Binomial Coefficients and Generating Functions

ITT9131 Konkreetne Matemaatika

Contents

Next section

Binomial coefficients

Definition

Let r be a real number and k an integer. The binomial coefficient "r choose k" is the real number

$$\binom{r}{k} = \begin{cases} \dfrac{r\cdot(r-1)\cdots(r-k+1)}{k!} = \dfrac{r^{\underline{k}}}{k!} & \text{if } k \geqslant 0\,, \\ 0 & \text{if } k < 0\,. \end{cases}$$

If $r = n$ is a natural number

In this case,

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$$

is the number of ways we can choose k elements from a set of n elements, in any order.

Consistently with this interpretation,

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leqslant k \leqslant n\,, \\ 0 & \text{if } k > n\,. \end{cases}$$

Binomial theorem

Theorem 1

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for any integer $n \geqslant 0$.

Proof. Expanding $(a+b)^n = (a+b)(a+b)\cdots(a+b)$ yields the sum of the 2^n products of the form $e_1 e_2 \cdots e_n$, where each e_i is a or b. These terms are composed by selecting from each factor $(a+b)$ either a or b. For example, if we select a k times, then we must choose b $n-k$ times. So, we can rearrange the sum as

$$(a+b)^n = \sum_{k=0}^{n} C_k a^k b^{n-k},$$

where the coefficient C_k is the number of ways to select k elements (k factors $(a+b)$) from a set of n elements (from the production of n factors $(a+b)(a+b)\cdots(a+b)$).
That is why the coefficient C_k is called "(from) n choose k" and denoted by $\binom{n}{k}$. Q.E.D.

Binomial coefficients and combinations

Theorem 2

The number of k-subsets of an n-set is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. At first, determine the number of k-element sequences: there are n choices for the first element of the sequence; for each, there are $n-1$ choices for the second; and so on, until there are $n-k+1$ choices for the k-th. This gives $n(n-1)...(n-k+1) = n^{\underline{k}}$ choices in all. And since each k-element subset has exactly $k!$ different orderings, this number of sequences counts each subset exactly $k!$ times. To get our answer, divide by $k!$:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

Q.E.D.

Some other notations used for the "n choose k" in literature: $C_k^n, C(n,k), {}_nC_k, {}^nC_k$.

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Properties of Binomial Coefficients

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$: A set of n elements has 2^n subsets.
2. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0]$: In a nonempty set, the number of subsets with odd cardinality is equal to the number of sets with even cardinality.

Proof.

- Take $a = b = 1$ in the binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

- Take $a = -1$ and $b = 1$:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = (-1+1)^n = 0$$

Q.E.D.

Another Property For $n \geqslant 0$ Integer

Symmetry of binomial coefficients

3 $\binom{n}{k} = \binom{n}{n-k}$ for every $n \geqslant 0$.

Proof. For $0 \leqslant k \leqslant n$ direct conclusion from Theorem 2:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k};$$

otherwise, both sides vanish. Q.E.D.

Another Property For $n \geqslant 0$ Integer

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Only if n is nonnegative!

For $n = -1$ and $k \geqslant 0$,

$$\binom{-1}{k} = \frac{(-1)^{\underline{k}}}{k!} = (-1)^k \text{ but } \binom{-1}{-1-k} = 0;$$

while for $k < 0$,

$$\binom{-1}{k} = 0 \text{ but } \binom{-1}{-1-k} = (-1)^{|k|-1}.$$

Yet Another Property

4 If $n, k > 0$, then $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

Proof.

- Note that

$$\frac{1}{n-k} + \frac{1}{k} = \frac{k+n-k}{k(n-k)} = \frac{n}{k(n-k)}.$$

- Multiplying this by $(n-1)!$ and dividing by $(k-1)!(n-k-1)!$, we get

$$\frac{(n-1)!}{(k-1)!(n-k)(n-k-1)!} + \frac{(n-1)!}{k(k-1)!(n-k-1)!} = \frac{n(n-1)!}{k(k-1)!(n-k)(n-k-1)!}$$

- This can be rewritten after some simplifying transformations as:

$$\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} = \frac{n!}{k!(n-k)!}$$

Q.E.D.

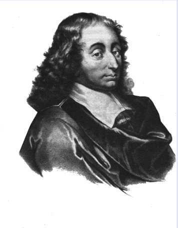
Pascal's Triangle

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Blaise Pascal
(1623–1662)

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$



Blaise Pascal
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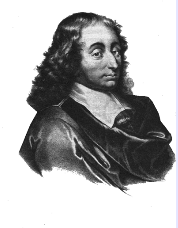
- Pascal Triangle is symmetric with respect to the vertical line through its apex.
- Every number is the sum of the two numbers immediately above it.

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1



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Warmup: The hexagon property

Statement

For every $n \geqslant 2$ and $0 < k < n$,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

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Interpretation

- Looking at Pascal's triangle in the previous slide, the six numbers in the expression above form a "hexagon" around $\binom{n}{k}$.
- Then the hexagon property says that the product of the odd-numbered corners of the hexagon equals that of the even-numbered corners.

Warmup: The hexagon property

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For every $n \geqslant 2$ and $0 < k < n$,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

Proof

Consider the expression of the binomial coefficients as a ratio of product of primes.
At the numerator, both sides contribute with $(n-1)! \cdot n! \cdot (n+1)!$
At the denominator:

- The left hand side contributes with:

$$(k-1)! \cdot (n-k)! \cdot (k+1)! \cdot (n-k-1)! \cdot k! \cdot (n+1-k)!$$

- The right-hand side contributes with:

$$k! \cdot (n-1-k)! \cdot (k-1)! \cdot (n-k+1)! \cdot (k+1)! \cdot (n-k)!$$

The contributions of the two sides are thus equal, and the thesis follows.

Special values

- $\binom{r}{0} = 1$ for every r real.
- $\binom{r}{1} = r$ for every r real.
- $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$ for every r real and $k \neq 0$ integer.
- $k\binom{r}{k} = r\binom{r-1}{k-1}$ for every r real and k integer (also 0).
- $\binom{n}{n} = [n \geqslant 0]$ for every n integer.
- $\binom{0}{k} = [k = 0]$ for every k integer.

The polynomial argument

Theorem

For every r real and k integer,

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

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$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}$$

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Wait! There's a problem:

We can have r appear in the lower index only if it is an integer!

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$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}$$

An issue which is only apparent

- $(r-k)\binom{r}{k}$ and $r\binom{r-1}{k}$ are polynomials in r of degree $k+1$.
- These two polynomials take equal values on each $r \geqslant 0$ integer.
- But two distinct polynomials of degree d can have at most d roots in common!

The polynomial argument

Theorem

For every r real and k integer,

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

Proof

$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}$$

Another use: The addition formula

As $\binom{r}{k}$ and $\binom{r-1}{k} + \binom{r-1}{k-1}$ are polynomials in r of degree k that take the same values in the $k+1$ points $r = 0, 1, \ldots, n$, by the polynomial argument we can conclude that:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \quad \forall r \in \mathbb{R} \;\; \forall k \in \mathbb{Z}.$$

Next section

Power series of functions

Example: Functions expanded as power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Power series of functions (2)

Power series of a function

- A power series of the function f is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

  where $c, a_0, a_1, \ldots$ are constants. (Taylor series)
- Special case $c = 0$ provides a Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

- Coefficients are defined as

$$a_n = \frac{f^{(n)}(c)}{n!}$$

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Power series of functions (3)

Example: Generating functions

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \qquad \langle 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \cdots \rangle$$

$$\sin(x) = x - \frac{x^3}{3} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \qquad \langle 0, 1, -\frac{1}{3}, 0, \frac{1}{120}, 0, -\frac{1}{5040}, \cdots \rangle$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \langle 1, 1, 1, 1, \cdots \rangle$$

Generating functions ...

... of the sequences

Generating Functions

Definition

The generating function for the infinite sequence $\langle g_0, g_1, g_2, \ldots \rangle$ is the power series

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots = \sum_{n=0}^{\infty} g_n x^n$$

Some simple examples

$$\langle 0,0,0,0,\ldots\rangle \quad \longleftrightarrow \quad 0 + 0x + 0x^2 + 0x^3 + \cdots = 0$$
$$\langle 1,0,0,0,\ldots\rangle \quad \longleftrightarrow \quad 1 + 0x + 0x^2 + 0x^3 + \cdots = 1$$
$$\langle 2,3,1,0,\ldots\rangle \quad \longleftrightarrow \quad 2 + 3x + 1x^2 + 0x^3 + \cdots = 2 + 3x + 1x^2$$

More examples (1)

$\langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots \\ xS &= \phantom{1 +{}} x + x^2 + x^3 + \cdots \end{aligned}$$

Subtract the equations:

$$(1-x)S = 1 \qquad \text{ehk} \qquad S = \frac{1}{1-x}$$

NB! This formula converges only for $-1 < x < 1$.

Actually, we don't worry about convergence issues.

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More examples (2)

$\langle a, ab, ab^2, ab^3, \dots \rangle \longleftrightarrow a + abx + ab^2x^2 + ab^3x^3 + \cdots = \frac{a}{1-bx}$

Like in the previous example:

$$\begin{aligned} S &= a + abx + ab^2x^2 + ab^3x^3 + \cdots \\ bxS &= \phantom{a + {}} abx + ab^2x^2 + ab^3x^3 + \cdots \end{aligned}$$

Subtract and get:

$$(1 - bx)S = a \qquad \text{ehk} \qquad S = \frac{a}{1 - bx}$$

More examples (3)

Taking in the last example $a = 0,5$ and $b = 1$ yields

$$0,5 + 0,5x + 0,5x^2 + 0,5x^3 + \cdots = \frac{0,5}{1-x} \tag{1}$$

Taking $a = 0,5$ and $b = -1$, gives

$$0,5 - 0,5x + 0,5x^2 - 0,5x^3 + \cdots = \frac{0,5}{1+x} \tag{2}$$

Adding equations (1) and (2), we get the generating function of the sequence $\langle 1, 0, 1, 0, 1, 0, \ldots \rangle$:

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{0,5}{1-x} + \frac{0,5}{1+x} = \frac{1}{(1-x)(1+x)} = \frac{1}{1-x^2}$$

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Next subsection

The complex derivative

Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, and z an internal point of A.
The complex derivative of f in z is (if exists) the quantity

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If $f'(z)$ exists, then:

- For $\Delta z = \Delta x$,

$$\frac{\partial f}{\partial x}(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = f'(z)$$

- For $\Delta z = i\Delta y$,

$$\frac{\partial f}{\partial y}(z) = i \lim_{\Delta x \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = if'(z)$$

As $i \cdot i = -1$, we get the Cauchy-Riemann condition

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

If A is open and f has complex derivative in every point of A, e say that f is holomophic in A.

Convergence of sequences of functions

Pointwise convergence

Let $f_n : A \to \mathbb{C}$ be functions. The (pointwise) limit of the sequence $\{f_n\}_{n\geqslant 0}$ is the function defined by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for every $z \in A$ where the limit exists.

For power series: $\sum_{n\geqslant 0} a_n z^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$.

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Uniform convergence

The sequence of functions $\{f_n\}_{n\geqslant 0}$ of functions converges uniformly to f in A if:

$$\forall \varepsilon > 0\ \exists n_\varepsilon \geqslant 0 \text{ such that } \forall n > n_\varepsilon\ \forall z \in A \,.\, |f_n(z) - f(z)| < \varepsilon :$$

that is, if pointwise convergence is independent of the point.

- The sequence $f_n(x) = e^{-x^2}[|x| \leqslant n]$ converges to $f(x) = e^{-x^2}$ uniformly in $\mathbb{R}$.
- The sequence $f_n(x) = [x > n]$ converges to zero in $\mathbb{R}$, but not uniformly.

Consequences of uniform convergence

Continuity of the limit

Uniform limit of continuous functions is continuous.

Not true for simply pointwise convergence:

$$\text{if } f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leqslant x \leqslant 1/n\,, \\ 0 & \text{if } 1/n \leqslant x \leqslant 1\,, \end{cases} \quad \text{then } \lim_{n\to\infty} f_n(x) = [x = 0]\,.$$

Swap limits

If f_n converges uniformly in A, then:

$$\lim_{z\to z_0} \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \lim_{z\to z_0} f_n(z)\ \forall z_0 \in A$$

Swap limit with differentiation

If $f_n \to f$ uniformly in A, all the f_n are differentiable, and f_n' converges uniformly, then f is differentiable and:

$$f'(z) = \lim_{n\to\infty} f_n'(z)$$

The convergence radius of a power series

Definition

The convergence radius of the power series

$$S(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$$

is:

$$R = \frac{1}{\limsup_{n \geqslant 0} \sqrt[n]{|a_n|}},$$

with the conventions $1/0 = \infty$, $1/\infty = 0$.

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Examples

- For $\alpha \in \mathbb{C}$, $\sum_{n \geqslant 0} \alpha^n z^n$ has convergence radius $1/|\alpha|$.
- $\sum_{n \geqslant 1} \dfrac{z^n}{n}$ has convergence radius 1.
- $\sum_{n \geqslant 0} \dfrac{z^n}{n!}$ has infinite convergence radius.

The Abel-Hadamard theorem

Statement

Let $S(z)$ be a power series of center z_0 and convergence radius R.

1. If $R > 0$, then $S(z)$ converges uniformly on every closed and bounded subset of the open disk $D_R(z_0)$ of center z_0 and radius R.
2. If $R < \infty$, then $S(z)$ does not converge at any point z such that $|z - z_0| > R$.

Examples

- $\sum_{n\geqslant 0} \dfrac{(-1)^n}{2^n(n+1)} z^n$ converges uniformly in $\{|z| \leqslant 1\}$.
- $\sum_{n\geqslant 0} \dfrac{(2i)^n}{n+1}$ does not exist.

Consequence for generating functions

If $\limsup_{n\geqslant 0} \sqrt[n]{|g_n|} < \infty$,
then the generating function of g_n is well defined in a neighborhood of 0.

The Abel-Hadamard theorem

Statement

Let $S(z)$ be a power series of center z_0 and convergence radius R.

1. If $R > 0$, then $S(z)$ converges uniformly on every closed and bounded subset of the open disk $D_R(z_0)$ of center z_0 and radius R.
2. If $R < \infty$, then $S(z)$ does not converge at any point z such that $|z - z_0| > R$.

Examples

- $\sum_{n\geqslant 0} \frac{(-1)^n}{2^n(n+1)} z^n$ converges uniformly in $\{|z| \leqslant 1\}$.
- $\sum_{n\geqslant 0} \frac{(2i)^n}{n+1}$ does not exist.

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Exploiting power series

Let $S(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$ for $|z - z_0| < r$.

1. For any such z we can approximate $S(z)$ with its partial sum

$$S_N(z) = \sum_{0 \leqslant n \leqslant N} a_n (z - z_0)^n$$

2. The quantity $|S(z) - S_N(z)|$ can be made arbitrarily small by setting N large enough.
3. The choice of n can be made good for every z such that $|z - z_0| \leqslant \rho < r$.
4. Arithmetic operations are sufficient to compute $S_N(z)$.

Power series are holomorphic functions

- Let $S(z) = \sum_{n\geqslant 0} a_n(z-z_0)^n$ and let $R > 0$ be its convergence radius.
- The function

$$T(z) = \sum_{n\geqslant 0} \frac{d}{dz}\left(a_n(z-z_0)^n\right) = \sum_{n\geqslant 1} na_n(z-z_0)^{n-1} = \sum_{n\geqslant 0}(n+1)a_{n+1}(z-z_0)^n$$

  is still a power series.
- But

$$\limsup_{n\geqslant 0} \sqrt[n]{|(n+1)a_{n+1}|} = \limsup_{n\geqslant 0} \sqrt[n]{|a_n|} :$$

  so $T(z)$ also has convergence radius R.
- By the Abel-Hadamard theorem, for every $z \in D_R(z_0)$,

$$S'(z) = \sum_{n\geqslant 0}(n+1)a_{n+1}(z-z_0)^n = T(z)$$

Holomorphic functions are power series *locally*

Laurent's theorem

Let f be holomorphic in a disk

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

Then there exist a sequence $\{a_n\}_{n \geqslant 0}$ of complex numbers such that:

1. The power series $S(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$ has convergence radius $R \geqslant r$.
2. For every $z \in D_r(z_0)$ we have $S(z) = f(z)$.

A function which is "locally a power series" at each point is called analytic.
For complex functions of a complex variable, analyticity is the same as holomorphy.

Holomorphic functions are power series *locally*

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For complex functions of a complex variable, analyticity is the same as holomorphy.

Counterexample in real analysis

Let $f(x) = e^{-1/x^2}$ for $x \neq 0$, $f(0) = 0$.

- Then f is infinitely differentiable in $\mathbb{R}$ ...
- ... but the Taylor series in $x = 0$ vanishes!

Holomorphic functions are power series *locally*

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A function which is "locally a power series" at each point is called analytic.
For complex functions of a complex variable, analyticity is the same as holomorphy.

Consequence for generating functions

Every function that is analytic in a neighborhood of the origin
is the generating function of some sequence.

The identity principle for analytic functions

Statement

- Let A be a connected open subset of the complex plane.
- Let $f : A \to \mathbb{C}$ be an analytic function.
- Suppose f is not identically zero in A.
- Then all the zeroes of f in A are isolated:
  If $z_0 \in A$ and $f(z_0) = 0$, then there exists $r > 0$ such that $f(z) \neq 0$ for every z such that $0 < |z - z_0| < r$.

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Corollary: Uniqueness of analytic continuation

Let:

- I a nonempty interval of the real line;
- A a connected open subset of the complex plane such that $I \subseteq A$; and
- $f : I \to \mathbb{R}$ a continuous function.

Then there exists at most one function analytic in A which coincides with f on I.

The identity principle for analytic functions

Statement

- Let A be a connected open subset of the complex plane.
- Let $f : A \to \mathbb{C}$ be an analytic function.
- Suppose f is not identically zero in A.
- Then all the zeroes of f in A are isolated:
  If $z_0 \in A$ and $f(z_0) = 0$, then there exists $r > 0$ such that $f(z) \neq 0$ for every z such that $0 < |z - z_0| < r$.

Consequence for generating functions

Every sequence $\{g_n\}_{n \geqslant 0}$ of complex numbers such that $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$ is uniquely determined by its generating function.

$1+2+3+4+\ldots = -1/12$!??

The series

$$\sum_{n\geqslant 1} n^{-s}$$

converges for every real value $s > 1$: for example, for $s = 2$,

$$\sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The Riemann zeta function is the unique analytic function $\zeta(s)$, defined for $s \in \mathbb{C} \setminus \{1\}$, such that $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ for every real $s > 1$.

- It happens that $\zeta(-1) = -1/12$.
- This does not mean that $\sum_{n\geqslant 1} n = -1/12$!
- Instead, it means that the formula $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ can be assumed valid only when s is real and greater than 1.

Basic generating functions

$G(z)$	z	$\langle g_0, g_1, g_2, g_3, \dots \rangle$	g_n
$z^m,\ m \in \mathbb{N}$	$z \in \mathbb{C}$	$\langle 0, \dots, 0, 1, 0, \dots, \rangle$	$[n = m]$
e^z	$z \in \mathbb{C}$	$\langle 1, 1, \frac{1}{2}, \frac{1}{6}, \dots \rangle$	$\dfrac{1}{n!}$
$\cos z$	$z \in \mathbb{C}$	$\langle 1, 0, -\frac{1}{2}, 0, \dots \rangle$	$\dfrac{(-1)^{\lfloor n/2 \rfloor}}{n!} \cdot [n \mod 2 = 0]$
$\sin z$	$z \in \mathbb{C}$	$\langle 0, 1, 0, -\frac{1}{6}, \dots \rangle$	$\dfrac{(-1)^{\lfloor n/2 \rfloor}}{n!} \cdot [n \mod 2 = 1]$
$(1+z)^\alpha$	$\lvert z \rvert < 1$	$\langle 1, \alpha, \frac{\alpha(\alpha-1)}{2}, \frac{\alpha^{\underline{3}}}{6}, \dots \rangle$	$\binom{\alpha}{n} = \dfrac{\alpha^{\underline{n}}}{n!}$
$\dfrac{1}{1-\alpha z}$	$\lvert z \rvert < 1/\lvert \alpha \rvert$	$\langle 1, \alpha, \alpha^2, \alpha^3, \dots \rangle$	α^n
$\ln \dfrac{1}{1-z}$	$\lvert z \rvert < 1$	$\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$	$\dfrac{1}{n} \cdot [n > 0]$, conv. $\dfrac{1}{0} \cdot 0 = 0$
$\ln(1+z)$	$\lvert z \rvert < 1$	$\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, \dots \rangle$	$\dfrac{(-1)^{n-1}}{n} \cdot [n > 0]$, conv. $\dfrac{1}{0} \cdot 0 = 0$

Analytic functions and generating functions: A summary

1. Every function that is analytic in a neighborhood of the origin of the complex plane is the generating function of some sequence.
   Reason why: Laurent's theorem.
2. Every sequence $\{g_n\}_{n \geqslant 0}$ of complex numbers such that
   $$\limsup_n \sqrt[n]{|g_n|} < \infty$$
   admits a generating function.
   Reason why: The Abel-Hadamard theorem.
3. Every such sequence is uniquely determined by its generating function.
   Reason why: Uniqueness of analytic continuation.

We can thus use all the standard operations on sequences and their generating functions, without caring about definition, convergence, etc., provided we do so under the tacit assumption that we are in a "small enough" circle centered in the origin of the complex plane.

Next subsection

Operations on Generating Functions

1. Scaling

If

$$\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x),$$

then

$$\langle cf_0, cf_1, cf_2, \ldots \rangle \longleftrightarrow c \cdot F(x),$$

for any $c \in \mathbb{R}$.

Proof.

$$\begin{aligned} \langle cf_0, cf_1, cf_2, \ldots \rangle &\longleftrightarrow & cf_0 + cf_1 x + cf_2 x^2 + \cdots = \\ &= & c(f_0 + f_1 x + f_2 x^2 + \cdots) = \\ &= & cF(x) \end{aligned}$$

Q.E.D.

Operations on Generating Functions (2)

2. Addition

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$ and $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(x)$, then

$$\langle f_0 + g_0, f_1 + g_1, f_2 + g_2, \ldots \rangle \longleftrightarrow F(x) + G(x).$$

Proof.

$$\langle f_0 + g_0,\ f_1 + g_1,\ f_2 + g_2,\ \ldots \rangle \longleftrightarrow \sum_{n=0}^{\infty} (f_n + g_n) x^n =$$

$$= \left(\sum_{n=0}^{\infty} f_n x^n \right) + \left(\sum_{n=0}^{\infty} g_n x^n \right) =$$

$$= F(x) + G(x)$$

Q.E.D.

Operations on Generating Functions (3)

3. Right-shift

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, then

$$\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeros}}, f_0, f_1, f_2, \ldots \rangle \longleftrightarrow x^k \cdot F(x).$$

Proof.

$$\begin{aligned} \langle 0, 0, \ldots, 0, f_0, f_1, f_2, \ldots \rangle \longleftrightarrow & f_0 x^k + f_1 x^{k+1} + f_2 x^{k+2} + \ldots = \\ & = x^k (f_0 + f_1 x + f_2 x^2 + \cdots) = \\ & = x^k \cdot F(x) \end{aligned}$$

Q.E.D.

Operations on Generating Functions (4)

4. Differentiation

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, then

$$\langle f_1, 2f_2, 3f_3, \ldots \rangle \longleftrightarrow F'(x).$$

Proof.

$$\begin{aligned}
\langle f_1, 2f_2, 3f_3, \ldots \rangle \longleftrightarrow f_1 + 2f_2x + 3f_3x^2 + \ldots &= \\
&= \frac{d}{dx}(f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots) = \\
&= \frac{d}{dx}F(x)
\end{aligned}$$

Q.E.D.

Operations on Generating Functions (4)

4. Differentiation

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, then

$$\langle f_1, 2f_2, 3f_3, \ldots \rangle \longleftrightarrow F'(x).$$

Example

- $\langle 1, 1, 1, \ldots \rangle \longleftrightarrow \frac{1}{1-x}$
- $\langle 1, 2, 3, \ldots \rangle \longleftrightarrow \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$

Operations on Generating Functions (5)

5. Integration

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, then

$$\langle 0, f_0, \frac{1}{2} f_1, \frac{1}{3} f_2, \frac{1}{4} f_3, \ldots \rangle \longleftrightarrow \int_0^x F(z)\mathrm{d}z.$$

Proof.

$$\begin{aligned}
\langle 0, f_0, \frac{1}{2} f_1, \frac{1}{3} f_2, \frac{1}{4} f_3, \ldots \rangle \longleftrightarrow f_0 x + \frac{1}{2} f_1 x^2 + \frac{1}{3} f_2 x^3 + \frac{1}{4} f_3 x^4 + \ldots &= \\
= \int_0^x (f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots)\mathrm{d}z &= \\
= \int_0^x F(z)\mathrm{d}z &
\end{aligned}$$

Q.E.D.

Operations on Generating Functions (5)

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If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$, then

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Example

- $\langle 1, 1, 1, 1 \ldots \rangle \longleftrightarrow \frac{1}{1-x}$
- $\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle \longleftrightarrow \int_0^x \frac{1}{1-z} \mathrm{d}z = -\ln(1-x) = \ln \frac{1}{(1-x)}$

Operations on Generating Functions (6)

6. Convolution (product)

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(z)$, $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, and

$$h_n = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0 = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=k} a_i b_j$$

then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Operations on Generating Functions (6)

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then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Proof.

$$\begin{aligned} F(x) \cdot G(x) &= (f_0 + f_1 x + f_2 x^2 + \ldots)(g_0 + g_1 x + g_2 x^2 + \ldots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0) x + (f_0 g_2 + f_1 g_1 + f_2 g_0) x^2 + \ldots \end{aligned}$$

Q.E.D.

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Proof.

$$\begin{aligned} F(x) \cdot G(x) &= (f_0 + f_1 x + f_2 x^2 + \ldots)(g_0 + g_1 x + g_2 x^2 + \ldots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0)x + (f_0 g_2 + f_1 g_1 + f_2 g_0)x^2 + \ldots \end{aligned}$$

Q.E.D. Notice that all terms involving the same power of x lie on a /sloped diagonal:

	$g_0 x^0$	$g_1 x^1$	$g_2 x^2$	$g_3 x^3$	$\ldots$
$f_0 x^0$	$f_0 g_0 x^0$	$f_0 g_1 x^1$	$f_0 g_2 x^2$	$f_0 g_3 x^3$	$\ldots$
$f_1 x^1$	$f_1 g_0 x^1$	$f_1 g_1 x^2$	$f_1 g_2 x^3$	$\ldots$	
$f_2 x^2$	$f_2 g_0 x^2$	$f_2 g_1 x^3$	$\ldots$		
$f_3 x^3$	$f_3 g_0 x^3$	$\ldots$			
$\vdots$	$\ldots$				

Operations on Generating Functions (6)

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then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Example

$$\begin{aligned}
\langle 1, 1, 1, 1 \ldots \rangle \cdot \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle &= \langle 1 \cdot 0,\ 1 \cdot 0 + 1 \cdot 1,\ 1 \cdot 0 + 1 \cdot 1 + 1 \cdot \frac{1}{2},\ \cdots \rangle \\
&= \langle 0,\ 1,\ 1 + \frac{1}{2},\ 1 + \frac{1}{2} + \frac{1}{3}, \cdots \rangle \\
&= \langle 0,\ H_1,\ H_2,\ H_3, \cdots \rangle
\end{aligned}$$

Hence

$$\sum_{k \geqslant 0} H_k x^k = \frac{1}{(1-x)} \ln \frac{1}{(1-x)}.$$

Operations on Generating Functions (7)

Example

$$
\begin{aligned}
\langle 1,1,1,1,\ldots\rangle &\longleftrightarrow \frac{1}{1-x} \\
\langle 1,2,3,4,\ldots\rangle &\longleftrightarrow \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \\
\langle 0,1,2,3,\ldots\rangle &\longleftrightarrow x\cdot\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2} \\
\langle 1,4,9,16,\ldots\rangle &\longleftrightarrow \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} \\
\langle 0,1,4,9,\ldots\rangle &\longleftrightarrow x\cdot\frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}
\end{aligned}
$$

Counting with Generating Functions

Choosing k-subset from n-set

Binomial theorem yields:

$$\left\langle \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots \right\rangle \longleftrightarrow$$
$$\longleftrightarrow \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$$

- Thus, the coefficient of x^k in $(1+x)^n$ is the number of ways to choose k distinct items from a set of size n.
- For example, the coefficient of x^2 is , the number of ways to choose 2 items from a set with n elements.
- Similarly, the coefficient of x^{n+1} is the number of ways to choose $n+1$ items from a n-set, which is zero.

Counting with Generating Functions

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$$\longleftrightarrow \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$$

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Next subsection

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is the function $A(x)$ that's expansion into power series has coefficient $a_i = 1$ of x^i iff i can be selected into the subset, otherwise $a_i = 0$.

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

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Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If any even number of elements can be selected:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

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Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If any even number of elements can be selected:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

- If any positive even number of elements can be selected:

$$A(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2}$$

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is the function $A(x)$ that's expansion into power series has coefficient $a_i = 1$ of x^i iff i can be selected into the subset, otherwise $a_i = 0$.

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If any number of elements multiple of 5 can be selected:

$$A(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$$

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is the function $A(x)$ that's expansion into power series has coefficient $a_i = 1$ of x^i iff i can be selected into the subset, otherwise $a_i = 0$.

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If at most four elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

- If at most one element can be selected:

$$A(x) = \frac{1-x^2}{1-x} = 1 + x$$

Counting elements of two sets

Convolution Rule

Let $A(x)$ be the generating function for selecting an item from (multi)set $\mathscr{A}$, and let $B(x)$, be the generating function for selecting an item from (multi)set $\mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ are disjoint, then the generating function for selecting items from the union $\mathscr{A} \cup \mathscr{B}$ is the product $A(x) \cdot B(x)$.

Proof. To count the number of ways to select n items from $\mathscr{A} \cup \mathscr{B}$ we have to select j items from $\mathscr{A}$ and $n-j$ items from $\mathscr{B}$, where $j \in \{0, 1, 2, \ldots, n\}$.
Summing over all the possible values of j gives a total of

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

ways to select n items from $\mathscr{A} \cup \mathscr{B}$. This is precisely the coefficient of x^n in the series for $A(x) \cdot B(x)$ Q.E.D.

How many positive integer solutions does the equation $x_1 + x_2 = n$ have?

- We accept any natural number can be solution for x_1, i.e generating function for selection a value for this variable is $A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$;
- same for variable x_2;

$$
\begin{aligned}
H(x) &= (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots) = \\
&= 1\cdot(1+x+x^2+x^3+\cdots) + x(1+x+x^2+x^3+\cdots) + \\
&\quad + x^2(1+x+x^2+x^3+\cdots) + x^3(1+x+x^2+x^3+\cdots) + \cdots = \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots = \frac{1}{(1-x)^2}
\end{aligned}
$$

Indeed, this equation has $n+1$ solutions:

$$
\begin{aligned}
0 + n &= n \\
1 + (n-1) &= n \\
2 + (n-2) &= n \\
\cdots & \quad \cdots \\
n + 0 &= n
\end{aligned}
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Number of solutions of the equation $x_1 + x_2 + \cdots + x_k = n$

Theorem

The number of ways to distribute n identical objects into k bins is $\binom{n+k-1}{n}$.

Proof.

The number of ways to distribute n objects equals to the number of solutions of $x_1 + x_2 + \cdots + x_k = n$ that is coefficient of x^n of the generating function $G(x) = 1/(1-x)^k = (1-x)^{-k}$.

For recollection: Maclaurin series (a Taylor series expansion of a function about 0):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

.

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.

Equation $x_1 + x_2 + \cdots + x_k = n$ (2)

Continuation of the proof ...

- let's differentiate $G(x) = (1-x)^{-k}$:

$$\begin{aligned} G'(x) &= k(1-x)^{-(k+1)} \\ G''(x) &= k(k+1)(1-x)^{-(k+2)} \\ G'''(x) &= k(k+1)(k+2)(1-x)^{-(k+3)} \\ G^{(n)}(x) &= k(k+1)\cdots(k+n-1)(1-x)^{-(k+n)} \end{aligned}$$

- Coefficient of x^n can be evaluated as:

$$\begin{aligned} \frac{G^{(n)}(0)}{n!} &= \frac{k(k+1)\cdots(k+n-1)}{n!} = \\ &= \frac{(k+n-1)!}{(k-1)!n!} = \\ &= \binom{n+k-1}{n} \end{aligned}$$

Q.E.D.

Distribute n objects into k bins so that there is at least one object in each bin

Theorem

The number of positive solutions of the equation
$x_1 + x_2 + \cdots + x_k = n$ is $\binom{n-1}{k-1}$.

Idea of the proof. Possible number of objects in a single bin ($x_i > 0$) could be generated by the function

$$C(x) = x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \cdots) = \frac{x}{1-x}$$

Similarly to the previous theorem, the number distributions is the coefficient of x^n of the generating function

$$H(X) = C^k(x) = \frac{x^k}{(1-x)^k}$$

Q.E.D.

Example: 100 Euro

How many ways 100 Euro can be changed using smaller banknotes?

Generating functions for selecting banknotes of 5, 10, 20 or 50 Euros:

$$A(x) = x^0 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$$
$$B(x) = x^0 + x^{10} + x^{20} + x^{30} + \cdots = \frac{1}{1-x^{10}}$$
$$C(x) = x^0 + x^{20} + x^{40} + x^{60} + \cdots = \frac{1}{1-x^{20}}$$
$$D(x) = x^0 + x^{50} + x^{100} + x^{150} + \cdots = \frac{1}{1-x^{50}}$$

Generating function for the task

$$P(x) = A(x)B(x)C(x)D(x) = \frac{1}{(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})}$$

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Example: 100 Euro (2)

1. Observation:

$$
\begin{aligned}
(1-x^5)(1 &+ x^5+\cdots+x^{45}+2x^{50}+2x^{55}+\cdots+2x^{95}+3x^{100}+3x^{105}+\cdots+3x^{145}+4x^{150}+\cdots) = \\
1 &+ x^5+\cdots+x^{45}+2x^{50}+2x^{55}+\cdots+2x^{95}+3x^{100}+3x^{105}+\cdots+3x^{145}+4x^{150}\cdots - \\
&- x^5-\cdots-x^{45}-x^{50}-2x^{55}-\cdots-2x^{95}-2x^{100}-3x^{105}-\cdots-3x^{145}-3x^{150}-4x^{155}-\cdots = \\
=1 &+ x^{50}+x^{100}+x^{150}+x^{200}+\cdots = \frac{1}{1-x^{50}}
\end{aligned}
$$

Thus:

$$F(x) = A(x)D(x) = \frac{1}{(1-x^5)(1-x^{50})} = \sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{10}\right\rfloor+1\right)x^{5k} = \sum_{k\geqslant 0} f_k x^{5k}$$

Example: 100 Euro (2)

1. Observation:

$$\begin{array}{rcl}
(1-x^5)(1 & + & x^5+\cdots+x^{45}+2x^{50}+2x^{55}+\cdots+2x^{95}+3x^{100}+3x^{105}+\cdots+3x^{145}+4x^{150}+\cdots) = \\
1 & + & x^5+\cdots+x^{45}+2x^{50}+2x^{55}+\cdots+2x^{95}+3x^{100}+3x^{105}+\cdots+3x^{145}+4x^{150}\cdots - \\
 & - & x^5-\cdots-x^{45}-x^{50}-2x^{55}-\cdots-2x^{95}-2x^{100}-3x^{105}-\cdots-3x^{145}-3x^{150}-4x^{155}-\cdots = \\
=1 & + & x^{50}+x^{100}+x^{150}+x^{200}+\cdots = \dfrac{1}{1-x^{50}}
\end{array}$$

Thus:

$$F(x)=A(x)D(x)=\frac{1}{(1-x^5)(1-x^{50})}=\sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{10}\right\rfloor+1\right)x^{5k}=\sum_{k\geqslant 0}f_k x^{5k}$$

2. Similarly:

$$G(x)=B(x)C(x)=\frac{1}{(1-x^{10})(1-x^{20})}=\sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)x^{10k}=\sum_{k\geqslant 0}g_k x^{10k}$$

Example: 100 Euro (3)

- Convolution:
$$P(x) = F(x)G(x) = \sum_{k\geqslant 0} c_n x^{5n}$$

- The coefficient of x^{100} equals to

$$\begin{aligned}
c_{20} &= f_0 g_{10} + f_2 g_9 + f_4 g_8 + \cdots + f_{20} g_0 \\
&= \sum_{k=0}^{10} f_{2k} g_{10-k} \\
&= \sum_{k=0}^{10} \left(\left\lfloor \frac{2k}{10} \right\rfloor + 1 \right) \left(\left\lfloor \frac{10-k}{2} \right\rfloor + 1 \right) \\
&= \sum_{k=0}^{10} \left\lfloor \frac{k+5}{5} \right\rfloor \left\lfloor \frac{12-k}{2} \right\rfloor \\
&= 1(6+5+5+4+4) + 2(3+3+2+2+1) + 3 \cdot 1 = \\
&= 24 + 22 + 3 = 49
\end{aligned}$$