Generating Functions

ITT9131 Konkreetne Matemaatika

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Next section

Generating functions and sequences

Let $\langle g_n \rangle = \langle g_0, g_1, g_2, \ldots \rangle$ be a sequence of complex numbers: for example, the solution of a recurrence equation.
We associate to $\langle g_n \rangle$ its generating function, which is the power series

$$G(z) = \sum_{n \geqslant 0} g_n z^n \,;$$

such series is defined in a suitable neighborhood of the origin.

Given a closed form for $G(z)$, we will see how to:

- Determine a closed form for g_n.
- Compute infinite sums.
- Solve recurrence equations.

If convenient, we will sum over all integers, under the tacit assumption that:

$$g_n = 0 \text{ whenever } n < 0.$$

Generating function manipulations

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Basic sequences and their generating functions

For $m \geqslant 0$ integer

- $\langle 1, 0, 0, 0, 0, 0, \ldots \rangle \quad \leftrightarrow \quad \sum_{n \geqslant 0} [n = 0] z^n = 1$

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- $\langle 1,2,4,8,16,32,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0} 2^n z^n = \frac{1}{1-2z}$

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For $m \geqslant 0$ integer and for $c \in \mathbb{C}$

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- $\left\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 1} \frac{1}{n} z^n = \ln \frac{1}{1-z}$

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- $\left\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \right\rangle \quad \leftrightarrow \quad \sum_{n \geqslant 1} \frac{(-1)^{n+1}}{n} z^n = \ln(1+z)$

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- $\left\langle 0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}z^n = \ln(1+z)$
- $\left\langle 1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\frac{1}{n!}z^n = \mathrm{e}^z$

Warmup: A simple generating function

Problem

Determine the generating function $G(z)$ of the sequence

$$g_n = 2^n + 3^n \,, n \geqslant 0$$

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Solution

- For $\alpha \in \mathbb{C}$, the generationg function of $\langle \alpha^n \rangle_{n \geqslant 0}$ is $G_\alpha(z) = \frac{1}{1-\alpha z}$.
- By linearity, we get

$$G(z) = G_2(z) + G_3(z) = \frac{1}{1-2z} + \frac{1}{1-3z}\,.$$

Extracting the even- or odd-numbered terms of a sequence

Let $\langle g_0, g_1, g_2, \ldots \rangle \leftrightarrow G(z)$.

Then

$$G(z) + G(-z) = \sum_n g_n \left(1 + (-1)^n\right) z^n = 2 \sum_n g_n \,[n \text{ is even}]\, z^n$$

Therefore

$$\langle g_0, 0, g_2, 0, g_4, \ldots \rangle \leftrightarrow \frac{G(z) + G(-z)}{2} = \sum_n g_{2n}\, z^{2n}$$

Similarly

$$\langle 0, g_1, 0, g_3, 0, g_5, \ldots \rangle \leftrightarrow \frac{G(z) - G(-z)}{2} = \sum_n g_{2n+1}\, z^{2n+1}$$

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Extracting the even- or odd-numbered terms of a sequence (2)

Example: $\langle 1,0,1,0,1,0,\ldots\rangle \leftrightarrow F(z) = \frac{1}{1-z^2}$

We have

$$\langle 1,1,1,1,1,\ldots\rangle \leftrightarrow G(z) = \frac{1}{1-z}.$$

Then the generating function for $\langle 1,0,1,0,1,0,\ldots\rangle$ is

$$\frac{1}{2}(G(z)+G(-z)) = \frac{1}{2}\left(\frac{1}{1-z}+\frac{1}{1+z}\right) = \frac{1}{2}\cdot\frac{1+z+1-z}{(1-z)(1+z)} = \frac{1}{1-z^2}$$

Extracting the even- or odd-numbered terms of a sequence (3)

Example: $\langle 0, 1, 3, 8, 21, \ldots\rangle = \langle f_0, f_2, f_4, f_6, f_8, \ldots\rangle$

We know that

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots\rangle \leftrightarrow F(z) = \frac{z}{1-z-z^2}.$$

Then the generating function for $\langle f_0, 0, f_2, 0, f_4, 0, \ldots\rangle$ is

$$\begin{aligned}\sum_n f_{2n} z^{2n} &= \frac{1}{2}\left(\frac{z}{1-z-z^2} + \frac{-z}{1+z-z^2}\right)\\ &= \frac{1}{2}\cdot\frac{z+z^2-z^3-z+z^2+z^3}{(1-z^2)^2-z^2}\\ &= \frac{z^2}{1-3z^2+z^4}\end{aligned}$$

This gives

$$\langle 0, 1, 3, 8, 21, \ldots\rangle \leftrightarrow \sum_n f_{2n}\, z^n = \frac{z}{1-3z+z^2}$$

Next subsection

Reviewing the convergence radius

Definition

The convergence radius of the power series $\sum_{n\geqslant 0} a_n(z-z_0)^n$ is the value R defined by:

$$\frac{1}{R} = \limsup_{n\geqslant 0} \sqrt[n]{|a_n|},$$

with the conventions $1/0=\infty$, $1/\infty=0$.

The Abel-Hadamard theorem

Let $\sum_{n\geqslant 0} a_n(z-z_0)^n$ be a power series of convergence radius R.

1. If $R>0$, then the series converges uniformly in every closed and bounded subset of the disk of center z_0 and radius R.
2. If $R<\infty$, then the series does not converge at any point z such that $|z-z_0|>R$.

Power series and infinite sums

The problem

- Consider an infinite sum of the form $\sum_{n\geqslant 0} a_n\beta^n$.
- Suppose that we are given a closed form for the generating function $G(z)$ of the sequence $\langle a_0, a_1, a_2, \ldots\rangle$.
- Can we deduce that $\sum_{n\geqslant 0} a_n\beta^n = G(\beta)$?

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- Can we deduce that $\sum_{n\geqslant 0} a_n\beta^n = G(\beta)$?

Answer: It depends!

Let R be the convergence radius of the power series $\sum_{n\geqslant 0} a_n z^n$.

- If $|\beta| < R$: YES
  by the Abel-Hadamard theorem and uniqueness of analytic continuation.
- If $|\beta| > R$: NO by the Abel-Hadamard theorem.
- If $|\beta| = R$: Sometimes yes, sometimes not!

Warmup: A sum with powers and harmonic numbers

The problem

Compute $\sum_{n\geqslant 0} H_n/10^n$

Warmup: A sum with powers and harmonic numbers

The problem

Compute $\sum_{n \geqslant 0} H_n / 10^n$

Solution

This looks like the sum of the power series $\sum_{n \geqslant 0} H_n z^n$ at $z = 1/10$ — if it exists ...

- For $n \geqslant 1$ it is $1 \leqslant H_n \leqslant n$: therefore, the convergence radius is 1.
- We know that the generating function of $g_n = H_n$ is $G(z) = \frac{1}{1-z} \ln \frac{1}{1-z}$.
- As we are within the convergence radius of the series, we *can* replace the sum of the series with the value of the function.

In conclusion,

$$\sum_{n \geqslant 0} \frac{H_n}{10^n} = \frac{1}{1 - \frac{1}{10}} \ln \frac{1}{1 - \frac{1}{10}} = \frac{10}{9} \ln \frac{10}{9} .$$

Abel's summation formula

Statement

- Let $S(z) = \sum_{n \geqslant 0} a_n z^n$ be a power series with center 0 and convergence radius 1.
- If

$$S = \sum_{n \geqslant 0} a_n = S(1)$$

  exists, then $S(z)$ converges uniformly in $[0, 1]$.
- In particular,

$$L = \lim_{x \to 1^-} S(x) \ , \ x \in [0, 1]$$

  also exists, and coincides with S.

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also exists, and coincides with S.

The converse does not hold!

For $|z| < 1$ we have:
$$\sum_{n \geqslant 0} (-1)^n z^n = \frac{1}{1+z}$$

Then $L = \dfrac{1}{2}$ but S does not exist.

Tauber's theorem

Statement

- Let $S(z) = \sum_{n \geqslant 0} a_n z^n$ be a power series with center 0 and convergence radius 1.
- If
$$L = \lim_{x \to 1^-} S(x)\ ,\ x \in [0,1]$$
exists and in addition
$$\lim_{n \to \infty} n a_n = 0$$
then $S = S(1)$ also exists, and coincides with L.

Next section

Solving recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n in terms of n.

"Algorithm"

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of z^n; this is a closed form for g_n.

Next subsection

Example: Fibonacci numbers revisited

Step 1 The recurrence

$$g_n = \begin{cases} 0, & \text{if } n \leqslant 0; \\ 1, & \text{if } n = 1; \\ g_{n-1} + g_{n-2} & \text{if } n > 1; \end{cases}$$

can be represented by the single equation

$$g_n = g_{n-1} + g_{n-2} + [n = 1],$$

where $n \in (-\infty, +\infty)$.
This is because the "simple" Fibonacci recurrence $g_n = g_{n-1} + g_{n-2}$ holds for every $n \geqslant 2$ by construction, and for every $n \leqslant 0$ as by hypothesis $g_n = 0$ if $n < 0$; but for $n = 1$ the left-hand side is 1 and the right-hand side is 0, so we need the correction summand $[n = 1]$.

Example: Fibonacci numbers revisited (2)

Step 2 For any n, multiply both sides of the equation by z^n …

$$
\begin{aligned}
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
g_{-2}z^{-2} &= g_{-3}z^{-2} + g_{-4}z^{-2} + [-2 = 1]z^{-2} \\
g_{-1}z^{-1} &= g_{-2}z^{-1} + g_{-3}z^{-1} + [-1 = 1]z^{-1} \\
g_0 &= g_{-1} + g_{-2} + [0 = 1] \\
g_1 z &= g_0 z + g_{-1} z + [1 = 1]z \\
g_2 z^2 &= g_1 z^2 + g_0 z^2 + [2 = 1]z^2 \\
g_3 z^3 &= g_2 z^3 + g_1 z^3 + [3 = 1]z^3 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

… and sum over all n.

$$\sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n = 1] z^n$$

Example: Fibonacci numbers revisited (3)

Step 3 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n=1] z^n =$$
$$= \sum_n g_n z^{n+1} + \sum_n g_n z^{n+2} + z =$$
$$= zG(z) + z^2 G(z) + z$$

Solving the equation yields

$$G(z) = \frac{z}{1 - z - z^2}$$

Step 4 Expansion the equation into power series $G(z) = \sum g_n z^n$ gives us the solution (see next slides):

$$g_n = \frac{\Phi^n - \widehat{\Phi}^n}{\sqrt{5}}$$

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Next section

Motivation

- A generating function is often in the form of a rational function

$$R(z) = \frac{P(z)}{Q(z)},$$

  where P and Q are polynomials.
- Our goal is to find "partial fraction expansion" of $R(z)$, i.e. represent $R(z)$ in the form

$$R(z) = S(z) + T(z),$$

  where $S(z)$ has known expansion into the power series, and $T(z)$ is a polynomial.
- A good candidate for $S(z)$ is a finite sum of functions like

$$S(z) = \frac{a_1}{(1-\rho_1 z)^{m_1+1}} + \frac{a_2}{(1-\rho_2 z)^{m_2+1}} + \cdots + \frac{a_\ell}{(1-\rho_\ell z)^{m_\ell+1}}$$

- We have proven the relation

$$\frac{a}{(1-\rho z)^{m+1}} = \sum_{n\geqslant 0}\binom{m+n}{m} a\rho^n z^n$$

- Hence, the coefficient of z^n in expansion of $S(z)$ is

$$s_n = a_1\binom{m_1+n}{m_1}\rho_1^n + a_2\binom{m_2+n}{m_2}\rho_2^n + \cdots a_\ell\binom{m_\ell+n}{m_\ell}\rho_\ell^n.$$

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Step 1: Finding $\rho_1, \rho_2, \ldots, \rho_m$

- Suppose $Q(z)$ has the form

$$Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_m z^m, \qquad \text{where } q_m \neq 0.$$

- The “reflected” polynomial Q^R has a relation to Q:

$$\begin{aligned} Q^R(z) &= z^m + q_1 z^{m-1} + q_2 z^{m-2} + \cdots + q_{m-1} z + q_m \\ &= z^m \left(1 + q_1 \frac{1}{z} + q_2 \frac{1}{z^2} + \cdots + q_{m-1} \frac{1}{z^{m-1}} + q_m \frac{1}{z^m}\right) \\ &= z^m Q\left(\frac{1}{z}\right) \end{aligned}$$

- If $\rho_1, \rho_2, \ldots, \rho_m$ are roots of Q^R, then $(z - \rho_i) | Q^R(z)$:

$$Q^R(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_m)$$

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Step 1: Finding $\rho_1, \rho_2, \ldots, \rho_m$ (2)

In all, we have proven

Lemma

$Q^R(z) = (z-\rho_1)(z-\rho_2)\cdots(z-\rho_m)$ iff $Q(z) = (1-\rho_1 z)(1-\rho_2 z)\cdots(1-\rho_m z)$

Example: $Q(z) = 1 - z - z^2$

$$Q^R(z) = z^2 - z - 1$$

This $Q^R(z)$ has roots

$$z_1 = \frac{1+\sqrt{5}}{2} = \Phi \qquad \text{and} \qquad z_2 = \frac{1-\sqrt{5}}{2} = \widehat{\Phi}$$

Therefore $Q^R(z) = (z-\Phi)(z-\widehat{\Phi})$ and $Q(z) = (1-\Phi z)(1-\widehat{\Phi} z)$.

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