# Sums

## ITT9131 Konkreete Matemaatika

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# Next section

# Sequences

## Definition

A sequence of elements of a set $A$ is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n\in\mathbb{N}}$
- $\{a_n\}$

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## Example

$$a_0=0,\ a_1=\frac{1}{2\cdot 3},\ a_2=\frac{2}{3\cdot 4},\ a_3=\frac{3}{4\cdot 5},\cdots$$

or

$$\left\langle 0,\ \frac{1}{6},\ \frac{1}{6},\ \frac{3}{20},\ \frac{2}{15},\cdots,\ \frac{n}{(n+1)(n+2)},\cdots\right\rangle$$

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## Notation

$$f(n) = \frac{n}{(n+1)(n+2)}$$

or

$$a_n = \frac{n}{(n+1)(n+2)}$$

# Sets of indexes

- $\mathbb{N}$ – set of indexes of the sequence $f = \{a_n\}_{n \in \mathbb{N}}$
- Any countably infinite set can be used for index. Examples of other frequently used indexes are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$
  - $\mathbb{N} - K \sim \mathbb{N}$, where $K$ is any finite subset of $\mathbb{N}$
  - $\mathbb{Z} \sim \mathbb{N}$
  - $\{1, 3, 5, 7, \ldots\} = ODD \sim \mathbb{N}$
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$A \sim B$ denotes that sets $A$ and $B$ are of the same cardinality, i.e. $|A| = |B|$.

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Two sets $A$ and $B$ have the same cardinality if there exists a bijection, that is, an injective and surjective function, from $A$ to $B$.

(See

`http://www.mathsisfun.com/sets/injective-surjective-bijective.html`

for detailed explanation)

# Finite sequence

- Finite sequence of elements of a set $A$ is a function $f : K \to A$, where $K$ is set a finite subset of natural numbers

  For example: $f : \{1, 2, 3, 4, \cdots, n\} n \to A$, $n \in \mathbb{N}$

  Special case: $n = 0$, i.e. empty sequence: $f(\emptyset) = e$

# Domain of the sequence

$$f : T \to A$$

$$a_n = \frac{n}{(n-2)(n-5)}$$

Domain of $f$ is $T = \mathbb{N} - \{2, 5\}$

# Next section

# Notation

For a finite set $K = \{1, 2, \cdots, m\}$ and a given sequence $f : K \to \mathbb{R}$ with $f(n) = a_n$ we write

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

Alternative notations

$$\sum_{k=1}^{m} a_k = \sum_{1 \leqslant k \leqslant m} a_k = \sum_{k \in \{1, \cdots, m\}} a_k = \sum_{K} a_k$$

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# Warmup: What does this notation mean?

$$\sum_{k=4}^{0} q_k$$

**Options:**

1. $\sum_{k=4}^{0} q_k = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{k \in \{4,3,2,1,0\}} q_k = \sum_{k=0}^{4} q_k$.
   This seems the sensible thing—**but:**
2. $\sum_{4 \le k \le 0} q_k = 0$ also looks like a feasible interpretation—**but:**
3. If

$$\sum_{k=m}^{n} q_k = \sum_{k \le n} q_k - \sum_{k<m} q_k \,,$$

(provided the two sums on the right-hand side exist finite) then $\sum_{k=4}^{0} q_k = \sum_{k \le 0} q_k - \sum_{k<4} q_k = -q_1 - q_2 - q_3$.

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# Warmup: Interpreting the $\Sigma$-notation

Compute $\sum_{\{0\leq k\leq 5\}} a_k$ and $\sum_{\{0\leq k^2\leq 5\}} a_{k^2}$.

## First sum

$$\{0 \leq k \leq 5\} = \{0, 1, 2, 3, 4, 5\} :$$

thus, $\sum_{\{0\leq k\leq 5\}} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

## Second sum

$$\{0 \leq k^2 \leq 5\} = \{0, 1, 2, -1, -2\} :$$

thus,

$\sum_{\{0\leq k\leq 5\}} a_{k^2} = a_{0^2} + a_{1^2} + a_{2^2} + a_{(-1)^2} + a_{(-2)^2} = a_0 + 2a_1 + 2a_2$.

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# Next section

# Sums and Recurrences

Computation of any sum

$$S_n = \sum_{k=1}^{n} a_k$$

can be presented in the recursive form:

$$S_0 = a_0$$
$$S_n = S_{n-1} + a_n$$

$\Rightarrow$ Techniques from CHAPTER ONE can be used for finding closed formulas for evaluating sums.

# Recalling repertoire method

- Given
$$g(0) = \alpha$$
$$g(n) = \Phi(g(n-1)) + \Psi(\beta, \gamma, \ldots) \qquad \text{for } n > 0.$$

  where $\Phi$ and $\Psi$ are linear, for example if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.

- Closed form is
$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \cdots \tag{1}$$

- Functions $A(n), B(n), C(n), \ldots$ could be found from the system of equations

$$\begin{aligned}
\alpha_1 A(n) + \beta_1 B(n) + \gamma_1 C(n) + \cdots &= g_1(n) \\
\vdots \qquad\qquad &= \vdots \\
\alpha_m A(n) + \beta_m B(n) + \gamma_m C(n) + \cdots &= g_m(n)
\end{aligned}$$

  where $\alpha_i, \beta_i, \gamma_i \cdots$ are constants committing (1) and recurrence relationship for the repertoire case $g_i(n)$ and any $n$.

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# Next subsection

# Example 1: arithmetic sequence

**Arithmetic sequence: $a_n = a + bn$**

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$:

$$S_0 = a$$
$$S_n = S_{n-1} + (a + bn)\ , \text{ for } n > 0.$$

Let's find a closed form for a bit more general recurrent equation:

$$R_0 = \alpha$$
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# Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

$$
\begin{aligned}
R_0 &= \alpha \\
R_1 &= \alpha + \beta + \gamma \\
R_2 &= \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma \\
R_3 &= \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma
\end{aligned}
$$

## Observation

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$A(n), B(n), C(n)$ can be evaluated using repertoire method:
we will consider three cases

1. $R_n = 1$ for all $n$
2. $R_n = n$ for all $n$
3. $R_n = n^2$ for all $n$

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# Repertoire method: case 1

**Lemma 1: $A(n) = 1$ for all $n$**

- $1 = R_0 = \alpha$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $1 = 1 + (\beta + \gamma n)$. This is possible only when $\beta = \gamma = 0$

Hence

$$1 = A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 0$$

$\square$

# Repertoire method: case 2

## Lemma 2: $B(n) = n$ for all $n$

- $\alpha = R_0 = 0$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $n = (n-1) + (\beta + \gamma n)$.
  I.e. $1 = \beta + \gamma n$.
  This gives that $\beta = 1$ and $\gamma = 0$

Hence

$$n = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0$$

$\square$

# Repertoire method: case 3

**Lemma 3:** $C(n) = \frac{n^2+n}{2}$ for all $n$

- $\alpha = R_0 = 0^2 = 0$
- Equation $R_n = R_{n-1} + (\beta + \gamma n)$ can be transformed as
  $n^2 = (n-1)^2 + \beta + \gamma n$
  $n^2 = n^2 - 2n + 1 + \beta + \gamma n$
  $0 = (1+\beta) + n(\gamma - 2)$
  This is valid iff $1 + \beta = 0$ and $\gamma - 2 = 0$

Hence

$$n^2 = A(n) \cdot 0 + B(n) \cdot (-1) + C(n)\gamma \cdot 2$$

Due to Lemma 2 we get

$$n^2 = -n + 2C(n)$$

□

# Repertoire method: summing up

According to Lemma 1, 2, 3, we get

1. $R_n = 1$ for all $n$ $\Longrightarrow$ $A(n) = 1$
2. $R_n = n$ for all $n$ $\Longrightarrow$ $B(n) = n$
3. $R_n = n^2$ for all $n$ $\Longrightarrow$ $C(n) = (n^2 + n)/2$

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That means that

$$R_n = \alpha + n\beta + \left(\frac{n^2 + n}{2}\right)\gamma$$

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The sum for arithmetic sequence we obtain taking $\alpha = \beta = a$ and $\gamma = b$:

$$S_n = \sum_{k=0}^{n} (a + bk) = (n+1)a + \frac{n(n+1)}{2} b$$

# Next subsection

# Perturbation method

**Finding the closed form for $S_n = \sum_{0 \leqslant k \leqslant n} a_k$:**

- Rewrite $S_{n+1}$ by splitting off first and last term:

$$S_n + a_{n+1} = a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k =$$

$$= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} =$$

$$= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

- Work on last sum and express in terms of $S_n$.
- Finally, solve for $S_n$.

# Example 2: geometric sequence

**Geometric sequence: $a_n = ax^n$**

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

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- Splitting off the first term gives

$$S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} =$$
$$= a + \sum_{0 \leqslant k \leqslant n} ax^{k+1} =$$
$$= a + x \sum_{0 \leqslant k \leqslant n} ax^k =$$
$$= a + xS_n$$

- Hence, we have the equation

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- Solution:

$$S_n = \frac{a - ax^{n+1}}{1 - x}$$

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Closed formula for geometric sum:

$$S_n = \frac{a(x^{n+1} - 1)}{x - 1}$$

# Next subsection

# Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

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T_0 &= 0 \\
T_n &= 2T_{n-1} + 1
\end{aligned}
$$

This sequence can be transformed into geometric sum using following manipulations:

- Divide equations by $2^n$:

$$
\begin{aligned}
T_0/2^0 &= 0 \\
T_n/2^n &= T_{n-1}/2^{n-1} + 1/2^n
\end{aligned}
$$

- Set $S_n = T_n/2^n$ to have:

$$
\begin{aligned}
S_0 &= 0 \\
S_n &= S_{n-1} + 2^{-n}
\end{aligned}
$$

(This is geometric sum with the parameters $a = 1$ and $x = 1/2$.)

# Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
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The Tower of Hanoi recurrence:

$$\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \frac{0.5(0.5^n - 1)}{0.5 - 1} \\ &= 1 - 2^{-n} \end{aligned} \qquad (a_0 = 0 \text{ has been left out of the sum})$$

$$T_n = 2^n S_n = 2^n - 1$$

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$$T_n = 2^n S_n = 2^n - 1$$

Just the same result we have proven by means of induction! :))

# Next subsection

# Linear recurrence in form $a_n T_n = b_n T_{n-1} + c_n$

Here $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are any sequences and initial value $T_0$ is a constant.

**The idea:**

Find a summation factor $s_n$ satisfying the property

$$s_n b_n = s_{n-1} a_{n-1} \qquad \text{for any } n$$

**If such a factor exists, one can do following transformations:**

- $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$
- Setting $S_n = s_n a_n T_n$, to rewrite the equation as

$$S_0 = s_0 a_0 T_0$$
$$S_n = S_{n-1} + s_n c_n$$

- Closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n}\left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k\right) = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

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$$T_n = \frac{1}{s_n a_n}(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k) = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$$

# Finding summation factor

Assuming that $b_n \neq 0$ for all $n$:

- Set $s_0 = 1$
- Compute next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$
\begin{aligned}
s_1 &= \frac{a_0}{b_1} \\
s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\
s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\
&\;\dots\dots\dots \\
s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n}
\end{aligned}
$$

(To be proved by induction!)

# Example: application of summation factor

**$a_n = c_n = 1$ and $b_n = 2$ gives Hanoi Tower sequence:**

- Evaluate summation factor

$$s_n = \frac{s_{n-1}a_{n-1}}{b_n} = \frac{a_0 a_1 \ldots a_{n-1}}{b_1 b_2 \ldots b_n} = \frac{1}{2^n}$$

- Solution is

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n(1 - 2^{-n}) = 2^n - 1$$

# YAE: constant coefficients

## Equation $Z_n = aZ_{n-1} + b$

### Taking $a_n = 1, b_n = a$ and $c_n = b$ :

- Evaluate summation factor

$$s_n = \frac{s_{n-1}a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n} = \frac{1}{a^n}$$

- Solution is

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1 b_1 Z_0 + \sum_{k=1}^{n} s_k c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b(1 + a + a^2 + \cdots + a^{n-1}) \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1} b \end{aligned}$$

# YAE : check up on results

**Equation $Z_n = aZ_{n-1} + b$**

$$
\begin{aligned}
Z_n &= aZ_{n-1} + b = \\
&= a^2 Z_{n-2} + ab + b = \\
&= a^3 Z_{n-3} + a^2 b + ab + b = \\
&\ldots\ldots \\
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b = \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a-1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

Continuing until $k = n$:

$$
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Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a-1} b = \\
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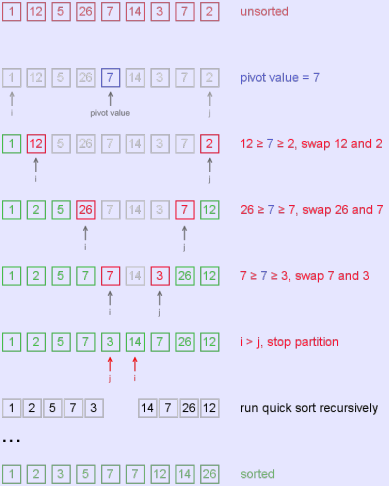
# Efficiency of quick sort

Average number of comparisons: $C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$



# Efficiency of quick sort (2)

**The average number of comparison steps when it is applied to $n$ items**

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k$$

**The following transformations reduce this equation**

$$nC_n = n^2 + n + 2 \sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{k=0}^{n-2} C_k$$

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$$nC_n - (n-1)C_{n-1} = n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1)$$

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$$nC_n - nC_{n-1} + C_{n-1} = n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1$$

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## Equation $nC_n = (n+1)C_{n-1} + 2n$

- Assuming $a_n = n, b_n = n+1$ and $c_n = 2n$ evaluate summation factor

$$s_n = \frac{a_1 a_2 \ldots a_{n-1}}{b_2 b_3 \ldots b_n} = \frac{1 \cdot 2 \cdot \ldots \cdot (n-1)}{3 \cdot 4 \cdot \ldots \cdot (n+1)} = \frac{2}{n(n+1)}$$

- Solution is

$$\begin{aligned} C_n &= \frac{1}{s_n a_n} \left( s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k \right) \\ &= \frac{n+1}{2} \sum_{k=1}^{n} \frac{4k}{k(k+1)} \\ &= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} = 2(n+1) \left( \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} - 1 \right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is $n$-th harmonic number.

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($k$-th harmonic produced by a violin string is the fundamental tone produced by a string that is $1/k$ times as long.)

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where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$.

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# Next section

# Manipulation of Sums

## Some properties of sums:

For $K$ being a finite set and $p(k)$ is any permutation of the set of all integers.

Distributive law

$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$$

Associative law

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$

Commutative law

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$$

## Application of these laws for $S = \sum_{0 \leqslant k \leqslant n} (a + bk)$

$$S = \sum_{0 \leqslant n-k \leqslant n} (a + b(n-k)) = \sum_{0 \leqslant k \leqslant n} (a + bn - bk) \qquad \text{(commutativity)}$$

$$2S = \sum_{0 \leqslant k \leqslant n} ((a + bk) + (a + bn - bk)) = \sum_{0 \leqslant k \leqslant n} (2a + bn) \qquad \text{(associativity)}$$

$$2S = (2a + bn) \sum_{0 \leqslant k \leqslant n} 1 = (2a + bn)(n + 1) \qquad \text{(distributivity)}$$

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# Yet another useful equality

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cup K'} a_k + \sum_{k\in K\cap K'} a_k$$

## Special cases:

a) for $1 \leqslant m \leqslant n$

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$$

b) for $n \geqslant 0$

$$\sum_{0\leqslant k\leqslant n} a_k = a_0 + \sum_{1\leqslant k\leqslant n} a_k$$

c) for $n \geqslant 0$

$$S_n + a_{n+1} = a_0 + \sum_{0\leqslant k\leqslant n} a_{k+1}$$

# Example: $S_n = \sum_{k=0}^{n} kx^k$

- For $x \neq 1$:

$$S_n + (n+1)x^{n+1} = \sum_{0 \leqslant k \leqslant n} (k+1)x^{k+1}$$

$$= \sum_{0 \leqslant k \leqslant n} kx^{k+1} + \sum_{0 \leqslant k \leqslant n} x^{k+1}$$

$$= xS_n + \frac{x(1 - x^{n+1})}{1 - x}$$

- $$\sum_{k=0}^{n} kx^k = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(x-1)^2}$$