

# Locators point-free

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Theory Lab Lunch, 7 May 2020

# Locators (Booij [Boo20]); supervised by Martín Escardó

A Dedekind real  $x$  is defined by –

- ▶ which rationals  $q$  are less than it ( $q < x$ )
- ▶ which rationals  $r$  are greater than it ( $x < r$ )

$x$  has the *locatedness* property:

$$q < r \vdash_{q,r:\mathbb{Q}} (q < x) \vee (x < r)$$

A *locator* for  $x$  is structure that, for all pairs  $q < r$ , chooses a valid one out of the properties  $q < x$  and  $x < r$ .

Thus with a locator we have

$$q < r \vdash_{q,r:\mathbb{Q}} (q < x) + (x < r)$$

$\mathbb{R}$  is space of (Dedekind) reals.

$\mathbb{R}^{\mathcal{L}}$  is space of reals equipped with locators.

$\text{pr}_1: \mathbb{R}^{\mathcal{L}} \rightarrow \mathbb{R}$  is projection.

# Analysis with locators (Booij)

Given a locator for  $x$  –

can compute decimal expansion without using Axiom of Choice.

Locator supplies the choices needed.

Analysis with locators –

tracks the usual real analysis.

Constructions on  $\mathbb{R}$  (including arithmetic, limits) lifted to  $\mathbb{R}^{\mathcal{L}}$ .

At the end, decimal expansion can be output.

# Univalent type theory (UTT) v. point-free topology

Booij's work was in setting of UTT.

- ▶ Allows for *discontinuous* functions.
- ▶ Type constructors include arbitrary function types.
- ▶ Not all reals have locators, so analysis with locators does not automatically restrict to real analysis.

We transfer it to *point-free topology*.

- ▶ All types are topologized, all maps continuous.
- ▶ Not all function types exist. ☹
- ▶  $\text{pr}_1 : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$  is a *point-free surjection*, and a coequalizer. Hence real analysis can be recovered from analysis with locators. 😊

# Point-free topology

- ▶ The points of a space are described as the models of a geometric theory.
- ▶ Continuity of maps is ensured by restricting to geometric constructions – colimits, finite limits, free algebras.
- ▶ A *bundle* over a space  $X$  is a space  $Y_x$  constructed (geometrically) out of points  $x$  of  $X$ .
- ▶ A *set*  $X$  is a discrete space.
- ▶ A *function* is a map between sets.

# Point-free topology in a context

Let  $x$  be a point of  $X$  {

In the scope of that declaration  $x : X$ , we are working

- ▶ in a geometric mathematics
- ▶ that includes a *generic* point  $x$  of  $X$ .

It is equivalent to working internally in the topos of sheaves over  $X$ .

Anything we do here with  $x$  can be transported to any specific point  $a$  of  $X$  by substitution.

(That's the universal property of classifying toposes.)

Let's define a space  $Y_x$ .

It constructs a bundle over  $X$ .

The substitution  $Y_x[a/x]$  gives the fibre over  $a$ .

}

Bundle space  $Y$  is dependent sum  $\sum_{x:X} Y_x$

$X$  is a geometric theory.

$Y_x$  is a geometric theory, defined using ingredients of model of  $X$ .

Combine them to get theory  $Y$ .

Its models are pairs  $(x, y)$ ,  $x : X$  and  $y : Y_x$

– just as they should be for a dependent sum  $\sum_{x:X} Y_x$ .

Projection  $p: Y \rightarrow X$  forgets  $y$ .

Fibre over  $a$  is space of points  $y$  of  $Y_x[a/x]$ .

# Locators using a pullback

Given a real  $x$  and rationals  $q < r$  –

two truthvalues  $\phi_l = (q < x)$  and  $\phi_r = (x < r)$ , disjunction is  $\top$ .

Write  $\Lambda$  for space of such pairs  $(\phi_l, \phi_r)$  of truthvalues,

$Q_<$  for set of pairs of rationals  $q < r$ .

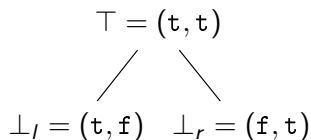
Get  $\mathbb{R} \times Q_< \rightarrow \Lambda$ , so  $\mathbb{R} \rightarrow \Lambda^{Q_<}$ .

(Sets are locally compact, so exponentiable.)

## Structure of $\Lambda$

Three classical solutions of

$\phi_l \vee \phi_r = \top$ :



For constructive solutions,  $\Lambda$  is the ideal completion of this 3-element poset.



# Locators using a pullback

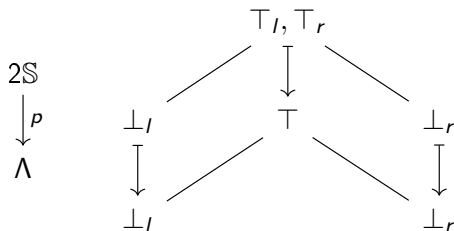
At  $x$  and  $q < r$ , locator provides element of  $\phi_I + \phi_r$

Two more truth values

- ▶  $\phi'_I = \phi_I$  chosen
- ▶  $\phi'_r = \phi_r$  chosen

Write  $2\mathbb{S}$  for space of quadruples  $(\phi'_I, \phi_I, \phi'_r, \phi_r)$

– 4 classical solutions.



# Locators using a pullback

A real equipped with a locator is –

- ▶ a real  $x$
- ▶ a map  $f: Q_{<} \rightarrow 2\mathbb{S}$
- ▶ if  $f(q < r) = (\phi'_l, \phi_l, \phi'_r, \phi_r)$  then  $\phi_l = (q < x)$  and  $\phi_r = (x < r)$

The following diagram is a pullback.

$$\begin{array}{ccc} \mathbb{R}^{\mathcal{L}} & \xrightarrow{\quad} & (2\mathbb{S})^{Q_{<}} \\ \text{pr}_1 \downarrow & & \downarrow p^{Q_{<}} \\ \mathbb{R} & \xrightarrow{x \mapsto (q < r) \mapsto (q < x, x < r)} & \Lambda^{Q_{<}} \end{array}$$

# Open surjections

A map  $f: Y \rightarrow X$  is *surjective* iff –

its inverse image function  $f^{-1}$  is one-one on opens.

This does *not* mean that every point of  $X$  has a pre-image.

It is a *conservativity* principle

– certain facts about  $X$  can be validly derived from facts about  $Y$ .

Surjections are better behaved if they are open

– ie direct images of opens are open (Joyal & Tierney [JT84])

Open surjections are preserved by pullback.

An open surjection is the coequalizer of its kernel pair.

# Open surjections

For a map  $f: Y \rightarrow X$ :

$f$  open  $\Leftrightarrow f$  fibrewise overt

$f$  open surjection  $\Leftrightarrow f$  fibrewise positive overt

A property holds “fibrewise” for  $f$  iff it holds for the fibre  $Y_x$  over the generic point.

This only makes sense if the property is geometric

– so if it holds for the generic fibre, it holds for all fibres.

# Overt spaces

A space  $X$  is *overt* if it has a positivity predicate  $\text{Pos}$  on its opens.

- ▶ Think:  $\text{Pos}(U)$  means  $U$  non-empty.
- ▶ It suffices to define  $\text{Pos}$  on a base of opens.
- ▶  $\text{Pos}$  must respect covers: if  $\text{Pos}(U)$ , and  $U$  is covered by a family  $\{V_i\}$ , then we must have  $\text{Pos}(V_i)$  for some  $i$ .
- ▶ Each basic  $U$  must be covered by the family  $\{\top \mid \text{Pos}(U)\}$ .

# The lower hyperspace (powerlocale) $P_L$

Points of  $P_L X$  = overt, “weakly closed” subspaces of  $X$ .

$X$  is itself overt iff it appears in  $P_L X$ , as the top point  $\top$ .

“Top” means right adjoint to  $!$ .

$$\begin{array}{ccc} & \top & \\ 1 & \xrightarrow{\quad} & P_L X \\ & \xleftarrow[\quad]{\top} & \\ & ! & \end{array}$$

- ▶  $P_L$  is a geometric construction (Vickers [Vic04])
- ▶ Hence overtiness is a geometric property
- ▶ – and we can talk about “fibrewise overtiness”.
- ▶ Also,  $P_L$  preserves coreflexive equalizers.

# Fibrewise overttness of $f: Y \rightarrow X$

Given  $x: X$ , we get two coreflexive equalizer diagrams:

$$Y_x \xrightarrow{i_x} Y \begin{array}{c} \xrightarrow{y \mapsto \langle x, y \rangle} \\ \xleftarrow{\pi_2} \\ \xrightarrow{\langle f, \text{Id}_Y \rangle} \end{array} X \times Y ,$$

$$P_L Y_x \xrightarrow{P_L i_x} P_L Y \begin{array}{c} \xrightarrow{P_L(y \mapsto \langle x, y \rangle)} \\ \xleftarrow{P_L \langle f, \text{Id}_Y \rangle} \end{array} P_L(X \times Y) .$$

Now apply  $\Sigma_{x:X}$  and simplify to get another equalizer.

$$(P_L/X) Y \longrightarrow X \times P_L Y \begin{array}{c} \xrightarrow{\tau} \\ \xleftarrow{\pi_2 P_L \langle f, \text{Id}_Y \rangle} \end{array} P_L(X \times Y)$$

$\tau$  is strength,  $(P_L/X) Y = \Sigma_{x:X} Y_x$ .

$f: Y \rightarrow X$  is fibrewise overt iff it is open

In context of  $x: X$ :

Overtness of  $Y_x$  is map  $1 \rightarrow P_L Y_x$  with certain properties  
– and can get it from map  $1 \rightarrow P_L Y$ .

Applying  $\Sigma_{x:X}$ :

Fibrewise overtness of  $f$  is a map  $X \rightarrow P_L Y$  with certain properties.

$f$  is open iff (Vickers [Vic95])

–  $P_L f$  has a right adjoint with certain properties  
This structure is equivalent to that for fibrewise overtness.

$f$  open iff fibrewise overt

Given that, further calculation shows surjection iff fibrewise positive.



$p^{Q<}$  is fibrewise positive overt ( $p: 2\mathbb{S} \rightarrow \Lambda$ )

More generally: if  $X$  is a set with decidable equality, then

$p^X: (2\mathbb{S})^X \rightarrow \Lambda^X$  is fibrewise positive overt

Point of  $\Lambda^X$  is decomposition  $X = X_l \cup X_r$ .

Fibre = set  $X_l + X_r$ .

For overttness, must say which opens are positive.

Suffices to do this for basic opens

– and check that covers are respected.

If  $U \leq \bigvee_i V_i$ , and  $U$  is positive, then one of the  $V_i$ s must be positive.

Subbasic opens  $x_l, x_r$  for  $x \in X_l, y \in X_r$ .

For  $x \in X_l \cap X_r$  have

$$x_l \wedge x_r \vdash \perp$$

$$\top \vdash x_l \vee x_r$$

$p^x$  is fibrewise positive overt

Basic opens (finite meets of subbasics) –

$$B = \{(S_l, S_r) \in \mathcal{F}X_l \times \mathcal{F}X_r \mid (\forall x \in S_l)(\forall y \in S_r)x \neq y\}$$

$$(S_l, S_r) = \bigwedge_{x \in S_l} x_l \wedge \bigwedge_{y \in S_r} y_r$$

$B$  is a set!

Positivity predicate: every basic in  $B$  is positive.

Respecting covers

$$(S_l, S_r) \vdash \bigvee (\{(S_l \cup \{x\}, S_r), (S_l, S_r \cup \{x\})\} \cap B)$$

Suppose  $(S_l, S_r) \in B, x \in X$ . Want at least one of  $(S_l \cup \{x\}, S_r), (S_l, S_r \cup \{x\})$  in  $B$ . Immediate if  $x \in S_l \cup S_r$ . Otherwise (and condition is decidable), use  $x \in X_l \cup X_r$ .

$\text{pr}_1 : \mathbb{R}^{\mathcal{L}} \rightarrow \mathbb{R}$  is an open surjection

- ▶ We now know  $p^X$  is fibrewise overt. (We've defined the positivity predicate.)
- ▶ It's fibrewise positive because  $(\emptyset, \emptyset)$ , which is  $\top$ , is positive.
- ▶ Hence it's an open surjection.
- ▶ Now put  $X = Q_{<}$
- ▶ and pull back to get  $\text{pr}_1$ .

# Open surjections are coequalizers of their kernel pairs

$K$  = kernel pair of  $\text{pr}_1$

Point = real  $x$  equipped with two locators.

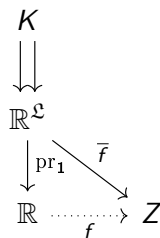
$\bar{f}$  depends only on  $x$

– gives same result for any locators

Then  $\bar{f}$  factors uniquely via  $\text{pr}_1$ .

In defining  $f$ , can pretend every real has a locator

– even though not all of them do.



# Examples of surjections

All these are *triquotient* maps (Plewe [Ple97]).

They are coequalizers of their kernel pairs, and preserved by pullback.

$$\text{pr}_1 : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$$

– open surjection

$$2^{\omega} \rightarrow [-1, 1]$$

$$(s_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} s_i / 2^i \text{ (where } s_i = \pm 1)$$

– proper surjection (Vickers [Vic17])






$$\lim : \text{Cauchy}_f X \rightarrow \overline{X}$$

Limits of Cauchy sequences in a generalized metric space  $X$ .

– triquotient (Vickers [Vic98])

Taking  $X = \mathbb{Q}$  gives surjection to  $\mathbb{R}$ .

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