Matroid*

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An Example

Example: Given an undirected graph $G = (V, E)$, Define $\mathbf{H}$ as:

$\mathbf{H} = \{F \subseteq E \mid F$ is a Hamiltonian circuit or a union of disjoint paths$\}$.

Then $(E, \mathbf{H})$ is an independent system.

Proof: (Hereditary)

Given any $F \in \mathbf{H}$ and $P \subset F$.

Since F is either a Hamiltonian circuit or a union of disjoint path, P must be a union of disjoint paths, which obviously belongs to $\mathbf{H}$. □

Independent System

Consider a finite set S and a collection $\mathbf{C}$ of subsets of S.

$(S, \mathbf{C})$ is called an independent system if

$$A \subset B,\ B \in \mathbf{C} \Rightarrow A \in \mathbf{C}.$$

We say that $\mathbf{C}$ is hereditary if it satisfies this property.

Each subset in $\mathbf{C}$ is called an independent subset.

Note that the empty set $\emptyset$ is necessarily a member of $\mathbf{C}$.

Matroid

An independent system $(S, \mathbf{C})$ is a matroid if it satisfies the exchange property:

$$A, B \in \mathbf{C} \text{ and } |A| < |B| \Rightarrow \exists\, x \in B \backslash A \text{ such that } A \cup \{x\} \in \mathbf{C}.$$

Thus a matroid should satisfy two requirements: **hereditary** and **exchange property**.

Matric Matroid

Matric Matroid: Consider a matrix M. Let S be the set of row vectors of M and $\mathbf{C}$ the collection of all linearly independent subsets of S. Then $(S, \mathbf{C})$ is a matroid.

Proof:

- Hereditary: If $A \subset B$ and $B \in \mathbf{C}$, meaning B is a linearly independent subset of row vectors of M, then A must be linearly independent.
- Exchange Property: The exchange property is a well known fact for linearly independence. Say, If A, B are sets of linearly independent rows of M, and $|A| < |B|$, then dim span(A) < dim span(B). Choose a row x in B that is not contained in span(A). Then $A \cup \{x\}$ is a linearly independent subset of rows of M. □

Graphic Matroid

Graphic Matroid M_G: Consider a (undirected) graph $G = (V, E)$. Let $S = E$ and $\mathbf{C}$ the collection of all edge sets each of which induces an acyclic subgraph of G. Then $M_G = (S, \mathbf{C})$ is a matroid.

Proof:

- Hereditary: If B is an edge set which induces an acyclic subgraph of G, obviously any $A \subset B$ induces an acyclic subgraph.
- Exchange Property: consider $A, B \in \mathbf{C}$ with $|A| < |B|$.

  Note that (V, A) has $|V| - |A|$ connected components and (V, B) has $|V| - |B|$ connected components.

  Hence, B has an edge e connecting two connected components of (V, A), which implies $A \cup \{e\} \in \mathbf{C}$. □

More Examples

Uniform matroid $U_{k,n}$: A subset $X \subseteq \{1, 2, \cdots, n\}$ is independent if and only if $|X| \leq k$.

Cographic matroid M_G^*: Let $G = (V, E)$ be an arbitrary undirected graph. A subset $I \subseteq E$ is independent if the complementary subgraph $(V, E \backslash I)$ of G is connected.

Matching matroid: Let $G = (V, E)$ be an arbitrary undirected graph. A subset $I \subseteq V$ is independent if there is a matching in G that covers I.

Disjoint path matroid: Let $G = (V, E)$ be an arbitrary directed graph, and let s be a fixed vertex of G. A subset $I \subseteq V$ is independent if and only if there are edge-disjoint paths from s to each vertex in I.

Notation

The word “matroid" is due to Hassler Whitney[1], who first studied matric matroid (1935).

Actually the greedy algorithm first appeared in the combinatorial optimization literature by Jack Edmonds[2] (1971).

An extension of matroid theory to **greedoid** theory was pioneered by Korte and Lovász, who greatly generalize the theory (1981-1984).

Hassler Whitney
(1907-1989)
Wolf Prize (1983)

[1] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, 57:509-533, 1935.

[2] Jack Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1:126-136, 1971.

Extension

An element x is called an extension of an independent subset I if $x \notin I$ and $I \cup \{x\}$ is independent.

An independent subset is maximal if it has no extension.

For any subset $F \subseteq S$, an independent subset $I \subseteq F$ is maximal in F if I has no extension in F.

Maximal Independent Subset

Consider an independent system $(S, \mathbf{C})$. For $F \subseteq S$, define

$$
\begin{aligned}
u(F) &= \min\{|I| \mid I \text{ is a maximal independent subset of } F\} \\
v(F) &= \max\{|I| \mid I \text{ is an independent subset of } F\}
\end{aligned}
$$

An Example: Maximal Independent Vertex Set

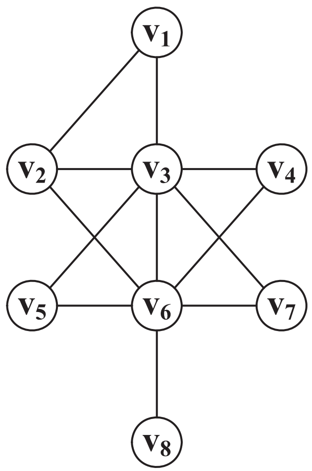
Independent Vertex Set: Given a graph $G = (V, E)$, an independent vertex set is a subset $I \subseteq V$ such that any two vertices in I are not directly connected.

$(V, \mathbf{I})$ is an independent system, where $\mathbf{I}$ is the collection of all independent vertex sets.

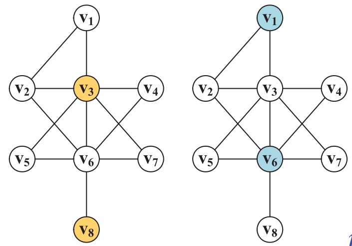
I is **maximal** if $\forall v \in V \backslash I, I \cup \{v\}$ is not an independent vertex set any more. ($u(V)$ is the cardinality value of such an I with minimum cardinality.)

I is **maximum** if it is with the largest cardinality among all maximal independent vertex set. ($v(V)$ is the cardinality value of any maximum independent vertex set I.)

An Independent Vertex Set Instance

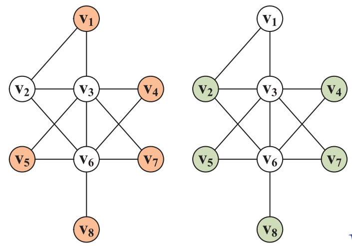


Maximal Independent Vertex Set



$u(V) = 2$

Maximum Independent Vertex Set



$v(V) = 5$

Matroid Theorem

Theorem: An independent system $(S, \mathbf{C})$ is a matroid if and only if for any $F \subseteq S$, $u(F) = v(F)$.

Proof: ($\Rightarrow$) For two maximal independent subsets A and B, if $|A| < |B|$, then there must exist an $x \in B$ such that $A \cup \{x\} \in \mathbf{C}$, contradicting the maximality of A.

($\Leftarrow$) Consider two independent subsets A and B with $|A| < |B|$. Set $F = A \cup B$. Then every maximal independent subset I of F has size $|I| \geq |B| > |A|$. Hence, A cannot be a maximal independent subset of F, so A has an extension in F. $\square$

Thus the definition of matroid could be either by exchange property or by $u(F) = v(F)$.

Corollary

Corollary: All maximal independent subsets in a matroid have the same size.

Proof: (Contradiction) Suppose A and B are two maximal independent subsets with $|A| < |B|$, then A must have an extension in $A \cup B$, which violates its maximality property. $\square$

Basis

In a matroid $(S, \mathbf{C})$, every maximal independent subset of S is called a **basis** (some reference call it **base**).

Example: In a graphic matroid $M_G = (S, \mathbf{C})$, $A \in \mathbf{C}$ is a basis if and only if A is a spanning tree.

Weighted Independent System

An independent system $(S, \mathbf{C})$ with a nonnegative function $c : S \to \mathbb{R}^+$ is called a weighted independent system.

In a weighted matroid, there is a maximum weight independent subset which is a basis.

Note: we can define the associated strictly positive weight function $c(\cdot)$ to each element $x \in S$. Thus the weight function extends to subsets of S by summation:

$$c(A) = \sum_{x \in A} c(x).$$

Greedy Algorithm for Independent System

We give a common greedy algorithm for any independent system $(S, \mathbf{C})$ with cost function c, solving a maximization problem as:

$$\begin{aligned} \text{maximize} \quad & c(I) \\ \text{subject to} \quad & I \in \mathbf{C} \end{aligned}$$

The algorithm is written as:

Algorithm 1: Greedy-MAX

```
Sort all elements in S into ordering c(x_1) ≥ c(x_2) ≥ ··· ≥ c(x_n);
A ← ∅;
for i = 1 to n do
    if A ∪ {x_i} ∈ C then
        A ← A ∪ {x_i};
output A;
```


Time Complexity

Let $n = |S|$ = number elements in S. Then sorting the elements of S requires $O(n \log n)$.

The **for-loop** iterates n times. In the body of the loop one needs to check whether $A \cup \{x\}$ is in $\mathbf{C}$. If each check takes $f(n)$ time, then the loop takes $O(nf(n))$ time.

Thus, Greedy-MAX takes $O(n \log n + nf(n))$ time.

Greedy Theorem for Independent System

Theorem: Consider a weighted independent system. Let A_G be obtained by the Greedy Algorithm. Let A^* be an optimal solution. Then

$$1 \leq \frac{c(A^*)}{c(A_G)} \leq \max_{F \subseteq S} \frac{v(F)}{u(F)}$$

where $v(F)$ is the maximum size of independent subset in F and $u(F)$ is the minimum size of maximal independent subset in F.

Proof

Denote $S_i = \{x_1, \ldots, x_i\}$. (Sorted in nonincreasing order). Then we prove that $S_i \cap A_G$ is a maximal independent subset of S_i.

(By Contradiction) If not, there exists an element $x_j \in S_i \backslash A_G$ such that $(S_i \cap A_G) \cup \{x_j\}$ is independent.

However, at the beginning of the jth iteration of the loop in the Greedy-Max, x_j must be selected into A_G^{j-1}. (Since $A_G^{j-1} \cup \{x_j\}$ must be a subset of $(S_j \cap A_G) \cup \{x_j\}$, and hence, is an independent set.)

Therefore we have $|S_i \cap A_G| \geq u(S_i)$.

Moreover, since $S_i \cap A^*$ is independent, we have $|S_i \cap A^*| \leq v(S_i)$.

Proof (2)

Now we express $c(A_G)$ and $c(A^*)$ in terms of $|S_i \cap A_G|$ and $|S_i \cap A^*|$.

$$\text{Firstly, } |S_i \cap A_G| - |S_{i-1} \cap A_G| = \begin{cases} 1, & \text{if } x_i \in A_G, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} c(A_G) &= \sum_{x_i \in A_G} c(x_i) \\ &= c(x_1) \cdot |S_1 \cap A_G| + \sum_{i=2}^{n} c(x_i) \cdot (|S_i \cap A_G| - |S_{i-1} \cap A_G|) \\ &= \sum_{i=1}^{n-1} |S_i \cap A_G| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A_G| \cdot c(x_n) \end{aligned}$$

Similarly,

$$c(A^*) = \sum_{i=1}^{n-1} |S_i \cap A^*| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A^*| \cdot c(x_n)$$

Proof (3)

Define $\rho = \max_{F \subseteq S} \frac{v(F)}{u(F)}$. Then we have

$$\begin{aligned} c(A^*) &= \sum_{i=1}^{n-1} |S_i \cap A^*| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A^*| \cdot c(x_n) \\ &\leq \sum_{i=1}^{n-1} v(S_i) \cdot (c(x_i) - c(x_{i+1})) + v(S_n) \cdot c(x_n) \\ &\leq \sum_{i=1}^{n-1} \rho \cdot u(S_i) \cdot (c(x_i) - c(x_{i+1})) + \rho \cdot u(S_n) \cdot c(x_n) \\ &\leq \sum_{i=1}^{n-1} \rho \cdot |S_i \cap A_G| \cdot (c(x_i) - c(x_{i+1})) + \rho \cdot |S_n \cap A_G| \cdot c(x_n) \\ &= \rho \cdot c(A_G). \end{aligned}$$

Proof (4)

Thus,

$$1 \leq \frac{c(A^*)}{c(A_G)} \leq \rho = \max_{F \subseteq S} \frac{v(F)}{u(F)}. \qquad \square$$

Note: This theorem implies that if we use Greedy-MAX to find a subset $I \in \mathbf{C}$ with the maximum weight, the result will not be that bad.

It is bounded by the size of the maximum size independent subset of S versus the minimum size maximal independent subset of S. Say,

$$\frac{1}{\rho} \cdot c(A^*) \leq c(A_G) \leq c(A^*).$$

Corollary for Matroid

Corollary: If $(S, \mathbf{C}, c)$ is a weighted matroid, then Greedy-MAX algorithm performs the optimal solution.

Proof: Since in a matroid for any $F \subseteq S$, $u(F) = v(F)$, the corollary can be directly derived from the previous theorem. $\square$

Minimizing or Maximizing?

Let $M = (S, \mathbf{C})$ be a matroid.

The algorithm Greedy-MAX(M, c) returns a set $I \in \mathbf{C}$ maximizing the weight $c(I)$.

If we would like to find a set $I \in \mathbf{C}$ with minimal weight, then we can use Greedy-MAX with weight function

$$c^*(x_i) = m - c(x_i), \qquad \forall x_i \in I,$$

where m is a real number such that $m \geq \max_{x_i \in S} c(x_i)$.

An Example: Graphic Matroid

Minimum Spanning Tree: For a connected graph $G = (V, E)$ with edge weight $c : E \to \mathbb{R}^+$, computing the minimum spanning tree.

If we set $c_{\max} = \max_{e \in E} c(e)$ and define $c^*(e) = c_{\max} - c(e)$, for every edge $e \in E$, then the MST problem is equivalent to find the maximum weight independent subset in the graphic matriod M_G.

This is because every maximum weight independent set is a base, i.e., a spanning tree which contains a fixed number of edges.

$$c^*(A) = (|V| - 1)c_{\max} - c(A).$$

An independent subset that maximizes the quantity $c^*(A)$ must minimize $c(A)$.

An Example (Cont.)

Thus if we implement Greedy-MAX to M_G, we will achieve a solution exactly the same as the Kruskal Algorithm.

We could also use the property of Greedy-MAX on Matroid to validate the correctness of the Kruskal algorithm.

More Examples

Matric matroid: Given a matrix M, compute a subset of vectors of maximum total weight that span the column space of M.

Uniform matroid: Given a set of weighted objects, compute its k largest elements.

Cographic matroid: Given a graph with weighted edges, compute its minimum spanning tree.

Matching matroid: Given a graph, determine whether it has a perfect matching.

Disjoint path matroid: Given a directed graph with a special vertex s, find the largest set of edge-disjoint paths from s to other vertices.

Matroid v.s. Greedy-MAX

Theorem: An independent system $(S, \mathbf{C})$ is a matroid if and only if for any cost function $c(\cdot)$, the Greedy-MAX algorithm gives an optimal solution.

Proof. ($\Rightarrow$) When $(S, \mathbf{C})$ is a matroid, $u(F) = v(F)$ for any $F \subseteq S$. Therefore, Greedy-MAX gives optimal solution.

Next, we show ($\Leftarrow$).

Sufficiency

($\Leftarrow$) For contradiction, suppose independent system $(S, \mathbf{C})$ is not a matroid. Then there exists $F \subseteq S$ such that F has two maximal independent sets I and J with $|I| < |J|$. Define

$$c(e) = \begin{cases} 1+\varepsilon & \text{if } e \in I \\ 1 & \text{if } e \in J \backslash I \\ 0 & \text{if } e \in S \backslash (I \cup J) \end{cases}$$

where ε is a sufficiently small positive number to satisfy $c(I) < c(J)$. Then the Greedy-MAX algorithm will produce I, which is not optimal! □

Unit-Time Task

A **unit-time task** is a job, such as a program to be run on a computer, that requires exactly one unit of time to complete.

Given a finite set S of unit-time tasks, a schedule for S is a permutation of S specifying the order in which to perform these tasks.

For example, the first task in the schedule begins at time 0 and finishes at time 1, the second task begins at time 1 and finishes at time 2, and so on.

Unit-time Task Scheduling Problem

The problem of **scheduling unit-time tasks with deadlines and penalties for a single processor** has the following inputs:

- a set $S = \{1, 2, ..., n\}$ of n unit-time tasks;
- a set of n integer deadlines $d_1, d_2, \dots, d_n$, such that each d_i satisfies $1 \le d_i \le n$ and task i is supposed to finish by time d_i;
- a set of n nonnegative weights or penalties $w_1, w_2, \dots, w_n$, such that a penalty w_i is incurred if task i is not finished by time d_i.

Requirement: find a schedule for S on a machine within time n that minimizes the total penalty incurred for missed deadline.

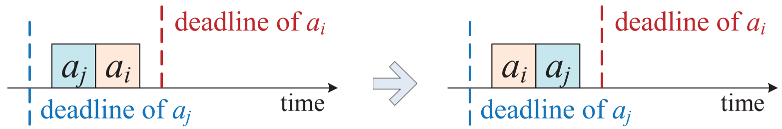
Properties of a Schedule

Given a schedule S, Define:

Early: a task is early in S if it finishes before its deadline.
Late: a task is late in S if it finishes after its deadline.

Early-First Form: S is in the early-first form if the early tasks precede the late tasks.

Claim: An arbitrary schedule can always be put into *early-first form* without changing its penalty value.

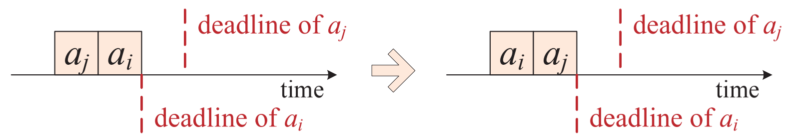


Properties of a Schedule (2)

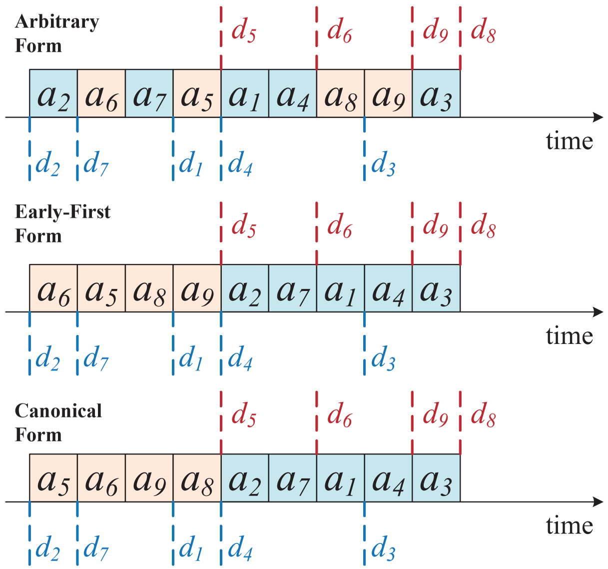
Canonical Form: An arbitrary schedule can always be transformed into *canonical form*, in which the early tasks precede the late tasks and are scheduled in order of monotonically increasing deadlines.

First put the schedule into early-first form.

Then swap the position of any consecutive early tasks a_i and a_j if $d_j > d_i$ but a_j appears before a_i.



An Example



Reduction

The search for an optimal schedule S thus reduces to finding a set A of tasks that we assign to be early in the optimal schedule.

To determine A, we can create the actual schedule by listing the elements of A in order of monotonically increasing deadlines, then listing the late tasks (i.e., $S - A$) in any order, producing a canonical ordering of the optimal schedule.

Independence

Independent: A set of tasks A is independent if there exists a schedule for these tasks without penalty.

Clearly, the set of early tasks for a schedule forms an independent set of tasks. Let $\mathbf{C}$ denote the set of all independent sets of tasks.

For $t = 0, 1, 2, \cdots, n$, let

$N_t(A)$ denote the number of tasks in A whose deadline is t or earlier.

Note that $N_0(A) = 0$ for any set A.

Lemma

Lemma: For any set of tasks A, the statements (1)-(3) are equivalent.
(1). The set A is independent.
(2). For $t = 0, 1, 2, \cdots, n$, $N_t(A) \leq t$.
(3). If the tasks in A are scheduled in order of monotonically increasing deadlines, then no task is late.

Proof:
$\neg(2) \Rightarrow \neg(1)$: if $N_t(A) > t$ for some t, then there is no way to make a schedule with no late tasks for set A, because more than t tasks must finish before time t. Therefore, (1) implies (2).

$(2) \Rightarrow (3)$: there is no way to "get stuck" when scheduling the tasks in order of monotonically increasing deadlines, since (2) implies that the ith largest deadline is at least i.

$(3) \Rightarrow (1)$: trivial. □

Greedy Approach

Use the previous lemma, we can easily compute whether or not a given set of tasks is independent.

The problem of minimizing the sum of the penalties of the late tasks is the same as the problem of maximizing the sum of the penalties of the early tasks.

Thus if $(S, \mathbf{C})$ is a matroid, then we can use Greedy-MAX to find an independent set A of tasks with the maximum total penalty, which is proved to be an optimal solution.

Matroid Theorem

Theorem: Let S be a set of unit-time tasks with deadlines and $\mathbf{C}$ the set of all independent tasks of S. Then $(S, \mathbf{C})$ is a matroid.

Proof: (Hereditary): Trivial.
(Exchange Property): Consider two independent sets A and B with $|A| < |B|$. Let k be the largest t such that $N_t(A) \geq N_t(B)$. Then $k < n$ and $N_t(A) < N_t(B)$ for $k+1 \leq t \leq n$. Choose $x \in \{i \in B \backslash A \mid d_i = k+1\}$.

$$\text{Then,} \quad N_t(A \cup \{x\}) = N_t(A) \leq t, \qquad \text{for } 1 \leq t \leq k,$$

$$\text{and} \quad N_t(A \cup \{x\}) = N_t(A) + 1 \leq N_t(B) \leq t, \qquad \text{for } k+1 \leq t \leq n.$$

Thus $A \cup \{x\} \in \mathbf{C}$. □

The Algorithm

Implementing Greedy-MAX, for any given set of tasks S, we could sort them by penalties and determine the best selections.

Time Complexity: $O(n^2)$.

Sort the tasks takes $O(n \log n)$.

Check whether $A \cup \{x\} \in \mathbf{C}$ takes $O(n)$.

There are totally $O(n)$ iterations of independence check.

Thus the finally complexity is $O(n \log n + n \cdot n) \to O(n^2)$.

An Example

Given an instance of 7 tasks with deadlines and penalties as follows:

a_i	1	2	3	4	5	6	7
d_i	4	2	4	3	1	4	6
w_i	70	60	50	40	30	20	10

Greedy-MAX selects a_1, a_2, a_3, a_4, then rejects a_5, a_6, and finally accepts a_7.

The final schedule is $\langle a_2, a_4, a_1, a_3, a_7, a_5, a_6 \rangle$.

The optimal penalty is $w_5 + w_6 = 50$.