Eigenvector-Based Semiparametric Filtering of Spatial Autocorrelation in Regression Models

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March 17, 2021

Motivation

- spatial misspecification can lead to inefficient or even biased and inconsistent parameter estimates in regression models
- social scientists predominantly (exclusively?) apply parametric spatial regression models to *control* for spatial confounding
- alternative: semiparametric filtering techniques – especially eigenvector-based spatial filtering (e.g., Griffith 1996 & 2003)
  - ease of estimation (standard OLS or ML estimators)
  - straightforward interpretation
  - accounts for SA at different scales/resolutions
  - easily adaptable to GLMs

Spatial Misspecification in Linear Models

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$$

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Spatial Error DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{\epsilon}}_{\boldsymbol{e}_{SEM}}$$

Spatial-X DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\rho\boldsymbol{W}\boldsymbol{X} + \boldsymbol{\epsilon}}_{\boldsymbol{e}_{SLX}}$$

Spatial Lag DGP

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\rho\boldsymbol{W}\boldsymbol{y} + \boldsymbol{\epsilon}}_{\boldsymbol{e}_{SAR}}$$

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$\boldsymbol{e}_{SEM}$

$$\underbrace{\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r \boldsymbol{\epsilon}}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

$\boldsymbol{e}_{SLX}$

$$\underbrace{\rho\boldsymbol{W}\boldsymbol{X}}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

$\boldsymbol{e}_{SAR}$

$$\underbrace{\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon})}_{\text{spatial part}} + \underbrace{\boldsymbol{\epsilon}}_{\text{noise}}$$

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Intuition:

- include a proxy variable for omitted spatial effects in the regression's systematic part
- ESF partitions the response variable into i) a systematic trend, ii) a stochastic spatial signal, and iii) *iid* disturbances

Example: SAR DGP

$$\boldsymbol{y}$$

(outcome)

$$\boldsymbol{X}\boldsymbol{\beta} \qquad \sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}) + \boldsymbol{\epsilon}$$

(systematic trend) (autocorrelated part)

$$\sum_{r=1}^{\infty} \rho^r \boldsymbol{W}^r (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}) \qquad \boldsymbol{\epsilon}$$

(stochastic spatial signal) (*iid* disturbances)

ESF is based on the spectral decomposition of a centered (and symmetric/ symmetrized) connectivity matrix $\boldsymbol{W}$:

$$\boldsymbol{MWM} = \boldsymbol{E\Lambda E}^{-1} = \boldsymbol{E\Lambda E}'$$

- demeaning projector $\boldsymbol{M} = (\boldsymbol{I} - \mathbf{11}'/n)$ – also ensures that all eigenvectors are orthogonal and uncorrelated
- $\boldsymbol{E}$ are all n eigenvectors
- $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues $\boldsymbol{\lambda}$

Spatial Eigenfunction Analysis

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Direct relationship to the numerator of the global Moran coefficient:

$$MC(\boldsymbol{x}) = \frac{n}{\boldsymbol{1}'\boldsymbol{W}\boldsymbol{1}} \frac{\boldsymbol{x}'\boldsymbol{MWM}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{M}\boldsymbol{x}}$$

Eigenvectors as Synthetic Map Patterns

- eigenvectors $\boldsymbol{E}$ depict distinct – and uncorrelated – synthetic map patterns
- $\boldsymbol{E}$ contains all possible spatial patterns permitted by $\boldsymbol{W}$
- corresponding eigenvalues indicate the level of SA

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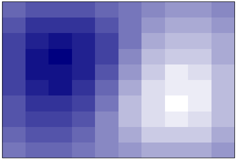
Eigenvector 1
(MC = 1)



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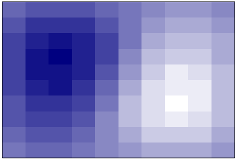
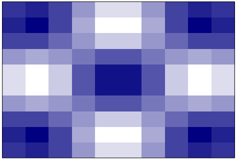


Eigenvector 10
(MC = 0.73)



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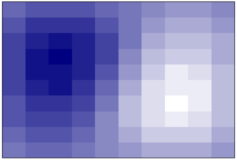

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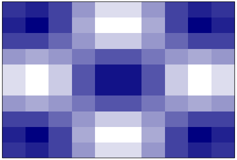
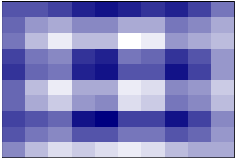
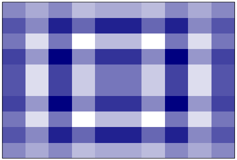


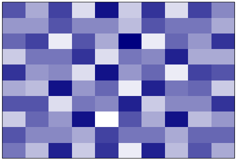
Eigenvector 20
(MC = 0.454)



Eigenvector 44
(MC = 0.066)



Eigenvector 80
(MC = -0.454)



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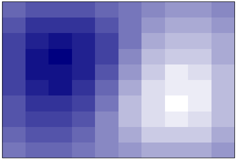

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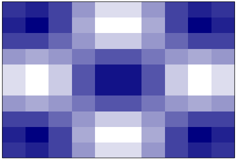

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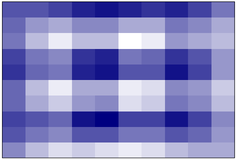

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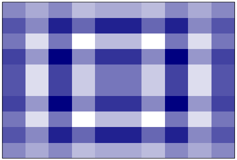

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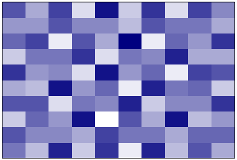
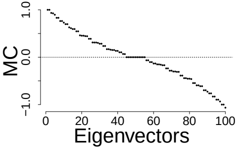


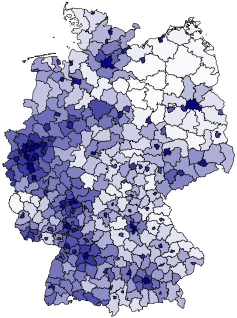
MCs of all
Eigenvectors



Real-World Map Patterns

More realistic (and complex) patterns can be obtained by a linear combination of eigenvectors:

Population Density



MC = 0.268 (p = 0.000)

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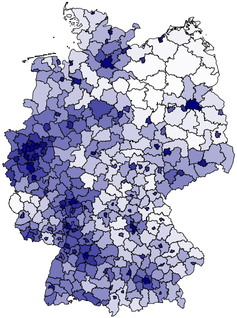

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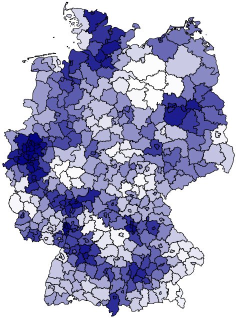
Spatial Filter



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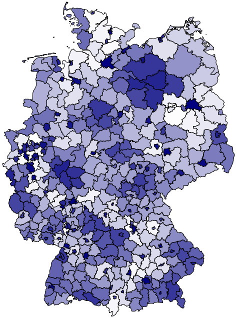
Filtered Residuals



MC = -0.078 (p = 0.486)

Eigenvector Selection for the Spatial Filter

However, it is impossible to include all n eigenvectors as regressors

1. identification of a candidate set $\boldsymbol{E}^C \subset \boldsymbol{E}$ based on
   - sign of SA
   - strength of SA
2. select relevant map patterns $\boldsymbol{E}^* \subset \boldsymbol{E}^C$ using supervised or unsupervised stepwise regression

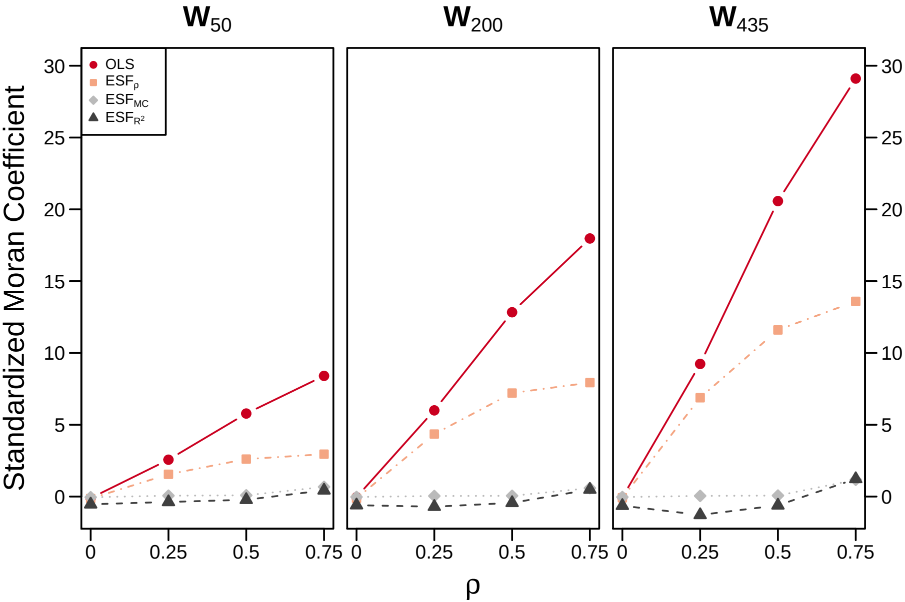
$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \overbrace{\boldsymbol{E}\boldsymbol{\gamma} + \boldsymbol{\epsilon}}^{\boldsymbol{e}_{OLS}} \\
&\approx \boldsymbol{X}\boldsymbol{\beta} + \underbrace{\boldsymbol{E}^*\boldsymbol{\gamma}}_{filter} + \underbrace{\boldsymbol{\epsilon}}_{noise}
\end{aligned}
$$

- different selection criteria can be used:
  - maximization of model fit (e.g., AIC, BIC, R^2, ...)
  - significance level of eigenvectors or residual SA
  - minimization of residual SA
  - LASSO
  - ...

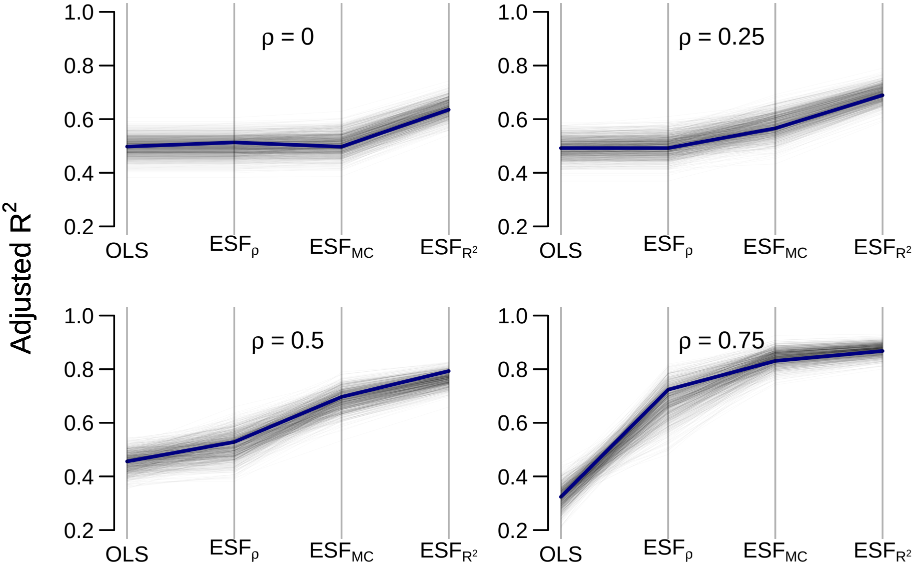
(Preliminary)

Monte Carlo Evidence

Reduction in Residual SA



Improved Model Fit



Conclusion

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- ESF offers numerous advantages for political scientists
- things to consider:

Pros	Cons
+ ease of model estimation	- "space" as misspecification problem
+ straightforward interpretation of parameters	- computationally demanding for large N
+ flexibility (no need to specify the spatial pattern in each variable)	- over- or undercorrection of SA possible
+ generalizability (also applicable to GLMs – with slight modifications)	

- applicability depends on research context (as always)
- ESF <u>is not</u> a replacement for parametric spatial regression models!

Additional Resources

- `spfilteR` package:
  CRAN: https://CRAN.R-project.org/package=spfilteR
  GitHub: https://github.com/sjuhl/spfilteR
- further projects & working papers
  1. introductory paper on the `spfilteR` package (under review)
  2. project on Moran eigenvector maps and spatial eigenfunction analysis

Feedback & suggestions are highly appreciated!

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