Real analysis via logic

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- Topology and continuity as *logical* phenomena (geometric logic).
- Now applying logical approach to real analysis.
- Gives fresh perspective on the analysis.

Locales

Formal topology

Constructivity

Point-free topology

Redeveloping real analysis (Ng, Vickers)

- exponentiation, logarithms [NV22]
- Fundamental Thm of Calculus [Vic23]
- trigonometry – WIP
- Fourier analysis – ambition

Topos theory

Bundles! – fibrewise topology

Geometric
LOGIC

Topology, continuity as *logical* phenomena

Usual approach: maths is discrete

Topological space (point-set) = set + extra structure

Continuous map = function respecting that structure

Geometric logic: "all" maths is continuous

Discrete maths of sets very restricted – eg $\mathbb{Q}$ is a set, real line $\mathbb{R}$ isn't. Can still access $\mathbb{R}$, by taking logical, *point-free* approach –

Topological space = logical theory (point = model of theory)

Continuity of map = definition respects logical constraints

Example: real line $\mathbb{R}$ as logical theory

Signature

For each rational $q \in \mathbb{Q}$: two propositional symbols $[\cdot < q]$, $[q < \cdot]$.

Axioms – eg

$$\begin{aligned} [\cdot < q'] &\vdash [\cdot < q] \quad (\text{if } q' < q) \\ [\cdot < q] \wedge [q < \cdot] &\vdash \bot \\ \top &\vdash [q < \cdot] \vee [\cdot < r] \quad (\text{if } q < r) \\ &\vdots \end{aligned}$$

Model x is real number as *Dedekind cut*:

Specify truth values $[x < q]$ and $[q < x]$ for every q,
ie which rationals are bigger than x, which are smaller.

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&\vdots \\
[\cdot < q] &\vdash \bigvee_{q' < q} [\cdot < q']
\end{aligned}
$$

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Defining maps

Think of maps $f : \mathbb{R} \to \mathbb{R}$ in a style of programming languages.

$$\boxed{\begin{array}{l} \text{Let } x{:}\mathbb{R} \\ \quad\vdots \\ f(x){:}\mathbb{R} := \cdots \\ \quad\vdots \end{array}}$$

Declare formal parameter x.
Do some auxiliary calulations.
Define result $f(x)$ as model:

- specify truth values $[f(x) < q]$, $[q < f(x)]$,
- prove that axioms hold.

eg absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}$

$$\boxed{\begin{array}{l} \text{Let } x{:}\mathbb{R} \\ \\ [|x| < q] := [x < q] \wedge [-q < x] \\ [q < |x|] := [q < x] \vee [x < -q] \\ \quad \text{... and prove axioms} \end{array}}$$

Inside the box, in scope of x, is a *different mathematics!*

1. Lots of non-standard truth values

$[x < q]$, $[q < x]$ (for each rational q)

Each expresses *where* (ie for which models x) something is true.

2. Continuity = different logic

Continuity: inverse image of open is open

$f^{-1}([x < q]) = [f(x) < q]$ is made from truth values of form $[x < r]$ and $[s < x]$ using $\wedge$ and $\bigvee$.
Similarly for $[q < f(x)]$.

We want continuity, therefore restrict mathematics inside box to limit how we construct $f(x)$.

Geometricity

Pure logic

Restricted formulae: $\bigvee$, $\wedge$, $=$, $\exists$.

Axioms as sequents:

$$\text{formula} \vdash_{\text{context}} \text{formula}$$

Context = finite stock of free variables, with implicit $\forall$.

Geometricity

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Corresponding maths

Restricted maths of sets:
Disjoint unions, quotients, finite products, equalizers, free algebras.

Function spaces Y^X, powersets $\mathcal{P}X$, the real line $\mathbb{R}$ are *not sets!* They must be dealt with as spaces.

Infinite $\bigvee$ can often be avoided by using $\exists$ with an infinite set. eg

$$[\cdot < q] \vdash_{q:\mathbb{Q}} (\exists q'{:}\mathbb{Q})(q' < q \wedge [\cdot < q']) \text{ for } [\cdot < q] \vdash \bigvee_{q' < q} [\cdot < q'].$$

Technicalities

- The "maths inside the box" is the geometric fragment of the internal mathematics of the classifying topos $\mathcal{S}[X]$.
  "Map" = geometric morphism.
- "Classifying topos" is slippery constructively – depends on choice of a base topos $\mathcal{S}$. To avoid that dependency, work without infinite disjunctions. [Vic17]

[Vic99] shows the technique in action in domain theory.
[Vic07] explains how standard topos results (eg [MLM92]) arrive at this point of view.
[Vic22] gives a more up-to-date discussion.

Logical manipulations I: Decomposing theories

Theory of Dedekind reals =

theory of *lower reals* to account for $[q < \cdot]$

$+$ theory of *upper reals* to account for $[\cdot < q]$

$+$ two axioms to relate them

Good strategy for point-free analysis (eg exp, log, integration)

1. Deal with lower and upper cases separately,
2. then combine them.

Can provide fresh insights – eg Ostrowski's Theorem in number theory (Ng [Ng22, NV]).

Logical manipulations II: Building up theories

Let $x{:}\mathbb{R}$ $\vdots$ $\vdots$ Space $\mathbb{T}(x) := \cdots$ $\vdots$	Declare formal parameter x. Do some auxiliary calculations. Define space $\mathbb{T}(x)$ as theory: ▶ define signature (as set) ▶ define axiom set (with appropriate structure).

Geometricity suggests $\mathbb{T}(x)$ depends *continuously* on x.

Logically – $(\mathbb{T}(x))_{x:\mathbb{R}}$ defines extension of theory of reals.
Models = pairs (x, y), $x{:}\mathbb{R}$, y model of $\mathbb{T}(x)$.
Dependent type theory: write $\sum_{x:\mathbb{R}} \mathbb{T}(x)$.
Forgetful map $\sum_{x:\mathbb{R}} \mathbb{T}(x) \to \mathbb{R}$, $(x, y) \mapsto x$, makes a *bundle* over $\mathbb{R}$.

None of this works satisfactorily in point-set topology

Can't describe bundle as continuously indexed family of spaces.

Logical approach works better!

Logical manipulations III: Modal logic $\to$ hyperspaces

Hyperspace[1] = space of subspaces

eg Use $\Box$ modality to construct new theory, models are to be subspaces W.
$\Box\phi$ assigned value true for those W in which every point has ϕ true in old theory.
Clearly $\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$.
For $\mathbb{R}$, signature has elements of the form $\Box\bot$, $\Box[q < \cdot]$, $\Box[\cdot < r]$ and $\Box([q < \cdot] \vee [\cdot < r])$.
Under suitable axioms, model = compact subspace of $\mathbb{R}$.

Application eg: Heine-Borel Theorem, closed intervals $[x, y]$ are compact

Using hyperspace, can demonstrate that $[x, y]$ depends *continuously* on x and y.

Other hyperspaces available; works for all spaces

[1] Point-free hyperspaces aka *powerlocales*. For geometricity see [Vic04].

Conclusion

Logic makes topology work
… better than *topology* does!

See [NV22] "Point-free construction of real exponentiation" for introduction to putting this into practice.

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