

Sums

ITT9131 Konkreetne Matemaatika

Chapter Two

Notation

Sums and Recurrences

Manipulation of Sums

Multiple Sums

General Methods

Finite and Infinite Calculus

Infinite Sums



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 - Guessing the answer
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Multiple sums

Definition

$$\sum_{i,j} a_{ij} = \sum_i \left(\sum_j a_{ij} [P(i,j)] \right)$$

where P is a predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$ for sets for indexes K_1 and K_2 .



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Generalisation of the law of associativity (law of *interchanging the order of summation*):

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j,k)]$$



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If $a_{jk} = a_j a_k$, then



$$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$$



Multiple sums with independent indices

If $P(j, k) = Q(j) \wedge R(k)$, where Q and R are properties and \wedge indicates the logical conjunction (AND), then the indices j and k are **independent** and the double sum can be rewritten:

$$\begin{aligned}\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k} ([Q(j) \wedge R(k)]) \\ &= \sum_{j,k} a_{j,k} [Q(j)][R(k)] \\ &= \sum_j [Q(j)] \sum_k a_{j,k} R(k) = \sum_j \sum_k a_{j,k} \\ &= \sum_k a_{j,k} [R(k)] \sum_j [Q(j)] = \sum_k \sum_j a_{j,k}\end{aligned}$$



Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j, k) = Q(j) \wedge R'(j, k) = R(k) \wedge Q'(j, k)$$

In this case, we can proceed as follows:

$$\begin{aligned}\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k} [Q(j)] [R'(j, k)] \\ &= \sum_j [Q(j)] \sum_k a_{j,k} [R'(j, k)] = \sum_{j \in J} \sum_{k \in K'} a_{j,k} \\ &= \sum_k [R(k)] \sum_j a_{j,k} [Q'(j, k)] = \sum_{k \in K} \sum_{j \in J'} a_{j,k}\end{aligned}$$

where:

- $J = \{j \mid Q(j)\}, K' = \{k \mid R'(j, k)\}$
- $K = \{k \mid R(k)\}, J' = \{j \mid Q'(j, k)\}$



Warmup: what's wrong with this sum?

$$\begin{aligned}\left(\sum_{j=1}^n a_j\right) \cdot \left(\sum_{k=1}^n \frac{1}{a_k}\right) &= \sum_{j=1}^n \sum_{k=1}^n \frac{a_j}{a_k} \\ &= \sum_{k=1}^n \sum_{j=1}^n \frac{a_k}{a_k} \\ &= \sum_{k=1}^n \sum_{j=1}^n 1 \\ &= n^2\end{aligned}$$



Warmup: what's wrong with this sum?

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Solution

The second passage is **seriously** wrong:

It is not licit to turn two **independent** variables into two **dependent** ones.



Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^n \sum_{k=j}^n a_j a_k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} a_j a_k.$



Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^n \sum_{k=j}^n a_j a_k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} a_j a_k$.

A crucial observation

$$[1 \leq j \leq n][j \leq k \leq n] = [1 \leq j \leq k \leq n] = [1 \leq k \leq n][1 \leq j \leq k]$$

Hence,

$$\sum_{j=1}^n \sum_{k=j}^n a_j a_k = \sum_{k=1}^n \sum_{j=1}^k a_j a_k$$

Also,

$$[1 \leq j \leq k \leq n] + [1 \leq k \leq j \leq n] = [1 \leq j, k \leq n] + [1 \leq j = k \leq n]$$



Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^n \sum_{k=j}^n a_j a_k = \sum_{1 \leq j \leq k \leq n} a_j a_k$.

A crucial observation (cont.)

This can also be understood by considering the following matrix:

$$\begin{pmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 & \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & a_2 a_3 & \dots & a_2 a_n \\ a_3 a_1 & a_3 a_2 & a_3 a_3 & \dots & a_3 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & a_n a_n \end{pmatrix}$$

and observing that $\sum_{j=1}^n \sum_{k=j}^n a_j a_k = S_U$ is the sum of the elements of the **upper triangular part** of the matrix.



Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^n \sum_{k=j}^n a_j a_k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} a_j a_k$.

A crucial observation (end)

If we add to S_U the sum $S_L = \sum_{k=1}^n \sum_{j=1}^k a_j a_k$ of the elements of the **lower triangular part** of the matrix, we count each element of the matrix once, **except those on the main diagonal**, which we count **twice**.

But the matrix is symmetric, so $S_U = S_L$, and

$$S_U = \frac{1}{2} \left(\left(\sum_{k=1}^n a_k \right)^2 + \sum_{k=1}^n a_k^2 \right)$$



Examples of multiple summing

Example 1

$$\begin{aligned} S_n &= \sum_{1 \leq k \leq n} \sum_{1 \leq j < k} \frac{1}{k-j} \\ &= \sum_{1 \leq k \leq n} \sum_{1 \leq k-j < k} \frac{1}{j} \\ &= \sum_{1 \leq k \leq n} \sum_{0 < j \leq k-1} \frac{1}{j} \\ &= \sum_{1 \leq k \leq n} H_{k-1} \\ &= \sum_{1 \leq k+1 \leq n} H_k \\ &= \sum_{0 \leq k < n} H_k \end{aligned}$$



Examples of multiple summing

Example 2

$$\begin{aligned} S_n &= \sum_{1 \leq j \leq n} \sum_{j < k \leq n} \frac{1}{k-j} \\ &= \sum_{1 \leq j \leq n} \sum_{j < k+j \leq n} \frac{1}{k} \\ &= \sum_{1 \leq j \leq n} \sum_{0 < k \leq n-j} \frac{1}{k} \\ &= \sum_{1 \leq j \leq n} H_{n-j} \\ &= \sum_{1 \leq n-j \leq n} H_j \\ &= \sum_{0 \leq j < n} H_j \end{aligned}$$



Examples of multiple summing

Example 3

$$\begin{aligned} S_n &= \sum_{1 \leq j < k \leq n} \frac{1}{k-j} \\ &= \sum_{1 \leq j < k+j \leq n} \frac{1}{k} \\ &= \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq n-k} \frac{1}{k} \\ &= \sum_{1 \leq k \leq n} \frac{n-k}{k} \\ &= \sum_{1 \leq k \leq n} \frac{n}{k} - \sum_{1 \leq k \leq n} 1 \\ &= n \left(\sum_{1 \leq k \leq n} \frac{1}{k} \right) - n = nH_n - n \end{aligned}$$

We have proved: $\sum_{0 \leq k < n} H_k = nH_n - n$



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General Methods: a Review

Example

$$\square_n = \sum_{0 \leq k \leq n} k^2 \quad \text{for } n \geq 0$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
\square_n	0	1	5	14	30	55	91	140	204	285	385	506	650



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Review: Method 0

Example: $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (<http://oeis.org/>)
- *etc*

Possible answer:

$$\square_n = \frac{n(n+1)(2n+1)}{6} \quad \text{for } n \geq 0$$



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Review: Method 1

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
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\square_n/n^2	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52



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\square_n/n^2	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\square_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56



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$n(n+1)$	0	2	6	12	20	30	42	56	72	90



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Hypothesis:

$$\frac{3\square_n}{n(n+1)} = n + \frac{1}{2} \implies \square_n = \frac{n(n+1/2)(n+1)}{3} = \frac{n(n+1)(2n+1)}{6}$$



Review: Method 1

Proof. $3\Box_n = n(n + \frac{1}{2})(n + 1)$

Assume that the formula is true for $n - 1$

We know that $\Box_n = \Box_{n-1} + n^2$

We have

$$\begin{aligned} 3\Box_n &= (n-1)(n - \frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n + \frac{1}{2})(n + 1) \end{aligned}$$



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Q.E.D.



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Review: Method 2

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Perturb the sum.

- Define a sum $\boxplus_n = 0^3 + 1^3 + 2^3 + \dots + n^3$.
- Then we have

$$\begin{aligned}\boxplus_n + (n+1)^3 &= \sum_{0 \leq k \leq n} (k+1)^3 = \sum_{0 \leq k \leq n} (k^3 + 3k^2 + 3k + 1) \\ &= \boxplus_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).\end{aligned}$$

- Delete \boxplus_n and extract \square_n

$$\begin{aligned}3\square_n &= (n+1)^3 - 3(n+1)n/2 - (n+1) \\ &= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)\end{aligned}$$



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Review: Method 3

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Build a repertoire.

Recurrence:
 $R_0 = 0$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$



Review: Method 3

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Build a repertoire.

$R_0 = 0$ Recurrence:

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n$

- Equation: $n = n - 1 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = 1; \beta = \gamma = 0$,
- and the solution has a form $n = A(n)$.



Review: Method 3

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Build a repertoire.

$R_0 = 0$ Recurrence:

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^2$

- Equation: $n^2 = (n-1)^2 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = -1; \beta = 2; \gamma = 0$,
- and hence, the solution has a form $n^2 = -A(n) + 2B(n) = -n + 2B(n)$.
- That gives $B(n) = \frac{n^2 + n}{2}$.



Review: Method 3

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Build a repertoire.

$R_0 = 0$ Recurrence:

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^3$

- Equation: $n^3 = (n-1)^3 + \alpha + \beta n + \gamma n^2 = n^3 - 3n^2 + 3n - 1 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = 1; \beta = -3; \gamma = 3$,
- hence, $n^3 = A(n) - 3B(n) + 3C(n) = n - 3 \frac{n^2+n}{2} + 3C(n)$.
- That gives $6C(n) = 2n^3 - 2n + 3n^2 + 3n = 2n^3 + 3n^2 + n = n(2n+1)(n+1)$.



Review: Method 3

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Build a repertoire.

$R_0 = 0$ Recurrence:

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

To resume

- $R_n = \square_n$ iff $\alpha = \beta = 0; \gamma = 1$
- The solution is

$$\square_n = \frac{n(n+1)(2n+1)}{6}$$



Next subsection

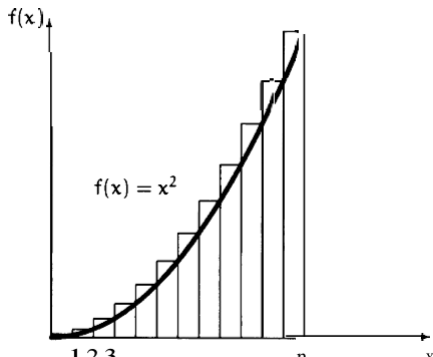
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Review: Method 4

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Replace sums by integrals.



$$\int_0^n x^2 dx = \frac{n^3}{3} \quad (1)$$

$$\square_n = \int_0^n x^2 dx + E_n \quad (2)$$

$$E_n = \sum_{k=1}^n \left(k^2 - \int_{k-1}^k x^2 dx \right) \quad (3)$$



Review: Method 4

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Replace sums by integrals.

Evaluate (3):

$$\begin{aligned} E_n &= \sum_{k=1}^n \left(k^2 - \int_{k-1}^k x^2 dx \right) \\ &= \sum_{k=1}^n \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\ &= \sum_{k=1}^n \left(k - \frac{1}{3} \right) \\ &= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6}. \end{aligned}$$

Finally, from (2) and (1) we get :

$$\square_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$



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Review: Method 5

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

Expand and Contract.

$$\begin{aligned}\square_n &= \sum_{0 \leq k \leq n} k^2 = \sum_{0 \leq j \leq k \leq n} k \\ &= \sum_{0 \leq j \leq n} \sum_{j \leq k \leq n} k \\ &= \sum_{0 \leq j \leq n} \frac{(j+n)(n-j+1)}{2} \\ &= \frac{1}{2} \sum_{0 \leq j \leq n} (j - j^2 + n(n+1)) \\ &= \frac{1}{4} n(n+1) - \frac{1}{2} \square_n + \frac{1}{2} n^2(n+1)\end{aligned}$$



Review: Method 5

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

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Review: Other methods

Example; $\square_n = \sum_{0 \leq k \leq n} k^2$ for $n \geq 0$

- Finite calculus
- Generating functions



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Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$Df(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = Df(x)$$

Leibnitz's notation

If $y = f(x)$, then

$$\frac{dy}{dx} = \frac{df}{dx}(x) = \frac{df(x)}{dx} = Df(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$



Derivative and Difference Operators

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Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$



Derivative and Difference Operators

Infinite calculus: derivative

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Central difference

$$\delta_h[f](x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

$$Df(x) = \lim_{h \rightarrow 0} \frac{\Delta_h[f](x)}{h}$$



Derivative of Power function

Example: $f(x) = x^3$

In this case

$$\begin{aligned}\Delta_h[f](x) &= f(x+h) - f(x) \\ &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= h(3x^2 + 3xh + h^2)\end{aligned}$$

Hence

$$Df(x) = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2$$



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In general:

$$D(x^m) = mx^{m-1}$$



(Forward) Difference of Power Function

Example: $f(x) = x^3$

In this case

$$\Delta f(x) = \Delta_1[f](x) = 3x^2 + 3x + 1$$



(Forward) Difference of Power Function

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$$\Delta f(x) = \Delta_1[f](x) = 3x^2 + 3x + 1$$

In general:

$$\Delta(x^m) = \sum_{k=1}^m \binom{m}{k} x^{m-k}$$



Falling and Rising Factorials

Definition

The **falling factorial power** (or simply **falling factorial**) is defined for $m \geq 0$ by

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

The **rising factorial power** (or simply **rising factorial**) is defined for $m \geq 0$ by

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Properties

$$x^{\underline{m}} = (-1)^m (-x)^{\overline{m}}$$

$$n! = n^{\underline{n}}$$

$$n! = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$x^{\underline{m}} \cdot x^{\underline{n}} = x^{\underline{n}} \cdot x^{\underline{m}} = (x^{\underline{m}})^2 (x-m)^{\underline{n-m}}, \text{ for } n > m$$

$$x^{\underline{m+n}} = x^{\underline{m}} (x-m)^{\underline{n}}$$

$$x^{\underline{m}} = \frac{x^{\underline{m+1}}}{x-m}$$

$$x^{-m} = \frac{x}{x^{\underline{m+1}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$



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$$x^{-m} = \frac{x}{x^{\overline{m+1}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$



Warmup: what is $0^{\underline{m}}$?

Case 1: $m > 0$

Then $0^{\underline{m}} = 0 \cdot (-1)^{\underline{m-1}} = 0$.

Case 2: $m = 0$

Then $0^{\underline{m}} = 1$ because it is defined as an empty product.

Case 3: $m < 0$

Then we want $1 = 0^{\underline{0}} = 0^{\underline{m}} \cdot (0 - m)^{\underline{-m}}$.

But as $m < 0$, $(-m)^{\underline{-m}} = |m|!$, and

$$0^{\underline{m}} = 1/|m|!$$



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Difference of Falling Power Function

$$\begin{aligned}\Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\&= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1) \\&= x(x-1)\cdots(x-m+2)((x+1) - (x-m+1)) \\&= mx(x-1)\cdots(x-m+2) \\&= mx^{\underline{m-1}}\end{aligned}$$



Difference of Falling Power Function

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Hence

$$\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$$



Difference of Falling Power Function (2)

Let's check this formula for negative power:

$$\begin{aligned}\Delta x^{-2} &= (x+1)^{-2} - x^{-2} \\&= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\&= \frac{(x+1) - (x+3)}{(x+1)(x+2)(x+3)} \\&= -\frac{2}{(x+1)(x+2)(x+3)} \\&= -2x^{-3}\end{aligned}$$



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Indefinite Integrals and Sums

Fundamental Theorem of Calculus

$$Df(x) = g(x) \quad \text{iff} \quad \int g(x)dx = f(x) + C$$

Definition

The indefinite sum of function $g(x)$ is a class of functions whose difference is $g(x)$:

$$\Delta f(x) = g(x) \quad \text{iff} \quad \sum g(x)\delta x = f(x) + C(x),$$

where $C(x)$ is a "periodic function" such that $C(x+1) = C(x)$ for any integer value of x .



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Definite Integrals and Sums

If $g(x) = Df(x)$, then

$$\int_a^b g(x) dx = f(x) \Big|_a^b = f(b) - f(a)$$

Analogously:

If $g(x) = \Delta f(x)$, then

$$\sum_a^b g(x) \delta x = f(x) \Big|_a^b = f(b) - f(a)$$



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Definite sums

Observations

- $\sum_a^a g(x) \delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x) \delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x) \delta x - \sum_a^b g(x) \delta x =$
 $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$



Definite sums

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 $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$

Hence

$$\begin{aligned}\sum_a^b g(x) \delta x &= \sum_{k=a}^{b-1} g(k) = \sum_{a \leq k < b} g(k) \\ &= \sum_{a \leq k < b} (f(k+1) - f(k)) = \\ &= (f(a+1) - f(a)) + (f(a+2) - f(a+1)) + \cdots \\ &\quad + (f(b-1) - f(b-2)) + (f(b) - f(b-1)) \\ &= f(b) - f(a)\end{aligned}$$



Integrals and Sums of Powers

If $m \neq -1$, then

$$\int_0^n x^m dx = \frac{x^{m+1}}{m+1} \Big|_0^n = \frac{n^{m+1}}{m+1}$$

Analogous finite case:

If $m \neq -1$, then

$$\sum_0^n k^m \delta x = \sum_{0 \leq k < n} k^m = \frac{k^{m+1}}{m+1} \Big|_0^n = \frac{n^{m+1}}{m+1}$$



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Sums of Powers: applications

Case $m = 1$

$$\sum_{0 \leq k < n} k = \frac{n^2}{2} = \frac{n(n-1)}{2}$$

Case $m = 2$ Due to $k^2 = k^2 + k^1$ we get

$$\begin{aligned}\sum_{0 \leq k < n} k^2 &= \frac{n^3}{3} + \frac{n^2}{2} \\&= \frac{1}{3}n(n-1)(n-2) + \frac{1}{2}n(n-1) \\&= \frac{1}{6}n(2(n-1)(n-2) + 3(n-1)) \\&= \frac{1}{6}n(n-1)(2n-4+3) \\&= \frac{1}{6}n(n-1)(2n-1)\end{aligned}$$

Taking $n+1$ instead of n gives

$$\square_n = \frac{(n+1)n(2n+1)}{6}$$



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Taking $n+1$ instead of n gives

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Sums of Powers (case $m = -1$)

Note

$$\begin{aligned}\Delta H_x &= H_{x+1} - H_x \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} + \frac{1}{x+1} - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{x} \\ &= \frac{1}{x+1}\end{aligned}$$

and

$$\sum_a^b x^{-1} \delta x = H_x \Big|_a^b$$



Sums of Discrete Exponential Functions

- We have

$$De^x = e^x.$$

Finite analogue should have $\Delta f(x) = f(x)$. That means:

$$f(x+1) - f(x) = f(x) \quad \text{iff} \quad f(x+1) = 2f(x) \quad \text{iff} \quad f(x) = 2^x$$

- The difference of c^x is

$$\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x$$

and anti-difference is then $c^x/(c-1)$, if $c \neq 1$ that gives the the sum of geometric progression

$$\sum_{a \leq k < b} c^k = \sum_a^b c^x \delta x = \left. \frac{c^x}{c-1} \right|_a^b = \frac{c^b - c^a}{c-1}$$



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Summation by Parts

Infinite analogue: integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Difference of product:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= u(x)\Delta v(x) + v(x+1)\Delta u(x) \\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the **shift operator** $Ef(x) = f(x+1)$. Taking the indefinite sum from both sides yields

Rule for summation by parts:

$$\sum u\Delta v = uv - \sum Ev\Delta u$$



Summation by Parts

Infinite analogue: integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Difference of product:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= u(x)\Delta v(x) + v(x+1)\Delta u(x) \\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the **shift operator** $Ef(x) = f(x+1)$. Taking the indefinite sum from both sides yields

Rule for summation by parts:

$$\sum u\Delta v = uv - \sum Ev\Delta u$$



Warmup: why the asymmetry?

How is it that, in the rule for summation by parts:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

the left-hand side is symmetric in u and v , but the right-hand side is not?



Warmup: why the asymmetry?

How is it that, in the rule for summation by parts:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

the left-hand side is symmetric in u and v , but the right-hand side is not?

Because the symmetry is elsewhere!

We can also write:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - \cancel{u(x+1)v(x)} + \cancel{u(x+1)v(x)} - u(x)v(x) \\ &= Eu(x)\Delta v(x) + v(x)\Delta u(x)\end{aligned}$$

So there actually is a symmetry—just not the one we thought:

$$u\Delta v + Ev\Delta u = v\Delta u + Eu\Delta v$$



Summation by Parts (2)

Example: $S = \sum_{k=0}^n k 2^k$

- Taking $u(x) = x$, $v(x) = 2^x$ and $Ev(x) = 2^{x+1}$:

$$\sum x 2^x \delta x = x 2^x - \sum 2^{x+1} \delta x = x 2^x - 2^{x+1} + C$$

This yields

$$\begin{aligned}\sum_{k=0}^n k 2^k &= \sum_0^{n+1} x 2^x \delta x \\ &= (x 2^x - 2^{x+1}) \Big|_0^{n+1} \\ &= ((n+1) 2^{n+1} - 2^{n+2}) - (0 \cdot 2^0 - 2) \\ &= (n-1) 2^{n+1} + 2\end{aligned}$$



Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

Continuous analogue:

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} \\ &= \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right)\end{aligned}$$



Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

- Taking $u(x) = H_x$, $v(x) = \frac{x^2}{2}$, $\Delta u(x) = \Delta H_x = x^{-1} = \frac{1}{x+1}$, $\Delta v(x) = x = x^1$ and $Ev(x) = \frac{(x+1)^2}{2}$, we get

$$\begin{aligned}\sum_{k=0}^{n-1} kH_k &= \sum_0^n xH_x \delta x = uv \Big|_0^n + \sum_0^n Ev \Delta u \delta x \\ &= \frac{x^2}{2} H_x \Big|_0^n - \sum_0^n \frac{(x+1)^2}{2} \cdot x^{-1} \delta x \\ &= \frac{x^2}{2} H_x \Big|_0^n - \frac{1}{2} \sum_0^n x \delta x \\ &= \left(\frac{x^2}{2} H_x - \frac{1}{2} \cdot \frac{x^2}{2} \right) \Big|_0^n \\ &= \frac{n^2}{2} \left(H_n - \frac{1}{2} \right)\end{aligned}$$



Next section

- 1 Multiple sums
- 2 General Methods
 - Looking up
 - Guessing the answer
 - Perturbation
 - Build a repertoire
 - Integrals
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 - More methods
- 3 Finite and Infinite Calculus
 - Derivative and Difference Operators
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- 4 Table of Differences
- 5 Infinite Sums



Instead of Conclusion: Table of Differences

$f = \Sigma g$	$\Delta f = g$	$f = \Sigma g$	$\Delta f = g$
$x^0 = 1$	0	2^x	2^x
$x^1 = x$	1	c^x	$(c-1)c^x$
$x^2 = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
x^m	mx^{m-1}	cf	$c\Delta f$
$x^{m+1}/(m+1)$	x^m	$f+g$	$\Delta f + \Delta g$
H_x	$x^{-1} = 1/(x+1)$	fg	$f\Delta g + Eg\Delta f$



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How to sum infinite number sequences?

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2.$$



How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \dots = T - 1,$$

and

$$T = -1.$$



How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geq 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots = ?$$

Different ways to sum

$$(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \dots = 0 + 0 + 0 + 0 + \dots = 0$$

and

$$1 - (1 - 1) - (1 - 1) - (1 - 1) - (1 - 1) - \dots = 1 - 0 - 0 - 0 - 0 - 0 - \dots = 1$$



Defining Infinite Sums: Nonnegative values

Definition 1

If $a_k \geq 0$ for every $k \geq 0$, then

$$\sum_{k \geq 0} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = \sup_{K \subseteq \mathbb{N} \text{ finite}} \sum_{k \in K} a_k$$

Commutative property for infinite sums of nonnegative values

If $a_k \geq 0$ for every $k \geq 0$ and $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ is a **permutation** of the set of natural numbers, then

$$\sum_{k \geq 0} a_k = \sum_{k \geq 0} a_{\sigma(k)}$$



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Defining Infinite Sums: Arbitrary values

Every real number x can be written in the form $x = x^+ - x^-$, where:

- $x^+ = x \cdot [x > 0] = \max(0, x)$ is the *positive part* of x .
- $x^- = -x \cdot [x < 0] = \max(0, -x)$ is the **negative part** of x .

Observe that $|x| = x^+ + x^-$.

Definition 2

Let K be a set and a_k a real number for every $k \in K$, then:

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$



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Convergent and divergent series

The sum $\sum_k a_k$ is:

- **absolutely convergent** if both $\sum a_k^+$ and $\sum a_k^-$ are finite.

In this case, $\sum_{k \in \mathbb{N}} a_k^+ - \sum_{k \in \mathbb{N}} a_k^- = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k$, and the infinite sum has the commutative property.

- **positively divergent** if $\sum a_k^+ = \infty$ and $\sum a_k^-$ is finite.

In this case, it is licit to put $\sum_k a_k = +\infty$.

- **negatively divergent** if $\sum a_k^- = \infty$ and $\sum a_k^+$ is finite.

In this case, it is licit to put $\sum_k a_k = -\infty$.

If both $\sum a_k^+ = \infty$ and $\sum a_k^- = \infty$, then "all bets are off":

Riemann series theorem

Let $\{a_k\}_{k \geq 0}$ a sequence of real numbers such that:

$$\sum_{k \geq 0} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = S \in \mathbb{R} \text{ but } \sum_{k \geq 0} |a_k| = +\infty$$

For every $M \in \mathbb{R}$ there exists a permutation σ of \mathbb{N} such that:

$$\sum_{k \geq 0} a_{\sigma(k)} = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_{\sigma(k)} = M$$



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