Implementation of “A Bayesian analysis of log-periodic precursors to financial crashes” paper in R

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I. Introduction

This report is about the implementation of algorithms described in the paper **A Bayesian Analysis of Log-Periodic Precursors to Financial Crashes** [1], utilizing the R programming language. Since the paper was published in 2006, the algorithms it proposes have not been available in a coded form. In response, I developed code to replicate the paper's results. My methodology relied on the Markov Chain Monte Carlo (MCMC) method, specifically utilizing the Metropolis-Hastings Algorithm within Gibbs Sampler (referred to as **Metropolis within Gibbs**), to accurately reproduce the findings of the original study. This document introduces the theoretical setting of Metropolis within Gibbs algorithm and provides a comparative analysis between the results obtained in the original paper and those derived from this implementation.

II. Theoretical Setting

A. *Johansen–Ledoit–Sornette (JLS) model of log periodicity*

Johansen–Ledoit–Sornette (JLS) model is called Log Periodic Power Law(LPPL) model. In the model, the S&P 500 index is fitted to the specification

$$\begin{aligned}\ln s(t) &= q(t)\\ &= A - B(t_c - t)^{\beta}\left[1 + \frac{C}{\sqrt{1+(\omega/\beta)^2}}\cos(\omega \ln(t_c - t) + \phi)\right]\end{aligned} \tag{1}$$

- $s(t)$: S&P 500 index
- A: log price at t_c(i.e. $\log s(t_c)$)
- B: This is a log periodic coefficient and determines if there is a log-periodicity or not.
- t_c: This is a critical time at which the potential for a crash subsides, and this should not be interpreted as the crash time.
- C: amplitude of oscillation
- ω: frequency of oscillation during the bubble period
- β: exponent
- ϕ: phase($0 \le \phi \le 2\pi$)

JLS claim that the hazard rate of a crash($h(t)$) will vary log-periodically. The hazard rate is the density that the event occurs at time t given it has not already occurred.

$$\begin{aligned}h(t) &= \frac{f(t)}{1-F(t)}\\ H(t) &= \bar{\kappa}\int_{t_0}^{t} h(t')dt'\\ \Delta H(t_1, t_2) &= H(t_2) - H(t_1)\\ q(t_{i+t}) - q(t_i) &\sim N(\Delta H(t_i, t_{i+1}), \sigma^2(t_{i+1} - t_i))\\ H(t) &= A - B(t_c - t)^{\beta}\left[1 + \frac{C}{\sqrt{1+(\omega/\beta)^2)}}\cos(\omega \ln(t_c - t) + \phi)\right]\end{aligned} \tag{2}$$

B. *Model without crash probabilities*

In the referenced paper, the parameters employed in the original Log-Periodic Power Law (LPPL) model are represented by $\xi = [B, C, \beta, \omega, \phi, t_c]$. Furthermore, the log price at time t_i, denoted as $\log s(t_i) = q(t_i)$, is assumed to follow a normal distribution in the paper.

$$\log s(t_i) = q(t_i) \sim N(mean, variance)$$

where

$$mean = q(t_{i-1}) + \mu(t_i - t_{i+1}) + \Delta H(t_{i-1}, t_i; \xi)$$
$$variance = \sigma^2(t_i - t_{i-1})$$

for $i = 1, \cdots, N$

C. Prior Distributions

- The scope of each parameter

Parameter	Scope
μ	$\mu \in \mathbb{R}$
τ	$\tau \in \mathbb{R}_+$
B	$B \in \mathbb{R}_+$
C	$C \in [0,1]$
β	$\beta \in [0,1]$
ω	$\omega \in \mathbb{R}_+$
ϕ	$\phi \in [0, 2\pi)$
t_c	$t_c \in [t_N, \infty)$

1) Diffuse Priors

$$\mu \sim N(0.0003, (0.01)^2)$$
$$\tau \sim \Gamma(1.0, 10^{-5})$$
$$B \sim \Gamma(1.0, 100)$$
$$C \sim U(0,1)$$
$$\beta \sim U(0,1)$$
$$\omega \sim \Gamma(16.0, 2.5)$$
$$\phi \sim U(0, 2\pi)$$
$$t_c - t_n \sim \Gamma(1.0, 0.001)$$

2) Tight Priors

$$\mu \sim N(0.0, 10^{-6})$$
$$\tau \sim \Gamma(1.0, 10^{-5})$$
$$B \sim \Gamma(100.0, 7613.8)$$
$$C \sim U(0,1)$$
$$\beta \sim B(41.3834, 29.9228)$$
$$\omega \sim \Gamma(16.0, 2.5)$$
$$\phi \sim U(0, 2\pi)$$
$$t_c - t_n \sim \Gamma(100.0, 25.0)$$

D. Posterior Density

Given $\theta = [\mu, \tau, B, C, \beta, \omega, \phi, t_c]$ and q_{t_i}, the probability density for $q_{t_{i+1}}$ is

$$p(q_{t_{i+1}}|q_{t_i}, \theta) = \sqrt{\frac{\tau}{2\pi(t_{i+1} - t_i)}} \times \exp\left[-\frac{\tau(q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1} - t_i) - \Delta H(t_{i+1}, t_i; \xi))^2}{2(t_{i+1} - t_i)}\right] \quad (3)$$

The posterior density is

$$p^{nc}(\theta|q_{t_0}, q_{t_1}, ..., q_{t_N}) \propto p(\theta) \prod_{i=0}^{N-1} p(q_{t_{i+1}}|q_{t_i}, \theta) \quad (4)$$

The log posterior density is

$$\log p^{nc}(\theta|q_{t_0}, q_{t_1}, ..., q_{t_N}) \propto \log p(\theta) + \sum_{i=0}^{N-1} \log p(q_{t_{i+1}}|q_{t_i}, \theta)$$

III. Metropolis within Gibbs

To obtain the posterior distribution of each parameter utilizing the Gibbs sampler, it is imperative to ascertain the full conditional distribution of each parameter. However, due to the complexity of the model's probability density, deriving the full conditional distribution of each parameter in a closed form proves to be infeasible. Consequently, the **Metropolis-Hastings Algorithm within the Gibbs Sampler** (referred to as **Metropolis within Gibbs**) is adopted. For the proposal distributions, normal distributions are employed. The notation $\mathcal{Q}_{t_0}^{t_N}$ represents a vector containing log prices from time t_0 to t_N.(i.e. $\mathcal{Q}_{t_0}^{t_N} = [q_0, q_1, \cdots, q_N]$)

A. Updating Rule for μ

The full conditional distribution is

$$\begin{aligned}
&P(\mu|\tau, \xi, \mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(\mu, \tau, \xi, \mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\tau, \xi, \mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(\mu, \tau, \xi|\mathcal{Q}_{t_0}^{t_N})/P(\tau, \xi|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\mu, \tau, \xi|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\mu)P(\tau)P(\xi) \prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i}, \theta) \\
&\propto P(\mu) \prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i}, \theta)
\end{aligned}$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(\mu|\tau,\xi,\mathcal{Q}_{t_0}^{t_N}) \propto \log P(\mu)+\sum_{i=0}^{N-1}\log P(q_{t_{i+1}}|q_{t_i},\theta)\\
&\propto \log P(\mu)+\sum_{i=0}^{N-1}[\frac{1}{2}\log\tau-\frac{1}{2}\log 2\pi(t_{i+1}-t_i)\\
&-\frac{\tau\{q_{t_{i+1}}-q_{t_i}-\mu(t_{i+1}-t_i)-\Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 1: Updating Rule for μ

Input: $\mu^{(t)}$, $\tau^{(t)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_μ^2 {The step size σ_μ^2 is a hyperparameter. $\xi^{(t)}=[B^{(t)},C^{(t)},\beta^{(t)},\omega^{(t)},\phi^{(t)},t_c^{(t)}]$}
Output: $\mu^{(t+1)}$
Sample $\mu^{'}\sim N(\mu^{(t)},\sigma_\mu^2)$.
$\log\alpha\leftarrow\log P(\mu^{'}|\tau^{(t)},\xi^{(t)},\mathcal{Q}_{t_0}^{t_N})-\log P(\mu^{(t)}|\tau^{(t)},\xi^{(t)},\mathcal{Q}_{t_0}^{t_N})$
Sample $U\sim U(0,1)$.
if $\log\alpha>\log U$ **then**
  $\mu^{(t+1)}\leftarrow\mu^{'}$ {Accept}
else
  $\mu^{(t+1)}\leftarrow\mu^{(t)}$ {Reject}
return $\mu^{(t+1)}$

B. Updating Rule for τ

The full conditional distribution is

$$
\begin{aligned}
&P(\tau|\mu,\xi,\mathcal{Q}_{t_0}^{t_N})\\
&=\frac{P(\mu,\tau,\xi,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\xi,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}\\
&=P(\mu,\tau,\xi|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\xi|\mathcal{Q}_{t_0}^{t_N})\\
&\propto P(\mu,\tau,\xi|\mathcal{Q}_{t_0}^{t_N})\\
&\propto P(\mu)P(\tau)P(\xi)\prod_{i=0}^{N-1}P(q_{t_{i+1}}|q_{t_i},\theta)\\
&\propto P(\tau)\prod_{i=0}^{N-1}P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(\tau|\mu,\xi,\mathcal{Q}_{t_0}^{t_N})\\
&\propto \log P(\tau)+\sum_{i=0}^{N-1}\log P(q_{t_{i+1}}|q_{t_i},\theta)\\
&\propto \log P(\tau)+\sum_{i=0}^{N-1}[\frac{1}{2}\log\tau-\frac{1}{2}\log 2\pi(t_{i+1}-t_i)\\
&-\frac{\tau\{q_{t_{i+1}}-q_{t_i}-\mu(t_{i+1}-t_i)-\Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 2: Updating Rule for τ

Input: $\mu^{(t+1)}$, $\tau^{(t)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_τ^2 {The step size σ_τ^2 is a hyperparameter. $\xi^{(t)}=[B^{(t)},C^{(t)},\beta^{(t)},\omega^{(t)},\phi^{(t)},t_c^{(t)}]$}
Output: $\tau^{(t+1)}$
Sample $\tau^{'}\sim N(\tau^{(t)},\sigma_\tau^2)$.
$\log\alpha\leftarrow\log P(\tau^{'}|\mu^{(t+1)},\xi^{(t)},\mathcal{Q}_{t_0}^{t_N})-\log P(\tau^{(t)}|\mu^{(t+1)},\xi^{(t)},\mathcal{Q}_{t_0}^{t_N})$
Sample $U\sim U(0,1)$.
if $\log\alpha>\log U$ **then**
  $\tau^{(t+1)}\leftarrow\tau^{'}$ {Accept}
else
  $\tau^{(t+1)}\leftarrow\tau^{(t)}$ {Reject}
return $\tau^{(t+1)}$

C. Updating Rule for B

Let's denote $\xi_B=[C,\beta,\omega,\phi,t_c]$. The full conditional distribution is

$$
\begin{aligned}
&P(B|\mu,\tau,\xi_B,\mathcal{Q}_{t_0}^{t_N})\\
&=\frac{P(B,\mu,\tau,\xi_B,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_B,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}\\
&=P(B,\mu,\tau,\xi_B|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_B|\mathcal{Q}_{t_0}^{t_N})\\
&\propto P(B,\mu,\tau,\xi_B|\mathcal{Q}_{t_0}^{t_N})\\
&\propto P(B)P(\mu)P(\tau)P(\xi_B)\prod_{i=0}^{N-1}P(q_{t_{i+1}}|q_{t_i},\theta)\\
&\propto P(B)\prod_{i=0}^{N-1}P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(B|\mu,\tau,\xi_B,\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(B) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(B) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&- \frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 3: Updating Rule for B

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_B^2 {The step size σ_B^2 is a hyperparameter. $\xi_B^{(t)} = [C^{(t)}, \beta^{(t)}, \omega^{(t)}, \phi^{(t)}, t_c^{(t)}]$}
Output: $B^{(t+1)}$
Sample $B^{'} \sim N(B^{(t)}, \sigma_B^2)$.
$\log\alpha \leftarrow \log P(B^{'}|\mu^{(t+1)},\tau^{(t+1)},\xi_B^{(t)},\mathcal{Q}_{t_0}^{t_N}) - \log P(B^{(t)}|\mu^{(t+1)},\tau^{(t+1)},\xi_B^{(t)},\mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $B^{(t+1)} \leftarrow B^{'}$ {Accept}
else
  $B^{(t+1)} \leftarrow B^{(t)}$ {Reject}
return $B^{(t+1)}$

D. Updating Rule for C

Let's denote $\xi_C = [B, \beta, \omega, \phi, t_c]$. The full conditional distribution is

$$
\begin{aligned}
&P(C|\mu,\tau,\xi_C,\mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(C,\mu,\tau,\xi_C,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_C,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(C,\mu,\tau,\xi_C|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_C|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(C,\mu,\tau,\xi_C|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(C)P(\mu)P(\tau)P(\xi_C)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto P(C)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(C|\mu,\tau,\xi_C,\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(C) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(C) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&- \frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 4: Updating Rule for C

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_C^2 {The step size σ_C^2 is a hyperparameter. $\xi_C^{(t)} = [B^{(t+1)}, \beta^{(t)}, \omega^{(t)}, \phi^{(t)}, t_c^{(t)}]$}
Output: $C^{(t+1)}$
Sample $C^{'} \sim N(C^{(t)}, \sigma_C^2)$.
$\log\alpha \leftarrow \log P(C^{'}|\mu^{(t+1)},\tau^{(t+1)},\xi_C^{(t)},\mathcal{Q}_{t_0}^{t_N}) - \log P(C^{(t)}|\mu^{(t+1)},\tau^{(t+1)},\xi_C^{(t)},\mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $C^{(t+1)} \leftarrow C^{'}$ {Accept}
else
  $C^{(t+1)} \leftarrow C^{(t)}$ {Reject}
return $C^{(t+1)}$

E. Updating Rule for β

Let's denote $\xi_\beta = [B, C, \omega, \phi, t_c]$. The full conditional distribution is

$$
\begin{aligned}
&P(\beta|\mu,\tau,\xi_\beta,\mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(\beta,\mu,\tau,\xi_\beta,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_\beta,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(\beta,\mu,\tau,\xi_\beta|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_\beta|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\beta,\mu,\tau,\xi_\beta|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\beta)P(\mu)P(\tau)P(\xi_\beta)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto P(\beta)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(\beta|\mu,\tau,\xi_\beta,\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(\beta) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(\beta) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&-\frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 5: Updating Rule for β

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_β^2 {The step size σ_β^2 is a hyperparameter. $\xi_\beta^{(t)} = [B^{(t+1)}, C^{(t+1)}, \omega^{(t)}, \phi^{(t)}, t_c^{(t)}]$}
Output: $\beta^{(t+1)}$
Sample $\beta' \sim N(\beta^{(t)}, \sigma_\beta^2)$.
$\log\alpha \leftarrow \log P(\beta'|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\beta^{(t)}, \mathcal{Q}_{t_0}^{t_N}) - \log P(\beta^{(t)}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\beta^{(t)}, \mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $\beta^{(t+1)} \leftarrow \beta'$ {Accept}
else
  $\beta^{(t+1)} \leftarrow \beta^{(t)}$ {Reject}
return $\beta^{(t+1)}$

F. Updating Rule for ω

Let's denote $\xi_\omega = [B, C, \beta, \phi, t_c]$. The full conditional distribution is

$$
\begin{aligned}
&P(\omega|\mu,\tau,\xi_\omega,\mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(\omega,\mu,\tau,\xi_\omega,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_\omega,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(\omega,\mu,\tau,\xi_\omega|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_\omega|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\omega,\mu,\tau,\xi_\omega|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\omega)P(\mu)P(\tau)P(\xi_\omega)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto P(\omega)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(\omega|\mu,\tau,\xi_\omega,\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(\omega) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(\omega) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&-\frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 6: Updating Rule for ω

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_ω^2 {The step size σ_ω^2 is a hyperparameter. $\xi_\omega^{(t)} = [B^{(t+1)}, C^{(t+1)}, \beta^{(t+1)}, \phi^{(t)}, t_c^{(t)}]$}
Output: $\omega^{(t+1)}$
Sample $\omega' \sim N(\omega^{(t)}, \sigma_\omega^2)$.
$\log\alpha \leftarrow \log P(\omega'|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\omega^{(t)}, \mathcal{Q}_{t_0}^{t_N}) - \log P(\omega^{(t)}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\omega^{(t)}, \mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $\omega^{(t+1)} \leftarrow \omega'$ {Accept}
else
  $\omega^{(t+1)} \leftarrow \omega^{(t)}$ {Reject}
return $\omega^{(t+1)}$

G. Updating Rule for ϕ

Let's denote $\xi_\phi = [B, C, \beta, \omega, t_c]$. The full conditional distribution is

$$
\begin{aligned}
&P(\phi|\mu,\tau,\xi_\phi,\mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(\phi,\mu,\tau,\xi_\phi,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_\phi,\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(\phi,\mu,\tau,\xi_\phi|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_\phi|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\phi,\mu,\tau,\xi_\phi|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(\phi)P(\mu)P(\tau)P(\xi_\phi)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto P(\phi)\prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(\phi|\mu,\tau,\xi_\phi,\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(\phi) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(\phi) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&- \frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 7: Updating Rule for ϕ

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t+1)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ_ϕ^2 {The step size σ_ϕ^2 is a hyperparameter. $\xi_\phi^{(t)} = [B^{(t+1)}, C^{(t+1)}, \beta^{(t+1)}, \omega^{(t+1)}, t_c^{(t)}]$}
Output: $\phi^{(t+1)}$
Sample $\phi^{'} \sim N(\phi^{(t)}, \sigma_\phi^2)$.
$\log\alpha \leftarrow \log P(\phi^{'}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\phi^{(t)}, \mathcal{Q}_{t_0}^{t_N}) - \log P(\phi^{(t)}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_\phi^{(t)}, \mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $\phi^{(t+1)} \leftarrow \phi^{'}$ {Accept}
else
  $\phi^{(t+1)} \leftarrow \phi^{(t)}$ {Reject}
return $\phi^{(t+1)}$

H. Updating Rule for t_c

Let's denote $\xi_{t_c} = [B, C, \beta, \omega, \phi]$. The full conditional distribution is

$$
\begin{aligned}
&P(t_c|\mu,\tau,\xi_{t_c},\mathcal{Q}_{t_0}^{t_N}) \\
&= \frac{P(t_c,\mu,\tau,\xi_{t_c},\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})}{P(\mu,\tau,\xi_{t_c},\mathcal{Q}_{t_0}^{t_N})/P(\mathcal{Q}_{t_0}^{t_N})} \\
&= P(t_c,\mu,\tau,\xi_{t_c}|\mathcal{Q}_{t_0}^{t_N})/P(\mu,\tau,\xi_{t_c}|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(t_c,\mu,\tau,\xi_{t_c}|\mathcal{Q}_{t_0}^{t_N}) \\
&\propto P(t_c)P(\mu)P(\tau)P(\xi_{t_c}) \prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto P(t_c) \prod_{i=0}^{N-1} P(q_{t_{i+1}}|q_{t_i},\theta)
\end{aligned}
$$

Then, the log full conditional distribution is

$$
\begin{aligned}
&\log P(t_c|\mu,\tau,\xi_{t_c},\mathcal{Q}_{t_0}^{t_N}) \\
&\propto \log P(t_c) + \sum_{i=0}^{N-1} \log P(q_{t_{i+1}}|q_{t_i},\theta) \\
&\propto \log P(t_c) + \sum_{i=0}^{N-1} [\frac{1}{2}\log\tau - \frac{1}{2}\log 2\pi(t_{i+1}-t_i) \\
&- \frac{\tau\{q_{t_{i+1}} - q_{t_i} - \mu(t_{i+1}-t_i) - \Delta H(t_i,t_{i+1};\xi)\}^2}{2(t_{i+1}-t_i)}]
\end{aligned}
$$

Algorithm 8: Updating Rule for t_c

Input: $\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t+1)}$, $\phi^{(t+1)}$, $t_c^{(t)}$, $\sigma_{t_c-t_N}^2$ {The step size $\sigma_{t_c-t_N}^2$ is a hyperparameter. $\xi_{t_c}^{(t)} = [B^{(t+1)}, C^{(t+1)}, \beta^{(t+1)}, \omega^{(t+1)}, \phi^{(t+1)}]$}
Output: $t_c^{(t+1)}$
Sample $\{t_c - t_N\}^{'} \sim N(t_c^{(t)}, \sigma_{t_c-t_N}^2)$.
$t_c^{'} \leftarrow \{t_c - t_N\}^{'} + t_N$
$\log\alpha \leftarrow \log P(t_c^{'}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_{t_c}^{(t)}, \mathcal{Q}_{t_0}^{t_N}) - \log P(t_c^{(t)}|\mu^{(t+1)}, \tau^{(t+1)}, \xi_{t_c}^{(t)}, \mathcal{Q}_{t_0}^{t_N})$
Sample $U \sim U(0,1)$.
if $\log\alpha > \log U$ **then**
  $t_c^{(t+1)} \leftarrow t_c^{'}$ {Accept}
else
  $t_c^{(t+1)} \leftarrow t_c^{(t)}$ {Reject}
return $t_c^{(t+1)}$

I. Metropolis within Gibbs

Employing algorithms 1 through 8, the detailed procedure of the **Metropolis Hastings Algorithm within Gibbs Sampler** (referred to as **Metropolis within Gibbs**) is explicated in Algorithm 9. This Metropolis within Gibbs approach iteratively samples each parameter up to a predefined maximum number of iterations ($itermax$). At each iteration t, the parameters ranging from μ to t_c are sequentially sampled.

Algorithm 9: Metropolis within Gibbs

Set up initial value, $\vec{\theta}^{(1)} = [\mu^{(1)}, \tau^{(1)}, B^{(1)}, C^{(1)}, \beta^{(1)}, \omega^{(1)}, \phi^{(1)}, t_c^{(1)}]$.
for $1 \leq i < itermax$ **do**
  $\mu^{(t+1)} \leftarrow$ Algorithm 1($\mu^{(t)}$, $\tau^{(t)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_μ)
  $\tau^{(t+1)} \leftarrow$ Algorithm 2($\mu^{(t+1)}$, $\tau^{(t)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_τ)
  $B^{(t+1)} \leftarrow$ Algorithm 3($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_B)
  $C^{(t+1)} \leftarrow$ Algorithm 4($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_C)
  $\beta^{(t+1)} \leftarrow$ Algorithm 5($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_β)
  $\omega^{(t+1)} \leftarrow$ Algorithm 6($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_ω)
  $\phi^{(t+1)} \leftarrow$ Algorithm 7($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t+1)}$, $\phi^{(t)}$, $t_c^{(t)}$, σ^2_ϕ)
  $t_c^{(t+1)} \leftarrow$ Algorithm 8($\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t+1)}$, $\phi^{(t+1)}$, $t_c^{(t)}$, $\sigma^2_{t_c - t_N}$)
  $\vec{\theta}^{(t+1)} \leftarrow [\mu^{(t+1)}$, $\tau^{(t+1)}$, $B^{(t+1)}$, $C^{(t+1)}$, $\beta^{(t+1)}$, $\omega^{(t+1)}$, $\phi^{(t+1)}$, $t_c^{(t+1)}]$
  $t \leftarrow t + 1$
return $[\vec{\theta}^{(1)}, \vec{\theta}^{(2)}, \cdots, \vec{\theta}^{(itermax)}]$

IV. Results

A. Market-Time Model with Diffuse Priors

The initial values of parameters are assigned using Non-Linear Least Squares (NLLS) estimators, with the exception of μ and τ. Since μ and τ are not components of the original LPPL model and there are no NLLS estimators for them, their initial values are chosen to approximate the posterior mean values derived from the algorithm in the paper.

- $\mu^{(1)} = 0$
- $\tau^{(1)} = 15000$
- $B^{(1)} = 0.0130$
- $C^{(1)} = 0.966$
- $\beta^{(1)} = 0.580$
- $\omega^{(1)} = 5.711$
- $\phi^{(1)} = 4.845$
- $t_c^{(1)} = 10/20/1987$

The determination of each parameter's step size was conducted empirically, targeting acceptance rates within the range of 20% to 40%.

Parameter	NLLS	$A^{nc}_{lp,m}$(Paper)	Metropolis within Gibbs
μ	-	0.000420 (4.78×10^{-6})	0.000381 (1.77×10^{-8})
τ	-	13437 (2.43)	13437.242886 (0.018)
B	0.0130	0.007359 (3.44×10^{-5})	0.007228 (2.67×10^{-7})
C	0.966	0.500030 (0.001618)	0.496158 $(9.667818e \times 10^{-6})$
β	0.580	0.384268 (0.001726)	0.394065 (8.80×10^{-6})
ω	5.711	6.425584 (0.006496)	6.419012 (5.37×10^{-5})
ϕ	4.845	3.149531 (0.009522)	3.142698 (6.09×10^{-5})
t_c	10/20/1987	3/31/1998 (0.436634)	$116.096927 \rightarrow 4/4/1987$ (0.003509316)

TABLE I: Estimates of parameters using nonlinear least-squares and from posterior means for the 1983 to 16 October 1987 data set. Standard errors for the posterior means are given underneath.

The chart presents the posterior mean values and standard errors for each parameter. The rightmost column displays the outcomes yielded by the Metropolis within Gibbs algorithm, and the central column exhibits the results from the paper ($A^{nc}_{lp,m}$). Regarding posterior mean values, there is a negligible difference between the results of the Metropolis within Gibbs algorithm and those reported in the paper ($A^{nc}_{lp,m}$). To ascertain whether the Metropolis within Gibbs algorithm reproduces posterior distributions that align with the paper, ensuing sections juxtapose the posterior distribution graphs of parameters from the paper with those from the Metropolis within Gibbs algorithm.

1) Market Time Model with Diffuse Priors - μ

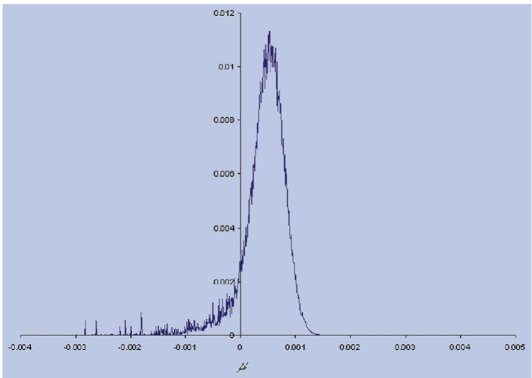

Fig. 1: Posterior density for the drift μ in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

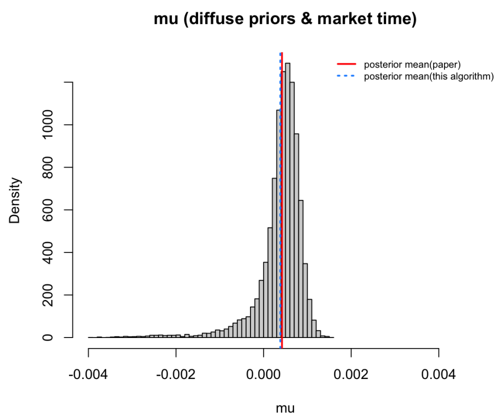

Fig. 2: Posterior density for the drift μ in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

2) Market Time Model with Diffuse Priors - τ

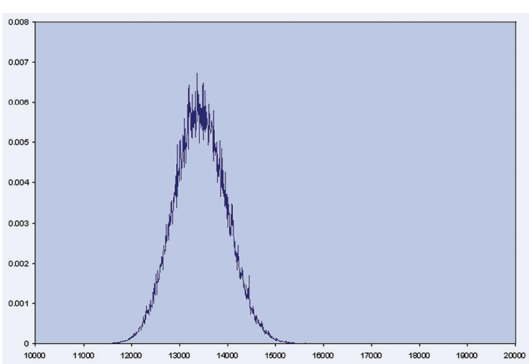

Fig. 3: Posterior density for the precision τ in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

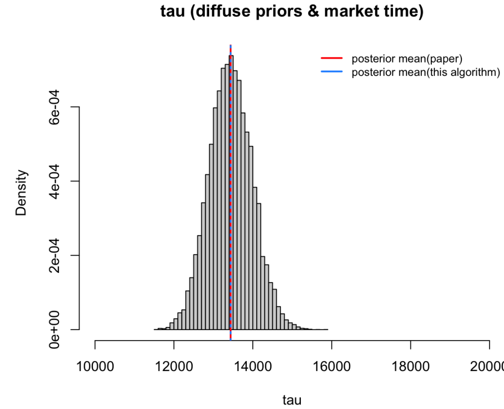

Fig. 4: Posterior density for the precision τ in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

3) Market Time Model with Diffuse Priors - B

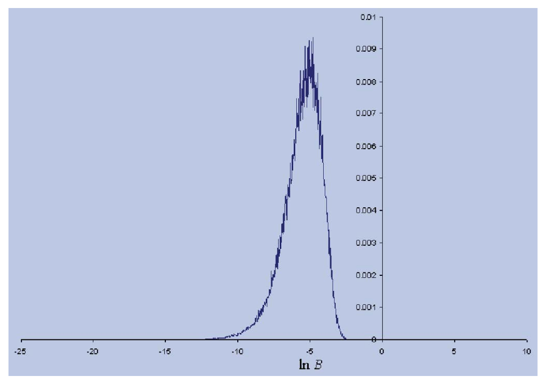

Fig. 5: Posterior density for the log of the log-periodic coefficient $\ln B$ in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

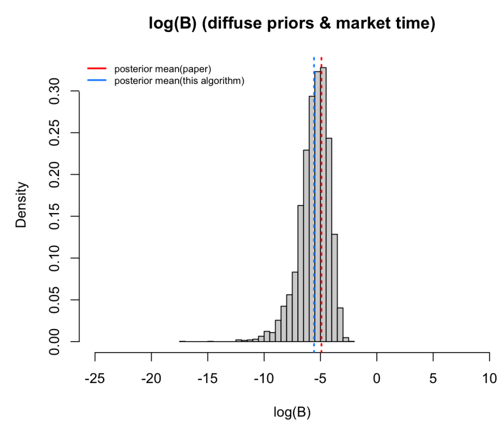

Fig. 6: Posterior density for the log of the log-periodic coefficient $\ln B$ in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

4) *Market Time Model with Diffuse Priors - C*

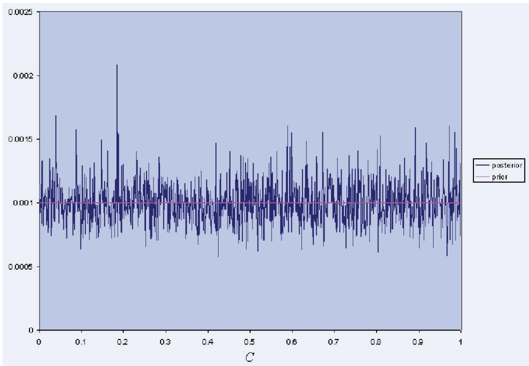

Fig. 7: Posterior density for the amplitude C in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

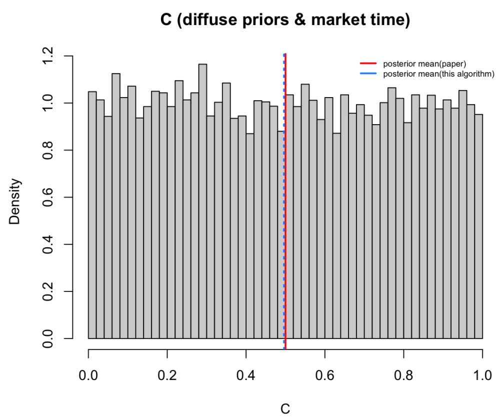

Fig. 8: Posterior density for the amplitude C in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

5) *Market Time Model with Diffuse Priors - β*

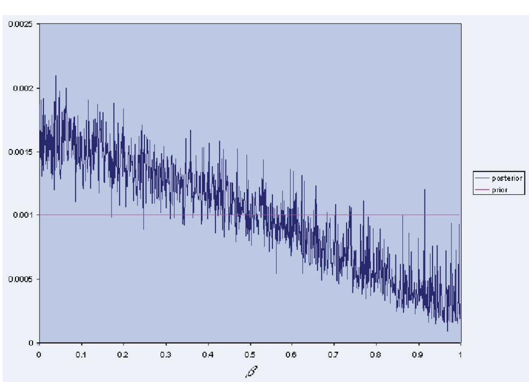

Fig. 9: Posterior density for the exponent β in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

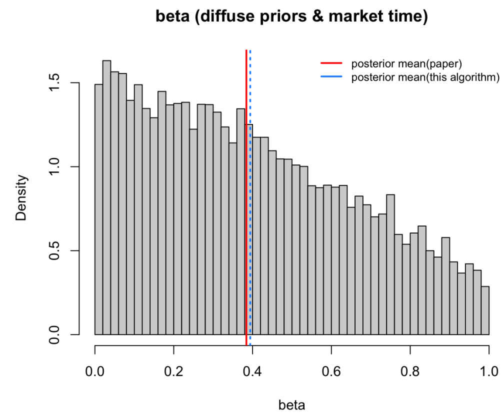

Fig. 10: Posterior density for the exponent β in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

6) *Market Time Model with Diffuse Priors - ω*

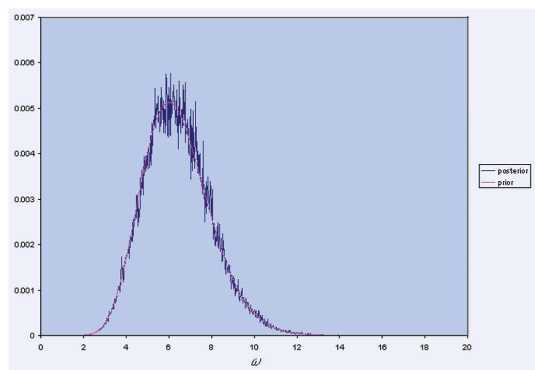

Fig. 11: Posterior density for the frequency ω in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

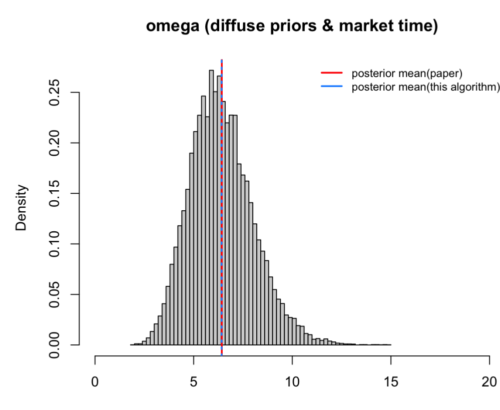

Fig. 12: Posterior density for the frequency ω in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

7) Market Time Model with Diffuse Priors - ϕ

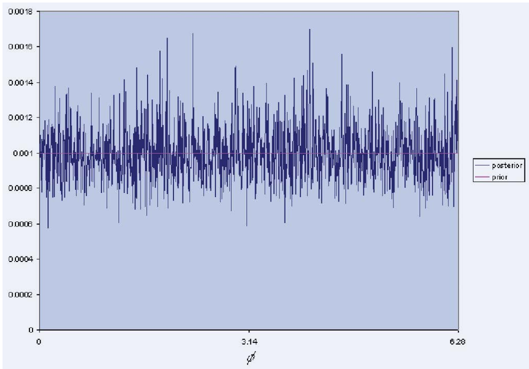

Fig. 13: Posterior density for the phase ϕ in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

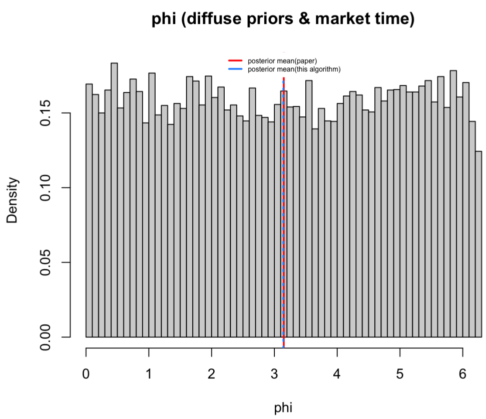

Fig. 14: Posterior density for the phase ϕ in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

8) Market Time Model with Diffuse Priors - $t_c - t_N$

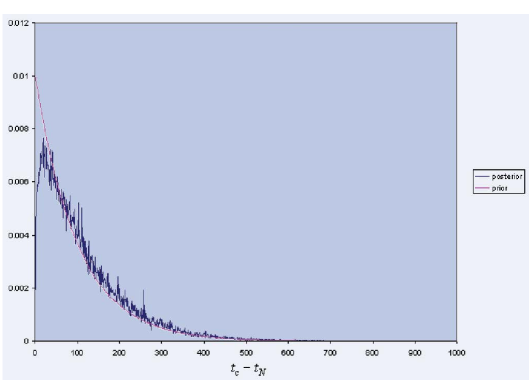

Fig. 15: Posterior density for the critical time t_c in the market-time model with diffuse priors and no crash probabilities, provided from the paper($A^{nc}_{lp,m}$).

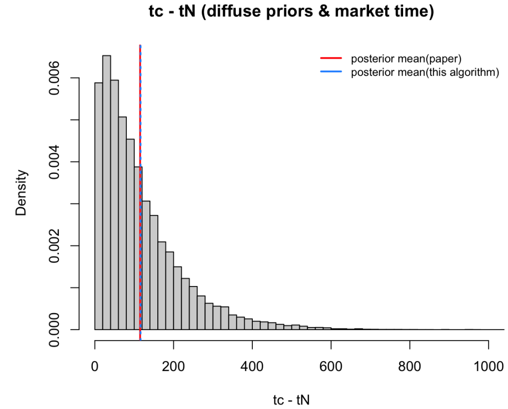

Fig. 16: Posterior density for the critical time t_c in the market-time model with diffuse priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

B. Calendar-Time Model with Tight Priors

The initial values of parameters are assigned using Non-Linear Least Squares (NLLS) estimators, with the exception of μ and τ. Since μ and τ are not components of the original LPPL model and there are no NLLS estimators for them, their initial values are chosen to approximate the posterior mean values derived from the algorithm in the paper.

- $\mu^{(1)} = 0$
- $\tau^{(1)} = 15000$
- $B^{(1)} = 0.0130$
- $C^{(1)} = 0.966$
- $\beta^{(1)} = 0.580$
- $\omega^{(1)} = 5.711$
- $\phi^{(1)} = 4.845$
- $t_c^{(1)} = 10/20/1987$

The determination of each parameter's step size was conducted empirically, targeting acceptance rates within the range of 20% to 40%.

The chart presents the posterior mean values and standard errors for each parameter. The rightmost column displays the outcomes yielded by the Metropolis within Gibbs algorithm, and the central column exhibits the results from the paper ($B^{nc}_{lp,c}$). Regarding posterior mean values, there is a negligible difference between the results of the Metropolis within Gibbs algorithm and those reported in the paper ($B^{nc}_{lp,c}$). To ascertain whether the Metropolis within Gibbs algorithm reproduces posterior

Parameter	NLLS	$B^{nc}_{lp,c}$(Paper)	Metropolis within Gibbs
μ	-	0.000032 (4.59×10^{-6})	0.000035 (7.78×10^{-9})
τ	-	15698 (11.70)	15696.134671 (0.022)
B	0.0130	0.012553 (2.77×10^{-5})	0.012522 (4.19×10^{-8})
C	0.966	0.721244 (0.005045)	0.727699 (8.16×10^{-6})
β	0.580	0.530319 (0.000998)	0.531670 (1.72×10^{-6})
ω	5.711	5.940680 (0.010669)	5.919239 (3.35×10^{-5})
ϕ	4.845	3.818973 (0.048272)	3.829750 (6.36×10^{-5})
t_c	10/20/1987	10/20/1987 (0.008439)	$4.079435 \rightarrow$ 10/20/1987 (1.30×10^{-5})

TABLE II: Estimates of parameters using nonlinear least-squares and from posterior means for the 1983 to 16 October 1987 data set. Standard errors for the posterior means are given underneath.

distributions that align with the paper, ensuing sections juxtapose the posterior distribution graphs of parameters from the paper with those from the Metropolis within Gibbs algorithm.

1) Calendar-Time Model with Tight Priors - μ

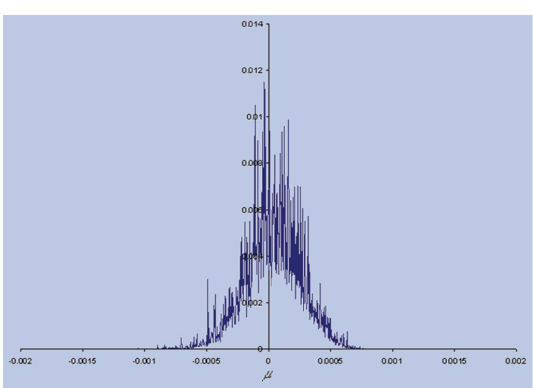

Fig. 17: Posterior density for the drift μ in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

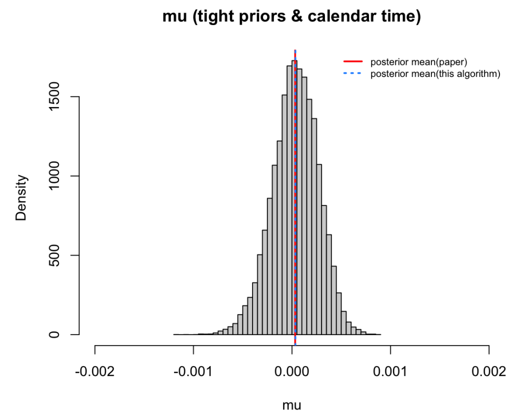

Fig. 18: Posterior density for the drift μ in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

2) Calendar-Time Model with Tight Priors - τ

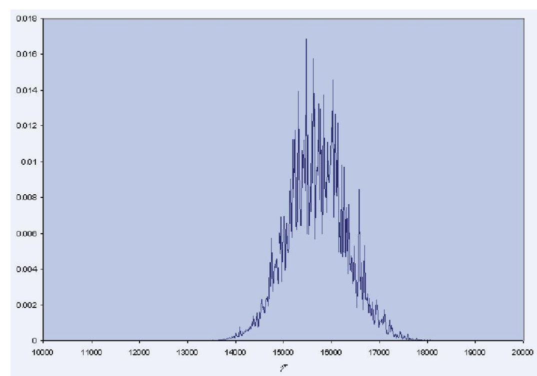

Fig. 19: Posterior density for the precision τ in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

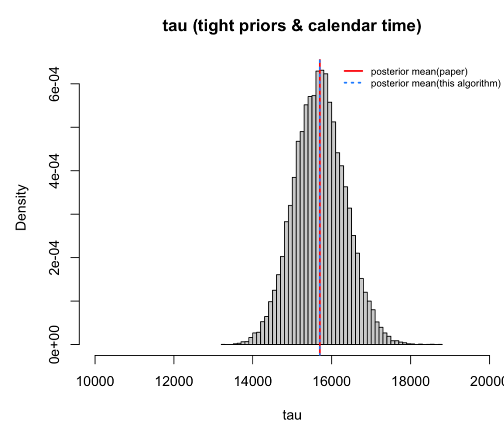

Fig. 20: Posterior density for the precision τ in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

3) Calendar-Time Model with Tight Priors - B

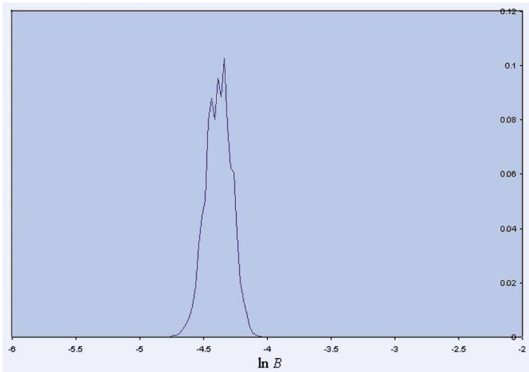

Fig. 21: Posterior density for the log of the log-periodic coefficient $\ln B$ in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

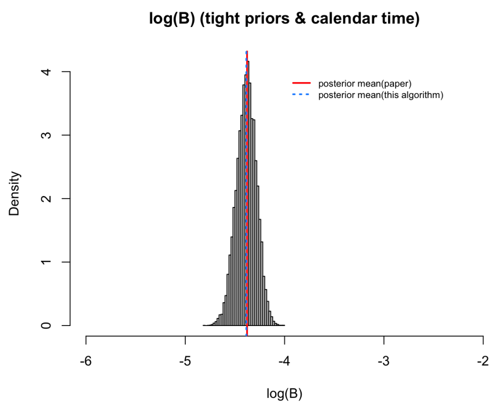

Fig. 22: Posterior density for the log of the log-periodic coefficient $\ln B$ in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

4) Calendar-Time Model with Tight Priors - C

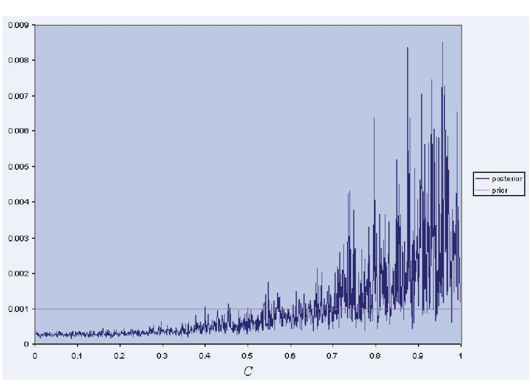

Fig. 23: Posterior density for the log-periodic amplitude C in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

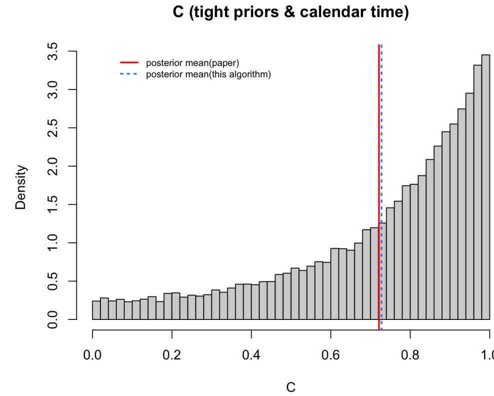

Fig. 24: Posterior density for the log-periodic amplitude C in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

5) Calendar-Time Model with Tight Priors - β

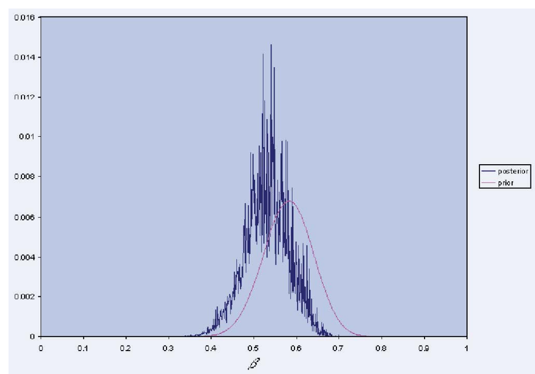

Fig. 25: Posterior density for the exponent β in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

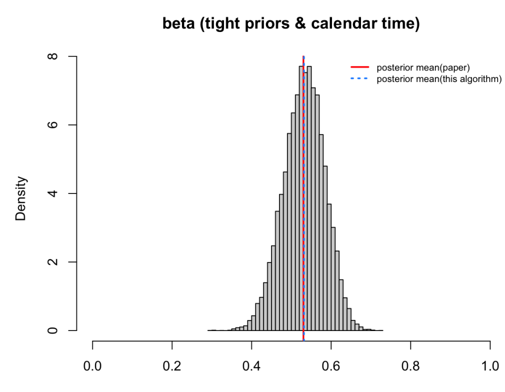

Fig. 26: Posterior density for the exponent β in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

6) *Calendar-Time Model with Tight Priors - ω*

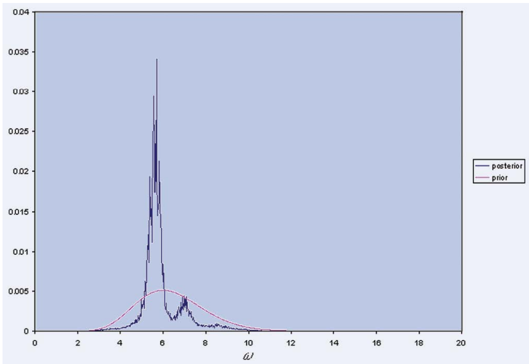

Fig. 27: Posterior density for the frequency ω in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

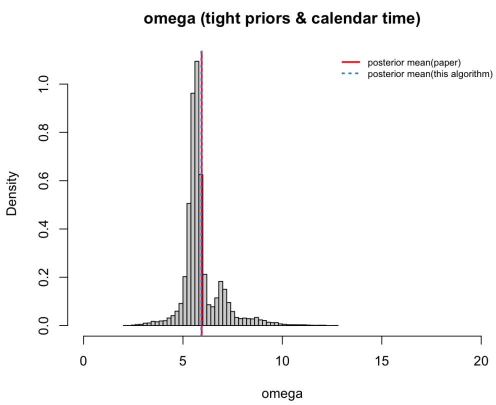

Fig. 28: Posterior density for the frequency ω in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

7) *Calendar-Time Model with Tight Priors - ϕ*

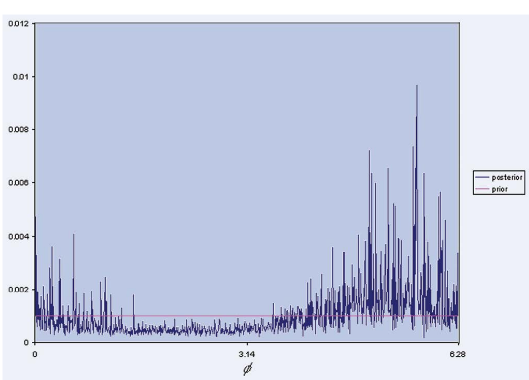

Fig. 29: Posterior density for the phase ϕ in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

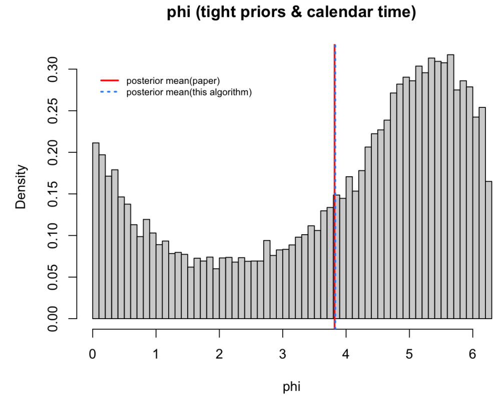

Fig. 30: Posterior density for the phase ϕ in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

8) *Calendar-Time Model with Tight Priors - $t_c - t_N$*

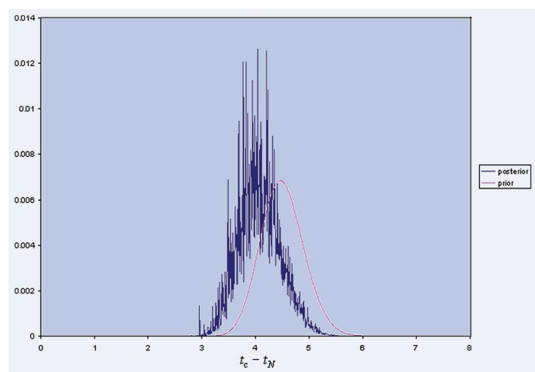

Fig. 31: Posterior density for the critical time t_c in the calendar-time model with tight priors and no crash probabilities, provided from the paper($B^{nc}_{lp,c}$).

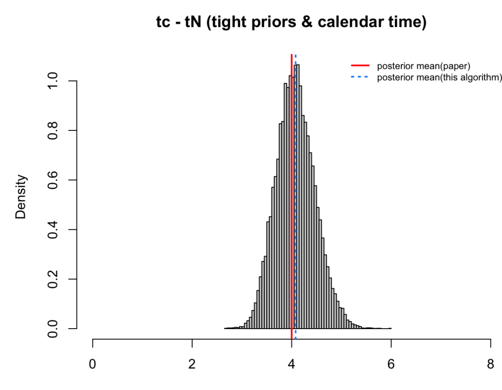

Fig. 32: Posterior density for the critical time t_c in the calendar-time model with tight priors and no crash probabilities, sampled by Metropolis within Gibbs algorithm.

REFERENCES

[1] George Chang and James Feigenbaum. A bayesian analysis of log-periodic precursors to financial crashes. *Quantitative Finance*, 6(1):15–36, 2006.