Spatial Eigenfunction Modeling of Geo-Referenced Data in the Social Sciences

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Motivation

- many phenomena of interest to social scientists cluster in space
  - economic development, regime type, voting behavior, religious beliefs, etc.
- geographic distribution of variables provide valuable information about the underlying mechanism of interest
- spatial autocorrelation (SA) causes severe problems for common econometric methods

Adequately accounting for spatial structures is necessary to guard against false inferences and to utilize spatial information!

This Project

- so far, social scientists use a limited subset of techniques suitable to handle SA
  - exploratory spatial analysis:
    - different types of maps
    - local indicators of SA (LISA)
  - inferential models
    - parametric spatial regression models

This Project

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- so far, social scientists use a limited subset of techniques suitable to handle SA
  - exploratory spatial analysis:
    - different types of maps
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  - inferential models
    - parametric spatial regression models

- spatial eigenfunction analysis – particularly Moran eigenvector maps (MEM) – is a simple yet powerful tool to analyze cross-sectional data structures
  1. identification & visualization of complex (multi-scale) spatial patterns
  2. specification, estimation, and interpretation of inferential models
  3. partitioning of the variation in $\boldsymbol{y}$ into individual components (space, covariates, joined)
  4. (structure-preserving) simulation of spatially autocorrelated data

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Spatial eigenfunction alanysis is based on the spectral decomposition of a centered (and symmetric/ symmetrized) connectivity matrix $\boldsymbol{W}$:

$$\boldsymbol{MWM} = \boldsymbol{E\Lambda E}^{-1} = \boldsymbol{E\Lambda E}'$$

- demeaning projector $\boldsymbol{M} = (\boldsymbol{I} - \mathbf{11}'/n)$ – also ensures that all eigenvectors are orthogonal and uncorrelated
- $\boldsymbol{E}$ are all n eigenvectors
- $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues $\boldsymbol{\lambda}$

SA & Spatial Eigenfunctions

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Direct relationship to the numerator of the global Moran coefficient:

$$MC(\boldsymbol{x}) = \frac{n}{\mathbf{1}'\boldsymbol{W}\mathbf{1}} \frac{\boldsymbol{x}'\boldsymbol{MWMx}}{\boldsymbol{x}'\boldsymbol{Mx}}$$

Eigenvectors as Synthetic Map Patterns

- $\boldsymbol{E}$ depict all possible distinct and mutually uncorrelated synthetic map patterns permitted by $\boldsymbol{W}$
- corresponding eigenvalues $\boldsymbol{\lambda}$ indicate the level of SA

Example: 401 German NUTS-3 regions

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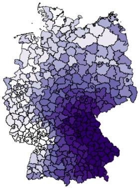

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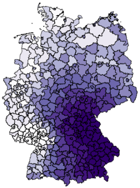
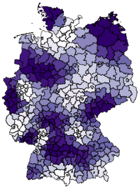


Eigenvector 25



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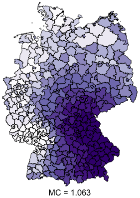

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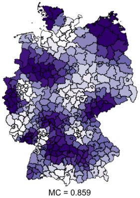
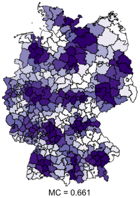


Eigenvector 50



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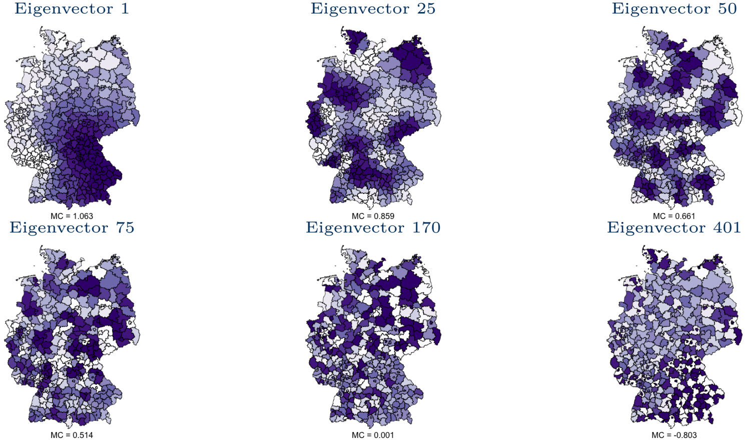

1. Identification of Complex Spatial Structures

Identification of Complex SA Structures

- global and local MC statistics might fail to identify mixtures of positive and negative SA
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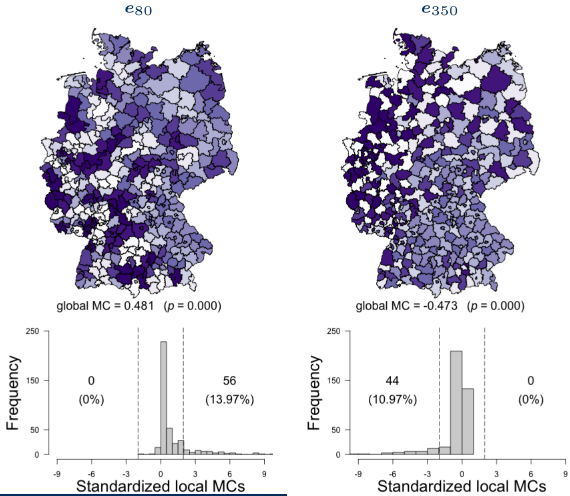

Identification of Complex SA Structures

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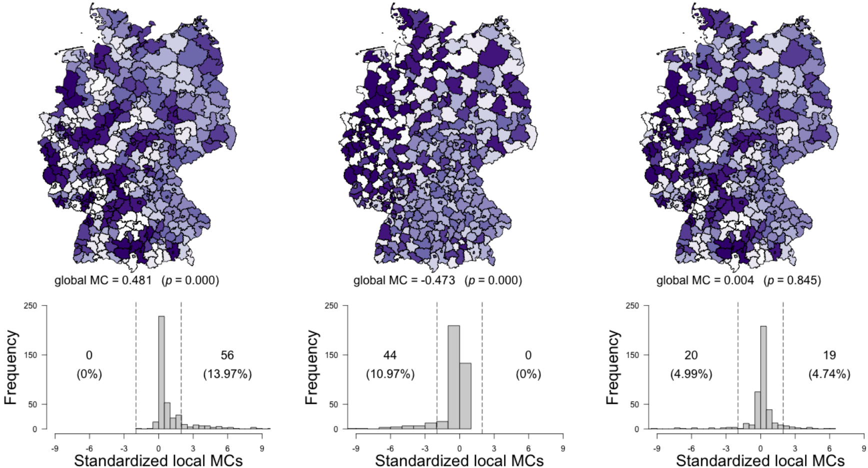
- Example:

e_{80} $\qquad$ e_{350} $\qquad$ $e_{80} + e_{350}$



Identification of Complex SA Structures

- eigenfunction analysis allows researchers to decompose the global MC into positively and negatively autocorrelated parts
  - global MC = $\mathrm{MC}^{+} + \mathrm{MC}^{-}$
- decomposing the global MC helps identifying complex non-random spatial patterns

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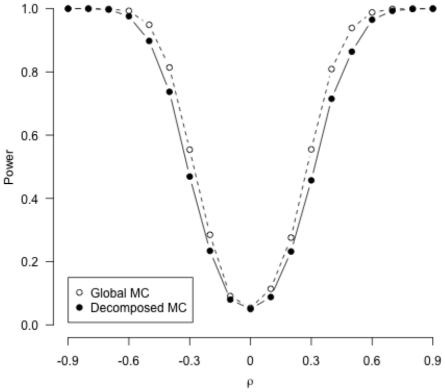
Monte Carlo setup:

- $N = 100$ units ordered on a regular 10×10 grid
- $\boldsymbol{W}$ is a symmetric contiguity matrix (rook scheme)
- ρ ranges from $-.9$ to $.9$ in steps of $.1$
- Scenario 1: simple spatial structure
  - $\boldsymbol{x} = (\boldsymbol{I} - \rho \boldsymbol{W}_g)^{-1} \boldsymbol{u}$
- Scenario 2: mixture of positive and negative SA
  - $\boldsymbol{x} = (\boldsymbol{I} - \rho \boldsymbol{W}_g)^{-1} \boldsymbol{u} + (\boldsymbol{I} - (-\rho) \boldsymbol{W}_g)^{-1} \boldsymbol{v}$

How does the decomposed MC based on spatial eigenfunctions performs compared to the global MC in terms of power?

Scenario 1: Simple Spatial Structure



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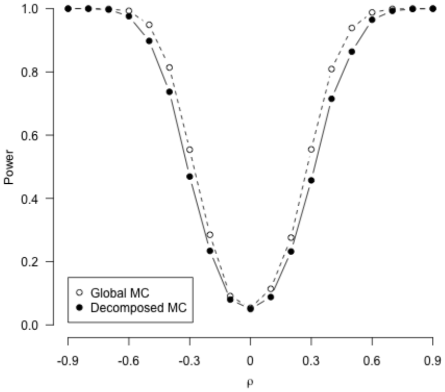
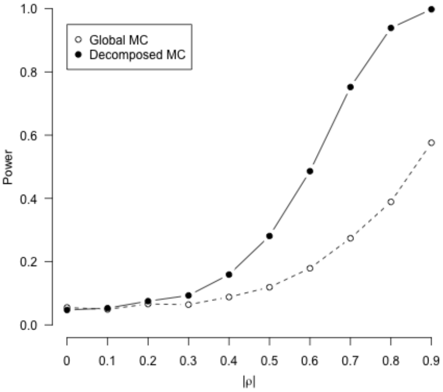


Scenario 2: Mixture of SA



2. Model Specification, Estimation, and Interpretation

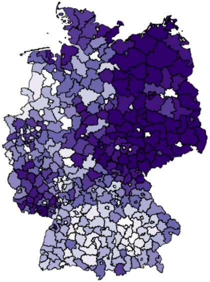
Specification, Estimation, and Interpretation

- SA causes severe problems for common econometric inferential techniques
- depending on the spatial DGP, SA can lead to
  1. incorrect standard errors
  2. biased and inconsistent parameter estimates
- spatial regression models address these problems but require many more or less rigid assumptions
  - knowledge of the true DGP
  - functional form assumptions
  - exact specification of SA in each regressor
  - difficult to estimate in a GLM framework

- semiparametric spatial filtering methods use Moran eigenvectors to construct a synthetic proxy variable that controls for SA

- a subset of eigenvectors can be combined to reproduce real-world map patterns
- Example: Median age in German NUTS-3 regions

Observed Median Age



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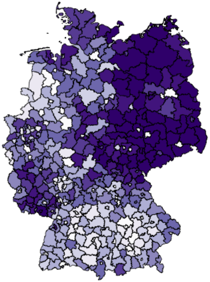
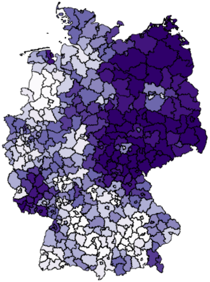


Synthetic Spatial Filter



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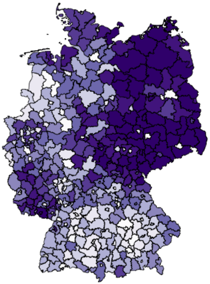

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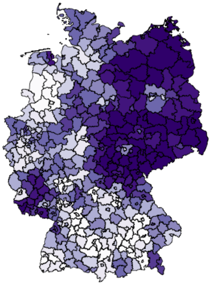
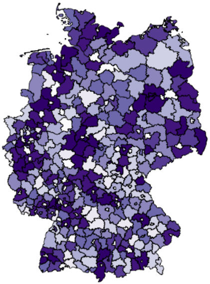


Filtered Residuals



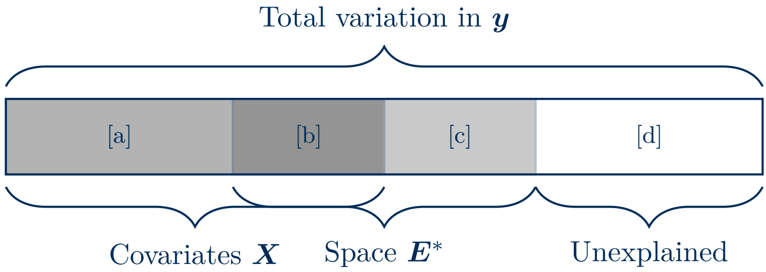
- a judiciously selected subset of eigenvectors controls for the underlying spatial pattern
- straightforward parameter estimation & interpretation

3. Variation Partitioning

Variation Partitioning

How much variation in $\boldsymbol{y}$ is caused by $\boldsymbol{X}$?

- a common spatial structure spuriously inflates the share of variation explained by the predictors
- disentangle the individual contribution of the covariates and the spatial structure



Variation Partitioning

1 identify a subset of eigenvectors $\boldsymbol{E}^*$ that serve as spatial predictors

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2 regress $\boldsymbol{y}$ on three sets of predictors and calculate R^2

 2.1 regress $\boldsymbol{y}$ on $\boldsymbol{X}$ (fraction $[a+b]$)

 2.2 regress $\boldsymbol{y}$ on $\boldsymbol{E}^*$ (fraction $[b+c]$)

 2.3 regress $\boldsymbol{y}$ on $\boldsymbol{X}$ and $\boldsymbol{E}^*$ (fraction $[a+b+c]$)

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3 using the results from step 2, calculate individual fractions

  3.1 $[a] = [a+b+c] - [b+c]$

  3.2 $[b] = [a+b] + [b+c] - [a+b+c]$

  3.3 $[c] = [a+b+c] - [a+b]$

  3.4 $[d] = 1 - [a+b+c]$

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4. use Moran spectral randomization to calculate R^2_{adj}

Variation Partitioning

Example:

- regress GDP ($\boldsymbol{y}$) on median age ($\boldsymbol{X}$)
- MC of (log) GDP: 0.347
- spatial filtering identifies 15 relevant eigenvectors ($\boldsymbol{E}^*$)

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	Joint Fractions			Individual Fractions			
	$[a+b]$	$[b+c]$	$[a+b+c]$	$[a]$	$[b]$	$[c]$	$[d]$
R^2	0.286	0.246	0.523	0.277	0.009	0.237	0.477
R^2_{adj}	0.284	0.217	0.492	0.275	0.009	0.208	0.508

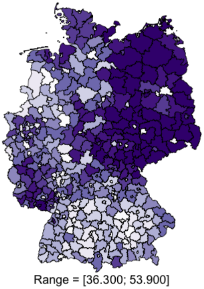
Note: Spatially constrained null model to calculate R^2_{adj} based on $1,000$ random permutations.

4. (Structure-Preserving) Simulation of Spatially Autocorrelated Data

Simulating Spatial Data

- spatial multipliers $(\boldsymbol{I} - \rho \boldsymbol{W})^{-1}$ are typically used to simulate SA data
  - fixed degree of SA across simulations
  - does not preserve spatial structure
- using MEMs for simulation exercises preserves the geographic distribution

Observed Median Age



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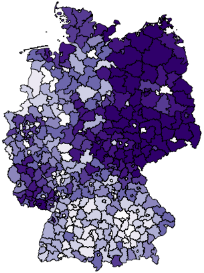
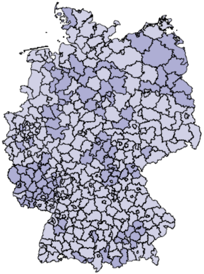


Range = [36.300; 53.900]

Spatial Multipliers (Mean)



Range = [45.549; 45.802]

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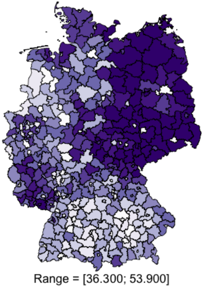

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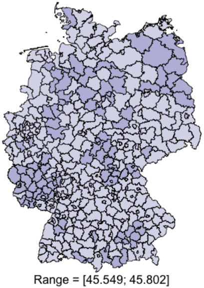
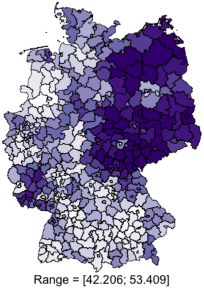


MEMs (Mean)



Conclusion

- spatial eigenfunction analysis complements the statistical repertoire
- it also helps addressing methodological problems caused by SA
- MEMs allow researchers to derive additional information from geo-referenced data
- improves exploratory and inferential analysis, especially w.r.t
  - identification & visualization of complex (multi-scale) spatial patterns
  - specification, estimation, and interpretation of inferential models
  - variation partitioning
  - simulation of SA data