# Locators point-free

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# Locators (Booij [Boo20]); supervised by Martín Escardó

A Dedekind real $x$ is defined by –

- which rationals $q$ are less than it ($q < x$)
- which rationals $r$ are greater than it ($x < r$)

$x$ has the *locatedness* property:

$$q < r \vdash_{q,r:\mathbb{Q}} (q < x) \vee (x < r)$$

A *locator* for $x$ is structure that, for all pairs $q < r$, chooses a valid one out of the properties $q < x$ and $x < r$.
Thus with a locator we have

$$q < r \vdash_{q,r:\mathbb{Q}} (q < x) + (x < r)$$

$\mathbb{R}$ is space of (Dedekind) reals.
$\mathbb{R}^{\mathfrak{L}}$ is space of reals equipped with locators.
$\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$ is projection.

# Analysis with locators (Booij)

Given a locator for $x$ –

can compute decimal expansion without using Axiom of Choice.

Locator supplies the choices needed.

Analysis with locators –

tracks the usual real analysis.

Constructions on $\mathbb{R}$ (including arithmetic, limits) lifted to $\mathbb{R}^{\mathfrak{L}}$.

At the end, decimal expansion can be output.

# Univalent type theory (UTT) v. point-free topology

## Booij's work was in setting of UTT.

- Allows for *dis*continuous functions.
- Type constructors include arbitrary function types.
- Not all reals have locators, so analysis with locators does not automatically restrict to real analysis.

## We transfer it to *point-free topology*.

- All types are topologized, all maps continuous.
- Not all function types exist. ☹
- $\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$ is a *point-free surjection*, and a coequalizer. Hence real analysis can be recovered from analysis with locators. ☺

# Point-free topology

- The points of a space are described as the models of a geometric theory.
- Continuity of maps is ensured by restricting to geometric constructions – colimits, finite limits, free algebras.
- A *bundle* over a space $X$ is a space $Y_x$ constructed (geometrically) out of points $x$ of $X$.
- A *set* $X$ is a discrete space.
- A *function* is a map between sets.

# Point-free topology in a context

## Let $x$ be a point of $X$ {

In the scope of that declaration $x : X$, we are working

- in a geometric mathematics
- that includes a *generic* point $x$ of $X$.

It is equivalent to working internally in the topos of sheaves over $X$.

## Anything we do here with $x$ can be transported to any specific point $a$ of $X$ by substitution.

(That's the universal property of classifying toposes.)

## Let's define a space $Y_x$.

It constructs a bundle over $X$.
The substitution $Y_x[a/x]$ gives the fibre over $a$.

}

# Bundle space $Y$ is dependent sum $\Sigma_{x:X} Y_x$

$X$ is a geometric theory.
$Y_x$ is a geometric theory, defined using ingredients of model of $X$.

Combine them to get theory $Y$.
Its models are pairs $(x, y)$, $x : X$ and $y : Y_x$
– just as they should be for a dependent sum $\Sigma_{x:X} Y_x$.
Projection $p\colon Y \to X$ forgets $y$.
Fibre over $a$ is space of points $y$ of $Y_x[a/x]$.

# Locators using a pullback

## Given a real $x$ and rationals $q < r$ –

two truthvalues $\phi_l = (q < x)$ and $\phi_r = (x < r)$, disjunction is $\top$.
Write $\Lambda$ for space of such pairs $(\phi_l, \phi_r)$ of truthvalues,
$Q_<$ for set of pairs of rationals $q < r$.
Get $\mathbb{R} \times Q_< \to \Lambda$, so $\mathbb{R} \to \Lambda^{Q_<}$.
(Sets are locally compact, so exponentiable.)

## Structure of $\Lambda$

Three classical solutions of
$\phi_l \vee \phi_r = \top$:

$$\begin{tikzcd}[column sep=small] & \top = (\mathtt{t}, \mathtt{t}) \arrow[dl, dash] \arrow[dr, dash] & \\ \bot_l = (\mathtt{t}, \mathtt{f}) & & \bot_r = (\mathtt{f}, \mathtt{t}) \end{tikzcd}$$

For constructive solutions, $\Lambda$ is the ideal completion of this 3-element poset.

# Locators using a pullback

At $x$ and $q < r$, locator provides element of $\phi_l + \phi_r$

Two more truth values

- $\phi'_l = \phi_l$ chosen
- $\phi'_r = \phi_r$ chosen

Write $2\mathbb{S}$ for space of quadruples $(\phi'_l, \phi_l, \phi'_r, \phi_r)$
– 4 classical solutions.

$$
\begin{tikzcd}
2\mathbb{S} \arrow[d, "p"] \\
\Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \top_l, \top_r \arrow[dl, dash] \arrow[d, mapsto] \arrow[dr, dash] & \\
\bot_l \arrow[d, mapsto] & \top \arrow[dl, dash] \arrow[dr, dash] & \bot_r \arrow[d, mapsto] \\
\bot_l & & \bot_r
\end{tikzcd}
$$

# Locators using a pullback

A real equipped with a locator is –

- a real $x$
- a map $f\colon Q_< \to 2\mathbb{S}$
- if $f(q<r) = (\phi'_l, \phi_l, \phi'_r, \phi_r)$ then $\phi_l = (q<x)$ and $\phi_r = (x<r)$

The following diagram is a pullback.

$$\begin{tikzcd}
\mathbb{R}^{\mathfrak{L}} \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & (2\mathbb{S})^{Q_<} \arrow[d, "p^{Q_<}"] \\
\mathbb{R} \arrow[r, "x \mapsto (q<r) \mapsto (q<x, x<r)"'] & \Lambda^{Q_<}
\end{tikzcd}$$

# Open surjections

A map $f\colon Y \to X$ is *surjective* iff –

its inverse image function $f^{-1}$ is one-one on opens.

This does *not* mean that every point of $X$ has a pre-image.

It is a *conservativity* principle

– certain facts about $X$ can be validly derived from facts about $Y$.

Surjections are better behaved if they are open

– ie direct images of opens are open (Joyal & Tierney [JT84])

Open surjections are preserved by pullback.

An open surjection is the coequalizer of its kernel pair.

# Open surjections

For a map $f \colon Y \to X$:

$f$ open $\Leftrightarrow$ $f$ fibrewise overt
$f$ open surjection $\Leftrightarrow$ $f$ fibrewise positive overt

A property holds "fibrewise" for $f$ iff it holds for the fibre $Y_x$ over the generic point.

This only makes sense if the property is geometric
– so if it holds for the generic fibre, it holds for all fibres.

# Overt spaces

A space $X$ is *overt* if it has a positivity predicate Pos on its opens.

- Think: $\mathrm{Pos}(U)$ means $U$ non-empty.
- It suffices to define Pos on a base of opens.
- Pos must respect covers: if $\mathrm{Pos}(U)$, and $U$ is covered by a family $\{V_i\}$, then we must have $\mathrm{Pos}(V_i)$ for some $i$.
- Each basic $U$ must be covered by the family $\{\top \mid \mathrm{Pos}(U)\}$.

# The lower hyperspace (powerlocale) $\mathrm{P}_L$

Points of $\mathrm{P}_L X$ = overt, "weakly closed" subspaces of $X$.

$X$ is itself overt iff it appears in $\mathrm{P}_L X$, as the top point $\top$.
"Top" means right adjoint to !.

$$
\begin{tikzcd}
1 \arrow[r, shift left=2, "\top"] \arrow[r, phantom, "\top"] & \mathrm{P}_L X \arrow[l, shift left=2, "!"]
\end{tikzcd}
$$

- $\mathrm{P}_L$ is a geometric construction (Vickers [Vic04])
- Hence overtness is a geometric property
- – and we can talk about "fibrewise overtness".
- Also, $\mathrm{P}_L$ preserves coreflexive equalizers.

# Fibrewise overtness of $f \colon Y \to X$

Given $x : X$, we get two coreflexive equalizer diagrams:

$$\begin{tikzcd} Y_x \arrow[r, "i_x"] & Y \arrow[r, shift left=2, "y \mapsto \langle x, y\rangle"] \arrow[r, shift right=2, "\langle f, \mathrm{Id}_Y\rangle"'] & X \times Y \arrow[l, "\pi_2"'] \end{tikzcd} ,$$

$$\begin{tikzcd} \mathrm{P}_L\, Y_x \arrow[r, "\mathrm{P}_L\, i_x"] & \mathrm{P}_L\, Y \arrow[r, shift left, "\mathrm{P}_L(y \mapsto \langle x, y\rangle)"] \arrow[r, shift right, "\mathrm{P}_L\langle f, \mathrm{Id}_Y\rangle"'] & \mathrm{P}_L(X \times Y) \end{tikzcd} .$$

Now apply $\Sigma_{x:X}$ and simplify to get another equalizer.

$$\begin{tikzcd} (\mathrm{P}_L/X)\, Y \arrow[r] & X \times \mathrm{P}_L\, Y \arrow[r, shift left, "\tau"] \arrow[r, shift right, "\pi_2\, \mathrm{P}_L\langle f, \mathrm{Id}_Y\rangle"'] & \mathrm{P}_L(X \times Y) \end{tikzcd}$$

$\tau$ is strength, $(\mathrm{P}_L/X)\, Y = \Sigma_{x:X} Y_x$.

# $f : Y \to X$ is fibrewise overt iff it is open

## In context of $x : X$:

Overtness of $Y_x$ is map $1 \to \mathrm{P}_L\, Y_x$ with certain properties
– and can get it from map $1 \to \mathrm{P}_L\, Y$.

## Applying $\Sigma_{x:X}$:

Fibrewise overtness of $f$ is a map $X \to \mathrm{P}_L\, Y$ with certain properties.

## $f$ is open iff (Vickers [Vic95])

– $\mathrm{P}_L\, f$ has a right adjoint with certain properties
This structure is equivalent to that for fibrewise overtness.

## $f$ open iff fibrewise overt

Given that, further calculation shows surjection iff fibrewise positive.

## $p^{Q_<}$ is fibrewise positive overt ($p\colon 2\mathbb{S} \to \Lambda$)

More generally: if $X$ is a set with decidable equality, then

$p^X\colon (2\mathbb{S})^X \to \Lambda^X$ is fibrewise positive overt

Point of $\Lambda^X$ is decomposition $X = X_l \cup X_r$.
Fibre = set $X_l + X_r$.

For overtness, must say which opens are positive.

Suffices to do this for basic opens
– and check that covers are respected.
If $U \leq \bigvee_i V_i$, and $U$ is positive, then one of the $V_i$s must be positive.

Subasic opens $x_l, y_r$ for $x \in X_l, y \in X_r$.

For $x \in X_l \cap X_r$ have

$$x_l \wedge x_r \vdash \bot$$

$$\top \vdash x_l \vee x_r$$

# $p^X$ is fibrewise positive overt

Basic opens (finite meets of subbasics) –

$$B = \{(S_l, S_r) \in \mathcal{F}X_l \times \mathcal{F}X_r \mid (\forall x \in S_l)(\forall y \in S_r) x \neq y\}$$

$(S_l, S_r) = \bigwedge_{x \in S_l} x_l \wedge \bigwedge_{y \in S_r} y_r$
$B$ is a set!

Positivity predicate: every basic in $B$ is positive.

Respecting covers

$$(S_l, S_r) \vdash \bigvee(\{(S_l \cup \{x\}, S_r), (S_l, S_r \cup \{x\})\} \cap B)$$

Suppose $(S_l, S_r) \in B, x \in X$. Want at least one of $(S_l \cup \{x\}, S_r)$, $(S_l, S_r \cup \{x\})$ in $B$. Immediate if $x \in S_l \cup S_r$. Otherwise (and condition is decidable), use $x \in X_l \cup X_r$.

## $\mathrm{pr}_1 : \mathbb{R}^{\mathcal{E}} \to \mathbb{R}$ is an open surjection

- We now know $p^X$ is fibrewise overt. (We've defined the positivity predicate.)
- It's fibrewise positive because $(\emptyset, \emptyset)$, which is $\top$, is positive.
- Hence it's an open surjection.
- Now put $X = Q_<$
- and pull back to get $\mathrm{pr}_1$.

# Open surjections are coequalizers of their kernel pairs

$K$ = kernel pair of $\mathrm{pr}_1$

Point = real $x$ equipped with *two* locators.

$\overline{f}$ depends only on $x$

– gives same result for any locators

Then $\overline{f}$ factors uniquely via $\mathrm{pr}_1$.

In defining $f$, can pretend every real has a locator

– even though not all of them do.

$$\begin{tikzcd}
K \arrow[d, shift left] \arrow[d, shift right] & \\
\mathbb{R}^{\mathfrak{L}} \arrow[d, "\mathrm{pr}_1"] \arrow[dr, "\overline{f}"] & \\
\mathbb{R} \arrow[r, dotted, "f"'] & Z
\end{tikzcd}$$

# Examples of surjections

All these are *triqotient* maps (Plewe [Ple97]).
They are coequalizers of their kernel pairs, and preserved by pullback.

$\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$

– open surjection

$2^\omega \to [-1, 1]$

$(s_i)_{i=1}^\infty \mapsto \sum_{i=1}^\infty s_i/2^i$ (where $s_i = \pm 1$)
– proper surjection (Vickers [Vic17])

$\lim : \mathrm{Cauchy}_f X \to \overline{X}$

Limits of Cauchy sequences in a generalized metric space $X$.
– triquotient (Vickers [Vic98])
Taking $X = \mathbb{Q}$ gives surjection to $\mathbb{R}$.

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