

Spatial Eigenfunction Modeling of Geo-Referenced Data in the Social Sciences

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- many phenomena of interest to social scientists cluster in space
 - economic development, regime type, voting behavior, religious beliefs, etc.
- geographic distribution of variables provide valuable information about the underlying mechanism of interest
- spatial autocorrelation (SA) causes severe problems for common econometric methods

Adequately accounting for spatial structures is necessary to guard against false inferences and to utilize spatial information!

- so far, social scientists use a limited subset of techniques suitable to handle SA
 - exploratory spatial analysis:
 - different types of maps
 - local indicators of SA (LISA)
 - inferential models
 - parametric spatial regression models

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 - exploratory spatial analysis:
 - different types of maps
 - local indicators of SA (LISA)
 - inferential models
 - parametric spatial regression models
- spatial eigenfunction analysis – particularly Moran eigenvector maps (MEM) – is a simple yet powerful tool to analyze cross-sectional data structures
 - ① identification & visualization of complex (multi-scale) spatial patterns
 - ② specification, estimation, and interpretation of inferential models
 - ③ partitioning of the variation in \mathbf{y} into individual components (space, covariates, joint)
 - ④ (structure-preserving) simulation of spatially autocorrelated data

Spatial eigenfunction analysis is based on the spectral decomposition of a centered (and symmetric/ symmetrized) connectivity matrix \mathbf{W} :

$$\mathbf{M}\mathbf{W}\mathbf{M} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' \quad (1)$$

- demeaning projector $\mathbf{M} = (\mathbf{I} - \mathbf{1}\mathbf{1}'/n)$ – also ensures that all eigenvectors are orthogonal and uncorrelated
- \mathbf{E} are all n eigenvectors
- $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues λ

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Direct relationship to the numerator of the global Moran coefficient:

$$MC(\mathbf{x}) = \frac{n}{\mathbf{1}'\mathbf{W}\mathbf{1}} \frac{\mathbf{x}'\mathbf{M}\mathbf{W}\mathbf{M}\mathbf{x}}{\mathbf{x}'\mathbf{M}\mathbf{x}} \quad (2)$$

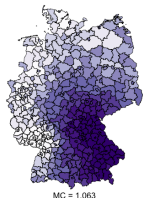
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- corresponding eigenvalues λ indicate the level of SA

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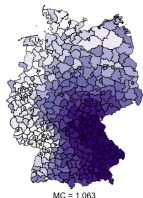
Eigenvector 1



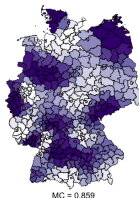
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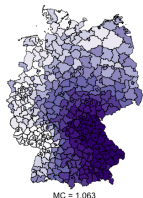
Eigenvector 25



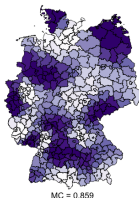
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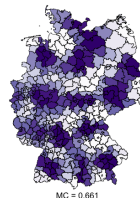
Eigenvector 1



Eigenvector 25



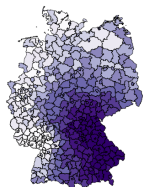
Eigenvector 50



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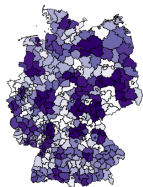
Example: 401 German NUTS-3 regions

Eigenvector 1



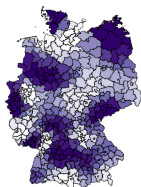
MC = 1.063

Eigenvector 75



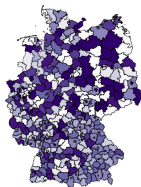
MC = 0.514

Eigenvector 25



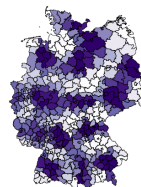
MC = 0.859

Eigenvector 170



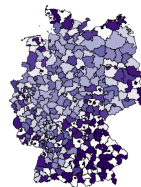
MC = 0.001

Eigenvector 50



MC = 0.661

Eigenvector 401

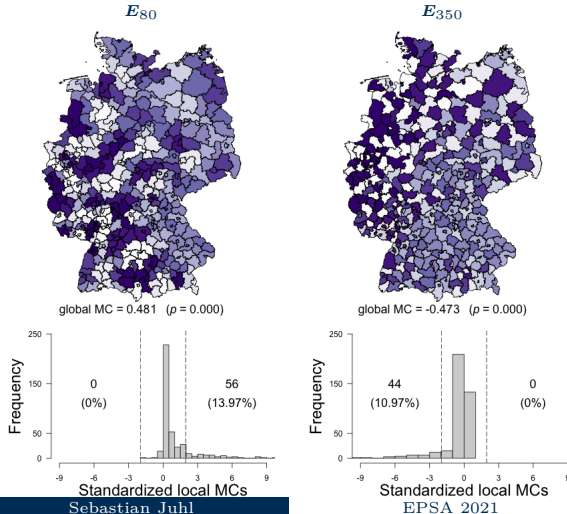


MC = -0.803

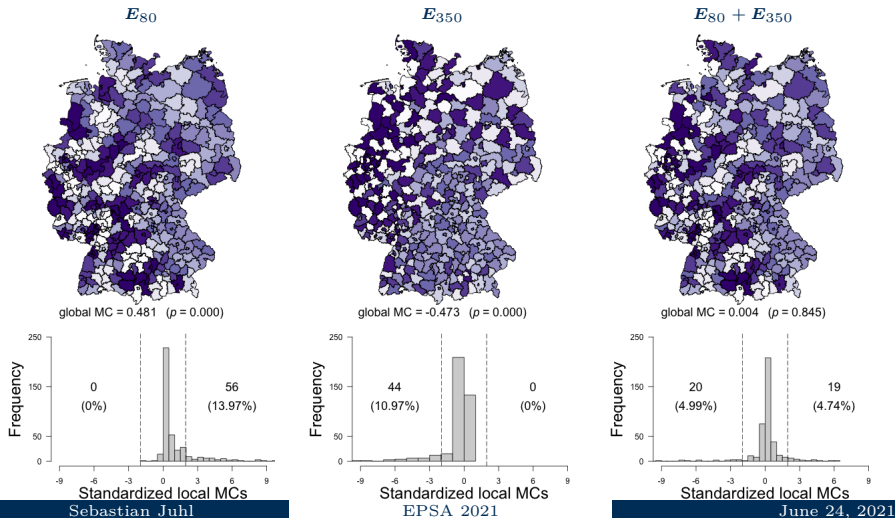
1. Identification of Complex Spatial Structures

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- eigenfunction analysis allows researchers to decompose the global MC into positively and negatively autocorrelated parts
 - $\text{global MC} = \text{MC}^+ + \text{MC}^-$
- decomposing the global MC helps identifying complex non-random spatial patterns

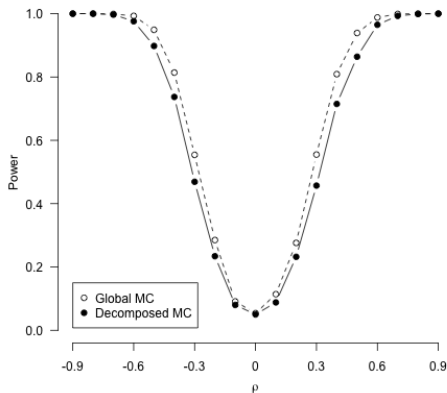
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Monte Carlo setup:

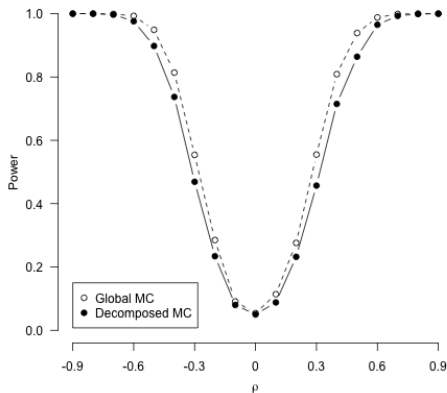
- $N = 100$ units ordered on a regular 10×10 grid
- \mathbf{W} is a row-normal symmetric contiguity matrix (rook scheme)
- ρ ranges from $-.9$ to $.9$ in steps of $.1$
- Scenario 1: simple spatial structure
 - $\mathbf{x} = (\mathbf{I} - \rho \mathbf{W}_g)^{-1} \mathbf{u}$
- Scenario 2: mixture of positive and negative SA
 - $\mathbf{x} = (\mathbf{I} - \rho \mathbf{W}_g)^{-1} \mathbf{u} + (\mathbf{I} - (-\rho) \mathbf{W}_g)^{-1} \mathbf{v}$

How does the decomposed MC based on spatial eigenfunctions performs compared to the global MC in terms of power?

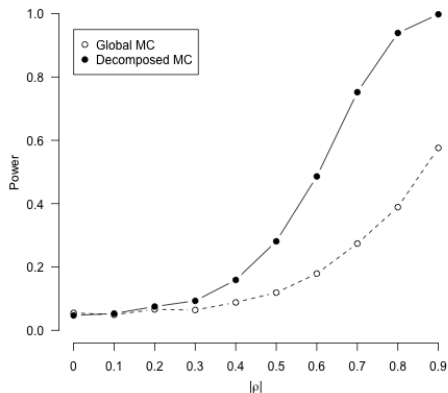
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Scenario 2: Mixture of SA

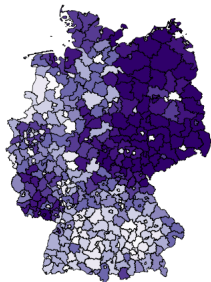


2. Model Specification, Estimation, and Interpretation

- SA causes severe problems for common econometric inferential techniques
- depending on the spatial DGP, SA can lead to
 - ① incorrect standard errors
 - ② biased and inconsistent parameter estimates
- spatial regression models address these problems but require many more or less rigid assumptions
 - knowledge of the true DGP
 - functional form assumptions
 - exact specification of SA in each regressor
 - difficult to estimate in a GLM framework
 - ...
- semiparametric spatial filtering methods use Moran eigenvectors to construct a synthetic proxy variable that controls for SA

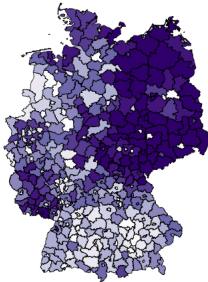
- a subset of eigenvectors E^* can be combined to reproduce real-world map patterns
- Example: Median age in German NUTS-3 regions (2017)

Observed Median Age

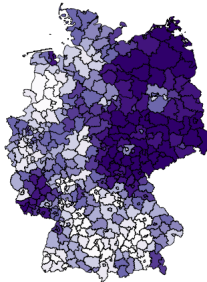


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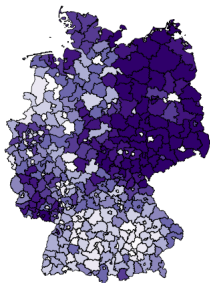


Synthetic Spatial Filter

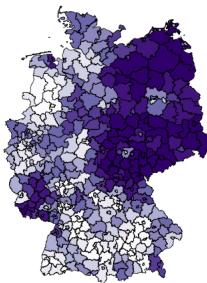


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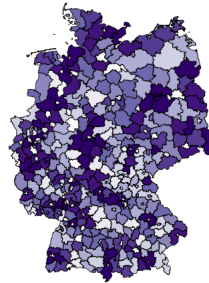
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Synthetic Spatial Filter



Filtered Residuals



- a judiciously selected subset of eigenvectors controls for the underlying spatial pattern
- straightforward parameter estimation & interpretation

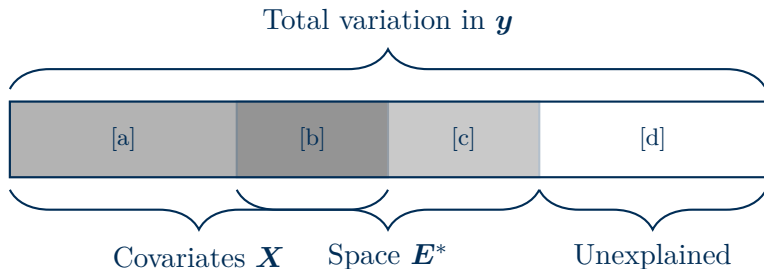
3. Variation Partitioning

How much variation in \mathbf{y} is caused by \mathbf{X} ?

- a common spatial structure spuriously inflates the share of variation explained by the predictors
- disentangle the individual contribution of the covariates and the spatial structure

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 - 2.1 regress \mathbf{y} on \mathbf{X} (fraction $[a + b]$)
 - 2.2 regress \mathbf{y} on \mathbf{E}^* (fraction $[b + c]$)
 - 2.3 regress \mathbf{y} on \mathbf{X} and \mathbf{E}^* (fraction $[a + b + c]$)

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- 3 using the results from step 2, calculate individual fractions
 - 3.1 $[a] = [a + b + c] - [b + c]$
 - 3.2 $[b] = [a + b] + [b + c] - [a + b + c]$
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- 4 use Moran spectral randomization to compute R_{adj}^2

Example:

- regress GDP (\mathbf{y}) on median age (\mathbf{X})
- MC of (log) GDP: 0.347 ($p = 0.000$)
- spatial filtering identifies 15 relevant eigenvectors (\mathbf{E}^*)

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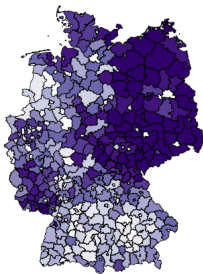
	<u>Joint Fractions</u>			<u>Individual Fractions</u>			
	$[a + b]$	$[b + c]$	$[a + b + c]$	$[a]$	$[b]$	$[c]$	$[d]$
R^2	0.286	0.246	0.523	0.277	0.009	0.237	0.477
R_{adj}^2	0.284	0.217	0.492	0.275	0.009	0.208	0.508

Note: Spatially constrained null model to calculate R_{adj}^2 based on 1,000 random permutations.

4. (Structure-Preserving) Simulation of Spatially Autocorrelated Data

- spatial multipliers $(\mathbf{I} - \rho\mathbf{W})^{-1}$ are typically used to simulate SA data
 - fixed degree of SA across simulations (controlled by ρ)
 - does not preserve spatial structure
- using MEMs for simulation exercises preserves the geographic distribution

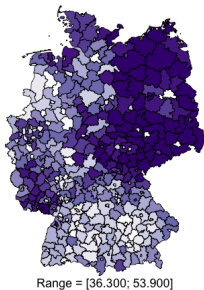
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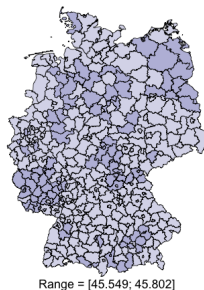
Range = [36.300; 53.900]

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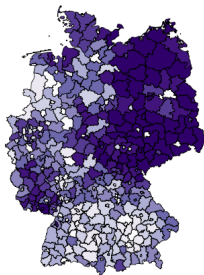


Spatial Multipliers (Means)



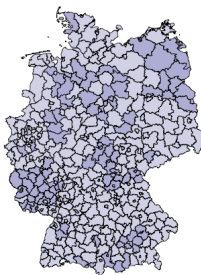
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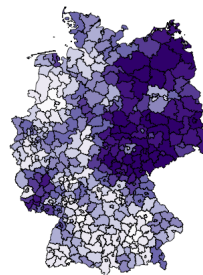
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Spatial Multipliers (Means)



Range = [45.549; 45.802]

MEMs (Means)



Range = [42.206; 53.409]

- spatial eigenfunction analysis complements the statistical repertoire
 - it helps addressing methodological problems caused by SA
 - MEMs allow researchers to derive additional information from geo-referenced data
- improves exploratory and inferential analysis, especially w.r.t
 - identification & visualization of complex (multi-scale) spatial patterns
 - specification, estimation, and interpretation of inferential models
 - variation partitioning
 - simulation of SA data

`spfilterR` package:

CRAN: <https://CRAN.R-project.org/package=spfilterR>

GitHub: <https://github.com/sjuhl/spfilterR>

Feedback & suggestions are highly appreciated!

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