Linear Programming *

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Outline

An Example : Profit Maximization

Suppose that a factory produces two products **1** and **2**, using resources **A**, **B**, and **C**, under following settings :

- We have 200 units of **A**, 300 units of **B**, and 400 units of **C**.
- Making a unit of product **1** requires a unit of **A** and **C**.
- Making a unit of product **2** requires a unit of **B** and **C**.
- The price for product **1** and **2** are respectively 1 and 6.

The factory aims to achieve the **maximum** profit, so how many units of product **1** and **2** the factory should produce ?

An Example : Profit Maximization

Suppose we produce x_1 and x_2 units for product **1** and **2**.

According to the settings, we have

- 200 units **A**, one for each product **1** : $x_1 \leq 200$.
- 300 units **B**, one for each product **2** : $x_2 \leq 300$.
- 400 units **C**, one for both **1** and **2** : $x_1 + x_2 \leq 400$.
- Nonnegative Production : $x_1, x_2 \geq 0$.

Maximizing Profits : $\max \quad f(x_1, x_2) = x_1 + 6x_2$.

Formulation of Linear Programming

$$\begin{aligned} \max \quad & f(x_1,x_2) = x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Linear Programming (LP) :
Both objective function and constraints are linear.

Standard Form of LP

Given

- n real numbers $c_1, c_2, \cdots, c_n$;
- m real numbers $b_1, b_2, \cdots, b_m$;
- $m \times n$ real numbers $\{a_{ij}\}_{i=1,2,\cdots,m;\ j=1,2,\cdots,n}$.

We wish to find n real numbers $x_1, x_2, \cdots, x_n$ such that

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\ & x_j \geq 0. \quad j = 1, 2, \cdots, n \end{aligned}$$

Variants of LP

An LP problem may NOT appear as its standard form initially :

- It can be a minimization problem. $\min \sum_{j=1}^{n} c_j x_j$
- It may have equality constraints. $\sum_{j=1}^{n} a_{ij} x_j = b_i$
- It may have inequality constraints with $\geq$. $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$
- It may have no restriction on the variables $\{x_i\}_{i=1}^{n}$.

Example :

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ s.t. \quad & x_1 + x_2 = 7, \\ & x_1 - 2x_2 \leq 4, \\ & x_1 \geq 0. \end{aligned}$$

However, we can equivalently transform different LP variants to the standard form.

Transformation to Standard Form

1. From min **to** max **:**

$$\min \quad \textstyle\sum_{j=1}^{n} c_j x_j \quad \Rightarrow \quad \max \quad -\textstyle\sum_{j=1}^{n} c_j x_j$$

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ s.t. \quad & x_1 + x_2 = 7, \\ & x_1 - 2x_2 \leq 4, \\ & x_1 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 2x_1 - 3x_2 \\ s.t. \quad & x_1 + x_2 = 7, \\ & x_1 - 2x_2 \leq 4, \\ & x_1 \geq 0. \end{aligned}$$

Transformation to Standard Form

2. Equality Constraint :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad \Rightarrow \quad \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le b_i \\ \sum_{j=1}^{n} a_{ij}x_j \ge b_i \end{cases}$$

$$\begin{array}{ll} \max & 2x_1 - 3x_2 \\ s.t. & x_1 + x_2 = 7, \\ & x_1 - 2x_2 \le 4, \\ & x_1 \ge 0. \end{array} \quad \Rightarrow \quad \begin{array}{ll} \max & 2x_1 - 3x_2 \\ s.t. & x_1 + x_2 \le 7, \\ & x_1 + x_2 \ge 7, \\ & x_1 - 2x_2 \le 4, \\ & x_1 \ge 0. \end{array}$$

Transformation to Standard Form

3. Inequality Constraint with $\ge$:

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i \quad \Rightarrow \quad -\sum_{j=1}^{n} a_{ij}x_j \le -b_i$$

$$\begin{array}{ll} \max & 2x_1 - 3x_2 \\ s.t. & x_1 + x_2 \le 7, \\ & x_1 + x_2 \ge 7, \\ & x_1 - 2x_2 \le 4, \\ & x_1 \ge 0. \end{array} \quad \Rightarrow \quad \begin{array}{ll} \max & 2x_1 - 3x_2 \\ s.t. & x_1 + x_2 \le 7, \\ & -x_1 - x_2 \le -7, \\ & x_1 - 2x_2 \le 4, \\ & x_1 \ge 0. \end{array}$$

Transformation to Standard Form

4. Variables without Constraints :

x_2 is without constraints. $\Rightarrow$ Introducing x_2^+ and x_2^-
$x_2 = x_2^+ - x_2^-, x_2^+, x_2^- \ge 0.$

$$\begin{array}{ll} \max & 2x_1 - 3x_2 \\ s.t. & x_1 + x_2 \le 7, \\ & -x_1 - x_2 \le -7, \\ & x_1 - 2x_2 \le 4, \\ & x_1 \ge 0. \end{array} \quad \Rightarrow \quad \boxed{\begin{array}{ll} \max & 2x_1 - 3x_2^+ + 3x_2^- \\ s.t. & x_1 + x_2^+ - x_2^- \le 7, \\ & -x_1 - x_2^+ + x_2^- \le -7, \\ & x_1 - 2x_2^+ + 2x_2^- \le 4, \\ & x_1, x_2^+, x_2^- \ge 0. \end{array}}$$

Standard Form

Slack Form of LP

In order to efficiently solve an LP problem, we express it in a form in which some of the constraints are **equality constraints**.

Specifically,

- $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ should be transformed to equality ;
- The only inequality constraints are $x_i \ge 0$.

Key : Introducing slack variables : s_i

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, s_i \ge 0$$

Slack Form of LP

Example : Profit Maximization

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \le 400, \\ & x_1 \le 200, \\ & x_2 \le 300, \\ & x_1, x_2 \ge 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 + x_3 = 400, \\ & x_1 + x_4 = 200, \\ & x_2 + x_5 = 300, \\ & x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

x_3, x_4, x_5 are slack variables.

Slack Form of LP

We can use a tuple $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$ to represent a slack form.

- N : Nonbasic Variable Set. $\{x_1, x_2\}$
- B : Basic Variable Set. $\{x_3, x_4, x_5\}$
- $\mathbf{A}, \mathbf{b}, \mathbf{c}$: Constant Terms and Coefficients.
- v : Optional Constant Term in Objective Function.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 400 & 200 & 300 \end{bmatrix}^T$$

$$\mathbf{c} = \begin{bmatrix} 1 & 6 \end{bmatrix}^T$$

Matrix-Vector Form of LP

Sometimes it is convenient to express an LP by matrix and vectors.

If we create

- an $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$;
- an m vector $\mathbf{b} = (b_1, b_2, \cdots, b_m)^T$;
- an n vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$;
- an n vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$,

then we can equivalently transform the standard form as

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \cdots, m \\ & x_j \ge 0. \quad j = 1, 2, \cdots, n \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

General Form of Programming

The general form of a programming is

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ s.t. \quad & g_i(\mathbf{x}) \le 0, \quad i = 1, 2, \cdots, m \\ & h_i(\mathbf{x}) = 0. \quad i = m+1, m+2, \cdots, n \end{aligned}$$

Linear Programming should satisfy that $f(\mathbf{x})$, $\{g_i(\mathbf{x})\}$, $\{h_i(\mathbf{x})\}$ are all linear functions.

Classification of Programming

$$
\begin{aligned}
\max \quad & f(\mathbf{x}) \\
s.t. \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \cdots, m \\
& h_i(\mathbf{x}) = 0. \quad i = m+1, m+2, \cdots, n
\end{aligned}
$$

We can classify programming as

- $\begin{cases} \text{Programming \textbf{with} constraints ; (We only study this.)} \\ \text{Programming \textbf{without} constraints.} \end{cases}$
- $\begin{cases} \text{\textbf{Linear} programming ;} \\ \text{\textbf{Nonlinear} programming. (including quadratic)} \end{cases}$
- $\begin{cases} \text{\textbf{Single}-objective programming ; (We only study this.)} \\ \text{\textbf{Multiple}-objective programming.} \end{cases}$
- $\cdots$

Integer Linear Programming (ILP)

ILP is an LP problem with an additional constraint that variables $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ must take on integral values.

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\
& x_j \in \mathbb{Z}. \quad j = 1, 2, \cdots, n.
\end{aligned}
$$

Examples : the amount of products, people, data packets,...

Note : ILP is an **NP** problem, which no efficient algorithms can solve *directly*.

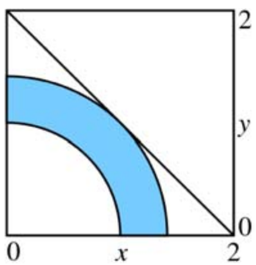
Nonlinear Programming

Nonlinear Programming : At least one of f_i, g_i, and h_i is nonlinear.

$$
\begin{aligned}
\max \quad & f(\mathbf{x}) \\
s.t. \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \cdots, m \\
& h_i(\mathbf{x}) = 0. \quad i = m+1, m+2, \cdots, n.
\end{aligned}
$$

Example :

$$
\begin{aligned}
\max \quad & x + y \\
s.t. \quad & x^2 + y^2 \geq 1, \\
& x^2 + y^2 \leq 2, \\
& x, y \geq 0.
\end{aligned}
$$



Quadratic Programming

Quadratic Programming (QP) is a special case of nonlinear programming in the form of

$$
\begin{aligned}
\max \quad & \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}
$$

where

- $\mathbf{Q}$: an $n \times n$ real symmetric matrix ;
- $\mathbf{c}$: an n vector ;
- $\mathbf{A}$: an $m \times n$ real matrix ;
- $\mathbf{b}$: an m vector.

Brief History of LP

- First formal application to problems in economics by Leonid Kantorovich in the 1930s, however the work was ignored ;
- Rediscovered by Tjalling Koopmans in the 1940s, along with applications to economics ;
- First algorithm (Simplex Algorithm) to solve linear programs by George Dantzig in 1947 ;
- Kantorovich and Koopmans receive Nobel Prize for economics in 1975 ; Dantzig, however, was ignored.
- LP was first shown to be solvable in polynomial time via Ellipsoid Method by Leonid Khachiyan in 1979, but a larger breakthrough came in 1984 when Narendra Karmarkar introduced a novel Interior-Point Method to solve LP.

Duality

Recall the Profit Maximization problem :

$$\begin{aligned} \max \quad & f(x_1, x_2) = x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Try to find the optimum.

(x_1, x_2)	$f(x_1, x_2)$
$(100, 200)$	1300
$(200, 200)$	1400
$\mathbf{(100, 300)}$	**1900**

However, we do NOT know whether $(100, 300)$ is exactly the optimal solution.

Duality enables us to prove that a solution is indeed optimal.

Intuitive Way to Show Optimality

Whether $(100, 300)$ is optimal ? (The objective value is 1900)

Upper bounding $x_1 + 6x_2$ by linearly combining constraints.

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Multiplier	Constraint
y_1	$x_1 + x_2 \leq 400$
y_2	$x_1 \leq 200$
y_3	$x_2 \leq 300$

When $(y_1, y_2, y_3) = (0, 1, 6)$, $x_1 + 6x_2 \leq 2000$;
When $(y_1, y_2, y_3) = (1, 0, 5)$, $x_1 + 6x_2 \leq 1900$! Optimal !

However, it is only a coincidence...

Rational Way to Show Optimality

$(y_1, y_2, y_3) = (1, 0, 5)$ serves as a certificate of the optimality, but how can we find it rationally ?

Multiplier	Constraint
y_1	$x_1 + x_2 \leq 400$
y_2	$x_1 \leq 200$
y_3	$x_2 \leq 300$

Multiply horizontally and add vertically, we obtain

$(y_1 + y_2)x_1 + (y_1 + y_3)x_2 \leq 400y_1 + 200y_2 + 300y_3$.

$y_1, y_2, y_3 \geq 0$ to ensure no flipping from $\leq$ to $\geq$ in constraints.

Rational Way to Show Optimality

$$(y_1 + y_2)x_1 + (y_1 + y_3)x_2 \leq 400y_1 + 200y_2 + 300y_3$$

We want the left-hand side to look like our objective function $x_1 + 6x_2$, so that the right-hand side becomes an upper bound on the objective function.

$$x_1 + 6x_2 \leq 400y_1 + 200y_2 + 300y_3 \text{ if } \begin{Bmatrix} y_1, y_2, y_3 \geq 0 \\ y_1 + y_2 \geq 1 \\ y_1 + y_3 \geq 6 \end{Bmatrix}$$

We should minimize $400y_1 + 200y_2 + 300y_3$ to get the tightest upper bound of $x_1 + 6x_2$. A new LP problem !

Dual LP

The new LP problem :

$$\begin{aligned} \min \quad & 400y_1 + 200y_2 + 300y_3 \\ s.t. \quad & y_1 + y_2 \geq 1, \\ & y_1 + y_3 \geq 6, \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

We call it the dual form of the original LP problem.

Primal and Dual Form

Correspondingly, we call the original LP problem as primal form.

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min \quad & 400y_1 + 200y_2 + 300y_3 \\ s.t. \quad & y_1 + y_2 \geq 1, \\ & y_1 + y_3 \geq 6, \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

Primal Form   Dual Form

Primal and Dual Form

Generally we have the primal and dual form of LP as

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \forall i \\ & x_j \geq 0. \quad \forall j \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \forall j \\ & y_i \geq 0. \quad \forall i \end{aligned}$$

It is obvious that the dual of dual form is the primal form.

Matrix-Vector Form

More generally, we can write primal and dual form in matrices and vectors.

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ s.t. & \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}. \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & \mathbf{y}^T\mathbf{b} \\ s.t. & \mathbf{y}^T\mathbf{A} \ge \mathbf{c}^T, \\ & \mathbf{y} \ge \mathbf{0}. \end{array}$$

Primal Form   Dual Form

Observations of Duality

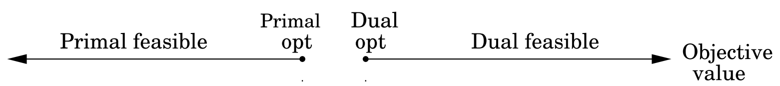
$$\begin{array}{ll} \max & x_1 + 6x_2 \\ s.t. & x_1 + x_2 \le 400, \\ & x_1 \le 200, \\ & x_2 \le 300, \\ & x_1, x_2 \ge 0. \end{array} \qquad \begin{array}{ll} \min & 400y_1 + 200y_2 + 300y_3 \\ s.t. & y_1 + y_2 \ge 1, \\ & y_1 + y_3 \ge 6, \\ & y_1, y_2, y_3 \ge 0. \end{array}$$

Recall $x_1 + 6x_2 \le 400y_1 + 200y_2 + 300y_3$. We observe that any feasible value of dual LP is an upper bound of the primal LP.

If we find a pair of primal and dual feasible objective values that are equal, then they must both be optimal.

One such pair, making the objective value both 1900, is :
Primal : $(x_1, x_2) = (100, 300)$; Dual : $(y_1, y_2, y_3) = (1, 0, 5)$.

Duality Theorem



Theorem (Weak Duality Theorem)

Let x be any feasible solution to the primal LP, and let y be any feasible solution to its dual LP. Then $\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$.

Theorem (Strong Duality Theorem)

x and y are optimal solutions to primal and dual LPs respectively if and only if $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$.

Proof of Weak Duality Theorem

Theorem (Weak Duality Theorem)

Let x be any feasible solution to the primal LP, and let y be any feasible solution to its dual LP. Then $\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$.

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j \le b_i, & \forall i \\ & x_j \ge 0. & \forall j \end{array} \quad \Rightarrow \quad \begin{array}{lll} \min & \sum_{i=1}^{m} b_i y_i & \\ s.t. & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, & \forall j \\ & y_i \ge 0. & \forall i \end{array}$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} a_{ij} y_i\Big) x_j = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} x_j\Big) y_i \le \sum_{i=1}^{m} b_i y_i.$$

Proof of Strong Duality Theorem

Theorem (Strong Duality Theorem)

x and y are optimal solutions to primal and dual LPs respectively if and only if $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$.

Proof.

"$\Leftarrow$" : By **Weak Duality Theorem**, $\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$. The primal LP is a maximization problem and the dual LP is a minimization problem. Thus, if feasible solutions x and y have the same objective value, neither can be improved.

"$\Rightarrow$" : It involves using Simplex Method, which contains complex mathematical derivations. We omit here.

Recall : Profit Maximization–2D Case

Products : **1**, **2** ;
Resources : **A**, **B**, **C**.

- 200 **A** ; 300 **B** ; 400 **C**.
- Product **1** = 1**A** + 1**C**.
- Product **2** = 1**B** + 1**C**.
- Price(**1**)=1, Price(**2**)=6.

Goal : Maximum profit.

Suppose : x_1 for **1** ; x_2 for **2**.

$$\begin{aligned} \max \quad & f(x_1, x_2) = x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal Solution :

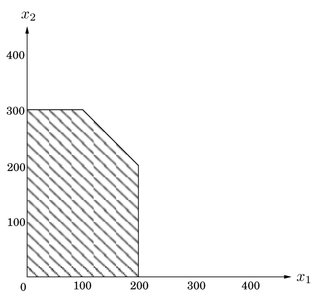
$$(x_1, x_2) = (100, 300)$$

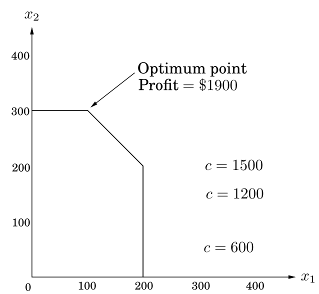
2D Case : Where is the Optimum ?

Where is the optimum in the feasible region ?

Constraints form the feasible region.

$$\begin{aligned} \max \quad & f(x_1, x_2) = x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$





On the Vertex !

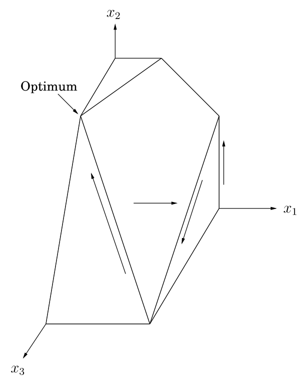
3D Case

Problem :

$$\begin{aligned} \max \quad & x_1 + 6x_2 + 13x_3 \\ s.t. \quad & x_1 + x_2 + x_3 \leq 400, \\ & x_2 + 3x_3 \leq 600, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Where is the optimum ?

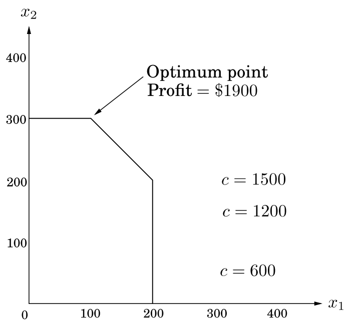
Feasible Region :

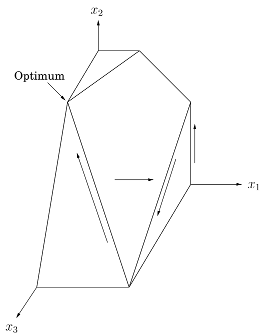


On the Vertex !

How to Find Optimum ?

Based on 2D and 3D situation, how can we find optimum of LP ?

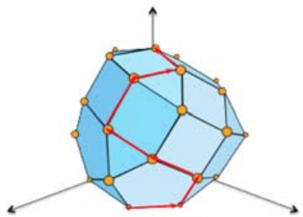




- Move on the boundary ;
- Find the vertex with largest (smallest) objective value.

Simplex Algorithm : For LP with arbitrary n variables.

General Description of Simplex Algorithm



Definitions :

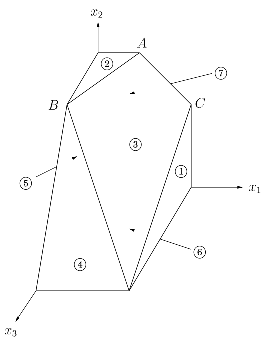
- Linear constraints form the boundary as a polyhedron, consisting of hyperplanes.
- Vertex is the point at which some hyperplanes meet.
- Two vertices are neighbors if they are adjacent on the polyhedron.

Observation : The optimal solution of LP exists on the vertex of the feasible region.

General Description of Simplex Algorithm

Example :

- A polyhedron defined by 7 inequalities (thus 7 hyperplanes).
- Vertices : A, B, C,...
- Neighbors : $\{A, B\}$, $\{A, C\}$,...



$$
\begin{array}{llr}
\max & x_1 + 6x_2 + 13x_3 & \\
s.t. & x_1 \leq 200, & ① \\
& x_2 \leq 300, & ② \\
& x_1 + x_2 + x_3 \leq 400, & ③ \\
& x_2 + 3x_3 \leq 600, & ④ \\
& x_1 \geq 0, & ⑤ \\
& x_2 \geq 0, & ⑥ \\
& x_3 \geq 0. & ⑦
\end{array}
$$

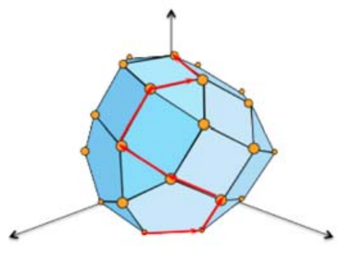
General Description of Simplex Algorithm

On each iteration, the Simplex Algorithm will do :

- Check whether the current vertex is optimal (if so, halt).
- Determine where to move next. → The one contributes to the increase of objective function.

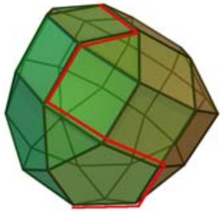
Initial Idea :

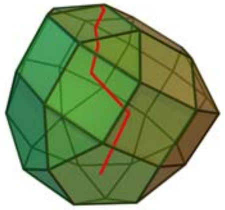
- Start at a vertex.
- Compare objective value with the neighbors.
- Move to neighbor that improves objective function, and repeat step 2.
- If no improving neighbor, stop.



Brief Overview of Algorithms for LP

- Simplex Algorithm
  - Efficient in practice, but exponential in worst case.
- Interior Point Algorithms
  - Ellipsoid Algorithm $O(n^4L)$ †
  - Karmarkar's Algorithm $O(n^{3.5}L)$
  - Path-Following Method (Barrier Function Method)


Simplex (Boundary)


Interior Point (Inside)

†L is bit length.

An Introductory Example

How to perform simplex algorithm on a specific LP problem ?

Example : Profit Maximization

$$\begin{aligned} \max \quad & f(x_1,x_2) = x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \le 400, \\ & x_1 \le 200, \\ & x_2 \le 300, \\ & x_1, x_2 \ge 0. \end{aligned}$$

An Introductory Example

Step 1 : Converting LP into slack form

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & x_1 + x_2 \le 400, \\ & x_1 \le 200, \\ & x_2 \le 300, \\ & x_1, x_2 \ge 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_2 = x_5, \\ & x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

x_3, x_4, x_5 are slack variables.

An Introductory Example

Step 2 : Obtaining Basic Solution

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_2 = x_5, \\ & x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

Define :

- nonbasic variables : x_1, x_2 (in the objective)
- basic variables : x_3, x_4, x_5 (others in the constraints)

There are infinite number of feasible solutions for $x_1, \cdots, x_5$. The basic solution is obtained by setting all nonbasic variables to be 0.

In this example, the basic solution is

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_5) = (0, 0, 400, 200, 300)$$

An Introductory Example

Step 3 : Selecting Nonbasic Variable

Our goal, in each iteration, is to reformulate the linear program so that the basic solution gives a greater value of objective function.

Currently for the basic solution $\bar{x} = (0, 0, 400, 200, 300)$, we have

$$f(\bar{x}_1, \bar{x}_2) = \bar{x}_1 + 6\bar{x}_2 = 0.$$

How to enhance $f(\bar{x}_1, \bar{x}_2)$?

- Select a nonbasic x_e with positive coefficient in $f(x_1, x_2)$;
- Increase the value of x_e without violating constraints.

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Step 3 : Selecting Nonbasic Variable

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_2 = x_5, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

- Choose the nonbasic variable x_2.
- When $x_2 \uparrow$, $x_3 \downarrow$ and $x_5 \downarrow$. However x_3 and x_5 should be nonnegative.
  - ▷ $x_3 \leq 0$ when $x_2 \geq 400$;
  - ▷ $x_5 \leq 0$ when $x_2 \geq 300$.
- $300 - x_2 = x_5$ is the tightest constraint for x_2.

  We transform it into $300 - x_5 = x_2$.

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Step 4 : Pivoting

In **pivoting**, we exchange a nonbasic and a basic variable.

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ s.t. \quad & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_2 = x_5, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & x_1 + 6(300 - x_5) \\ s.t. \quad & 100 - x_1 + x_5 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

$$\bar{x} = \{0, 0, 400, 200, 300\} \quad \Rightarrow \quad \bar{x} = \{0, 300, 100, 200, 0\}.$$

- The nonbasic variables become x_1 and x_5.
- In next round we select a new nonbasic variable with positive coefficient. (x_1 in this example).

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Step 5 : Repeat Step 2 to Step 4

- Repeat pivoting until all coefficients in the objective are negative.

$$\begin{aligned} \max \quad & x_1 + 6(300 - x_5) \\ s.t. \quad & 100 - x_1 + x_5 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & \boxed{1900 - x_3 - 5x_5} \\ s.t. \quad & 100 - x_3 + x_5 = x_1, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

$$\bar{x} = \{0, 300, 100, 200, 0\} \quad \Rightarrow \quad \bar{x} = \{100, 300, 0, 100, 0\}.$$

An Introductory Example

Step 5 : Repeat Step 2 to Step 4

- We can proved that when all coefficients are negative, the basic solution $\bar{x}$ is an optimal solution.

$$\begin{aligned} \max \quad & \boxed{1900 - x_3 - 5x_5} \\ s.t. \quad & 100 - x_3 + x_5 = x_1, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

- The maximum of the objective function is 1900, when $x_3 = x_5 = 0$;
- $\bar{x} = \{100, 300, 0, 100, 0\}$ is the optimal solution ;

The optimal solution for original problem $f(x_1, x_2)$ is :

$$(x_1^*, x_2^*) = (100, 300)$$

Summarization and Further Topics

Summarization :

- Simplex Algorithm searches the optimal vertex on the boundary of feasible region ;
- Simplex Algorithm iteratively exchanges the nonbasic and basic variables until the objective function cannot be further enhanced.

Further Topics :

- How to implement Simplex Algorithm ?
  - How to do the **pivoting** ?
  - How to find an initial **basic solution** ?
- Can Simplex Algorithm always find the optimal solution ?
- What will happen when the feasible region is unbounded ?

Please refer to Chapter 29.3 and 29.5 in “Introduction to Algorithms” (CLRS) for details.

Tools for Solving LP

Common Softwares and Toolboxes for solving LP :

MATLAB : Toolboxes

Mathematica : Toolboxes

Lingo : Large-scale LP (ILP)

Cplex : Large-scale ILP (Mixed ILP)

Gurobi : Similar to but better than Cplex synthetical.

Yalmip : Toolbox used by Matlab.