

Divide and Conquer*

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Outline

- 1 Divide-and-Conquer
 - Basic Technique
 - An Introductory Example: Multiplication
 - Recurrence Relations
- 2 Applications
 - Binary Search
 - Merge Sort
 - Matrix Multiplication

Divide-and-Conquer Strategy

The **divide-and-conquer strategy** solves a problem P by:

- (1) Breaking P into smaller subproblems of the same type.
- (2) Recursively solving these subproblems.
- (3) Appropriately combining their answers.

The **key works** lay in three different places:

- (1) How to partition problem into subproblems.
- (2) At the very tail end of the recursion, how to solve the smallest subproblems outright.
- (3) How to glue together the partial answers.

Johann C.F. Gauss



Johann Carl Friedrich Gauss

1777 - 1855

$$1 + 2 + \cdots + 100 = \frac{100 \cdot (1 + 100)}{2} = 5050.$$

Multiplication for Complex Numbers

Gauss once noticed that although the product of two complex numbers

$$(a + bi)(c + di) = ac - bd + (bc + ad)i$$

seems to involve **four** real-number multiplications, it can in fact be done with just **three**: ac , bd , and $(a + b)(c + d)$, since

$$bc + ad = (a + b)(c + d) - ac - bd.$$

In our big-O way of thinking, reducing the number of multiplications from four to three seems wasted ingenuity. However, this modest improvement becomes *very significant when applied recursively*.

Multiplication for Integers

Our method for multiplying n -bit numbers starts by making recursive calls to multiply these four pairs of $n/2$ -bit numbers, and then evaluates the preceding expression in $O(n)$ time.

Writing $T(n)$ for the overall running time on n -bit inputs, we get **the recurrence relation**:

$$T(n) = 4T(n/2) + O(n)$$

Optimization: By **Gauss's** trick, three multiplications, $x_L y_L$, $x_R y_R$, and $(x_L + x_R)(y_L + y_R)$, suffice, as

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R.$$

Multiplication for Integers

Suppose x and y are two n -bit integers, and assume for convenience that n is **a power of 2**.

Lemma: $\forall n \in \mathbb{N}, \exists n' \text{ with } n \leq n' \leq 2n \text{ such that } n' \text{ is a power of 2.}$

As a first step toward multiplying x and y , we split each of them into their **left and right halves**, which are $n/2$ bits long:

$$\begin{aligned} x &= \boxed{x_L} \boxed{x_R} = 2^{n/2} x_L + x_R \\ y &= \boxed{y_L} \boxed{y_R} = 2^{n/2} y_L + y_R. \end{aligned}$$

$$xy = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) = 2^n x_L y_L + 2^{n/2} (x_L y_R + x_R y_L) + x_R y_R.$$

The additions take linear time, as do the multiplications by powers of 2 (merely left-shifts). The significant operations are the four $n/2$ -bit **multiplications**; these we can handle by **four recursive calls**.

A Divide-and-Conquer Algorithm for Integer Multiplication

Algorithm 1: MULTIPLY(x, y)

Input: Positive integers x and y , in binary.

Output: Their product xy .

- 1 $n = \max(\text{size of } x, \text{size of } y)$ rounded as a power of 2 ;
- 2 **if** $n = 1$ **then**
- 3 \quad **return** xy ;
- 4 $x_L, x_R =$ leftmost $n/2$, rightmost $n/2$ bits of x ;
- 5 $y_L, y_R =$ leftmost $n/2$, rightmost $n/2$ bits of y ;
- 6 $P_1 = \text{MULTIPLY}(x_L, y_L)$;
- 7 $P_2 = \text{MULTIPLY}(x_R, y_R)$;
- 8 $P_3 = \text{MULTIPLY}(x_L + x_R, y_L + y_R)$;
- 9 **return** $P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$

New recurrence relation: $T(n) = 3T(n/2) + O(n) \rightarrow$ How well?

Master Theorem

If

$$T(n) = aT(\lceil n/b \rceil) + O(n^d)$$

for some constants $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a. \end{cases}$$

Proof of Master Theorem

Assume that n is a power of b . This will not influence the final bound in any important way: n is at most a multiplicative factor of b away from some power of b .

The size of the subproblems decreases by a factor of b with each level of recursion, and reaches the base case when

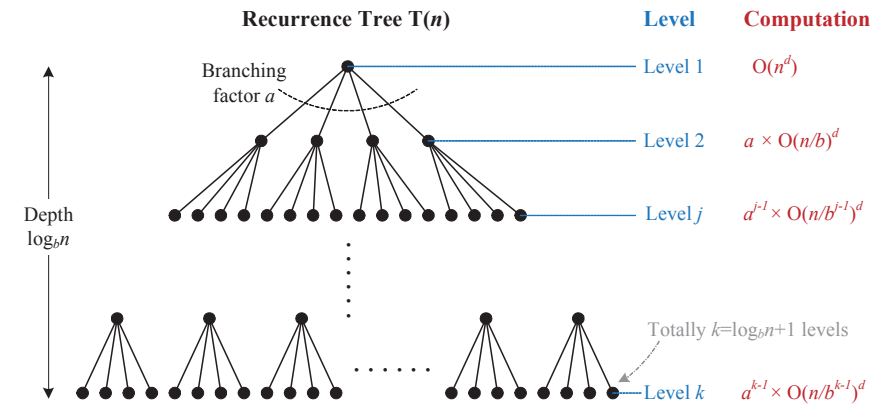
$$\frac{n}{b^{k-1}} = 1 \Rightarrow k = \log_b n + 1$$

(k is the level of the recursion tree, which equals to tree height + 1.)

The branching factor of the recursion tree is a , so the j -th level of the tree is made up of a^{j-1} subproblems, each of size n/b^{j-1} .

The total work done at the j -th level is

$$a^{j-1} \times O\left(\frac{n}{b^{j-1}}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^{j-1}.$$

Proof of Master Theorem: $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ 

Complexity of $T(n)$ = Sum up all computations at each level.

Proof of Master Theorem

The total work done is

$$\sum_{j=1}^{\log_b n + 1} \left(a^{j-1} \times O\left(\frac{n}{b^{j-1}}\right)^d \right) = \sum_{j=0}^{\log_b n} \left(O(n^d) \times \left(\frac{a}{b^d}\right)^j \right) = O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^j.$$

It's the sum of a *geometric series (GS)* with **ratio a/b^d** .

$$(1) \quad \frac{a}{b^d} < 1 \Rightarrow d > \log_b a:$$

$$O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^j \leq O(n^d) \frac{1}{1 - \frac{a}{b^d}} = O(n^d).$$

$$(\text{Sum of GS: } S_n = \sum_{j=1}^n a_1 q^{j-1} = a_1 \frac{1-q^n}{1-q} \leq a_1 \frac{1}{1-q} \text{ if } q < 1)$$

Proof of Master Theorem

$$(2) \frac{a}{b^d} = 1 \Rightarrow d = \log_b a:$$

$$O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^j = O(n^d)(\log_b n + 1) = O(n^d \log_b n) = O(n^d \log n).$$

$$(\log_b n = \frac{\log n}{\log b} = \frac{1}{\log b} \log n = O(\log n) \text{ by changing the base})$$

Proof of Master Theorem

$$(3) \frac{a}{b^d} > 1 \Rightarrow d < \log_b a: \text{ (reverse the GS in decreasing order)}$$

$$\begin{aligned} O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^j &= O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^{\log_b n} \cdot \left(\frac{b^d}{a}\right)^j \\ &= O(n^d) \sum_{j=0}^{\log_b n} \frac{a^{\log_b n}}{(b^{\log_b n})^d} \cdot \left(\frac{b^d}{a}\right)^j \\ &\leq O(n^d) \frac{n^{\log_b a}}{n^d} \cdot \frac{1}{1 - \frac{b^d}{a}} \\ &= O(n^{\log_b a}) \end{aligned}$$

$$(a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a})$$

Time Complexity of Multiplication

Original recurrence relation: $T(n) = 4T(n/2) + O(n)$

$$a = 4, b = 2, d = 1, d < \log_b a.$$

\Rightarrow **Time complexity:** $O(n^{\log_b a}) = O(n^2)$.

Optimized recurrence relation: $T(n) = 3T(n/2) + O(n)$

$$a = 3, b = 2, d = 1, d < \log_b a.$$

\Rightarrow **Time complexity:** $O(n^{\log_2 3}) \approx O(n^{1.59})$.

Binary Search

Algorithm 2: BinarySearch

Input: An array $A[1..n]$ of n elements sorted in nondecreasing order and an element x .

Output: j if $x = A[j]$, $1 \leq j \leq n$, and 0 otherwise.

```

1 low ← 1; high ← n; j ← 0;
2 while low ≤ high and j = 0 do
3   mid ← ⌊(low + high)/2⌋;
4   if x = A[mid] then
5     | j ← mid break;
6   else if x < A[mid] then
7     | high ← mid - 1;
8   else
9     | low ← mid + 1;
10 return j;
```

Time Complexity

To find a key x in $A[1, \dots, n]$ in sorted order, we first compare x with $A[n/2]$, and depending on the result we recurse either on the first half $A[1, \dots, n/2 - 1]$, or on the second half $A[n/2 + 1, \dots, n]$.

The recurrence function is

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(1),$$

By Master Theorem, $a = 1$, $b = 2$, $d = 0$, and thus the running time should be $O(\log n)$.

Analysis of Merge

Suppose $A[p..q]$ has m elements and $A[q + 1..r]$ has n elements. The number of comparisons done by Algorithm Merge is

- at least $\min\{m, n\}$;

E.g.

2	3	6
---	---	---

 and

7	11	13	45	57
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- at most $m + n - 1$.

E.g.

2	3	66
---	---	----

 and

7	11	13	45	57
---	----	----	----	----

If the two array sizes are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, the number of comparisons is between $\lfloor n/2 \rfloor$ and $n - 1$.

Merging Two Sorted Lists

Algorithm 3: Merge

Input: $A[1..m]$, p , q and r with $1 \leq p \leq q < r \leq m$.

Output: $A[p..r]$ (merging two sorted subarrays $A[p..q]$, $A[q + 1..r]$).

```

1  $s \leftarrow p$ ;  $t \leftarrow q + 1$ ;  $k \leftarrow p$ ;
2 while  $s \leq q$  and  $t \leq r$  do
3   if  $A[s] \leq A[t]$  then
4      $B[k] \leftarrow A[s]$ ;  $s \leftarrow s + 1$ ; ( $B[p..r]$  is an auxiliary array)
5   else  $B[k] \leftarrow A[t]$ ;  $t \leftarrow t + 1$ ;
6    $k \leftarrow k + 1$ ;
7 if  $s = q + 1$  then
8    $B[k..r] \leftarrow A[t..r]$ ;
9 else  $B[k..r] \leftarrow A[s..q]$ ;
10 return  $A[p..r] \leftarrow B[p..r]$ ;
```

Bottom-Up MergeSort Algorithm

Algorithm 4: MergeSort

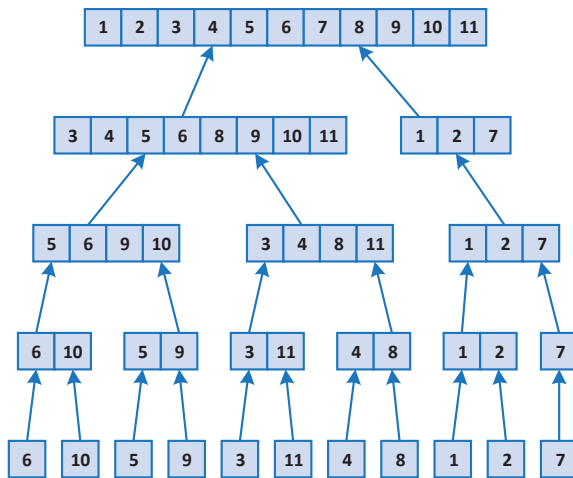
Input: An array $A[1..n]$ of n elements.

Output: $A[1..n]$ sorted in nondecreasing order.

```

1  $t \leftarrow 1$ ;
2 while  $t < n$  do
3    $s \leftarrow t$ ;  $t \leftarrow 2s$ ;  $i \leftarrow 0$ ;
4   while  $i + t \leq n$  do
5      $\text{Merge}(A, i + 1, i + s, i + t)$ ;
6      $i \leftarrow i + t$ ;
7   if  $i + s < n$  then
8      $\text{Merge}(A, i + 1, i + s, n)$ ;
9 return  $A[1..n]$ ;
```

An Example

**Time Complexity:**

Recurrence:

$$T(n) = 2T(n/2) + O(n);$$

By Master Theorem

$$T(n) = O(n \log n).$$

An $n \log n$ Lower Bound for Sorting

Sorting algorithms can be depicted as **trees**.

The **depth** of the tree – the number of comparisons on the longest path from root to leaf, is exactly the worst-case time complexity of the algorithm.

Consider any such tree that sorts an array of n elements. Each of its leaves is labeled by a **permutation** of $\{1, 2, \dots, n\}$.

every permutation must appear as the label of a leaf.

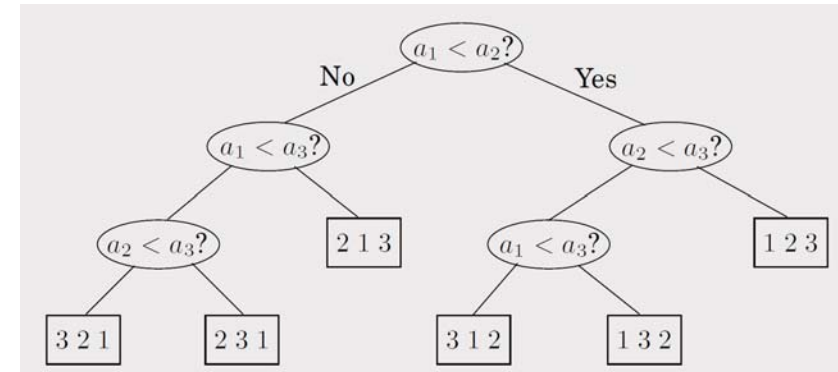
This is a binary tree with $n!$ leaves. Thus, the depth of our tree – and the complexity of our algorithm – must be at least

$$\log(n!) \approx \log\left(\sqrt{\pi(2n+1/3)} \cdot n^n \cdot e^{-n}\right) = \Omega(n \log n),$$

where we use **Stirling's formula**.

An $n \log n$ Lower Bound for Sorting

An example **sorting permutation tree** for $\{a_1, a_2, a_3\}$:



The product of two $n \times n$ matrices X and Y is a third $n \times n$ matrix $Z = XY$, with (i, j) th entry

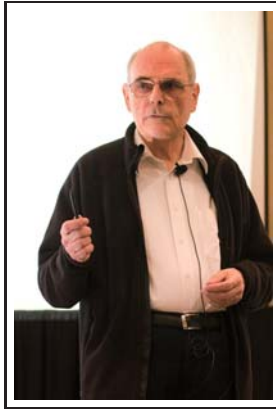
$$Z_{ij} = \sum_{k=1}^n X_{ik} Y_{kj}$$

That is, Z_{ij} is the **dot product** of the i th row of X with the j th column of Y .

In general, XY is not the same as YX ; **matrix multiplication is not commutative**.

The preceding formula implies an $O(n^3)$ algorithm for matrix multiplication.

Volker Strassen



Volker Strassen (1936 –)

In 1969, the German mathematician **Volker Strassen** announced a surprising $O(n^{2.81})$ algorithm.

Divide and conquer

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Then

$$XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

To compute the size- n product XY , recursively compute eight size- $n/2$ products $AE, BG, AF, BH, CE, DG, CF, DH$ and then do some $O(n^2)$ -time addition.

The recurrence is

$$T(n) = 8T(n/2) + O(n^2)$$

with solution $O(n^3)$.

Strassen's trick

$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

where

$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) \end{aligned}$$

The recurrence is

$$T(n) = 7T(n/2) + O(n^2)$$

with solution $O(n^{\log_2 7}) \approx O(n^{2.81})$.