Dynamic Programming*

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Algorithm Course: Shanghai Jiao Tong University

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Outline

Outline

Algorithmic Paradigms

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming: Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

History

Richard E. Bellman (1920-1984): Pioneered the systematic study of dynamic programming in 1950s.

Etymology:

- Dynamic programming = planning over time
- Secretary of Defense had pathological fear of mathematical research.
- Bellman sought a "dynamic" adjective to avoid conflict.

Applications

Areas: Bioinformatics, Control Theory, Information Theory, Operations Research, Computer Science (Theory, Graphics, AI, Compilers, Systems, $\cdots$)

Some Famous Algorithms

- Avidan-Shamir for seam carving.
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- De Boor for evaluating spline curves.
- Knuth-Plass for word wrapping text in TEX.
- Smith-Waterman for genetic sequence alignment.
- Bellman-Ford-Moore for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.
- Needleman-Wunsch/Smith-Waterman for sequence alignment.

Dynamic Programming Books

Outline

Weighted Interval Scheduling Problem

Job j starts at s_j, finishes at f_j, and has weight or value $w_j > 0$.

Two jobs are compatible if they don't overlap.

Goal: find maximum weight subset of mutually compatible jobs.

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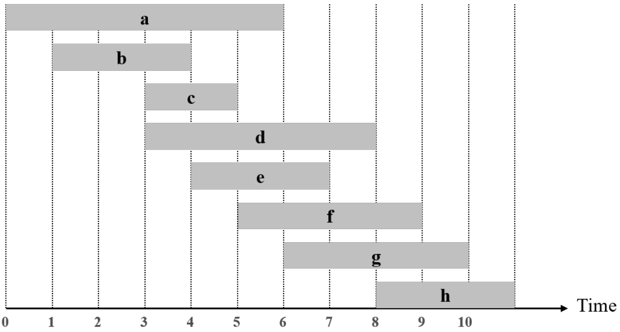

Unweighted Interval Scheduling Review

Recall: Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Unweighted Interval Scheduling Review

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- Consider jobs in ascending order of finish time.
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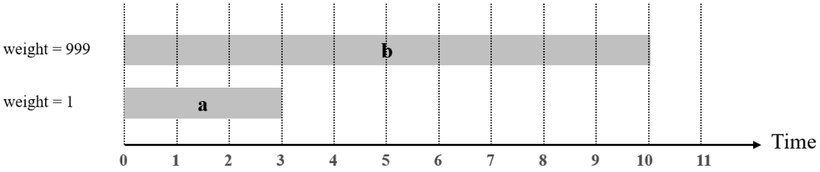
Observation: Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Definition: $p(j)$ = largest index $i < j$ such that job i is compatible with j.

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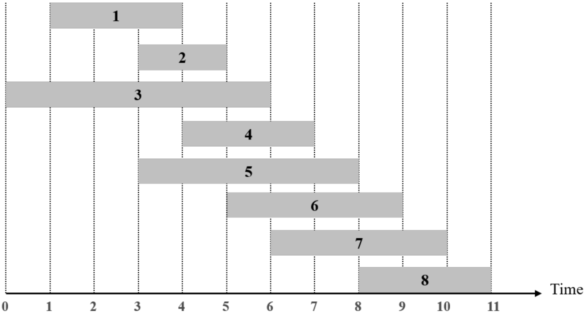
Example: $p(8) = 5, p(7) = 3, p(2) = 0$.



Binary Choice

Recurrence template: $OPT(j)$ = value of optimal solution to the problem consisting of job requests $1, 2, \cdots, j$.

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Optimal substructure:

Case 1: OPT selects job j.

- collect weight w_j,
- can't use incompatible jobs $\{p(j)+1, p(j)+2, \cdots, j-1\}$,
- must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \cdots, p(j)$.

Case 2: OPT does not select job j.

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Case 2: OPT does not select job j.

- must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \cdots, j - 1$.

$$OPT(j) = \begin{cases} 0, & j = 0, \\ \max\{w_j + OPT(p(j)), OPT(j-1)\}, & otherwise \end{cases}$$

Brute Force Algorithm

Algorithm 1: Weighted Interval Scheduling – Brute Force

Input: $n; s_1, \cdots, s_n; f_1, \cdots, f_n; w_1, \cdots, w_n;$
Output: Optimal weight $OPT(n)$.

Sort jobs by finish times so that $f_1 \le f_2 \le \cdots \le f_n$;
Compute $p(1), p(2), \cdots, p(n)$;
return `B-Sched` (n);

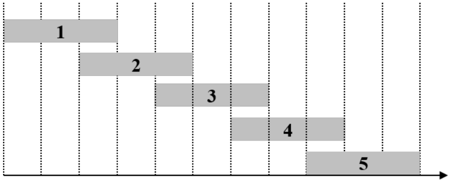
Algorithm 2: `B-Sched` (j)

if $j = 0$ **then**
    return 0;
else
    return $\max\{w_j +$ `B-Sched` $(p(j))$, `B-Sched` $(j-1)\}$;

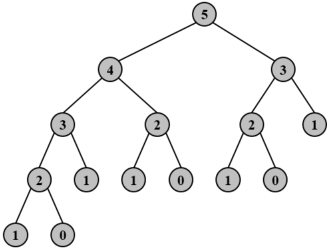
Brute Force Algorithm

Observation: Recursive algorithm fails spectacularly because of redundant sub-problems ⇒ exponential algorithms.

Example: Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Memoization: Store sub-results in cache; lookup as needed

Algorithm 3: Weighted Interval Scheduling – Memoization

Input: n; $s_1, \cdots, s_n$; $f_1, \cdots, f_n$; $w_1, \cdots, w_n$;
Output: Optimal weight $OPT(n)$.

```
Sort jobs by finish times so that f_1 ≤ f_2 ≤ ··· ≤ f_n;
Compute p(1), p(2), ···, p(n);
M[0] = 0;                                  // global array
return M-Sched(n);
```

Algorithm 4: `M-Sched(j)`

```
if M[j] is uninitialized then
    M[j] = max{w_j + M-Sched(p(j)), M-Sched(j - 1)};
return M[j];
```

Running Time

Claim: Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n \log n)$ via sorting by start time.
- `M-Sched`(j): each invocation takes $O(1)$ time and either
  (1) returns an existing value $M[j]$
  (2) initializes $M[j]$ and makes two recursive calls
- Progress measure Φ = number nonempty entries of $M[\cdot]$.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (2) increases Φ by 1 $\Rightarrow$ at most $2n$ recursive calls.
- Overall running time of `M-Sched`(n) is $O(n)$.

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Remark: $O(n)$ if jobs are pre-sorted by start and finish times.

Finding a Solution from the OPT Value

Algorithm 5: `Find-Solution` (j)

if $j = 0$ **then**
    return $\emptyset$;
else if $w_j + M[p(j)] > M[j-1]$ **then**
    return $\{j\} \cup$ `Find-Solution` $(p(j))$;
else
    return `Find-Solution` $(j-1)$;

Finding a Solution from the OPT Value

Algorithm 5: `Find-Solution` (j)

```
if j = 0 then
    return ∅;
else if w_j + M[p(j)] > M[j − 1] then
    return {j} ∪ Find-Solution(p(j));
else
    return Find-Solution(j − 1);
```

- Run `Find-Solution`(n) to find optimal schedule;
- $\#$ of recursive calls $1 \leq n \Rightarrow O(n)$;

Tabulation: Bottom-Up Dynamic Programming

Algorithm 6: Weighted Interval Scheduling – Tabulation

Input: n; $s_1, \cdots, s_n$; $f_1, \cdots, f_n$; $w_1, \cdots, w_n$;
Output: Optimal weight $OPT(n)$.

```
Sort jobs by finish times so that f_1 ≤ f_2 ≤ ··· ≤ f_n;
Compute p(1), p(2), ···, p(n);
M[0] = 0;
for j = 1 → n do
    M[j] = max{w_j + M[p(j)], M[j − 1]};
```

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Input: n; $s_1, \cdots, s_n$; $f_1, \cdots, f_n$; $w_1, \cdots, w_n$;
Output: Optimal weight $OPT(n)$.

1 Sort jobs by finish times so that $f_1 \leq f_2 \leq \cdots \leq f_n$;
2 Compute $p(1), p(2), \cdots, p(n)$;
3 $M[0] = 0$;
4 **for** $j = 1 \rightarrow n$ **do**
5 $\quad M[j] = \max\{w_j + M[p(j)], M[j-1]\}$;

Running Time: $O(n \log n)$.

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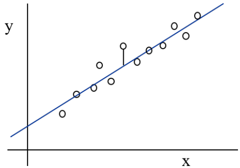
Running Time: $O(n \log n)$.

Those who cannot remember the past are condemned to repeat it.
—— Kevin Wayne@Princeton

Outline

Segmented Least Squares

- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$.
- Find a line $y = ax + b$ to minimize the sum of the squared error:



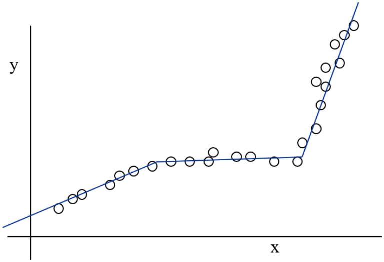
Solution: Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n\sum_i x_i^2 - (\sum_i x_i)^2}, b = \frac{\sum_i y_i - a\sum_i x_i}{n}$$

Segmented Least Squares

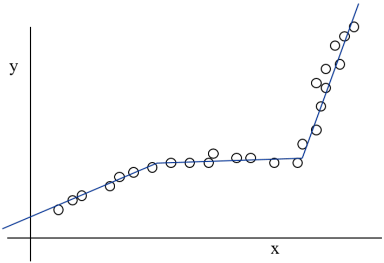
- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ with $x_1 < x_2 < \cdots < x_n$, find a sequence of lines that minimizes $f(x)$.

Question: What's a reasonable choice for $f(x)$ to balance accuracy (goodness of fit) and parsimony (number of lines)?



Segmented Least Squares

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: (x_1, y_1), (x_2, y_2) , $\cdots$, (x_n, y_n) with $x_1 < x_2 < \cdots < x_n$, find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors E in each segment
  - the number of lines L
- Tradeoff function: $E + cL$, for some constant $c > 0$.



Multiway Choice

Notation:

- $OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \cdots, p_j$.
- $e(i,j)$ = minimum sum of squares for points $p_i, p_{i+1}, \cdots, p_j$.

Compute $OPT(j)$:

- Last segment uses points $p_i, p_{i+1}, \cdots, p_j$ for some i.
- $Cost = e(i,j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0, & j = 0, \\ \min\limits_{1 \le i \le j} \{e(i,j) + c + OPT(i-1)\}, & otherwise \end{cases}$$

Segmented Least Squares

Algorithm 7: Segmented Square Error (SSE)

Input: n; $p_1, \cdots, p_n$; c;

Output: Optimal square error for $p_1, \cdots, p_n$.

```
for j = 1 → n do
    for i = 1 → j do
        compute least square error e_ij for segment p_i, ···, p_j;
M[0] = 0;
for j = 1 → n do
    M[j] = min_{1≤i≤j} {e_ij + c + M[i − 1]};
return M[n];
```

$$M[j] = \min_{1 \leq i \leq j} \{e_{ij} + c + M[i-1]\}$$

Segmented Least Squares

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return M[n];
```

Time Complexity: $O(n^3)$ (can be improved to $O(n^2)$)

Space Complexity: $O(n^2)$.

Algorithm Analysis

Theorem (Bellman, 1961) `SSE` solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.

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Proof: Bottleneck = computing e_{ij} for $O(n^2)$ pairs,

$$a = \frac{n\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n\sum_i x_i^2 - (\sum_i x_i)^2}, \; b = \frac{\sum_i y_i - a\sum_i x_i}{n},$$

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$O(n)$ per pair e_{ij} using previous formula. $\square$

Remark: Can be improved to $O(n^2)$ time.

- $\forall i$: precompute cumulative sums $\sum_{k=1}^{i} x_k, \sum_{k=1}^{i} y_k, \sum_{k=1}^{i} x_k^2, \sum_{k=1}^{i} x_k y_k$,
- Using cumulative sums, we can compute e_{ij} in $O(1)$ time.

Outline

Knapsack Problem

Given n objects and a "knapsack".

Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.

Knapsack has capacity of W kilograms.

Goal: fill knapsack so as to maximize total value.

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Example: $\{3, 4\}$ has value 40.

W = 11

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Example: $\{3, 4\}$ has value 40.

Greedy: repeatedly add item with maximum ratio v_i/w_i.

Example: $\{5, 2, 1\}$ achieves only value = 35 $\Rightarrow$ greedy not optimal.

First Attempt

Definition: $OPT(i)$ = max profit subset of items $1, \cdots, i$.

Case 1: OPT does not select item i.

- OPT selects best of $\{1, 2, \cdots, i-1\}$.

Case 2: OPT selects item i.

- accepting item i does not immediately imply that we will have to reject other items,
- without knowing what other items were selected before i, we don't even know if we have enough room for i.

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Conclusion: Need more sub-problems!

Adding a New Variable

Definiton: $OPT(i)$ = max profit subset of items $1, \cdots, i$ with weight limit w.

Case 1: OPT does not select item i.

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Case 2: OPT selects item i.

- new weight $limit = w - w_i$
- OPT selects best of using $\{1, 2, \cdots, i-1\}$ this new weight limit

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$$OPT(i, w) = \begin{cases} 0, & j = 0, \\ OPT(i-1, w), & w_i > w, \\ \max\{OPT(i-1, w), v_i + OPT(i-1, w-w_i)\}, & otherwise \end{cases}$$

Bottom-Up Algorithm (Fill up an n-by-W array)

Algorithm 8: Knapsack Algorithm using n-by-W Array

Input: $n, W, w_1, \cdots, w_n, v_1, \cdots, v_n$;

Output: Optimal value of knapsack with W.

```
for w = 0 → W do
    M[0, w] = 0;
for i = 1 → n do
    for w = 1 → W do
        if w_i > w then
            M[i, w] = M[i − 1, w];
        else
            M[i, w] = max{M[i − 1, w], v_i + M[i − 1, w − w_i]};
return M[n, W];
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

OPT: $\{4, 3\}$

value $= 22 + 18 = 40$

Running Time

Running time: $\Theta(nW)$.

- Not polynomial in input size!
- "Pseudo-polynomial".
- Decision version of Knapsack is NP-complete.

Knapsack approximation algorithm: There exists a poly-time algorithm that produces a feasible solution that has value within 0.01% of optimum.

Outline

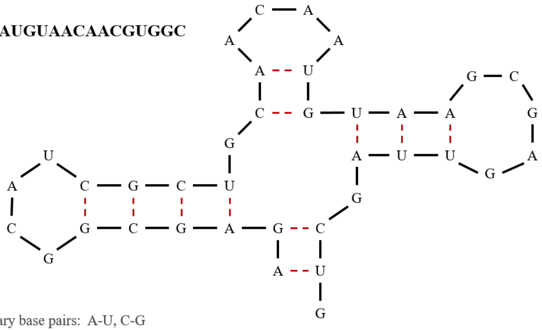
RNA Secondary Structure

RNA: String $B = b_1 b_2 \cdots b_n$ over alphabet $\{A, C, G, U\}$.

Secondary structure: RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

Example: **GUCGAUUGAGCGAAUGUAACAACGUGGC**

UACGGCGAGA



complementary base pairs: A-U, C-G

RNA Secondary Structure

Secondary structure: A set of pairs $S = \{(b_i, b_j)\}$ that satisfy:

[Watson-Crick] S is a matching and each pair in S is a Watson-Crick complement: A-U, U-A, C-G, or G-C.

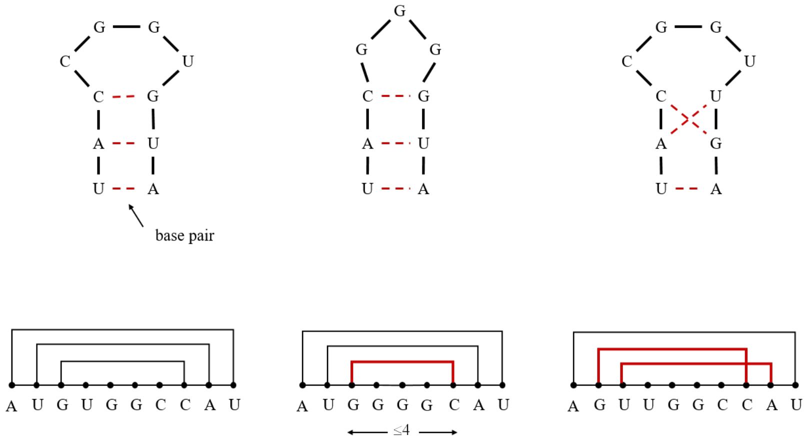
[No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.

[Non-crossing] If (b_i, b_j) and (b_k, b_l) are two pairs in S, then we cannot have $i < k < j < l$.

Free energy: Usual hypothesis is that an RNA molecule will form the secondary structure with the optimum total free energy.

Goal: Given an RNA molecule $B = b_1 b_2 \cdots b_n$, find a secondary structure S that maximizes the number of base pairs

Examples



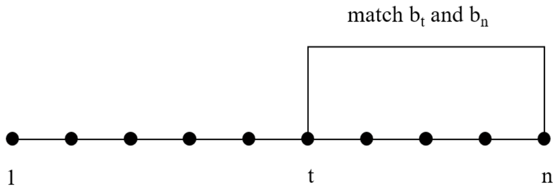
ok

sharp turn

crossing

Subproblems

First attempt: $OPT(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1 b_2 \cdots b_j$.



Difficulty: Results in two sub-problems.

- Finding secondary structure in: $b_1 b_2 \cdots b_{t-1}$.
- Finding secondary structure in: $b_{t+1} b_{t+2} \cdots b_{n-1}$.

Dynamic Programming Over Intervals

Notation: $OPT(j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \cdots b_j$.

Case 1: If $i \geq j-4$.

- $OPT(i,j) = 0$ by no-sharp turns condition.

Case 2: Base b_j is not involved in a pair.

- $OPT(i,j) = OPT(i,j-1)$

Case 3: Base b_j pairs with b_t for some $i \leq t < j-4$.

- non-crossing constraint decouples resulting sub-problems
- $OPT(i,j) = 1 + \max_t\{OPT(i,t-1) + OPT(t+1,j-1)\}$

Remark: Same core idea in CKY algorithm to parse context-free grammars.

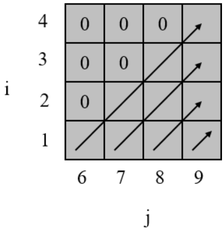
Bottom Up Dynamic Programming Over Intervals

Question: What order to solve the sub-problems?

Answer: Do shortest intervals first.

```
RNA(b1,…,bn) {
  for k = 5, 6, …, n-1
    for i = 1, 2, …, n-k
      j = i + k
      Compute M[i, j]

  return M[1, n]            using recurrence
}
```



Running time: $O(n^3)$.

Dynamic Programming Summary

Recipe

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques

- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares.
- Adding a new variable: knapsack.
- Dynamic programming over interval

Top-down vs. bottom-up: different people have different intuitions.

Outline

String Similarity: How similar are two strings?

How similar are two strings?

- ocurrance
- occurrence

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

Edit Distance

Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

-	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	-	T	A	C	A	T

Sequence Alignment

Goal: Given two strings $X = x_1x_2\cdots x_m$ and $Y = y_1y_2\cdots y_n$ find alignment of minimum cost.

Definiton: An alignment M is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

Definiton: The pair $x_i - y_j$ and $x_{i'} - y_{j'}$ cross if $i < i'$, but $j > j'$.

$$M = \sum_{(x_i,y_j)\in M} \alpha_{x_iy_j} + \sum_{i:x_i\text{ unmatched}} \delta + \sum_{j:y_j\text{unmatched}} \delta$$

mismatch

gap

Example: *CTACCG* vs. *TACATG*.

Solution: $M = x_2 - y_1, x_3 - y_2, x_4 - y_3, x_5 - y_4, x_6 - y_6$.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G
-	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

Problem Structure

Definition: $OPT(i,j)$ = min cost of aligning strings $x_1x_2\cdots x_i$ and $y_1y_2\cdots y_j$.

Case 1: OPT matches $x_i - y_j$.
pay mismatch for $x_i - y_j$ + min cost of aligning two strings $x_1x_2\cdots x_{i-1}$ and $y_1y_2\cdots y_{j-1}$

Case 2a: OPT leaves x_i unmatched.
pay gap for x_i and min cost of aligning $x_1x_2\cdots x_{i-1}$ and $y_1y_2\cdots y_j$

Case 2b: OPT leaves y_j unmatched.
pay gap for y_j and min cost of aligning $x_1x_2\cdots x_i$ and $y_1y_2\cdots y_{j-1}$

Sequence Alignment

Algorithm 9: Sequence Alignment

Input: m, n, $x_1x_2\cdots x_m$, $y_1y_2\cdots y_n$, α, δ;

```
for i = 0 → m do  M[i,0] = iδ ;
for j = 0 → n do  M[0,j] = jδ ;
for i = 1 → m do
    for j = 1 → n do
        M[i,j] = min(α[x_i, y_j] + M[i−1, j−1], δ + M[i−1, j],
          δ + M[i, j−1]);
return M[m,n];
```

Sequence Alignment

Algorithm 9: Sequence Alignment

Input: $m, n, x_1x_2\cdots x_m, y_1y_2\cdots y_n, \alpha, \delta$;

```
for i = 0 → m do  M[i, 0] = iδ ;
for j = 0 → n do  M[0, j] = jδ ;
for i = 1 → m do
    for j = 1 → n do
        M[i, j] = min(α[x_i, y_j] + M[i − 1, j − 1], δ + M[i − 1, j],
          δ + M[i, j − 1]);
return M[m, n];
```

Analysis: $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10$.

Computational biology: $m = n = 100,000$. 10 billions ops OK, but 10GB array?

Outline

Linear Space

Question: Can we avoid using quadratic space?

Easy. Optimal value in $O(m+n)$ space* and $O(mn)$ time.

- Compute $OPT(i,\cdot)$ from $OPT(i-1,\cdot)$.
- No longer a simple way to recover alignment itself.

*including space storing original strings

Linear Space

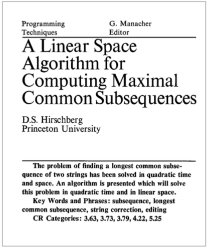
Question: Can we avoid using quadratic space?

Easy. Optimal value in $O(m+n)$ space* and $O(mn)$ time.

- Compute $OPT(i,\cdot)$ from $OPT(i-1,\cdot)$.
- No longer a simple way to recover alignment itself.

Theorem. [Hirschberg 1975] Optimal alignment in $O(m+n)$ space and $O(mn)$ time.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.


Programming Techniques　G. Manacher Editor

A Linear Space Algorithm for Computing Maximal Common Subsequences

D.S. Hirschberg
Princeton University

The problem of finding a longest common subsequence of two strings has been solved in quadratic time and space. An algorithm is presented which will solve this problem in quadratic time and in linear space.

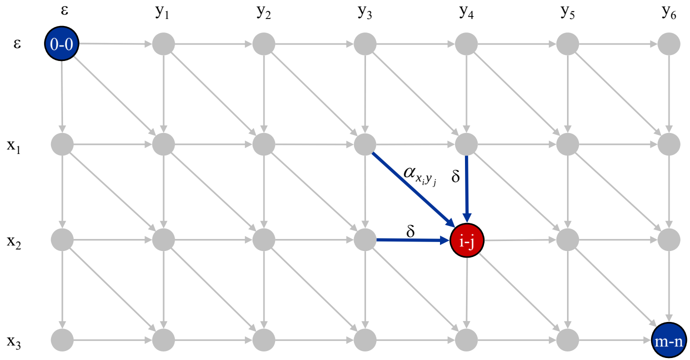
Key Words and Phrases: subsequence, longest common subsequence, string correction, editing

CR Categories: 3.63, 3.73, 3.79, 4.22, 5.25

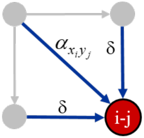
*including space storing original strings

Edit Distance Graph

- Let $f(i,j)$ be shortest path from $(0,0)$ to (i,j).
- Observation: $f(i,j) = OPT(i,j)$.



Edit Distance Graph



- Let $f(i,j)$ be shortest path from $(0,0)$ to (i,j).
- Observation: $f(i,j) = OPT(i,j)$.

Proof: (by strong induction on $i+j$)

Base case: $f(0,0) = OPT(0,0) = 0$

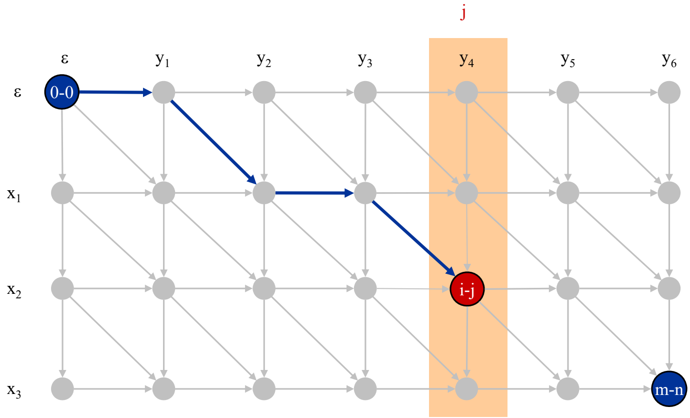
Inductive hypothesis: assume true for all (i',j') with $i'+j' < i+j$.

Induction: Last edge on shortest path to (i,j) is from $(i-1,j-1)$, $(i-1,j)$, or $(i,j-1)$.

$$\begin{aligned} f(i,j) &= \min\{a_{x_i y_i} + f(i-1,j-1), \delta + f(i-1,j), \delta + f(i,j-1)\} \\ &= \min\{a_{x_i y_i} + OPT(i-1,j-1), \delta + OPT(i-1,j), \delta + OPT(i,j-1)\} \\ &= OPT(i,j) \end{aligned}$$

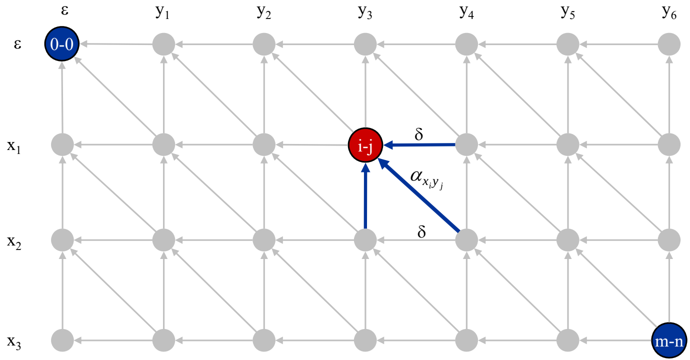
Edit Distance Graph

- Let $f(i,j)$ be shortest path from $(0,0)$ to (i,j).
- Can compute $f(\cdot,j)$ for any j in $O(mn)$ time and $O(m+n)$ space.



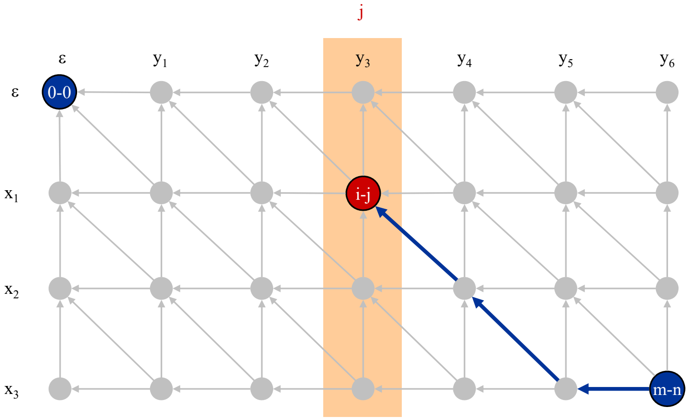
Edit Distance Graph

- Let $g(i,j)$ be shortest path from (i,j) to (m,n).
- Can compute by reversing the edge orientations and inverting the roles of $(0,0)$ and (m,n)



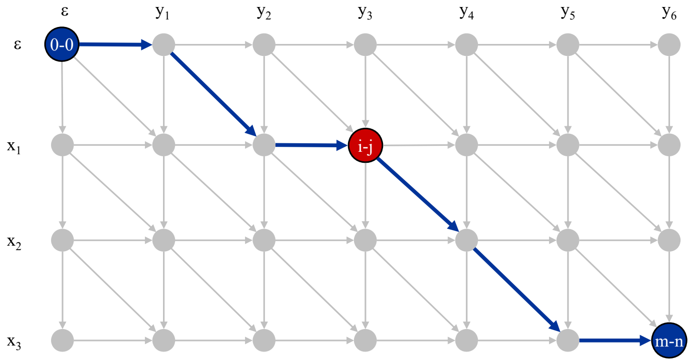
Edit Distance Graph

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\cdot, j)$ for any j in $O(mn)$ time and $O(m+n)$ space.



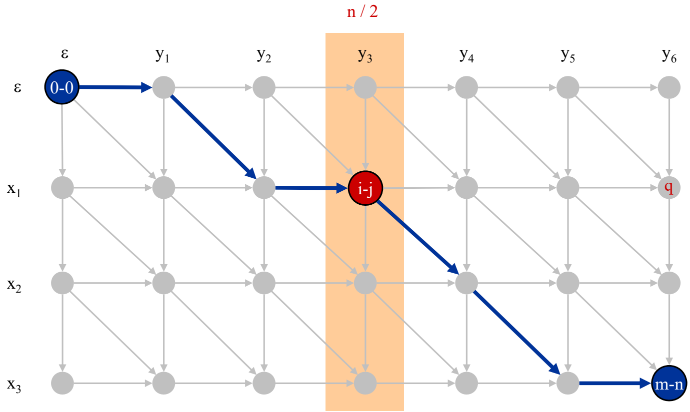
Edit Distance Graph

Observation 1: The cost of the shortest path that uses (i,j) is $f(i,j)+g(i,j)$.



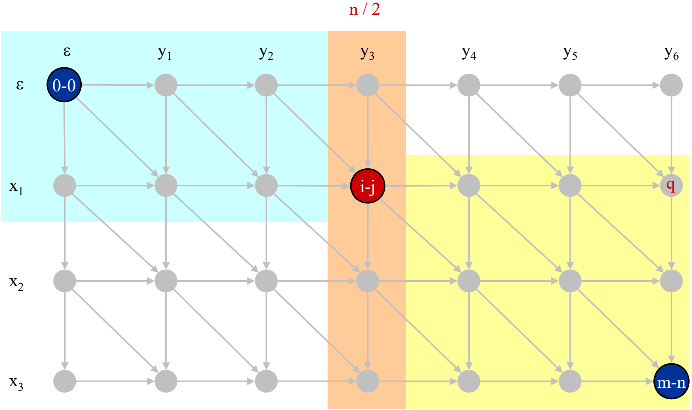
Edit Distance Graph

Observation 2: Let q be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, the shortest path from $(0, 0)$ to (m, n) uses $(q, n/2)$.



Edit Distance Graph

Divide: find index q that minimizes $f(q, n/2) + g(q, n/2)$ using DP.
Align x_q and $y_{n/2}$.
Conquer: recursively compute optimal alignment in each piece.



Running Time Analysis Warmup

Theorem: Let $T(m,n)$ = max running time of algorithm on strings of length at most m and n. $T(m,n) = O(mn\log n)$.

$$T(m,n) \leq 2T(m,n/2) + O(mn) \Rightarrow T(m,n) = O(mn\log n)$$

Remark: Analysis is not tight because two sub-problems are of size $(q,n/2)$ and $(m-q,n/2)$. In next slide, we save $\log n$ factor.

Running Time Analysis

Theorem. Let $T(m,n)$ = max running time of algorithm on strings of length m and n. $T(m,n) = O(mn)$

Proof: (by induction on n)

- $O(mn)$ time to compute $f(\cdot, n/2)$ and $g(\cdot, n/2)$ and find index q.
- $T(q, n/2) + T(m-q, n/2)$ time for two recursive calls
- Choose constant c so that:

$$T(m,2) \leq cm$$

$$T(2,n) \leq cn$$

$$T(m,n) \leq cmn + T(q,n/2) + T(m-q,n/2)$$

Running Time Analysis (Continued)

Theorem. Let $T(m,n)$ = max running time of algorithm on strings of length m and n. $T(m,n) = O(mn)$

Proof:

- Base cases: $m = 2$ or $n = 2$.
- Inductive hypothesis: $T(m,n) \leq 2cmn$.

$$
\begin{aligned}
T(m,n) &\leq T(q,n/2) + T(m-q,n/2) + cmn \\
&\leq 2cqn/2 + 2c(m-q)n/2 + cmn \\
&= cqn + cmn - cqn + cmn \\
&= 2cmn
\end{aligned}
$$