On constructivity

of

the notion of formal space

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Abstract of our talk

- why formal spaces in **MF**

- notion of constructive/strong constructive foundation

- strong constructivity of
  - the **Minimalist Foundation** (**MF**)
  - **MF** + **generated Positive Topologies**

- construction of predicative versions of Hyland's Effective Topos

- Open problems

Fundamental issue

$\boxed{\text{What is a space?}}$

What is a space ?

in Sambin's talk:

the answer depends

from the underlying conception of mathematics

$\Longrightarrow$ it depends from the chosen foundation

Key issue

the notion of Formal Topology/Positive Topology

is a notion of space

in the **Minimalist Foundation** (for short **MF**)

with NO known alternatives....

Why we need formal spaces in **MF**

contrary to most know constructive foundations: Aczel's **CZF**, Martin-Löf's type theory,...

the **Minimalist Foundation**

is a foundation for constructive mathematics

compatible with **Weyl's classical predicative mathematics**

$\Rightarrow$ in **MF** the Continuum **must** be represented in a pointfree way

Characteristics of predicative definitions

in the sense of *Russell-Poincarè*

> Whatever involves an apparent variable
>
> must *not be among the possible values* of that variable.

Necessity of a base to describe a point-free topology (=locale) predicatively!

even in *strong constructive predicative* theories like **Aczel's CZF** (+REA)

based on work by Moerdijk-van den Berg-Rathjen and Curi

Theorem:

No complete suplattice is a set

(unless it is the trivial one!)

in **Aczel's CZF** (and hence in **MF**)

reason:

consistency with variations of Troelstra's Uniformity Principle

$$\forall x \in \mathcal{P}(1)\ \exists y \varepsilon a\ R(x, y) \ \rightarrow\ \exists y \varepsilon a\ \forall x \in \mathcal{P}(1)\ R(x, y)$$

$\Downarrow$

need of two size entities: **collections**/**sets**

to represent a locale as a **collection**

closed under suprema indexed on a **set**

---

alternatively:

---

work with set of generators + relations

as in Vicker's development

Fundamental issues

What is constructive mathematics?

From **Bishop**'s *"Mathematics as a numerical language"*

[Constructive]

"**Mathematics** describes and predicts the results of certain finitely.. **computations** within the **set of integers**"

Essence of Constructive mathematics

= maths which admits a COMPUTATIONAL intepretation

Constructive Mathematics is a

bridge

abstract maths COMPUTATIONAL maths

Why developing constructive mathematics?

to EXTRACT the computational contents

i.e **the meaning** of abstract mathematics

in Bishop's words

what is constructive mathematics?

CONSTRUCTIVE mathematics
=
IMPLICIT COMPUTATIONAL mathematics

$\Downarrow$

| constructive mathematician is an implicit programmer!! |

[G. Sambin] Doing Without Turing Machines: Constructivism and Formal Topology.
In "Computation and Logic in the Real World". LNCS 4497, 2007

CONSTRUCTIVE proofs

=

SOME programs

Fundamental issues

What is a constructive foundation?

Need of a two level **constructive Foundation** *(j.w.w. G. Sambin)*

a Constructive Foundation should

bridge

LANGUAGE of abstract maths

(of commonly used **extensional** set theory)

to communicate with people

PROGRAMMING LANGUAGE

(of **intensional** type theory)

to extract programs

from constructive proofs

to communicate with machines

From **Bishop***'s "Schizophrenia in contemporary mathematics"*

informal mathematics	must be written in the appropriate language for communicating with people,
formal mathematics	must be written in the appropriate language for communicating with machines..

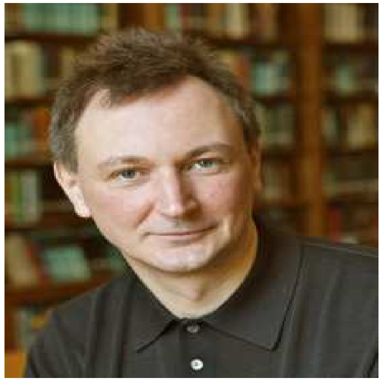
Use an interactive theorem prover...

in order to *speak to machines* as in **Bishop's view**....

+

in order to *check* correctness of mathematical proofs

as strongly advocated by **V. Voevodsky**



better to **built**

an interactive theorem prover

on an intensional type theory

like done with proof-assistants:

Coq, Agda, Matita

Problem: how to model **extensional concepts** in an intensional theory?

What **foundation** *for COMPUTER-AIDED formalization of proofs?*

(j.w.w. G. Sambin)

a constructive foundation should be equipped with

Foundation	extensional level (used by mathematicians to do their proofs) $\Downarrow$ interpreted via a **QUOTIENT model** intensional level (language of computer-aided formalized proofs)

$\Downarrow$

realizability level (used by computer scientists to extract programs) our FOUNDATION = ONLY the first TWO LEVELS

linked by a quotient completion

where

our extensional sets = quotients of intensional sets
only **implicitely**
being formalized in an abstract extensional language
as the usual one of common practice!

may LEVELS in our notion of constructive foundation collapse ?

YES, for example in the following two-level foundation

Aczel's CZF (usual math language)

$\Downarrow$ (GLOBALLY interpreted in)

Martin-Löf's type theory **MLTT**

which serves as the intensional and the realizability levels

may LEVELS in our notion of constructive foundation collapse ?

Can all levels be modelled within a single theory?

what about

MLTT + Univalence axiom ??

our notion of FOUNDATION combines different languages

language of *(local) AXIOMATIC SET THEORY*	at extensional level
language of *CATEGORY THEORY*	algebraic structure to link intensional/extensional levels via a quotient completion
language of TYPE THEORY	at intensional level

Our use of category theory

to express the abstract link between extensional/intensional levels:

use

notion of ELEMENTARY QUOTIENT COMPLETION

$$\mathcal{Q}(P)$$

(in the language of CATEGORY THEORY)

relative to a suitable **Lawvere's elementary doctrine** P

in:

[M.E.M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis, 2013

[M.E.M.-Rosolini'13] "Elementary quotient completion", Theory and Applications of Categories, 2013

[M.E.M.-Rosolini'15] "Unifying exact completions", Applied Categorical Structures, 2015

About the plurality of foundations of mathematics

classical mathematics	constructive mathematics
one *standard* impredicative foundation	NO standard foundation
ZFC axiomatic set theory	but different incomparable foundations

Plurality of foundations $\Rightarrow$ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory HoTT and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

what common core ??

Our TWO-LEVEL Minimalist Foundation

from [Maietti'09] according to requirements in [M.E.M, G. Sambin05]

- its intensional level

  = a PREDICATIVE VERSION of Coquand's Calculus of Constructions (Coq).

  = a FRAGMENT of Martin-Löf's intensional type theory + one UNIVERSE

- its extensional level

  has a PREDICATIVE LOCAL set theory

  (NO choice principles)

ENTITIES in the Minimalist Foundation

$$\begin{tikzcd}
\text{small propositions} \arrow[r, hook] \arrow[d, hook] & \text{sets} \arrow[d, hook] \\
\text{propositions} \arrow[r, hook] & \text{collections}
\end{tikzcd}$$

Why we need to have both classes/collections and sets

in **MF** and in *Aczel*'s **CZF**

Constructive predicative notion of Locale = Formal Topology by *P. Martin-Löf* and *G. Sambin*
represented by the fixpoints of a closure operator on a base of opens B assumed to be a preorder set: $\mathcal{A}_{\lhd}: \quad \mathcal{P}(B) \longrightarrow \mathcal{P}(B)$ $\quad\quad U \mapsto \{x \in B \mid x \lhd U\}$ satisfying a convergence property: $\mathcal{A}_{\lhd}(U \downarrow V) = \mathcal{A}_{\lhd}(U) \cap \mathcal{A}_{\lhd}(V)$ $U \downarrow V \equiv \{a \in B \mid \exists u \varepsilon U\, a \leq u \,\&\, \exists v \varepsilon V\, a \leq v\}$

NO restriction to inductively generated formal topologies

Why being predicative?

for a finer analysis of mathematical concepts and proofs

cfr. *H. Friedman*'s "Reverse mathematics"

On the intensional level of MF

Theorem: the intensional level of **MF** extended with the following resizing rule

$$\frac{A \text{ proposition}}{A \text{ small proposition}}$$

becomes equivalent to the **Coquand's Calculus of Constructions** with list types.

On the intensional level of MF

Theorem: the extensional level of **MF** extended with the following resizing rule

$$\frac{A \text{ proposition}}{A \text{ small proposition}}$$

becomes equivalent to the generic internal language

of **quasi-toposes** with a Natural Numbers Object.

What is the third level of MF?

an extension of **Kleene realizability**

as required in [M.E.M., G.Sambin05]

provided in

[H. Ishihara, M.E.M., S. Maschio, T. Streicher, 2018]

Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice

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This **Kleene realizability** semantics for **MF**

shows that

MF is a **strong constructive foundation**

What is the role of the **third level** *of a constructive foundation ?*

it provides a realizability model

of the extensional level

where to extact **programs**

from **constructive proofs** of the **extensional level**

i.e. satisfying:

- **the choice rule** (**CR**)

$\exists x \varepsilon A \, \phi(x)$ **true** under hypothesis Γ

$$\Downarrow$$

there exists a function calculating $f(x)$ such that

$\phi(f(x))$ **true** under hypothesis $x \in \Gamma$

- "its functions represents computable functions"

Notion of **strong constructive** *foundation*

a *two-level foundation* is a **strong constructive foundation**

iff

its intensional level is consistent with

the **axiom of choice** (**AC**) + **formal Church's thesis** (**CT**)

i.e. it is a **proofs-as-programs theory**

as in [M. Sambin-2005]

paradigmatic example:

Heyting arithmetics with finite types with Kleene realizability semantics axiom of choice

$$(AC) \qquad \forall x \in A \ \exists y \in B \ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B \ \ \forall x \in A \ \ R(\, x \,,\ f\,(x)\,)$$

formal Church's thesis

$$(CT) \qquad \forall f \in \mathsf{N}at \to \mathsf{N}at \qquad \exists e \in \mathsf{N}at$$
$$(\, \forall x \in \mathsf{N}at \ \ \exists y \in \mathsf{N}at \quad T(e,x,y) \ \& \ U(y) =_{\mathsf{N}at} f\,(x)\,)$$

NON examples of strongly constructive theories

NO classical theory

NO theory with extensionality of functions

can be strongly constructive

NON examples of strongly constructive theories

A theory consistent with AC+ CT

CAN NOT BE

- classical

$$\text{Peano Arithmetics} + \text{AC} + \text{CT} \vdash \perp$$

  (because we can define characteristic functions of non-computable predicates)

- extensional even with intuitionistic logic

$$\text{Intuitionistic arithmetics with finite types} + \text{AC} + \text{CT} + \text{extfun} \vdash \perp$$

$$\text{extfun} \;=\; \text{extensionality of functions}$$

$$\text{extfun}\ \frac{f(x) =_B g(x)\ \ true\ \ [x \in A]}{\lambda x.f(x) =_{A\to B} \lambda x.g(x)\ \ true}$$

extensionality

of functions

TECHNICAL DIFFICULT QUESTION

is **Martin-Löf's intensional type theory strongly constructive**?

i.e. consistent with formal Church's thesis??

key issue: the presence of the so called ξ-rule for lambda terms.

A realizability semantics for the extensional level

$\mathsf{T}_{\mathbf{i}MF}$	$\longrightarrow$	$\mathcal{T}_{eff}$
predicative tripos		predicative realizability tripos
		model to view **iMF** *proofs-as-programs*

$\Downarrow$

extensional level **eMF**		effective model of **eMF** proofs
$\Downarrow$ (interpreted)		
$\mathcal{Q}(T_{\mathbf{i}MF})$	$\longrightarrow$	$\mathcal{Q}(T_{eff})$
elementary quotient completion		elementary quotient completion
of $\mathsf{T}_{\mathbf{i}MF}$		of $\mathcal{T}_{eff}$
quotient model of **iMF**		*predicative Hyland's* **Eff**

Crucial categorical tool

the **exact completion of a lex category**
is represented an elementary completion $\mathcal{Q}(P)$
of an **elementary Lawvere doctrine** P

see

[M.E.M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis, 2013.

Predicative Generalization of Elementary topos

A **predicatively generalized** elementary topos is given by

- a finite limit category $\mathcal{C}$;
- a **FULL sub-fibration** of the codomain fibration on $\mathcal{C}$

$$\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{C}$$

such that

$$\begin{tikzcd} \mathcal{S} \arrow[rr, hook, "i"'] \arrow[dr, "\pi_{\mathcal{S}}"'] & & \mathcal{C}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

where i is an inclusion functor preserving cartesian morphisms and making the diagram commute satisfying a series of properties:

$$\pi_{\mathcal{S}} : \mathcal{S} \to \mathcal{C}$$

satisfies the following:

- each fibre in $\mathcal{S}$ is an LCC pretopos preserved by the inclusion in $\mathcal{C}$ and by base change functors;
- the **subobject doctrine** associated to $\mathcal{C}$ is a first order Lawvere hyperdoctrine (represents the logic over collections)
- there is a $\mathcal{C}$-object Ω **classifying** $\mathcal{S}$- subobjects of $\mathcal{C}$-objects:

  i.e.

  $$\mathsf{Sub}_{\mathcal{S}} \simeq \mathcal{C}(-, \Omega)$$

  where $\mathsf{Sub}_{\mathcal{S}}(A)$ is the full subcategory of $\mathsf{Sub}_{\mathcal{C}}(A)$ of those subobjects which are represented by objects in $\mathcal{S}$;
- there exist **power-objects** of $\pi_{\mathcal{S}}$ fibre objects

for every $\mathcal{C}$-object A,

for every object $\alpha: X \to A$ in $\mathcal{S}$,

there is an *exponential object* $(\pi_\Omega)^\alpha$ in $\mathcal{C}/A$

where $\pi_\Omega: A \times \Omega \to A$ is the first projection, i.e. there is a natural isomorphism

$$\mathcal{C}/A(- \times \alpha\,,\, \pi_\Omega) \simeq \mathcal{C}/A(-\,,\,(\pi_\Omega)^\alpha)$$

as functors on $\mathcal{C}/A$.

Our meta-language: Feferman's Theory of NON-iterative fixpoints $\widehat{ID_1}$

(j.w.w. S. Maschio)

we build a **predicative** version of *Hyland's Effective Topos*

by formalizing it into

the PREDICATIVE fragment of 2nd order arithmetics

of **Feferman's Theory of NON-iterative fixpoints** $\widehat{ID_1}$

motivation:

fixpoints are needed to interpret **iMF**-sets

as in

[I. Ishihara, M.E.M., S. Maschio, T.Streicher'18]

"Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice", *AML*.

A predicative version of Hyland's Effective Topos

(j.w.w S. Maschio)

---

it is built as the exact completion $\mathcal{C}_{pEff}$

of the (lex) category $\mathbf{Rec}^{I\hat{D}_1}$ of recursive classes + recursive morphisms

(with extensional function equality)

in **Feferman's Theory of NON-iterative fixpoints** $\widehat{ID_1}$

and the *objects of the subfibration of sets* are families of set-theoretic quotients

related to a universe of sets defined by a fix-point in $\widehat{ID_1}$

---

observe that:

---

our predicative effective topos

$\mathcal{C}_{pEff}$ = $\mathcal{Q}(\mathbf{wSub_{Rec}})$ is the elementary quotient completion

of the weak subobjects **doctrine** of $\mathbf{Rec}^{I\hat{D}_1}$

thought of as a predicative tripos $\mathcal{T}_{eff}$

---

the interpretation of the *logical connectives and quantifiers*

in the hyperdoctrine structure of the subobject functor

is equivalent to Kleene realizability interpretation of intuitionistic logic.

in [M.E. Maietti and S. Maschio'18] "A predicative variant of Hyland's Effective Topos" on ArXiv

Embedding in **Hyland's Effective topos**

our predicative effective topos

$$\mathcal{C}_{pEff} = \mathcal{Q}(\mathbf{wSub}_{\mathbf{Rec}})$$

can be embedded in *Hyland*'s Effective Topos Eff

$$\mathcal{Q}(\mathbf{wSub}_{\mathbf{Rec}}) \cong (\mathbf{Rec})_{ex/lex} \hookrightarrow (\mathbf{pAsm})_{ex/lex} \cong \mathbf{Eff}$$

because Eff is an exact on lex completion on partioned assemblies

by embedding the category $\mathbf{Rec}^{I\hat{D}_1}$ of recursive functions in $\widehat{ID_1}$

in the corresponding category of subsets of natural numbers and recursive functions in Eff.

Key peculiarity of **MF***: two notions of function*

in both levels of **MF**

a *primitive notion* of type-theoretic function

$$f(x) \in B \; [x \in A]$$

$\neq$ (syntactically)

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

$\Downarrow$

NO axiom of unique choice/ NO choice rules in **MF**

$\Downarrow$

MF needs a third level for extraction of programs from proofs

from [M.17]

Axiom of unique choice

$$\forall x \in A\ \exists! y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

Key peculiarities of **MF**

CONTRARY to **Martin-Löf's type theory** and to **Aczel's CZF**

for A, B **MF**-sets:

Functional relations from A to B do NOT always form a set

=Exponentiation $Fun(A, B)$ of functional relations is not always a set

$$\neq$$

Operations (typed-theoretic terms) from A to B do form a set

= Exponentiation $Op(A, B)$ is a set

three different kinds of real numbers

in the extensional **MF**

(even with classical logic!)

in accordance with **Weyl's notion of continuum**

reals as Dedekind cuts NOT a set
reals as Cauchy sequences or Brower's reals NOT a set
reals as Cauchy sequences as our operations(=Bishop's reals) form a set

recall:

Aczel's CZF+ classical logic = IMPREDICATIVE Zermelo Fraenkel theory

Martin-Löf's type theory + classical logic = IMPREDICATIVE

How to represent real numbers in the Minimalist Foundation ?

Dedekind reals

can be represented only

in a point-free way

via

via Martin-Löf-Sambin's FORMAL TOPOLOGY

including inductive methods to generate point -free topology

by [Coquand,Sambin,Smith,Valentini2003]

$\Rightarrow$ $\boxed{\text{we need to extend } \textbf{MF} + \text{inductive/coinductive definitions to represent Sambin's } \textbf{generated Positive Topologies}}$

Dedekind reals as ideal points of point-free topology

$$\boxed{\begin{array}{c}\text{Dedekind reals}\\ = \\ \text{ideal points (= constructive completely prime filters)}\\ \text{of Joyal's inductively generated formal topology}\end{array}}$$

pointfree presentation of Dedekind reals

Joyal's formal topology $\mathcal{R}_d \equiv (\mathbb{Q} \times \mathbb{Q}, \lhd_{\mathcal{R}}, \mathsf{Pos}_{\mathcal{R}})$

Basic opens are pairs $\langle p, q\rangle$ of rational numbers

whose cover $\lhd_{\mathcal{R}}$ is inductively generated as follows:

$$\frac{q \leq p}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{\langle p, q\rangle \in U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{p' \leq p \langle q \leq q' \qquad \langle p', q'\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

$$\frac{p \leq r \langle s \leq q \qquad \langle p, s\rangle \lhd_{\mathcal{R}} U \quad \langle r, q\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \mathsf{wc}\ \frac{wc(\langle p, q\rangle) \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

where $$wc(\langle p, q\rangle) \equiv \{\, \langle p', q'\rangle \in \mathbb{Q} \times \mathbb{Q} \mid p \langle p' \langle q' \langle q \}$$

on the Continuum in the **Minimalist Foundation MF**

The NON equivalence of the different representations of real numbers in **MF**

provides a paradigmatic example

of the peculiar characteristics

of **MF** itself

Positive Topologies

in **MF**

$$(B, \quad \mathcal{A}_{\triangleleft}, \quad \mathcal{J}_{\mathsf{Pos}} \,)$$

defined by

- B is a preordered set (base of opens)
- $\mathcal{A}_{\triangleleft}$ is a closure operator on B

$$\begin{array}{llll} \mathcal{A}_{\triangleleft}: & \mathcal{P}(B) & \longrightarrow & \mathcal{P}(B) \\ & U & \mapsto & \{x \in B \mid x \triangleleft U\,\} \end{array}$$

  satisfying a convergence property:

  $\mathcal{A}_{\triangleleft}(U \downarrow V) = \mathcal{A}_{\triangleleft}(U) \cap \mathcal{A}_{\triangleleft}(V)$

  $U \downarrow V \;\equiv\; \{a \in B \mid \exists u \varepsilon U\; a \leq u \;\&\; \exists v \varepsilon V\; a \leq v\;\}$

- $\mathcal{J}_{\mathsf{Pos}}$ is an interior operator

$$
\begin{array}{rccc}
\mathcal{J}_{\mathsf{Pos}}: & \mathcal{P}(B) & \longrightarrow & \mathcal{P}(B) \\
& W & \mapsto & \{x \in B \mid \mathsf{Pos}(a, W)\}
\end{array}
$$

satisfying

$$
\text{(compatibility)} \quad \frac{\mathsf{Pos}(a, W) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u, W)}
$$

related to locale theory to *overt weakly closed subspaces*

in [Vickers'07, Ciraulo-Vickers'16]

The two level extension **MF**$_{ind}$

we extend the extensional level **eMF** of **MF** as follows:

eMF$_{ind}$ = **eMF**+ inductive covers $a \lhd_{I,C} W$ as small propositions

+ coinductive positivity predicates $\mathsf{Pos}_{I,C}(a, W)$ as small propositions

defining a **generated Positive Topology**

for any axiom-set

$A\ set_{eMF} \qquad (generators)$

$I(a)\ set_{eMF}\ [a \in A] \qquad\qquad C(a,j)\ set_{eMF}\ [a \in A, j \in I(a)]$

$\Rightarrow$ without iterating the topological generation

$$\text{small propositions}_{eMF} \subseteq \text{small propositions}_{eMF_{ind}}$$

$$\text{sets}_{eMF} \subseteq \text{sets}_{eMF_{ind}}$$

we then extend **iMF** to **iMF**$_{ind}$

with fix primitive proofs for rules of $(-) \lhd_{I,C} (-)$ $\mathsf{Pos}_{\mathsf{I,C}}(-,-)$

by preserving the interpretation in [M.E.M.09]

of the extensional level **eMF**

into the intensional level **iMF**

A Kleene realizability semantics for **iMF**$_{ind}$

as suggested by M. Rathjen

we can extend Kleene realizability semantics via suitable inductive definitions

to validate

iMF$_{ind}$ with Church's thesis and Axiom of Choice

in CZF+ REA

following [Griffor-Rathjen94]

How to interpret coinductive definitions

the Positivity predicate defined by coinduction

on a set A with axiom-set

$$A\ set_{\mathbf{eMF}} \qquad (\textit{generators})$$

$$I(a)\ set_{eMF}\ [a \in A] \qquad\qquad C(a,j)\ set_{eMF}\ [a \in A, j \in I(a)]$$

is an interior operator defined as the maximum fix point of an operator of the form for any fixed a subset W

$$\tau : \mathcal{P}(A) \to \mathcal{P}(A)$$

$$\tau(X) = \{x \in A \mid x \varepsilon W \ \&\ (\forall i \in I(x))(\exists y \varepsilon C(x,i))\ y \varepsilon W \ \cap X\ \}$$

which can be defined as

$$\mathsf{Pos}(W) = \bigcup \{\ K \in \mathcal{P}_{\mathbf{eMF}}(A) \mid K \subseteq \tau(K)\ \&\ K \subseteq W\}$$

if the axiom of choice is valid for A

following a suggestion by T. Coquand in [M.E.M., S. Valentini 04]

Key issue to interpret coinduction via inductive definitions

coinductive definitions for generated Positive topologies

can be reduced to

suitable to inductive definitions

if the base A satisfies the axiom of choice

A predicative version of Hyland's Effective Topos for generated topologies

for **eMF**$_{ind}$

---

we can extend the construction of the predicative effective topos $\mathcal{C}_{peff}$ for **eMF**

to one, called d $\mathcal{C}_{peffind}$ for **eMF**$_{ind}$

thanks to the tool of elementary quotient completions

on the predicative realizability tripos for **iMF**$_{ind}$+AC+ CT

Story of Joyal's arithmetic universes

introduced in 70's

$(Pred(\mathcal{S}))_{ex}$ = exact completions of

a category of predicates $Pred(\mathcal{S})$

for a Skolem theory $\mathcal{S}$

to prove *Gödel's incompleteness theorems* from Wraith's notes

conjecture: abstract arithmetic universe = arithmetic pretopos ?

= pretopos + free monoid actions?

in [M.E.M 2010]

a generic arithmetic universe

=

a list-arithmetic pretopos

shown by using

extensional dependent type theory à la Martin-Löf

in [M.E.M05]

initial Joyal's arithmetic universes $\simeq$ initial list-arithmetic pretopos

Joyal's initial arithmetic universe $\mathcal{A}_0$ embeds in the predicative **Eff**

via universal properties:

given a Skolem theory $\mathcal{S}$

(= *cartesian category whose objects are finite products of a Natural numbers object*)

we can define a Lawvere doctrine

with decidable predicates $P\colon Nat^n \to Nat$

as fibre objects

defined as primitive recursive morphisms of $\mathcal{S}$ such that $P \cdot P = P$:

$$\begin{array}{rccc} \widehat{Pred}_{\mathcal{S}}: & \mathcal{S}^{OP} & \longrightarrow & InfSL \\ & Nat^n & \mapsto & \text{predicates } P \text{ over } Nat^n \\ & & \mapsto & \text{with pointwise order} \end{array}$$

Joyal's category of predicates $\mathsf{Pred}(\mathcal{S})$

=

base of the extensional completion

of the free comprehension completion

of $\widehat{Pred_{\mathcal{S}}}$

as in

[MR13] Maietti M.E. , Rosolini G. : Elementary quotient completion. *Theory and Applications of Categories* 2013

$\Downarrow$

by universal properties

$$\widehat{Pred_{\mathcal{S}}} \quad \hookrightarrow \quad Sub_{\mathcal{C}_{peff}}$$

and hence

$$\mathcal{A}_0 = \mathsf{Pred}(\mathcal{S})_{ex/lex} \quad \hookrightarrow \quad \mathcal{C}_{peff}$$

Open problems

- the exact proof-theoretic strenght of $\mathbf{MF}_{ind}$
- provide a type theoretic formulation of Aczel's Presentation Axiom *without the existence of universes of sets*