Eigenvector-Based Semiparametric Filtering of Spatial Autocorrelation in Regression Models

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Motivation

- spatial misspecification can lead to inefficient or even biased and inconsistent parameter estimates in regression models
- social scientists predominantly (exclusively?) apply parametric spatial regression models to *control* for spatial misspecification
- semiparametric filtering techniques can offer an attractive alternative:
  - ease of estimation (standard OLS or ML estimators)
  - straightforward interpretation
  - accounts for SA at different scales/resolutions
  - easily adaptable to GLMs
- especially eigenvector-based spatial filtering (e.g., Griffith 1996 & 2003) is of practical utility

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$$

Spatial Misspecification in Linear Models

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$$

Spatial Error DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{\epsilon}}_{\boldsymbol{e}_{SEM}}$$

Spatial-X DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\rho\boldsymbol{W}\boldsymbol{X} + \boldsymbol{\epsilon}}_{\boldsymbol{e}_{SLX}}$$

Spatial Lag DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\rho\boldsymbol{W}\boldsymbol{y} + \boldsymbol{\epsilon}}_{\boldsymbol{e}_{SAR}}$$

$$\boldsymbol{y} = (\boldsymbol{I} - \rho\boldsymbol{W})^{-1}(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon})$$

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$$(\boldsymbol{I} - \rho\boldsymbol{W})^{-1} = (\boldsymbol{I} + \rho\boldsymbol{W} + \rho^2\boldsymbol{W} + ...)$$

$$\boldsymbol{y} = \boldsymbol{X\beta} + \left.\begin{array}{l} +\rho\boldsymbol{W\epsilon} \\ +\rho^2\boldsymbol{W}^2\boldsymbol{\epsilon} \\ +... + \boldsymbol{\epsilon} \end{array}\right\} \boldsymbol{\epsilon}_{SEM}$$

$$\boldsymbol{y} = \boldsymbol{X\beta} + \left.\begin{array}{l} +\rho\boldsymbol{W}(\boldsymbol{X\beta} + \boldsymbol{\epsilon}) \\ +\rho^2\boldsymbol{W}^2(\boldsymbol{X\beta} + \boldsymbol{\epsilon}) \\ +... + \boldsymbol{\epsilon} \end{array}\right\} \boldsymbol{\epsilon}_{SAR}$$

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$\underline{\boldsymbol{e}_{SEM}}$

$$\underbrace{\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r \boldsymbol{\epsilon}}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

$\underline{\boldsymbol{e}_{SLX}}$

$$\underbrace{\rho\boldsymbol{W}\boldsymbol{X}}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

$\underline{\boldsymbol{e}_{SAR}}$

$$\underbrace{\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon})}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

Eigenvector-Based Spatial Filtering

UNIVERSITY OF MANNHEIM

Intuition:

- include a proxy variable for spatial effects in the regression's systematic part
- ESF partitions the response variable into i) a systematic trend, ii) a stochastic spatial signal, and iii) *iid* disturbances

Example: SAR DGP

$$\boldsymbol{y}$$

(outcome)

$$\boldsymbol{X\beta}$$

(systematic trend)

$$\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r(\boldsymbol{X\beta} + \boldsymbol{\epsilon}) + \epsilon$$

(autocorrelated part)

$$\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r(\boldsymbol{X\beta} + \boldsymbol{\epsilon})$$

(stochastic spatial signal)

$$\epsilon$$

(*iid* disturbances)

Spatial Eigenfunction Analysis

ESF is based on the spectral decomposition of a centered (and symmetric/ symmetrized) connectivity matrix: $\boldsymbol{W}$:

$$\boldsymbol{MWM} = \boldsymbol{E\Lambda E}^{-1} = \boldsymbol{E\Lambda E}'$$

- demeaning projector $\boldsymbol{M} = (\boldsymbol{I} - \boldsymbol{11}'/n)$ – also ensures that all eigenvectors are orthogonal and uncorrelated
- $\boldsymbol{E}$ are all n eigenvectors
- $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues $\boldsymbol{\lambda}$

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Direct relationship to the numerator of the global Moran coefficient:

$$MC(\boldsymbol{x}) = \frac{n}{\mathbf{1}'\boldsymbol{W}\mathbf{1}} \frac{\boldsymbol{x}'\boldsymbol{MWMx}}{\boldsymbol{x}'\boldsymbol{Mx}}$$

Eigenvectors as Synthetic Map Patterns

- eigenvectors $\boldsymbol{E}$ depict distinct – and uncorrelated – synthetic map patterns
- corresponding eigenvalues indicate the level of SA

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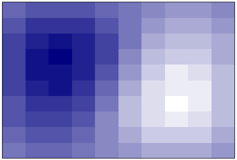
Eigenvector 1
(MC = 1)



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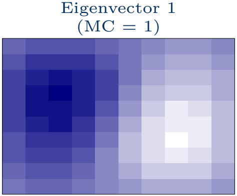

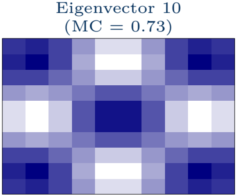

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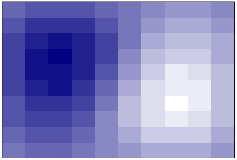
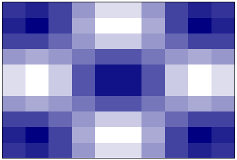
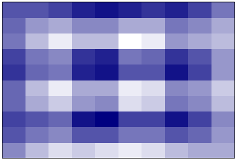


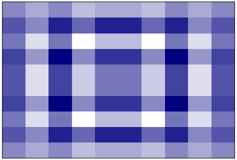
Eigenvector 10
(MC = 0.73)



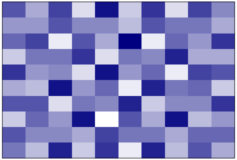
Eigenvector 20
(MC = 0.454)



Eigenvector 44
(MC = 0.066)



Eigenvector 80
(MC = -0.454)



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- corresponding eigenvalues indicate the level of SA

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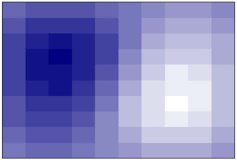

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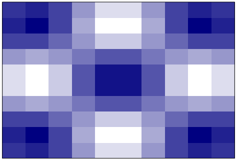

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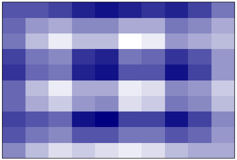

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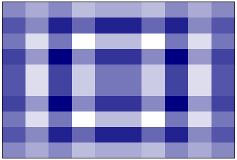

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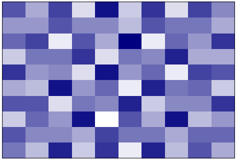
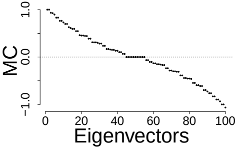


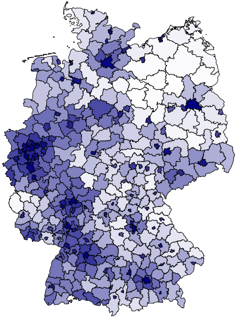
MCs of all
Eigenvectors



Real-World Map Patterns

- $\boldsymbol{E}$ depict <u>all</u> possible spatial patterns permitted by $\boldsymbol{W}$
- more complex patterns can be obtained by a linear combination of eigenvectors

Population Density



MC = 0.268 (p = 0)

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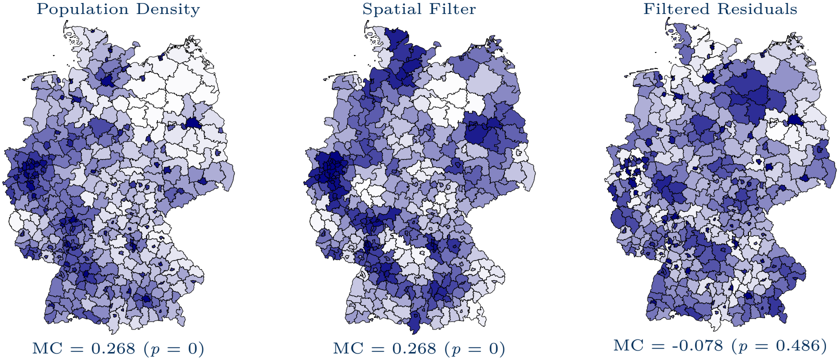


Eigenvector Selection for the Spatial Filter

However, it is impossible to include all n eigenvectors as regressors

1. identification of a candidate set $\boldsymbol{E}^C \subset \boldsymbol{E}$ based on
   - sign of SA
   - strength of SA
2. select relevant map patterns $\boldsymbol{E}^* \subset \boldsymbol{E}^C$ using supervised or unsupervised stepwise regression

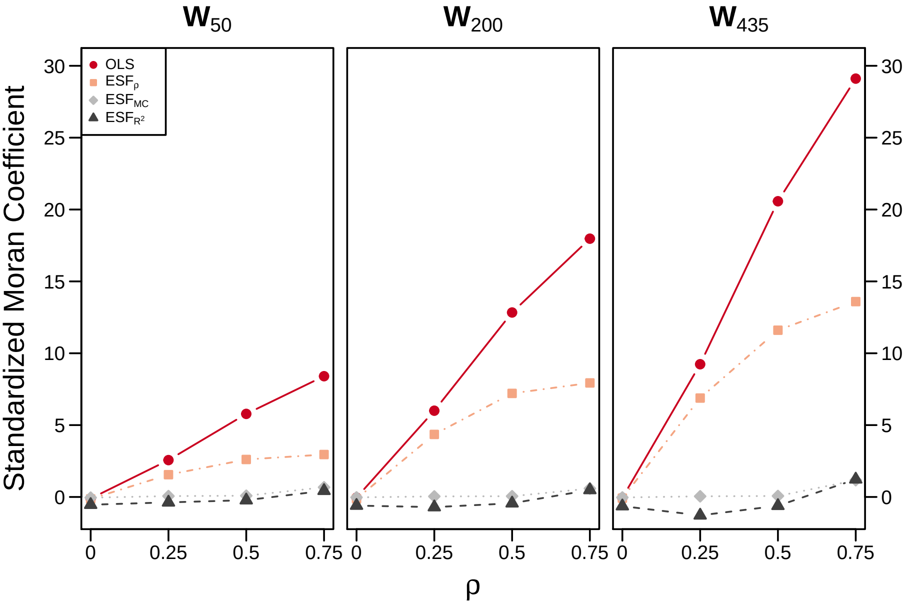
$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \overbrace{\boldsymbol{E}\boldsymbol{\gamma} + \boldsymbol{\epsilon}}^{\boldsymbol{e}_{OLS}} \\ &\approx \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\boldsymbol{E}^*\boldsymbol{\gamma}}_{filter} + \underbrace{\boldsymbol{\epsilon}}_{noise}\end{aligned}$$

- different selection criteria can be used:
  - maximization of model fit (e.g., AIC, BIC, R^2, ...)
  - significance level of eigenvectors or residual SA
  - minimization of residual SA
  - LASSO
  - ...

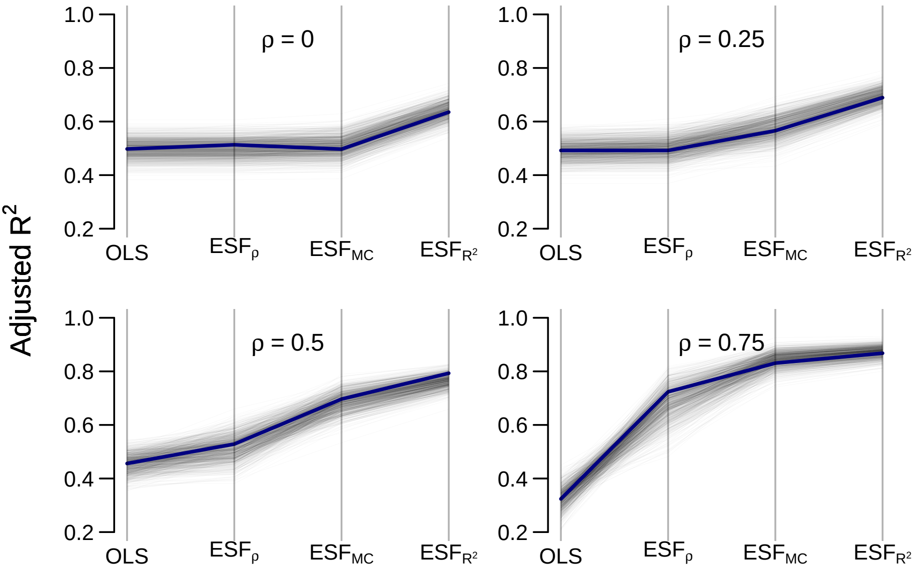
(Preliminary)

Monte Carlo Evidence

Reduction in Residual SA



Improved Model Fit



Conclusion

- ESF offers numerous advantages for political scientists
- things to consider:

Pros

\+ ease of model estimation
\+ straightforward interpretation of parameters
\+ flexibility
(no need to specify the spatial pattern in each variable)
\+ generalizability
(also applicable to GLMs – with some modifications)

Cons

\- "space" as misspecification problem
\- computationally demanding for large N
\- over- or undercorrection of SA possible

- applicability depends on research context (as always)
- ESF is not a replacement for parametric spatial regression models!

Additional Resources

- `spfilteR` package:
  CRAN: https://CRAN.R-project.org/package=spfilteR
  GitHub: https://github.com/sjuhl/spfilteR
- further projects & working papers
  1. introductory paper on the `spfilteR` package (under review)
  2. project on Moran eigenvector maps and spatial eigenfunction analysis

Feedback & suggestions are highly appreciated!

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