Graph Decomposition*

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*Special Thanks is given to Prof. Yijia Chen for sharing his teaching materials.

Outline

Exploring Graphs

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Algorithm 1: EXPLORE(G, v)

Input: $G = (V, E)$ is a graph; $v \in V$

Output: VISITED(u) = *true* for all nodes u **reachable** from v

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VISITED(v) = true;
PREVISIT(v);
foreach edge (v, u) ∈ E do
    if not VISITED(u) then
        EXPLORE(G, u);
POSTVISIT(v);
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- PREVISIT, POSTVISIT procedures are optional.
- work on a vertex when first discovered and left for the last time.

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Theorem: EXPLORE(G, v) is correct (it visits all nodes reachable from v).

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Every node reachable from v must be visited:
If $\exists u$ that EXPLORE misses, choose a path from v to u. Let z be the last vertex on that path that EXPLORE visited. Let w be the node immediately after it on this path.

So z was visited but w was not. This is a contradiction: while EXPLORE was at node z, it would have noticed w and moved on to it.

Depth-First Search

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Algorithm 2: DFS(G)

Input: $G = (V, E)$ is a graph

Output: VISITED(v) is set to *true* for all nodes $v \in V$

```
foreach v ∈ V do
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- A loop in which adjacent edges are scanned, to see if they lead somewhere new.

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Thus the depth-first search has a running time of $O(|V| + |E|)$.

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   - Exploring Graphs
   - Connectivity in Undirected Graphs
   - Previsit and Postvisit Orderings
2. Depth-First Search in Directed Graphs
   - Types of Edges
   - Directed Acyclic Graphs
   - Strongly Connected Components
3. Breadth-First Search
   - Correctness and Efficiency

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Initially, $cc = 0$, will increment each time DFS calls EXPLORE.

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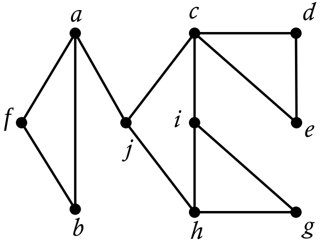
Lemma: $\forall u, v \in V$, intervals $[\text{PRE}(u), \text{POST}(u)]$, $[\text{PRE}(v), \text{POST}(v)]$ are either *disjoint* or *one is contained within the other*.

An executing example

Assume we use alphabetical order to explore G:



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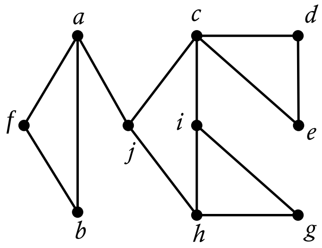

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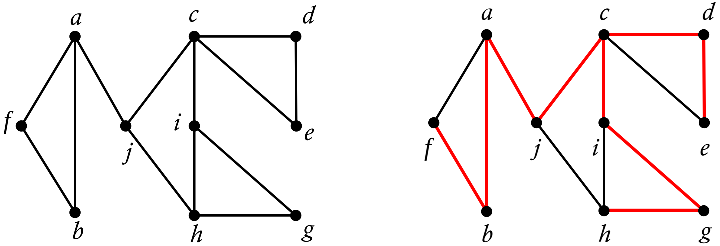

a	b	f	b	a	j	c	d	e	d	c	i	g	h	g	i	c	j	a
													g					
								d				i	i	i				
							c	c	c		c	c	c	c	c			
		b				j	j	j	j	j	j	j	j	j	j	j		
	a	a	a		a	a	a	a	a	a	a	a	a	a	a	a	a	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

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DFS yields a **search tree/forests**: root; parent and child; descendant and ancestor.

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PRE/POST ordering for (u, v)	Edge type
$[_u \quad [_v \quad\quad]_v \quad]_u$	Tree/forward
$[_v \quad [_u \quad\quad]_u \quad]_v$	Back
$[_v \quad]_v \quad\quad [_u \quad]_u$	Cross

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All the other v_j on the cycle are reachable from it and will therefore be its descendants in the search tree.

In particular, the edge $v_{i-1} \rightarrow v_i$ (or $v_k \rightarrow v_0$ if $i = 0$) is a back edge.

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The guaranteed existence of a source suggests an alternative approach to linearization:

1. Find a source, output it, and delete it from the graph.
2. Repeat until the graph is empty.

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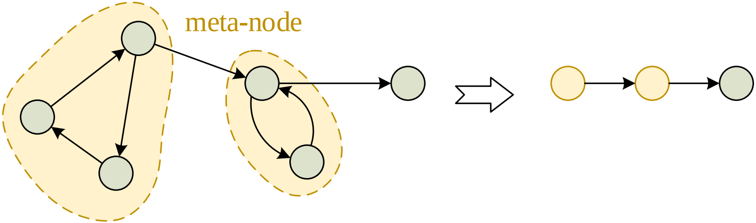
Lemma: Every directed graph is a dag of its strongly connected components.



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Therefore, if we call EXPLORE on a node that lies in a sink strongly connected component (a strongly connected component that is a sink in the meta-graph), then we will retrieve exactly that component.

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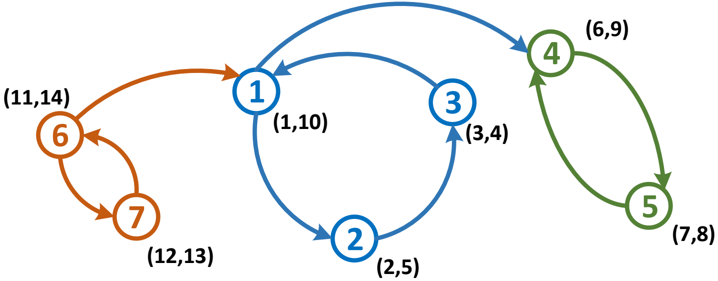
Lemma: The node that receives the highest POST number in a depth-first search must lie in a *source strongly connected component*.

Investigation (Cont')

Note: The smallest POST number in a depth-first search may NOT lie in a *sink strongly connected component*!

An Counter Example: (Node ID denotes the explore order)



The smallest POST number is Node 3, NOT in the sink strongly connected component (green).

An Efficient Algorithm

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(B) How do we continue once this first component has been discovered?

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Solving Problem A:

Consider the **reverse graph** G^R, the same as G but with all edges reversed (has exactly the same strongly connected components as G).

So, if we do a depth-first search of G^R, the node with the highest POST number will come from a source strongly connected component in G^R, which is a sink strongly connected component in G.

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Solving Problem B:

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Thus we can keep using the post numbering from our initial depth-first search on G^R to successively output the second strongly connected component, the third strongly connected component, and so on.

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The Linear-Time Algorithm:

1. Run depth-first search on G^R.
2. Run depth-first search on G, and process the vertices in decreasing order of their POST numbers from step 1.

Outline

Breadth-First Search

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Algorithm 3: BFS(G, s)

Input: Graph $G = (V, E)$, directed or undirected; vertex $s \in V$

Output: DIST(u) is set to the distance from s to all reachable u

foreach $u \in V$ **do**
    DIST$(u) = \infty$;
DIST$(s) = 0$;
$Q = [s]$ (queue containing just s);
while *Q is not empty* **do**
    $u =$ EJECT(Q);
    foreach *edge* $(u, v) \in E$ **do**
        if DIST$(v) = \infty$ **then**
            INJECT(Q, v);
            DIST$(v) =$ DIST$(u) + 1$;

Correctness and efficiency

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Lemma: For each $d = 0, 1, 2, \ldots$, there is a moment at which

(1) all nodes at distance $\leq d$ from s have their distances correctly set;

(2) all other nodes have their distances set to ∞; and

(3) the queue contains exactly the nodes at distance d.

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Lemma: BFS has a running time of $O(|V| + |E|)$.

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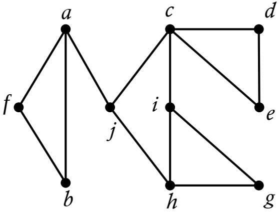

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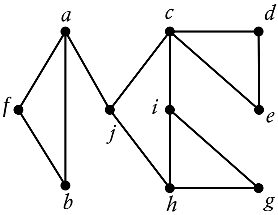

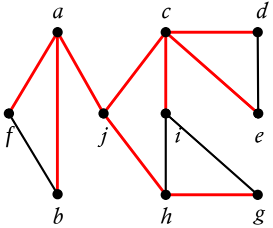

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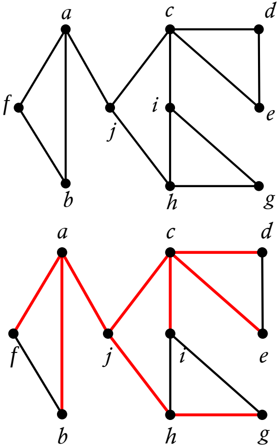

1		a					
2	a	b	f	j			
3	b	f	j				
4	f	j					
5	j	c	h				
6	c	h	d	e	i		
7	h	d	e	i	g		
8	d	e	i	g			
9	e	i	g				
10	i	g					
11	g						