$t \leftarrow 0$;
for $j = 1$ to $n$ do
    Assign job $j$ to interval $[t, t + t_j]$;
    $s_j \leftarrow t, f_j \leftarrow t + t_j$;
    $t \leftarrow t + t_j$;
return intervals $[s_j, f_j]$;
```

Implementation: $O(n \log n)$.



Correctness Proof: Reduce Optimal Solution

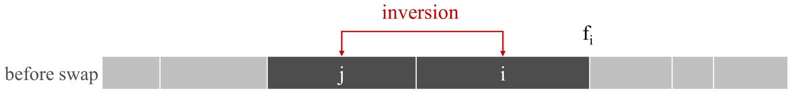
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Correctness Proof: Optimal Solution vs Algorithm Solution

Definition. Given a schedule S, an inversion is a pair of jobs i and j such that: $i < j$ but j scheduled before i.



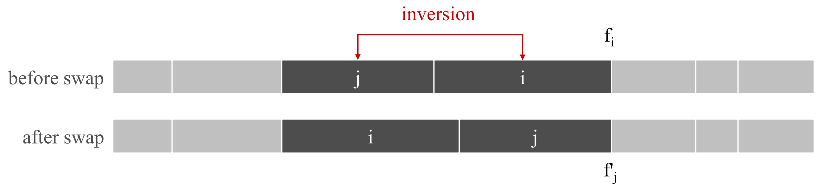
[as before, we assume jobs are numbered so that $\mathbf{d_1 \leq d_2 \leq \ldots \leq d_n}$]

Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Correctness Proof: Inversions

Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.



Proof. Let l be the lateness before the swap, and let l' be it afterwards.

- $l'_k = l_k$ for all $k \neq i, j$
- $l'_i \leq l_i$
- If job j is late:

$$\begin{aligned} l'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq l_i && \text{(definition)} \end{aligned}$$

Correctness Proof

Theorem. Greedy schedule S is optimal.

Proof. Define S^* to be an optimal schedule that has the fewest number of inversions. (We can assume S^* has no idle time.)

- If S^* has no inversions, then $S = S^*$.
- If S^* has an inversion, let $i - j$ be an adjacent inversion.

  Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions. This contradicts definition of S^*. □

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, ...

Optimal Offline Caching

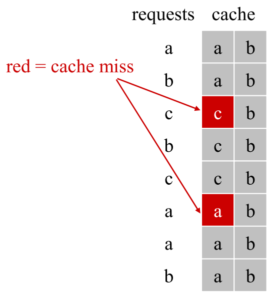
- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, ..., d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Example. $k = 2$, initial cache = ab,

requests: a, b, c, b, c, a, a, b.

Optimal eviction schedule: 2 cache misses.



red = cache miss

requests	cache	
a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b

Optimal Strategy: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache: a b c d e f

future queries: g a b c e d a b b a c d e a f a d e f g h ...

↑ cache miss

↑ eject this one

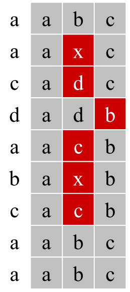
Theorem. [Bellady, 1960s] FF is an optimal eviction schedule.

Proof. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

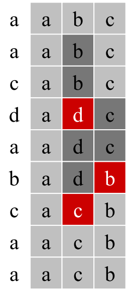
Definition. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses (means no more cache insertion/replacement here).



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule



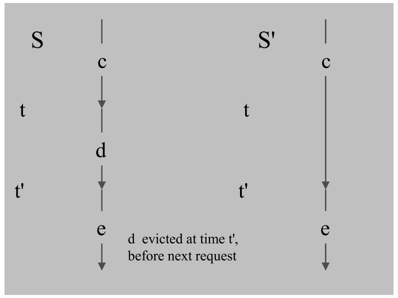
a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

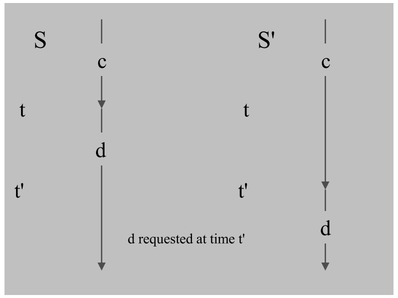
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, we can transform it into a reduced schedule S' with no more cache replacement (insertion).

Proof. (by induction on number of unreduced[†] items)
Suppose S brings d into the cache at time t, without a request. Let c be the item S evicts when it brings d into the cache.



Case 1



Case 2

[†] doesn't enter cache at requested time

Theorem. FF is an optimal eviction algorithm

Proof. (by induction on number of requests j)

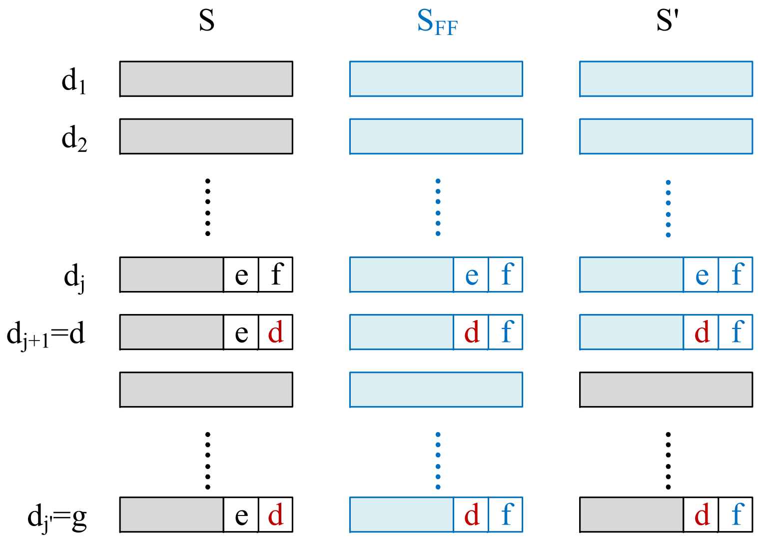
Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first $j+1$ requests.

Let S be reduced schedule that satisfies invariant through j requests. We produce S' that satisfies invariant after $j+1$ requests.

Consider $(j+1)^{th}$ request $d = d_{j+1}$. Since S and S_{FF} have agreed up until now, they have the same cache contents before request $j+1$.

- Case 1: (d is already in the cache) $S' = S$ satisfies invariant.
- Case 2: (d is not in the cache; S, S_{FF} evict same element) $S' = S$ satisfies invariant.
- Case 3: (d is not in the cache; S_{FF} evicts e; S evicts $f \neq e$) Let S' agree with S_{FF} at the $j+1$ requests; we show that having element f in cache is no worse than having element e.

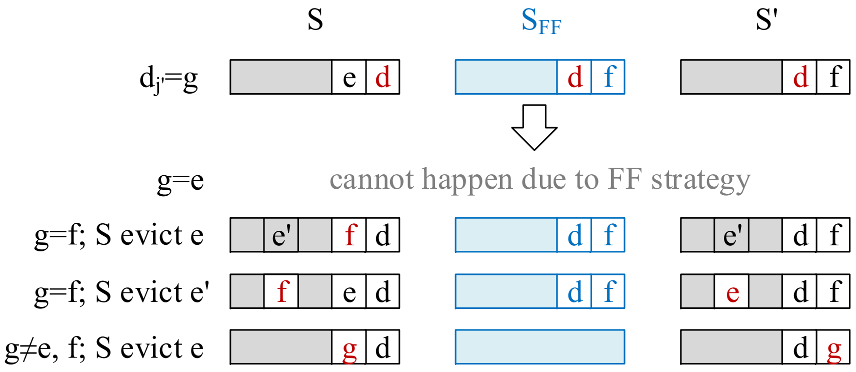
An Illustration of Case 3



	S	S_{FF}	S'
d_1			
d_2			
⋮	⋮	⋮	⋮
d_j	e f	e f	e f
d_{j+1}=d	e d	d f	d f
	⋮	⋮	⋮
$d_{j'}$=g	e d	d f	d f

Correctness Proof (Continued)

Let j' be the first time after $j+1$ that S and S' take a different action (must involve e or f or both), and let g be item requested at time j'.



	S	S_{FF}	S'
$d_{j'}$=g	e d	d f	d f
g=e	cannot happen due to FF strategy		
g=f; S evict e	e' f d	d f	e' d f
g=f; S evict e'	f e d	d f	e d f
g≠e, f; S evict e	g d		d g

Correctness Proof (Continued)

Case 3*a*: $g=e$. Can't happen with Farthest-In-Future since there must be a request for f before e.

Case 3*b*: $g=f$. Element f can't be in cache of S, so let e' be the element that S evicts.

- if $e'=e$, S' accesses f from cache; now S and S' have same cache
- if $e'\neq e$, S' evicts e' and brings e into the cache; now S and S' have the same cache.

Case 3*c*: $g\neq e,f$. S must evict e (otherwise S' would take the same action). Make S' evict f; now S and S' have the same cache. ☐

Caching Perspective

Online vs. Offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.
LRU. Evict page whose most recent access was earliest[‡].

Theorem. FF is optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8 in Cornell Book]
- LIFO is arbitrarily bad.

[‡] FF with direction of time reversed!

Coin Changing

Goal. Given US currency denominations:

1 (cent), 5 (nickel), 10 (dime), 25 (quarter), 100 (dollar),

devise a changing method using fewest number of coins.

Example. 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Example. $2.89.

Cashier's Algorithm

Algorithm 4: Cashier's Algorithm

```
Sort coins denominations by value: $c_1 < c_2 < \ldots < c_n$;
$S \leftarrow \emptyset$ ;                                    // coins selected
while $x \neq 0$ do
    let $k$ be largest integer such that $c_k < x$;
    if $k = 0$ then
        return "no solution found";
    $x \leftarrow x - c_k$;
    $S \leftarrow S \cup \{k\}$;
return $S$;
```

Question. Is cashier's algorithm optimal?

Properties of Optimal Solution

Property. Number of pennies ≤ 4.
Proof. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.

Property. Number of quarters ≤ 3.

Property.Number of nickels + Number of dimes ≤ 2.
Proof.

- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.
- Recall: at most 1 nickel.

Correctness Proof

Theorem. Greedy algorithm is optimal for U.S. coinage.

Proof. (by induction on x) Consider an optimal way to change $c_k < x < c_{k+1}$: greedy takes coin k. We claim that any optimal solution must also take coin k.

If not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x.

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	P ≤ 4	-
2	5	N ≤ 1	4
3	10	N + D ≤ 2	4 + 5 = 9
4	25	Q ≤ 3	20 + 4 = 24
5	100	no limit	75 + 24 = 99

Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm.

Is Cashier's Algorithm Work for Any Denominations?

Observation 1. Greedy is sub-optimal for US postal denominations:

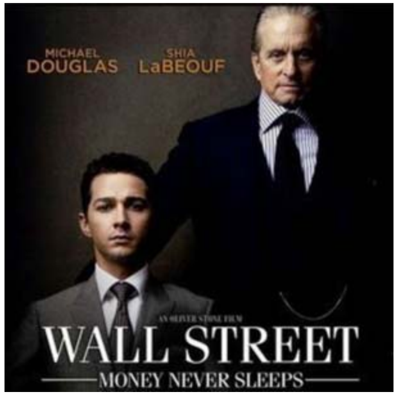
$$1, 10, 21, 34, 70, 100, 350, 1225, 1500.$$

Counter example. 140¢. (Greedy: 100, 34, six 1's; Optimal: 70, 70.)

Observation 2. Even no feasible solution with system ¢={7, 8, 9}.

- Cashier's algorithm: 15¢= 9+???
- Optimal: 15¢= 7 + 8.

Movie: Wall Street (1987)



[§]

Greed is good.

Greed is right.

Greed works.

Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.

— Gordon Gecko
(Michael Douglas)

[§] Watch the movie segment at the class webpage.