# GEOMETRIC CONSTRUCTIONS PRESERVE FIBRATIONS

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ABSTRACT. Let $\mathcal{C}$ be a representable 2-category, and $\mathfrak{T}_\bullet$ a 2-endofunctor of the arrow 2-category $\mathcal{C}^\downarrow$ such that (i) $\operatorname{cod} \mathfrak{T}_\bullet = \operatorname{cod}$ and (ii) $\mathfrak{T}_\bullet$ preserves proneness of morphisms in $\mathcal{C}^\downarrow$. Then $\mathfrak{T}_\bullet$ preserves fibrations and opfibrations in $\mathcal{C}$.

The proof takes Street's characterization of (e.g.) opfibrations as pseudoalgebras for 2-monads $\mathfrak{L}_B$ on slice categories $\mathcal{C}/B$ and develops it by defining a 2-monad $\mathfrak{L}_\bullet$ on $\mathcal{C}^\downarrow$ that takes change of base into account, and uses known results on the lifting of 2-functors to pseudoalgebras.

## 1. INTRODUCTION

The results reported here arose out of an analysis (Fauser, Raynaud & Vickers 2012) of existing topos approaches to quantum foundations as either fibrational or opfibrational. There the toposes and their spectral gadgets are interpreted as point-free bundles $p\colon \Sigma \to B$, in other words maps in the category **Loc** of locales. Points of the base space $B$ are *contexts*, or classical perspectives on a quantum system, and each fibre is a classical state space for its context. Instances of the specialization order in $B$ (representing context refinement) induce maps between the corresponding fibres, in either a covariant (opfibrational) or contravariant (fibrational) way. Moreover, the fibre maps have a canonical property that makes them either opfibrations or fibrations in the 2-categorical sense of (Street 1974), when one considers **Loc** as a 2-category, with its order enrichment using the specialization order.

Our techniques in various places make use of features of the so-called "geometric reasoning", and in particular the way it allows a bundle such as $p$ to be defined geometrically as a transformation of points of $B$ into point-free spaces, the fibres. Then a typical point $y$ of $\Sigma$ can be expressed as a pair $(x, y')$ where $x$ is a point of $B$ and $y'$ is a point of the fibre over it. This is brought out in some detail in (Spitters, Vickers & Wolters 2014). From this it is immediately evident what is the specialization order within each fibre, but it is harder to identify the general specialization across fibres in $\Sigma$.

At this point it is very helpful to know if $p$ is a fibration or an opfibration. Suppose, for instance, it is known that $p$ is an opfibration. We want to know when $(x_1, y_1') \sqsubseteq (x_2, y_2')$. Certainly we must have $x_1 \sqsubseteq x_2$, since $p$ must, like any continuous map, preserve specialization. Hence there is a fibre map $r_{x_1 x_2}\colon \Sigma_{x_1} \to \Sigma_{x_2}$. It then turns out that $(x_1, y_1') \sqsubseteq (x_2, y_2')$ iff $r_{x_1 x_2}(y_1') \sqsubseteq y_2'$.

For the spectral bundles these facts arose from rather deep topological facts (Johnstone 1993): in the opfibrational case $p$ was a local homeomorphism, and in the fibrational case $p$ corresponds

to an internal compact regular locale in the topos of sheaves over $B$. There are some bundle constructions that do not preserve those topological properties but for which it is still going to be interesting to know whether they result in fibrations or opfibrations. An example is the valuation locale – see (Coquand & Spitters 2009) for an account of how it relates to the quantum discussion.

The fundamental result that we shall prove is that *any* geometric construction preserves both opfibrations and fibrations.

In locale theory, the definition of "geometric construction" (of locales) is as follows.

(1) Suppose we have an endofunctor $T$ of **Loc**. On the face of it, it is given as an endofunctor of the category **Fr** of frames, although we shall move away from that view.

(2) Suppose also that the definition of $T$ can be applied to internal frames in any topos. Since (Joyal & Tierney 1984) internal frames in the topos of sheaves over a locale $B$ are dual to bundles over $B$, it follows that $T$ induces an endofunctor $T_B$ on the slice **Loc**$/B$.

(3) Geometricity is the property that the construction is stable (up to isomorphism) under change of base, i.e. pullback $f^*\colon \mathbf{Loc}/B \to \mathbf{Loc}/B'$ along a map $f\colon B' \to B$. This is often proved by showing that $T$ can be described by a construction on presentations of frames. See, e.g., (Vickers 2004), or (Vickers 2011) for an application to the valuation locale.

These can be expressed in the language of indexed category theory (see, e.g., (Johnstone 2002, B1)): the slice category **Loc**$/B$, for a variable locale $B$, is "indexed over **Loc**", and that notion includes the reindexing functors $f^*$. Moreover, the geometricity of $T$ then amounts to it being an indexed endofunctor of the indexed category – this notion includes the $T_B$s' commuting with reindexing up to coherent isomorphisms.

Although this captures the intuitions, the overall structure is complicated and we shall find it convenient to use the alternative fibrational structure (see, again, (Johnstone 2002, B1)). For this, all the slices are bundled together into a single category, the arrow category $\mathbf{Loc}^\downarrow$,[1] and then the endofunctors $T$ and $T_B$ as given by a single endofunctor $T_\bullet$ on $\mathbf{Loc}^\downarrow$. Indeed, it seems that we have to go to the fibrational setting, since we also need to use slicewise endofunctors (some monads defined by Street) that are not indexed.
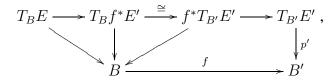
It is geometricity that allows the $T_B$s to take in change of base. We can factor an arbitrary bundle morphism via a pullback square – this is the "vertical-prone" factorization for the codomain fibration $\mathrm{cod}\colon \mathbf{Loc}^\downarrow \to \mathbf{Loc}$.

$$
\begin{array}{ccc}
E \xrightarrow{\ g\ } E' & \qquad & E \longrightarrow f^*E' \longrightarrow E' \\
\Big\downarrow{\scriptstyle p} \quad \Big\downarrow{\scriptstyle p'} & & {\scriptstyle p}\searrow \ \Big\downarrow \qquad \Big\downarrow{\scriptstyle p'} \\
B \xrightarrow{\ f\ } B' & & \qquad B \xrightarrow{\ f\ } B'
\end{array} \quad .
\tag{1-1}
$$

---

[1] This is usually written $\mathbf{Loc}^\to$, but we use a down arrow to reinforce an idea that the objects are bundles.

Then geometricity gives us an isomorphism in

$$T_B E \longrightarrow T_B f^* E' \xrightarrow{\cong} f^* T_{B'} E' \longrightarrow T_{B'} E' \;,$$

thus enabling us to define $T_f g \colon T_B E \to T_{B'} E'$.

The only extra condition needed on $T_\bullet$ is that if a bundle morphism as on the left-hand side of (1-1) happens to be a pullback square, then so is the result of applying $T_\bullet$ to it:

$$
\begin{array}{ccc}
T_B(E) & \xrightarrow{T_f(g)} & T_{B'}(E') \\
\downarrow{\scriptstyle T_B(p)} & & \downarrow{\scriptstyle T_{B'}(p')} \\
B & \xrightarrow{f} & B'
\end{array}
$$

In (Vickers & Townsend 2012) are given sufficient conditions for an endofunctor on **Loc** (or, more generally, any cartesian category $\mathcal{C}$) to be part of an indexed endofunctor on slices. This covers coherence of the geometricity isomorphisms, or, equivalently, the strict functoriality of $T_\bullet$. The conditions are verified there for the powerlocales, and in (Vickers 2011) for valuation locales.

With this formulation of geometricity, we are able to generalize from **Loc** to a very general class of 2-categories, and prove (Theorem 37) that if $T_\bullet$ is geometric, then for each base object $B$, $T_B$ preserves the properties of being fibration or opfibration over $B$.

Note that we shall be using fibrations and opfibrations in two different settings. In a given 2-category $\mathcal{C}$ (generalizing **Loc**), with suitable limits, we are interested in when a morphism in it – an object of $\mathcal{C}^\downarrow$ – is a fibration or an opfibration. On the other hand, we also use extensively the fact that, if we forget the 2-cells in $\mathcal{C}$, then the codomain functor $\mathrm{cod} \colon \mathcal{C}^\downarrow \to \mathcal{C}$ is a bifibration – both a fibration and an opfibration – in the 2-category of categories.

We use some results from Street (Street 1974). There, for each base object $B$, a 2-monad $\mathfrak{L}_B$ is defined on $\mathcal{C}/B$ which characterizes the opfibrations as those bundles $p \colon E \to B$ that support the structure of a normalized $\mathfrak{L}_B$-pseudoalgebra. There is also a dual result (reversing 2-cells) that the fibrations are the normalized pseudoalgebras for a 2-monad $\mathfrak{R}_B$. Hence to prove that $T_B$ preserves opfibrations it is natural to prove that it lifts to the pseudoalgebra category of $\mathfrak{L}_B$. From the 1-categorical theory one would expect this to be equivalent to defining a natural transformation that, with $T_B$, makes a monad opfunctor, and the 2-categorical issues have already been taken care of in (Marmolejo & Wood 2008). However, we also find that we must extend Street's technique slightly by defining a 2-monad $\mathfrak{L}_\bullet$ on the whole of $\mathcal{C}^\downarrow$. We then show that $T_\bullet$ lifts to the pseudoalgebra category.

## 2. BACKGROUND

2.1. **Some results from Street and 2-tangle notation.** We now recall some definitions and results from (Street 1974). However, we shall also replace Street's diagrams with a dual "2-tangle"

notation, which seems to be more compact for handling 2-categories and lax or pseudoalgebra structures.

We use the following different interpretations for tangles to work with 2-categories. In a 2-category we have 0-cells (objects), 1-cells (arrows) and 2-cells, which, in the canonical example of **Cat** are categories, functors and natural transformations. In our intended application of **Loc** they are locales, continuous maps, and instances of the specialization order. In an ordinary diagram an $n$-cell is represented by an $n$-dimensional object, hence 0-cell = vertex, 1-cell = (oriented) edge, 2-cell = (oriented) area connecting paths of edges. Tangles invert this association so that 0-cells are areas, 1-cells are vertical edges, with right-to-left reading order *across* the edges, and 2-cells are "coupons", with downward reading order from the edges attached at the top to those at the bottom. We introduce a special coupon for trivial (equality) 2-cells, representing commutative diagrams. By way of example we provide here an introduction to this notion. A formal way to use these diagrams is given in (Melliès 2006).

Commutative diagrams are made out of $n$-gons, and we discuss how 2-cells for bi-, tri- and tetra-gons are described. First we look at a 2-cell $\alpha$ of a bigon (parallel pair of arrows $f, g$ from $A$ to $B$). The vertices $A, B$ become areas in the tangle picture, which we shade here for clarity. Later we will drop these area labels when they are reconstructible from the tangle.



Correspondence between a bigon diagram and a tangle.                                                    (2-1)

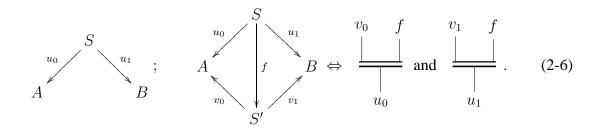For trigons and tetragons we find:



(2-2)

(In practice all our coupons are drawn as rectangles, however many inputs or outputs they have.)

The coupons for identity 2-cells, for ordinary commutative diagrams, are depicted as double lines representing an equality symbol.



(2-3)

A general commutative $n$-gon may have $i$ input and $j$ output lines (with $i + j = n$). Such an $n$-gon may need further manipulation of its internal structure, which we lose (on purpose, and to compactify) in our notation. Furthermore, we can merge, but have to be careful when disconnecting, such diagrams. We drop from here onwards the shading and labelling of the 0-cells.

$$\text{(2-4)}$$

A collection of identities with one input $f$ and one output $f$ is equivalent to a trivial identity 2-cell.
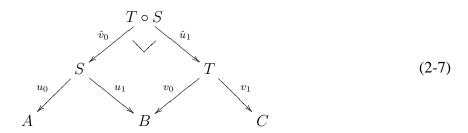
$$\text{(2-5)}$$

The last two equalities are due to the invertibility of the identity 2-cell, when one edge of the commutative $n$-gon (here a trigon) is identity. A similar equation holds for non-trivial 2-cells (e.g. triangles) iff they are invertible. The right-hand side of the final equation is an identity 2-cell on an identity 1-cell. This is a void diagram and vanishes.

We continue to present some definitions and results from Street (Street 1974) in 2-tangle notation. Let $\mathcal{C}$ be a category, $A, B$ be objects in $\mathcal{C}$. The category $\mathcal{SPN}(A, B)$ has as objects spans $(u_0, S, u_1)$ from $A$ to $B$ and arrows are arrows of spans $f\colon (u_0, S, u_1) \to (v_0, S', v_1)$

$$\text{(2-6)}$$

For a span $S$, the *reverse span* $S^*$ form $B$ to $A$ is given by $(u_1, S, u_0)$. If $\mathcal{C}$ has pullbacks, then a span $(u_0, S, u_1)$ from $A$ to $B$ and a span $(v_0, T, v_1)$ from $B$ to $C$ has a composite span

$(u_0 \hat{v}_0, T \circ S, v_1 \hat{u}_1)$ from $A$ to $C$ defined as

$$
\begin{array}{c}
T \circ S \\
\overset{\hat{v}_0}{\swarrow} \quad \vee \quad \overset{\hat{u}_1}{\searrow} \\
S \qquad\qquad T \\
\overset{u_0}{\swarrow} \quad \overset{u_1}{\searrow} \quad \overset{v_0}{\swarrow} \quad \overset{v_1}{\searrow} \\
A \qquad\qquad B \qquad\qquad C
\end{array}
\tag{2-7}
$$

where, following Street, we decorate pulled back arrows with hats. Similarly, given composable spans and arrows $f, g$ of such spans then the arrow $g \circ f \colon T \circ S \to T' \circ S'$ is induced on pullbacks is an arrow of the composite spans. An *opspan* from $A$ to $B$ in $\mathcal{C}$ is a span from $A$ to $B$ in $\mathcal{C}^{op}$, but arrows of opspans are arrows from $\mathcal{C}$.

**Definition 1.** *A* comma object *for the opspan* $(r, D, s)$ *from* $A$ *to* $B$ *is a span* $(d_0, r/s, d_1)$ *from* $A$ *to* $B$ *together with a 2-cell*

$$
\begin{array}{ccc}
r/s \overset{d_1}{\longrightarrow} B & & \\
\;\;{\scriptstyle d_0}\downarrow \quad \overset{\lambda}{\Longrightarrow} \quad \downarrow{\scriptstyle s} & \simeq & \boxed{\lambda} \\
A \overset{r}{\longrightarrow} D & & 
\end{array}
\tag{2-8}
$$

*satisfying the two following universality conditions.*

- *For any span* $(u_0, S, u_1)$ *from* $A$ *to* $B$, *composition with* $\lambda$ *yields a bijection between arrows of spans* $f$ *(2-6) and 2-cells* $\sigma$ *given by (either one of)*

$$
\boxed{\lambda}\,f = \boxed{\sigma} \;;\quad \boxed{\lambda}\;\Big| = \boxed{\sigma}\;.
\tag{2-9}
$$

- *Given 2-cells* $\xi, \eta$ *such that the two composites*

$$
\boxed{\xi}\,,\,\boxed{\lambda} = \boxed{\lambda}\,,\,\boxed{\eta}
\tag{2-10}
$$

*are equal, then there exists a unique 2-cell $\phi$ such that $\xi = d_0\phi$, $\eta = d_1\phi$.*

$$\exists! \; \boxed{\phi} \; \text{s.t.} \quad \begin{array}{c} f \\ \boxed{\xi} \\ f' \end{array} = \begin{array}{cc} d_0 & f \\ & \boxed{\phi} \\ d_0 & f' \end{array} \; \text{and} \; \begin{array}{c} d_0 \; f \\ \boxed{\eta} \\ d_0 \; f' \end{array} = \begin{array}{cc} d_1 & f \\ & \boxed{\phi} \\ d_1 & f' \end{array} \tag{2-11}$$

The comma object of the identity opspan $(1, A, 1)$ from $A$ to $A$ is denoted by $\Phi A = A/A$. When $\Phi A$ exists for each object $A$ and if $\mathcal{C}$ has 2-pullbacks, then $\mathcal{C}$ is a *representable* 2-category. In a representable 2-category every opspan $(r, D, s)$ has a comma object via span composition $r/s = s^* \circ \Phi D \circ r$.

An intuitive way to think about the 2-category $\mathcal{C}$ is that each 0-cell $A$ itself has objects and morphisms, but in the generalized sense of generalized elements in a topos. Given a "stage of definition", another 0-cell $W$, the generalized objects and morphisms of $A$ are the objects and morphisms of the category $\mathcal{C}(W, A)$. 1-cells $A \to A'$ and 2-cells between them then give functors $\mathcal{C}(W, A) \to \mathcal{C}(W, A')$ and natural transformations between them. Hiding $W$ we can then understand Definition 1 as saying that the (generalized) objects of $r/s$ are triples $(u_0, u_1, \sigma)$ where $u_0, u_1$ are objects of $A$ and $B$ and $\sigma \colon ru_0 \to su_1$ is a morphism. A morphism from $(u_0, u_1, \sigma)$ to $(u_0', u_1', \sigma')$ is a pair $(\xi, \eta)$ where $\xi \colon u_0 \to u_0'$ and $\eta \colon u_1 \to u_1'$, and there is a commutative square

$$\begin{array}{ccc} ru_0 & \xrightarrow{\;\sigma\;} & su_1 \\ {\scriptstyle r\xi}\downarrow & & \downarrow{\scriptstyle s\eta} \\ ru_0' & \xrightarrow[\;\sigma'\;]{} & su_1' \end{array}$$

It is often helpful to calculate 1-cells and 2-cells in terms of their action on generalized objects and morphisms.

If $p \colon E \to B$ is a bundle (1-cell) in $\mathcal{C}$, and $u$ is a generalized object of $B$, then the pullback $u^*E$ is the *generalized fibre* of $p$ over $u$. The reason for this point of view can be understood by considering the case where the stage of definition $W$ is the terminal object 1 in **Loc** or in the 2-category of categories, when pullback along points gives actual fibres. Thus geometricity of a construction as preservation under pullback can be understood as the property that the construction works fibrewise, and hence is compatible with viewing a bundle as a dependent type.

Let $\mathfrak{D}$ be a 2-monad on a 2-category $\mathcal{C}$, with unit $i \colon 1 \to \mathfrak{D}$ and multiplication (composition) $c \colon \mathfrak{D}\mathfrak{D} \to \mathfrak{D}$.

**Definition 2.** *A* lax $\mathfrak{D}$*-algebra is a quadruple* $(E, c, \zeta, \theta)$ *where* $E$ *(the carrier) is an object of* $\mathcal{C}$*,* $c$ *(the structure map) is an arrow* $c\colon \mathfrak{D}E \to E$ *and* $\zeta, \theta$ *are 2-cells*

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\simeq
\begin{array}{c}
\text{(diagram)}
\end{array}
\quad ; \qquad
\begin{array}{c}
\text{(diagram)}
\end{array}
\simeq
\begin{array}{c}
\text{(diagram)}
\end{array}
\quad ; \qquad (2\text{-}12)
$$

*such that the following 3 equations hold.*

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\overset{(1)}{=}
\overset{(2)}{=}
\begin{array}{c}
\text{(diagram)}
\end{array}
\; ; \qquad
\begin{array}{c}
\text{(diagram)}
\end{array}
\overset{(3)}{=}
\begin{array}{c}
\text{(diagram)}
\end{array}
\qquad (2\text{-}13)
$$

*A* $\mathfrak{D}$*-pseudoalgebra has* $\zeta, \theta$ *invertible; a* normalized $\mathfrak{D}$*-algebra has* $\zeta = 1$*, a* $\mathfrak{D}$*-algebra has* $\zeta = 1 = \theta$*.* $(\mathfrak{D}E, c, \zeta = 1, \theta = 1)$ *is the* free $\mathfrak{D}$*-algebra on* $E$*.*

**Definition 3.** *A* lax homomorphism *of lax* $\mathfrak{D}$*-algebras from* $E$ *to* $E'$ *is a pair* $(f, \theta_f)$ *of an arrow* $f\colon E \to E'$ *and a 2-cell*

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\simeq
\begin{array}{c}
\text{(diagram)}
\end{array}
\qquad (2\text{-}14)
$$

*such that the following equations hold*

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\overset{(5)}{=}
\begin{array}{c}
\text{(diagram)}
\end{array}
\; ; \qquad
\begin{array}{c}
\text{(diagram)}
\end{array}
\overset{(6)}{=}
\begin{array}{c}
\text{(diagram)}
\end{array}
\qquad (2\text{-}15)
$$

*A lax homomorphism is called a* pseudo-homomorphism *if* $\theta_f$ *is invertible, and is called a* homomorphism *when* $\theta_f$ *is identity.*

2.1.1. *The monads $\mathfrak{L}_B$ and $\mathfrak{R}_B$.* Street defines (Street 1974, pp.118, 122) two 2-monads for which pseudoalgebra structure is related to opfibrations and fibrations. This is done on the slice 2-category $\mathcal{C}/B$, with objects $(E, p\colon E \to B)$, arrows $f\colon E \to E'$ in $\mathcal{C}$ such that $p'f = p$,

and 2-cells $(E,p) \overset{f}{\underset{g}{\Rightarrow^\sigma}} (E',p')$ in $\mathcal{C}$ such that $p'\sigma = 1_p$.

The first monad, for opfibrations, is defined on a 2-functor $\mathfrak{L}_B\colon \mathcal{C}/B \to \mathcal{C}/B$ given by

$$(E,p) \overset{f}{\underset{g}{\Rightarrow^\sigma}} (E',p') \longmapsto (p/B, d_1) \overset{f/B}{\underset{g/B}{\Rightarrow^{\sigma/B}}} (p'/B, d_1) \tag{2-16}$$

Thus a generalized object of $\mathfrak{L}_B E$ is a triple $(e, b, \alpha)$ with $e, b$ objects of $E$ and $B$, and $\alpha\colon pe \to b$. $\mathfrak{L}_B$ can also be expressed in terms of $\Phi$ and span composition as $\mathfrak{L}_B(E, p) = (\Phi B \circ p, d_1\hat{p})$, $\mathfrak{L}(f) = 1 \circ f$, and $\mathfrak{L}_B(\sigma) = 1 \circ \sigma$. We shall see these in more detailed tangle form when generalized in Section 4.

We now have a diagram

$$\begin{array}{ccccc}
\mathfrak{L}_B^2 E & \xrightarrow{d_0} & \mathfrak{L}_B E & \xrightarrow{d_0} & E \\
d_1 \downarrow & \overset{\lambda}{\Leftarrow} & d_1 \downarrow & \overset{\lambda}{\Leftarrow} & \downarrow p \\
B & = & B & = & B
\end{array}$$

The unit $i$ and multiplication $c$ of Street's monad can be defined in terms of tangles as follows.

**Definition 4.** *The components on $(E, p)$ of $i\colon 1 \to \mathfrak{L}_B$ and $c\colon \mathfrak{L}_B^2 \to \mathfrak{L}_B$ are defined by*

$$d_0 i = 1_E \quad d_1 i = p \quad d_0 c = d_0 d_0 \quad d_1 c = d_1$$

*and*



A generalized object of $\mathfrak{L}_B^2 E$ is $(e, b_1, b_2, \alpha, \alpha_1)$ with $\alpha\colon pe \to b_1$ and $\alpha_1\colon b_1 \to b_2$, and then $c(e, b_1, b_2, \alpha, \alpha_1) = (e, b_2, \alpha_1\alpha)$; also, $i(e) = (e, pe, 1)$.

Street shows that $i$ and $c$ are 2-natural transformations fulfilling the monad equations, and hence making $(\mathfrak{L}_B, i, c)$ a 2-monad on $\mathcal{C}/B$. Moreover, although we shall not use this, $\mathfrak{L}_B$ has the 'Kock property', that is $c \dashv i\mathfrak{L}_B$ in the 2-functor 2-category $[\mathcal{C}, \mathcal{C}]$ with identity counit.

Similarly one defines the monad $\mathfrak{R}_B$ which is the monad $\mathfrak{L}_B$ in the category $\mathcal{C}^{co}/B$ where the direction of 2-cells are reversed. This amounts to using comma objects $(B/p)$, arrows $B/f$ and 2-cells $B/\sigma$ in $\mathcal{C}/B$. The further development is dual.

**Definition 5.** *An arrow* $p\colon E \to B$ *is called a* pseudo-opfibration *(Street:* 0-fibration*) when* $(E, p)$ *supports the structure of an* $\mathfrak{L}_B$*-pseudoalgebra. It is an* opfibration *(Street:* normal 0-fibration*) when it supports the structure of a normalized* $\mathfrak{L}_B$*-pseudoalgebra.*

*Analogously,* $p\colon E \to B$ *is called a* pseudofibration *(Street:* 1-fibration*) or a* fibration *(Street:* normal 1-fibration*) if it supports the corresponding structures for* $\mathfrak{R}_B$.

*We shall not use this, but an opfibration or fibration is said to* split *if the normal pseudoalgebra structure can be chosen to be an algebra.*

In terms of generalized objects, the pseudoalgebra structure map $c\colon \mathfrak{L}_B E \to E$ maps $(e, b, \alpha)$ to some object in the fibre over $b$. In doing so, it lifts $\alpha\colon pe \to b$ to a fibre map from $(pe)^*E$ to $b^*E$.

In practice the pseudo- opfibrations and fibrations are too weak to be useful for us, as the lifting to fibre maps (reindexing) need not be functorial in any sense. For good properties capturing the standard notions of fibration we need the normal versions; for these reindexing is at least pseudo-ofunctorial. Note that all the notions are inherently cloven, since the adjoint in the Chevalley criterion chooses supine or prone liftings, and hence amounts to a cleavage or cocleavage.

**Proposition 6** (Chevalley criterion)**.**

(1) *The arrow* $p\colon E \to B$ *is a pseudo-opfibration over* $B$ *if and only if the arrow* $\tilde{p}\colon \Phi E \to p/B$ *corresponding to the 2-cell*

$$
\begin{array}{ccc}
\Phi E & \xrightarrow{pd_1} & B \\
{\scriptstyle d_0}\downarrow & {\scriptstyle p\lambda}\nearrow\!\!\!\!\!\!/ & \downarrow{\scriptstyle 1} \\
E & \xrightarrow{p} & B
\end{array}
\quad \simeq \quad
\begin{array}{cc}
p & d_0 \\
\boxed{p\lambda} & \\
1 & pd_1
\end{array}
\tag{2-17}
$$

*has a left adjoint with unit an isomorphism.*

(2) *The arrow* $p$ *is an opfibration iff* $\tilde{p}$ *has a left adjoint with unit an identity.*

(3) *The corresponding statements hold for pseudofibrations and fibrations, replacing* $p/B$ *by* $B/p$ *and requiring a right adjoint with counit either an isomorphism or an identity.*

*Proof.* (1) is (Street 1974, proposition 9), and (2) is the remarks following it. (3) follows by duality on 2-cells. □

## 2.2. Liftings of 2-transitions.

The classical lifting result seems to go back to Applegate (Applegate 1963), see (Johnstone 1975) and (Manes 1976). Let $(\mathfrak{D}_1, i_1, c_1)$ be a monad on the (ordinary) category $\mathcal{D}_1$ and $(\mathfrak{D}_2, i_2, c_2)$ be a monad on $\mathcal{D}_2$. Denote as usual the category of Eilenberg-Moore algebras as $\mathcal{D}_1^{\mathfrak{D}_1}$ or $\mathfrak{D}_1$-Alg etc. One has forgetful functors $U_i\colon \mathcal{D}_i^{\mathfrak{D}_i} \to \mathcal{D}_i$ forgetting

the algebra structure map. Given a functor $T\colon \mathcal{D}_1 \to \mathcal{D}_2$, then a functor $\overline{T}\colon \mathcal{D}_1^{\mathfrak{D}_1} \to \mathcal{D}_2^{\mathfrak{D}_2}$ is a *lifting* of $T$ such that

$$
\begin{array}{ccc}
\mathcal{D}_1^{\mathfrak{D}_1} & \xrightarrow{\ \overline{T}\ } & \mathcal{D}_2^{\mathfrak{D}_2} \\
{\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle U_2} \\
\mathcal{D}_1 & \xrightarrow{\ T\ } & \mathcal{D}_2
\end{array}
\tag{2-18}
$$

commutes.

**Lemma 7** (Applegate). *Let $\mathfrak{D}_i$ be monads on categories $\mathcal{D}_i$ as above, and $T\colon \mathcal{D}_1 \to \mathcal{D}_2$ a functor. Then the functors $\overline{T}\colon \mathcal{D}_1^{\mathfrak{D}_1} \to \mathcal{D}_2^{\mathfrak{D}_2}$ which are liftings of $T$ are in 1-to-1 correspondence with natural transformations $\psi\colon \mathfrak{D}_2 T \to T\mathfrak{D}_1$ such that the following diagram commutes.*

$$
\begin{array}{ccccc}
T & \xrightarrow{\ i_2 T\ } & \mathfrak{D}_2 T & \xleftarrow{\ c_2 T\ } & \mathfrak{D}_2^2 T \\
& {\scriptstyle T i_1}\searrow & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathfrak{D}_2 \psi} \\
& & T\mathfrak{D}_1 & & \mathfrak{D}_2 T\mathfrak{D}_1 \\
& & & {\scriptstyle Tc_1}\nwarrow & \downarrow{\scriptstyle \psi\mathfrak{D}_1} \\
& & & & T\mathfrak{D}_1^2
\end{array}
\tag{2-19}
$$

Such pairs $(T, \psi)$ are commonly known as *monad functors* from $\mathfrak{D}_1$ to $\mathfrak{D}_2$, but we shall follow (Marmolejo & Wood 2008) in referring to $\psi$ as a *transition from $\mathfrak{D}_1$ to $\mathfrak{D}_2$ along $T$*.

The commutative diagram (2-19) can also be written in tangle form as



$$
\tag{2-20}
$$

Our previous tangle diagrams have been within a 2-category $\mathcal{C}$. These here are different in that they are in the 3-category of 2-categories, and the 3-cells – which relate to 2-cells in individual 2-categories – are thus inaccessible without developing a 3-dimensional calculus. The tangle diagrams have shifted a dimension, and much of what follows addresses the interplay between the two levels, and using the 2-dimensional calculus at different levels to capture the whole system.

Given $\psi$, one defines $\overline{T}$ as $\overline{T}(\mathfrak{D}_1 E \xrightarrow{c_1} E) = (\mathfrak{D}_2 T E \xrightarrow{\psi E} T\mathfrak{D}_1 E \xrightarrow{Tc_1} TE)$ defining the algebra structure $c_2 = c_1 \circ \psi$ on $TE$. We need a similar result for 2-categories and lax- or pseudo-algebras, which can for the pseudo case be derived from Marmolejo and Wood (Marmolejo & Wood 2008).[2] That paper deals with pseudomonads, where the three monad equations hold

---

[2]The analogous result for lax algebras was obtained by N. Gambino (private communication).

only up to isomorphism subject to coherence conditions. Hence it already covers our case of 2-monads. For two pseudomonads $\mathfrak{D}_i$ on 2-categories $\mathcal{D}_i$ ($i = 1, 2$), and a 2-functor $T: \mathcal{D}_1 \to \mathcal{D}_2$, the paper defines the notion of *transition from $\mathfrak{D}_1$ to $\mathfrak{D}_2$ along $T$* as a strong transformation (the "pseudo" generalization of a 2-natural transformation) $\psi: \mathfrak{D}_2 T \to T\mathfrak{D}_1$ together with two invertible modifications that relax the commutativities in diagram (2-19), subject to some coherence conditions. (Marmolejo & Wood 2008) goes on to show that the existence of a transition along $T$ implies that $T$ lifts to the pseudoalgebra categories. (The definition of pseudoalgebra for a pseudomonad appears in (Marmolejo 1997).)

In our situation we have 2-monads, but still need to use pseudoalgebras in order to connect with Street's result. We can summarize the restricted results of (Marmolejo & Wood 2008) as follows.

**Definition 8.** *Let $\mathfrak{D}_i = (D_i, i_i, c_i)$ ($i = 1, 2$) be 2-monads on 2-categories $\mathcal{D}_i$, and let $T: \mathcal{D}_1 \to \mathcal{D}_2$ be a 2-functor. Then a* 2-transition from $\mathfrak{D}_1$ to $\mathfrak{D}_2$ along $T$ *is a 2-natural transformation $\psi: \mathfrak{D}_2 T \to T\mathfrak{D}_1$ such that the equations* (2-20) *hold.*

**Proposition 9.** *Let $\mathfrak{D}_i = (D_i, i_i, c_i)$ ($i = 1, 2$) be 2-monads on 2-categories $\mathcal{D}_i$, let $T: \mathcal{D}_1 \to \mathcal{D}_2$ be a 2-functor, and let $\psi$ be a 2-transition from $\mathfrak{D}_1$ to $\mathfrak{D}_2$ along $T$.*

(1) *$T$ lifts to a 2-functor between the pseudoalgebra categories, $\widehat{T}: \mathcal{D}_1^{\mathfrak{D}_1} \to \mathcal{D}_2^{\mathfrak{D}_2}$, such that diagram* (2-18) *commutes, with $\overline{T}$ replaced by $\widehat{T}$.*

(2) *$\widehat{T}$ preserve normality.*

*Proof.* (1) is in (Marmolejo & Wood 2008). For (2), let $(E, c, \zeta = 1, \theta)$ be a normal $\mathfrak{D}_1$-pseudoalgebra as in Definition 2. Its image under $\widehat{T}$ is carried by $TE$, its structure map is the bottom line in the following diagram, and its $\zeta$ is got by pasting the square and the triangle. The square commutes by (2-19), and $T\zeta$ is the identity because $E$ is normalized.

$$
\begin{array}{ccccc}
TE & \xrightarrow{\quad 1 \quad} & TE & & \\
{\scriptstyle i_2 TE}\downarrow & & \downarrow{\scriptstyle Ti_1 E} & \overset{1}{\searrow} & \\
& & & {\scriptstyle T\zeta}\nearrow & \\
\mathfrak{D}_2 TE & \xrightarrow[\psi E]{} & T\mathfrak{D}_1 E & \xrightarrow[Tc]{} & TE
\end{array}
$$

$\square$

## 3. THE ARROW CATEGORY $\mathcal{C}^{\downarrow}$

For the rest of the paper $\mathcal{C}$ will be a representable 2-category.

In this section we summarize some general features of the arrow category $\mathcal{C}^{\downarrow}$. An object of $\mathcal{C}^{\downarrow}$ is a bundle (an arbitrary morphism in $\mathcal{C}$), and a morphism $f$ is a commutative square in which the vertical arrows are bundles, the domain and codomain of $f$. We shall write $\overline{f}$ and $\underline{f}$ for the "upstairs" and "downstairs" parts of $f$, the horizontal arrows in the square. Our general presumption is that $\mathcal{C}$ is in fact a 2-category, and then $\mathcal{C}^{\downarrow}$ is too. We use similar notation for the components of a 2-cell $\alpha: f \to g$, got as the results of applying the 2-functors $\mathrm{dom}, \mathrm{cod}: \mathcal{C}^{\downarrow} \to$

$\mathcal{C}$. The following diagram commutes at the 2-cell level, in the sense that $p'\overline{\alpha} = \underline{\alpha}p$.

$$
\begin{array}{ccc}
E & \overset{\overline{f}}{\underset{\overline{g}}{\Downarrow\overline{\alpha}}} & E' \\
\Big\downarrow{\scriptstyle p} & \overset{f}{\underset{g}{}} & \Big\downarrow{\scriptstyle p'} \\
B & \underset{g}{\overset{f}{\Downarrow\underline{\alpha}}} & B'
\end{array}
$$

For any 2-category $\mathcal{D}$, a 2-functor $F\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$ can be understood as a pair of functors $\overline{F} = \operatorname{dom} F$ and $\underline{F} = \operatorname{cod} F$ from $\mathcal{D}$ to $\mathcal{C}$, together with a 2-natural transformation $F{\downarrow}\colon \overline{F} \to \underline{F}$. Given two such 2-functors, $F$ and $G$, a 2-natural transformation $\alpha\colon F \to G$ can be understood as a pair of 2-natural transformations $\overline{\alpha}\colon \overline{F} \to \overline{G}$ and $\underline{\alpha}\colon \underline{F} \to \underline{G}$ such that $G{\downarrow} \circ \overline{\alpha} = \underline{\alpha} \circ F{\downarrow}$. Thus a 2-natural transformation is a commutative square of 2-functors to $\mathcal{C}$ and 2-natural transformations.

We note some basic 2-functors and 2-natural transformations associated with $\mathcal{C}^{\downarrow}$.

**Definition 10.** $\mathfrak{I}\colon \mathcal{C} \to \mathcal{C}^{\downarrow}$ *is given by* $\overline{\mathfrak{I}} = \underline{\mathfrak{I}} = \operatorname{Id}_{\mathcal{C}}$, *with* $\mathfrak{I}{\downarrow}$ *the identity 2-natural transformation. Thus* $\mathfrak{I}B = \begin{array}{c} B \\ \| \\ B \end{array}$ .

*The 2-natural transformation* $\mathsf{cc}\colon \operatorname{Id}_{\mathcal{C}^{\downarrow}} \to \mathfrak{I}\operatorname{cod}$ *is defined by*

$$
\mathsf{cc}\left(\begin{array}{c} E \\ \Big\downarrow{\scriptstyle p} \\ B \end{array}\right) = 
\begin{array}{ccc}
E & \overset{p}{\longrightarrow} & B \\
\Big\downarrow{\scriptstyle p} & & \| \\
B & == & B
\end{array}
$$

We note immediately a "yanking" formula, easily proved:

$$
\begin{array}{c}
\operatorname{cod} \quad \boxed{\mathsf{cc}} \quad \operatorname{cod} \\
\underline{\phantom{xx}}\big|\; {\scriptstyle \mathfrak{I}} \;\big|\phantom{x} \\
\overline{\underline{\phantom{xxxx}}} \quad \operatorname{cod}
\end{array}
\quad = \quad
\begin{array}{c}
\operatorname{cod} \\
| \\
\operatorname{cod}
\end{array}
\tag{3-1}
$$

We next remark on the fact that $\operatorname{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$ is a bifibration (fibration and opfibration) in the 2-category $\operatorname{Cat}$. Here there are more concrete characterizations.

**Definition 11.** *Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a functor between categories. A morphism* $f\colon X \to Y$ *in* $\mathcal{C}$ *is* prone[3] *(with respect to* $F$ *) if for every* $g\colon Z \to Y$ *such that* $Fg$ *factors via* $Ff$, *as*

---

[3]We have adopted Paul Taylor's terms; prone and supine morphisms are also commonly called *cartesian* and *cocartesian*.

$Fg = (Ff)h'$, there is a unique $h\colon Z \to X$ such that $g = fh$ and $Fh = h'$.



Dually, $f$ is supine *if for every* $g\colon X \to Z$ *such that* $Fg$ *factors via* $Ff$, *as* $Fg = h'(Ff)$, *there is a unique* $h\colon Y \to Z$ *such that* $g = hf$ *and* $Fh = h'$.

Then $F$ is a fibration if for every object $Y$ in $\mathcal{C}$, and every morphism $f'\colon X' \to F(Y)$ in $\mathcal{D}$, there is a prone morphism $f\colon X \to Y$ such that $Ff = f'$. $F$ is an opfibration if for every object $Y$, and every morphism $f'\colon X' \to FY$ in $\mathcal{D}$, there is a supine morphism $f\colon X \to Y$ such that $Ff = f'$.

For $\mathrm{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$ it is well known that a morphism of $\mathcal{C}^{\downarrow}$ is prone iff, as commutative square in $\mathcal{C}$, it is a pullback; and $\mathrm{cod}$ is a fibration if $\mathcal{C}$ has pullbacks.

If in addition $\mathcal{C}$ is (as in our situations) a representable 2-category, then the pullbacks are 2-pullbacks, and the prone morphisms in $\mathcal{C}^{\downarrow}$ also allow lifting of 2-cells: if we have $(g_i, h'_i)$ $(i = 1, 2)$ lifting to $h_i$, and in addition we have $\alpha\colon g_1 \to g_2$ and $\beta'\colon h'_1 \to h'_2$ such that $F\alpha = (Ff)\beta'$, then there is a unique $\beta\colon h_1 \to h_2$ such that $\alpha = f\beta$ and $F\beta = \beta'$.

It is also easy to show that $f$ is supine with respect to $\mathrm{cod}$ iff $\overline{f}$ is an isomorphism, and $\mathrm{cod}$ is always an opfibration. Trivially, the supine morphisms allow lifting of 2-cells.

We can extend this to $\mathrm{cod}\colon 2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}^{\downarrow}] \to 2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}]$. The discussion at the start of this section shows that $2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}^{\downarrow}]$ is isomorphic to $2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}]^{\downarrow}$, so we know that the prone morphisms in $2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}^{\downarrow}]$ correspond to the pullback squares in $2\text{-}\mathrm{fun}[\mathcal{D}, \mathcal{C}]$.

**Lemma 12.** *Let* $v\colon \mathfrak{F} \to \mathfrak{G}$ *be a 2-natural transformations between 2-functors* $\mathfrak{F}, \mathfrak{G}\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$, *and suppose also that for each object* $X$ *of* $\mathcal{D}$ *the square*

$$
\begin{array}{ccc}
\overline{\mathfrak{F}}X & \xrightarrow{\overline{v}X} & \overline{\mathfrak{G}}X \\
{\scriptstyle \mathfrak{F}\downarrow(X)}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \mathfrak{G}\downarrow(X)} \\
\underline{\mathfrak{F}}X & \xrightarrow{\underline{v}X} & \underline{\mathfrak{G}}X
\end{array}
$$

*is a pullback in* $\mathcal{C}$.

*Then the corresponding square of functors from* $\mathcal{D}$ *to* $\mathcal{C}$ *is a pullback in* $2\text{-}fun[\mathcal{D}, \mathcal{C}]$, *and hence* $v$ *is prone in* $2\text{-}fun[\mathcal{D}, \mathcal{C}^{\downarrow}]$.

*Proof.* If $\mathfrak{H}$ is a 2-functor from $\mathcal{D}$ to $\mathcal{C}$, then constructing a pullback fill-in from $\mathfrak{H}$ to $\overline{\mathfrak{F}}$ amounts to constructing the components for every $X$, which are determined uniquely by the pullback squares in $\mathcal{C}$. Hence we have the uniqueness part of the pullback property. Proving that it gives a 2-natural transformation follows routinely from the pullback property. $\qquad\square$

Proneness of $v$ in the above lemma amounts to the following: 2-natural transformations $u\colon \mathfrak{H} \to \mathfrak{F}$ are uniquely determined by $\underline{u}$ and $vu$.

## 4. THE MONADS $\mathfrak{L}_\bullet$ AND $\mathfrak{R}_\bullet$ ON $\mathcal{C}^\downarrow$

In this section we shall define two monads $\mathfrak{L}_\bullet$ and $\mathfrak{R}_\bullet$ on the arrow 2-category $\mathcal{C}^\downarrow$ extending Street's monads $\mathfrak{L}_B$ and $\mathfrak{R}_B$ on the 2-categories $\mathcal{C}/B$. This will allow us to have general base change morphisms for bundles, and not only bundles over a fixed base $B$ as in the slice 2-category $\mathcal{C}/B$. We develop the theory for $\mathfrak{L}_\bullet$ as the $\mathfrak{R}_\bullet$ case is obtained by working in $\mathcal{C}^{co}$ with reversed 2-cells.

**Definition 13.** *Let $\mathcal{C}^\downarrow$ be the arrow 2-category over the representable 2-category $\mathcal{C}$, and let $\mathfrak{L}_B\colon \mathcal{C}/B \to \mathcal{C}/B$ be the 2-monad defined by Street. We define the 2-functor $\mathfrak{L}_\bullet\colon \mathcal{C}^\downarrow \to \mathcal{C}^\downarrow$ on the arrow 2-category as follows:*

$$
\mathfrak{L}_\bullet \left(
\begin{array}{c}
E \overset{\overline{f}}{\underset{\overline{g}}{\Rightarrow \overline{\alpha}}} E' \\
p \downarrow \quad \underset{\underline{f}}{} \quad \downarrow p' \\
B \overset{}{\underset{g}{\Rightarrow \underline{\alpha}}} B'
\end{array}
\right)
=
\begin{array}{c}
\mathfrak{L}_B E \overset{\mathfrak{L}_{\underline{f}}\overline{f}}{\underset{\mathfrak{L}_{\underline{g}}\overline{g}}{\Rightarrow \mathfrak{L}_{\underline{\alpha}}\overline{\alpha}}} \mathfrak{L}_{B'} E' \\
d_1 \downarrow \quad \underset{\underline{f}}{} \quad \downarrow d_1' \\
B \overset{}{\underset{g}{\Rightarrow \underline{\alpha}}} B'
\end{array}
\tag{4-1}
$$

*For 0-cells, $\mathfrak{L}_B E$ is as defined in Section 2.1.1. On 1-cells, $\mathfrak{L}_{\underline{f}}\overline{f}$ is uniquely defined such that a), b) and c) hold:*



$$
a)\colon \;\; \frac{d_0' \quad \mathfrak{L}_{\underline{f}}\overline{f}}{\overline{f} \quad d_0} \quad ; \quad
b)\colon \;\; \frac{d_1' \quad \mathfrak{L}_{\underline{f}}\overline{f}}{\underline{f} \quad d_1} \quad ; \quad
\begin{array}{c} p' \quad d_0' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \boxed{\lambda'} \\ d_1' \quad \mathfrak{L}_{\underline{f}}\overline{f} \end{array}
\overset{c)}{=}
\begin{array}{c} p' \quad d_0' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \underline{f} \;\; \boxed{\lambda} \\ d_1' \quad \mathfrak{L}_{\underline{f}}\overline{f} \end{array}
\tag{4-2}
$$

*On 2-cells $\mathfrak{L}_{\underline{\alpha}}\overline{\alpha}$ is uniquely defined such that d) and e) hold:*

$$
\begin{array}{c} d_0' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \boxed{\mathfrak{L}_{\underline{\alpha}}\overline{\alpha}} \\ d_0' \quad \mathfrak{L}_{\underline{g}}\overline{g} \end{array}
\overset{d)}{=}
\begin{array}{c} d_0' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \overline{f} \\ \boxed{\overline{\alpha}} \quad d_0 \\ \overline{g} \\ d_0' \quad \mathfrak{L}_{\underline{g}}\overline{g} \end{array}
\;\; ; \quad
\begin{array}{c} d_1' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \boxed{\mathfrak{L}_{\underline{\alpha}}\overline{\alpha}} \\ d_1' \quad \mathfrak{L}_{\underline{g}}\overline{g} \end{array}
\overset{e)}{=}
\begin{array}{c} d_1' \quad \mathfrak{L}_{\underline{f}}\overline{f} \\ \underline{f} \\ \boxed{\underline{\alpha}} \quad d_1 \\ \underline{g} \\ d_1' \quad \mathfrak{L}_{\underline{g}}\overline{g} \end{array}
\;\; .
\tag{4-3}
$$

Note that the object part of $\mathfrak{L}_\bullet$ on an object $(E, p\colon E \to B)$ is Street's monad $\mathfrak{L}_B$ on the slice $\mathcal{C}/B$. The extension to the arrow category includes base change.

**Proposition 14.** *The map $\mathfrak{L}_\bullet\colon \mathcal{C}^\downarrow \to \mathcal{C}^\downarrow$ of Definition 13 is a 2-functor.*

*Proof.* For functoriality on 1-cells, suppose we have bundle morphisms $f\colon (E,p) \to (E',p')$ and $f'\colon (E',p') \to (E'',p'')$. We must show that

$$
\begin{array}{c|c|c|c}
p'' & d_0'' & \mathfrak{L}_{\underline{f'}}\overline{f'} & \mathfrak{L}_{\underline{f}}\overline{f} \\
\boxed{\lambda''} & & | & | \\
& d_1'' & \mathfrak{L}_{\underline{f'}}\overline{f'} & \mathfrak{L}_{\underline{f}}\overline{f}
\end{array}
\quad = \quad
\begin{array}{c}
p'' \quad d_0'' \quad \mathfrak{L}_{\underline{f'}}\overline{f'} \quad \mathfrak{L}_{\underline{f}}\overline{f} \\
\underline{f'} \quad \underline{f} \quad \boxed{\lambda} \quad \begin{array}{c} p \\ d_0 \\ d_1 \end{array} \\
d_1'' \quad \mathfrak{L}_{\underline{f'}}\overline{f'} \quad \mathfrak{L}_{\underline{f}}\overline{f}
\end{array}
$$

This is straightforwardly shown by applying the definition of $\mathfrak{L}_{\underline{f}}\overline{f}$ and $\mathfrak{L}_{\underline{f'}}\overline{f'}$. Functoriality on 2-cells is easier. □

**Proposition 15.** *There is a 2-natural transformation* $d_0\colon \operatorname{dom}\mathfrak{L}_\bullet \to \operatorname{dom}$ *whose component at* $p\colon E \to B$ *is* $d_0\colon \mathfrak{L}_B E \to E$.

*Proof.* The 2-naturality follows from equations (4-2) (a) and (4-3) (d). □

We now define the components of $i\colon 1 \to \mathfrak{L}_\bullet$ and $c\colon \mathfrak{L}_\bullet^2 \to \mathfrak{L}_\bullet$ exactly as in Definition 4.

**Proposition 16.** *We obtain a 2-monad* $(\mathfrak{L}_\bullet, i, c)$ *on* $\mathcal{C}^\downarrow$.

*Proof.* The monad equations for $i$ and $c$ are applied slicewise, in other words through their restrictions to Street's monad $\mathfrak{L}_B$ on $\mathcal{C}/B$, and so are already known to hold. It remains to show that $i, c$ are 2-natural transformations on $\mathcal{C}^\downarrow$, hence work as expected under base change. We give the proof for $c$, which is the more complicated: we show that $c'\overline{\mathfrak{L}_\bullet^2 f} = \overline{\mathfrak{L}_\bullet f}\,c$.

$$
\begin{array}{c|c|c|c}
p' & d_0' & c' & \overline{\mathfrak{L}_\bullet^2 f} \\
\boxed{\lambda'} & & | & | \\
& d_1' & c' & \overline{\mathfrak{L}_\bullet^2 f}
\end{array}
\;=\;
\begin{array}{c}
p' \quad d_0' \quad c' \quad \overline{\mathfrak{L}_\bullet^2 f} \\
\begin{array}{c|c} p' & d_0' \\ \boxed{\lambda'} & \\ & d_1' \\ \boxed{\lambda'} & \end{array} \; d_0' \; d_1' \\
d_1' \quad c' \quad \overline{\mathfrak{L}_\bullet^2 f}
\end{array}
\;=\;
\begin{array}{c}
p' \quad d_0' \quad c' \quad \overline{\mathfrak{L}_\bullet^2 f} \\
\begin{array}{c|c} p' & d_0' \\ \boxed{\lambda'} & \\ & d_1' \end{array} \; \boxed{\mathfrak{L}_\bullet f} \; d_0 \\
\begin{array}{c|c} d_1' & d_0' \\ \boxed{\lambda'} & \\ & d_1' \end{array} \; \overline{\mathfrak{L}_\bullet^2 f} \\
d_1' \quad c' \quad \overline{\mathfrak{L}_\bullet^2 f}
\end{array}
$$

**Proposition 17.** *Let $\mathcal{C}$ be representable category. Then a 1-cell $p\colon E \to B$ is a pseudo-opfibration iff, as an object in $\mathcal{C}^{\downarrow}$, it can be equipped with pseudoalgebra structure $(c, \zeta, \theta)$ for $\mathfrak{L}_{\bullet}$ such that $\underline{c} = 1_B$. It is an opfibration iff, in addition, the pseudoalgebra structure can be chosen to be normalized ($\zeta$ is an identity 2-cell).*

*Proof.* The condition $\underline{c} = 1_B$ is what is needed to have the $\mathfrak{L}_{\bullet}$-pseudoalgebra structure restrict to $\mathfrak{L}_B$-pseudoalgebra structure. All the pseudoalgebra conditions then live entirely inside the fibre of $\mathcal{C}^{\downarrow}$ over $B$, and we have reduced to Proposition 6. □

Dually, $p$ is a pseudofibration iff it has $\mathfrak{R}_{\bullet}$-pseudoalgebra structure with $\underline{c} = 1_B$, a fibration iff the pseudoalgebra can be chosen to be normalized.

## 5. THE 2-FUNCTORS $\mathfrak{K}_n$

In this section we introduce a family of 2-functors $\mathfrak{K}_n\colon \mathcal{C}^{\downarrow} \to \mathcal{C}^{\downarrow}$. Upstairs they are equal to $\mathfrak{L}_{\bullet}^n$, but their downstairs parts are not the identity. The reason for doing this arises in Section 6, where the definition of the natural transformation $\Psi_{\bullet}$ relies on an isomorphism upstairs that does not correspond to one downstairs. By modifying the downstairs part we can use natural isomorphisms between endofunctors on $\mathcal{C}^{\downarrow}$, and this makes it much easier to calculate with the 2-dimensional calculus.

Note in the following that it is immediate from the definition that $\Phi B = \mathfrak{L}_B B$. In fact, by composing the 2-functor $\overline{\mathfrak{L}_{\bullet}}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$ with the 2-functor $\mathfrak{I}\colon \mathcal{C} \to \mathcal{C}^{\downarrow}$ (Definition 10), we obtain the 2-functor $\Phi\colon \mathcal{C} \to \mathcal{C}$. Now from the morphism $\begin{pmatrix} p \\ B \end{pmatrix}\colon (E, p) \to (B, 1)$ we get $\mathfrak{L}_B p\colon \mathfrak{L}_B E \to \Phi B$.

**Definition 18.** *For each non-negative integer $n$, we define a 2-endofunctor $\mathfrak{K}_n\colon \mathcal{C}^{\downarrow} \to \mathcal{C}^{\downarrow}$. As a 2-natural transformation between 2-functors to $\mathcal{C}$, it is defined as $\mathsf{dom}\ \mathfrak{L}_{\bullet}^n\ \mathsf{cc}$,*

With notation as in Definition 13, $\mathfrak{K}_n$ can be calculated as follows.

$$\mathfrak{K}_n \left( \begin{array}{c} \overset{\overline{f}}{\underset{E}{\overbrace{\quad}}} \overset{\Downarrow \overline{\alpha}}{\phantom{x}} E' \\ p \downarrow \quad \overset{\overline{g}}{\underset{\underline{f}}{\phantom{xx}}} \quad \downarrow p' \\ B \quad \Downarrow \underline{\alpha} \quad B' \\ \underset{\underline{g}}{\phantom{xx}} \end{array} \right) = \begin{array}{ccc} \mathfrak{L}^n_B E & \overset{\mathfrak{L}^n_f \overline{f}}{\underset{\Downarrow \mathfrak{L}^n_{\underline{\alpha}} \overline{\alpha}}{\longrightarrow}} & \mathfrak{L}^n_{B'} E' \\ \mathfrak{L}^n_B p \downarrow & \overset{\mathfrak{L}^n_{\underline{g}} \overline{g}}{\underset{\mathfrak{L}^n_{\underline{f}} \underline{f}}{\phantom{xx}}} & \downarrow \mathfrak{L}^n_{B'} p' \\ \mathfrak{L}^n_B B & \overset{\Downarrow \mathfrak{L}^n_{\underline{\alpha}} \underline{\alpha}}{\underset{\mathfrak{L}^n_{\underline{g}} \underline{g}}{\longrightarrow}} & \mathfrak{L}^n_{B'} B' \end{array} \tag{5-1}$$

We shall in fact only need $\mathfrak{K}_1$ and $\mathfrak{K}_2$ (and note that $\mathfrak{K}_0$ is the identity endofunctor). However, some results have uniform proofs for all $n$.

**Definition 19.** *We define a 2-natural transformation $d_0\colon \mathfrak{K}_{n+1} \to \mathfrak{K}_n$ from the following commutative square of 2-functors from $\mathcal{C}^\downarrow$ to $\mathcal{C}$.*

$$\begin{array}{ccc} \mathsf{dom}\, \mathfrak{L}_\bullet \mathfrak{L}^n_\bullet & \xrightarrow{\ d_0 \mathfrak{L}^n_\bullet\ } & \mathsf{dom}\, \mathfrak{L}^n_\bullet \\ {\scriptstyle \mathsf{dom}\, \mathfrak{L}_\bullet \mathfrak{L}^n_\bullet\, \mathsf{cc}} \downarrow & & \downarrow {\scriptstyle \mathsf{dom}\, \mathfrak{L}^n_\bullet\, \mathsf{cc}} \\ \mathsf{dom}\, \mathfrak{L}_\bullet \mathfrak{L}^n_\bullet \mathfrak{I}\, \mathsf{cod} & \xrightarrow[\ d_0 \mathfrak{L}^n_\bullet \mathfrak{I}\, \mathsf{cod}\ ]{} & \mathsf{dom}\, \mathfrak{L}^n_\bullet \mathfrak{I}\, \mathsf{cod} \end{array}$$

*(The $d_0$ in the diagram is that of Proposition 15.)*

**Proposition 20.** *In $\mathcal{C}/B$ let $p\colon E \to B$ and $p'\colon E' \to B$ be objects, and let $k$ be a morphism from the first to the second over $B$. Then the following square is a pullback in $\mathcal{C}$.*

$$\begin{array}{ccc} \mathfrak{L}_B E & \xrightarrow{\ d_0\ } & E \\ {\scriptstyle \mathfrak{L}_B k} \downarrow \ \ \lrcorner & & \downarrow {\scriptstyle k} \\ \mathfrak{L}_B E' & \xrightarrow[\ d'_0\ ]{} & E' \end{array}$$

*Proof.* First, note that $\mathfrak{L}_B k$ is uniquely determined by the properties that $d'_0(\mathfrak{L}_B k) = k d_0$, $d'_1(\mathfrak{L}_B k) = d_1$, and

$$\begin{array}{c} \overset{p'\quad d'_0\quad \mathfrak{L}_B k}{\phantom{x}} \\ \boxed{\lambda'} \quad \Big| \\ \overset{}{d'_1 \quad \mathfrak{L}_B k} \end{array} = \begin{array}{c} \overset{p'\quad d'_0\quad \mathfrak{L}_B k}{\phantom{x}} \\ \overset{p\quad d_0}{\boxed{\lambda}} \\ \overset{d_1}{\phantom{x}} \\ \overset{}{d'_1 \quad \mathfrak{L}_B k} \end{array} \tag{5-2}$$

Now suppose we have an object $X$ with morphisms $f\colon X \to E$, $g\colon X \to \mathfrak{L}_B E'$ such that $kf = d'_0 g$. We require a unique morphism $h\colon X \to \mathfrak{L}_B E$ satisfying the pullback conditions

$d_0 h = f$ and $(\mathfrak{L}_B k)h = g$. We shall show that these conditions are equivalent to the conditions

$$d_0 h = f \tag{5-3a}$$

$$d_1 h = d_1' g \tag{5-3b}$$



$$\tag{5-3c}$$

Note that (5-2) is just the case of (5-3c) where $g$ is replaced by $\mathfrak{L}_B k$ and $h$ by the identity 1-cell. By inverting the identity 2-cells on the right hand side of (5-3c) we see that the equations (5-3) suffice to define $h$ uniquely using the fact that $\mathfrak{L}_B E$ is a comma object.

Assuming the pullback equations, we can derive (5-3) by substituting $g = (\mathfrak{L}_B k)h$ and using (5-2). For the converse, we use the fact that $\mathfrak{L}_B E'$ is a comma object to prove $(\mathfrak{L}_B k)h = g$. We have $d_0' g = kf = k d_0 h = d_0'(\mathfrak{L}_B k)h$ and $d_1' g = d_1 h = d_1'(\mathfrak{L}_B k)h$. It remains to show $\lambda' g = \lambda'(\mathfrak{L}_B k)h$, for which we calculate



$\square$

**Corollary 21.** *For any 2-functor $\mathfrak{F}\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$, we have that $d_0\mathfrak{F}\colon \mathfrak{K}_{n+1}\mathfrak{F} \to \mathfrak{K}_n\mathfrak{F}$ satisfies the conditions of Lemma 12 and hence is prone.*

*Proof.* Take Proposition 20, substituting $\mathfrak{L}_\bullet^n \mathfrak{F}X$ for $p$, $\mathfrak{L}_\bullet^n \mathfrak{I} \operatorname{cod} \mathfrak{F}X$ for $p'$, and $\mathfrak{L}_\bullet^n \operatorname{cc}\mathfrak{F}X$ for $k$. Then the pullback diagram there is that of Definition 19, applied to $\mathfrak{F}X$. We can now apply Lemma 12. $\square$

**Definition 22.** *We define a 2-natural transformation $d_1\colon \mathfrak{K}_{n+1} \to \mathfrak{K}_n\mathfrak{L}_\bullet$ as follows. Consider a square of 2-endomorphisms on $\mathcal{C}^{\downarrow}$,*



$$\tag{5-4}$$

*The 2-natural transformation on the bottom is* $\mathsf{cc}\,\mathfrak{L}_\bullet\mathfrak{I}\,\mathsf{cod}$ :

$$
\begin{array}{ccccc}
\boxed{\mathsf{CC}} & \mathfrak{L}_\bullet & \mathfrak{I} & \mathsf{cod} \\
& \mathsf{cod} & & \\
& \mathsf{cod} & & \\
\mathfrak{I} & & & \mathsf{cod} & \mathfrak{L}_\bullet
\end{array}
$$

*The square commutes, because, using equation* (3-1)*, we have*

$$
\begin{array}{c}
\boxed{\mathsf{CC}}\;\mathfrak{L}_\bullet \quad \boxed{\mathsf{CC}} \\
\mathsf{cod} \qquad \mathsf{cod} \quad \mathfrak{I} \\
\mathsf{cod} \\
\mathfrak{I} \qquad\qquad \mathsf{cod}\;\mathfrak{L}_\bullet
\end{array}
=
\begin{array}{c}
\boxed{\mathsf{CC}}\;\mathfrak{L}_\bullet \\
\mathsf{cod} \\
\mathsf{cod} \\
\mathfrak{I}\quad \mathsf{cod}\quad \mathfrak{L}_\bullet
\end{array}
=
\begin{array}{c}
\boxed{\mathsf{CC}} \qquad \mathfrak{L}_\bullet \\
\\
\mathfrak{I}\quad \mathsf{cod}\quad \mathfrak{L}_\bullet
\end{array}
$$

*Applying* $\mathsf{dom}\,\mathfrak{L}_\bullet^n$ *to the whole square, we get a commutative square of 2-functors from* $\mathcal{C}^\downarrow$ *to* $\mathcal{C}$ *in which the left and right sides correspond to* $\mathfrak{K}_{n+1}$ *and* $\mathfrak{K}_n\mathfrak{L}_\bullet$ *as 2-functors from* $\mathcal{C}^\downarrow$ *to itself. Thus we have a 2-natural transformation* $d_1\colon \mathfrak{K}_{n+1} \to \mathfrak{K}_n\mathfrak{L}_\bullet$ .

The notation $d_1$ is used because its downstairs part, applied to a bundle $p\colon E \to B$, is $\mathfrak{L}_B^n d_1\colon \mathfrak{L}_B^{n+1}B \to \mathfrak{L}_B^n B$ .

**Lemma 23.**

$$
\begin{array}{c}
\mathfrak{K}_{n+2} \\
\boxed{d_1} \\
\mathfrak{K}_{n+1} \\
\boxed{d_0} \\
\mathfrak{K}_n \quad \mathfrak{L}_\bullet
\end{array}
=
\begin{array}{c}
\mathfrak{K}_{n+2} \\
\boxed{d_0} \\
\mathfrak{K}_{n+1} \\
\boxed{d_1} \\
\mathfrak{K}_n \quad \mathfrak{L}_\bullet
\end{array}
$$

*Proof.* We must show that the equation holds when composed on the left with $\mathsf{dom}$ and with $\mathsf{cod}$. $\mathsf{dom}$ is easy, given that each $\mathsf{dom}\,d_1$ is an identity 2-cell, and $\mathsf{dom}\,d_0\mathfrak{L}_\bullet = \mathsf{dom}\,d_0$ (with different values of $n$ for the two instances of $d_0$). $\mathsf{cod}$ is straightforward from the definitions. Modulo appropriate identity 2-cells at top and bottom, both sides reduce to

$$
\begin{array}{cccccc}
\mathsf{dom} & \mathfrak{L}_\bullet & \mathfrak{L}_\bullet^n & \boxed{\mathsf{CC}} & \mathfrak{L}_\bullet & \mathfrak{I} & \mathsf{cod} \\
\boxed{d_0} & & & & \mathsf{cod} & & \\
\mathsf{dom} & & \mathfrak{L}_\bullet^n & \mathfrak{I} & & & \mathsf{cod} \quad \mathfrak{L}_\bullet
\end{array}\;.
$$

$\square$

The following lemma is used in Definition 31.

**Lemma 24.** *Let* $\mathfrak{F}$ *be a 2-functor from* $\mathcal{D}$ *to* $\mathcal{C}^\downarrow$. *Then* $d_1\mathfrak{F}\colon \mathfrak{K}_{n+1}\mathfrak{F} \to \mathfrak{K}_n\mathfrak{F}$ *is supine over* $[\mathcal{D},\mathsf{cod}]\colon [\mathcal{D},\mathcal{C}^\downarrow] \to [\mathcal{D},\mathcal{C}]$.

*Proof.* In diagram (5-4), the top arrow is an isomorphism, so each component of $d_1$ is a supine morphism in $\mathcal{C}^{\downarrow}$. □

**Definition 25.** *Let* $\alpha\colon \mathfrak{L}_{\bullet}^m \to \mathfrak{L}_{\bullet}^n$ *be a 2-natural transformation. Consider the square of 2-functors*

$$
\begin{array}{ccc}
\operatorname{dom} \mathfrak{L}_{\bullet}^m & \xrightarrow{\ \operatorname{dom}\alpha\ } & \operatorname{dom} \mathfrak{L}_{\bullet}^n \\
{\scriptstyle \operatorname{dom} \mathfrak{L}_{\bullet}^m \, \mathrm{cc}}\Big\downarrow & & \Big\downarrow{\scriptstyle \operatorname{dom} \mathfrak{L}_{\bullet}^n \, \mathrm{cc}} \\
\operatorname{dom} \mathfrak{L}_{\bullet}^m \mathfrak{I}\operatorname{cod} & \xrightarrow[\ \operatorname{dom}\alpha\mathfrak{I}\operatorname{cod}\ ]{} & \operatorname{dom} \mathfrak{L}_{\bullet}^n \mathfrak{I}\operatorname{cod}
\end{array}
$$

*It commutes, and so defines a 2-natural transformation* $\alpha\colon \mathfrak{K}_m \to \mathfrak{K}_n$.

We use this to define 2-natural transformations $i\colon \operatorname{Id}_{\mathcal{C}^{\downarrow}} \to \mathfrak{K}_1$ and $c\colon \mathfrak{K}_2 \to \mathfrak{K}_1$.

**Lemma 26.**



*(The right-hand side of (c) is the empty diagram, i.e. an identity 2-cell on an identity 1-cell.)*

*Proof.* In each part we must check the equation "upstairs and downstairs", i.e. when left composed with $\operatorname{dom}$ and with $\operatorname{cod}$.

(a): For the upstairs part, after composing with an appropriate identity 2-cell at the top, we find we require



This is just a rearrangement of the condition $d_0 c = d_0 d_0$ in Definition 4.

The downstairs part follows from the upstairs, because composing with $\operatorname{cod}$ on the left is equal to composing with $\operatorname{dom}$ on the left and $\mathfrak{I}\operatorname{cod}$ on the right.

(b): The upstairs part is clear, bearing in mind that each $\overline{d_1}$ is an identity morphism. For the downstairs part, composing with an appropriate identity 2-cell at the top, we calculate as below. Note that for the $c$ belonging to $\mathfrak{L}_{\bullet}$ we have $\operatorname{cod} c = \operatorname{cod}$. This is used in the second and fourth

equations. The third uses equation (3-1).

$$\text{LHS} \quad = \quad \begin{array}{c}\text{dom} \qquad\qquad \mathfrak{L}_\bullet \quad \mathfrak{L}_\bullet \quad \mathfrak{I} \quad \text{cod} \\ \boxed{\text{CC}} \qquad \boxed{\mathcal{C}} \\ \mathfrak{I} \quad\ \text{cod} \quad \mathfrak{L}_\bullet \qquad\qquad \text{cod} \quad \mathfrak{L}_\bullet \end{array}$$

$$= \quad \begin{array}{c}\text{dom} \quad \boxed{\text{CC}} \quad \mathfrak{L}_\bullet \quad \mathfrak{L}_\bullet \quad \mathfrak{I} \quad \text{cod} \\ \mathfrak{I} \quad\ \text{cod} \\ \text{cod} \quad \mathfrak{L}_\bullet \qquad\qquad \text{cod} \quad \mathfrak{L}_\bullet \end{array}$$

$$= \quad \begin{array}{c}\text{dom} \quad \boxed{\text{CC}} \quad \mathfrak{L}_\bullet \quad \boxed{\text{CC}} \quad \mathfrak{L}_\bullet \quad \mathfrak{I} \quad \text{cod} \\ \mathfrak{I} \quad \text{cod} \quad\ \mathfrak{I} \quad \text{cod} \\ \text{cod} \qquad\qquad \text{cod} \qquad\qquad \text{cod} \quad \mathfrak{L}_\bullet \end{array} \quad = \text{RHS.}$$

(c) and (d) are analogous to (a) and (b), but simpler. □

## 6. FUNCTORS $\mathfrak{T}_\bullet$ PRESERVING (OP)FIBRATIONS

In this section we present our main technical result, Theorem 37: any indexed bundle 2-endomorphism $\mathfrak{T}_\bullet$ preserves both opfibrations and fibrations (as well as the pseudo- versions). The main part of the argument is to show that $\mathfrak{T}_\bullet$ lifts to the 2-categories of pseudoalgebras and normalized pseudoalgebras of $\mathfrak{L}_\bullet$, by defining a 2-transition $\Psi_\bullet$ from $\mathfrak{L}_\bullet$ to itself along $\mathfrak{T}_\bullet$. A dual argument shows that it also lifts to the corresponding 2-categories for $\mathfrak{R}_\bullet$.

**Definition 27.** *Let $\mathcal{C}^\downarrow$ be the arrow 2-category over a representable 2-category $\mathcal{C}$.*

*A* bundle 2-endomorphism *for $\mathcal{C}$ is a 2-endofunctor $\mathfrak{T}_\bullet$ of $\mathcal{C}^\downarrow$ such that $\operatorname{cod}\mathfrak{T}_\bullet = \operatorname{cod}$. The $\bullet$ used in the notation for bundle endofunctors indicates the ability to restrict to $\mathfrak{T}_B$ on each slice $\mathcal{C}/B$, and use notation similar to that already introduced for $\mathfrak{L}_\bullet$. Thus $\mathfrak{T}_f \overline{f}$ denotes $\overline{\mathfrak{T}_\bullet f}$. (The "bundle endomorphism" condition already tells us that $\underline{\mathfrak{T}_\bullet f} = \underline{f}$.)*

*We say that $\mathfrak{T}_\bullet$ is* indexed *if it preserves proneness – whenever a morphism $f$ in $\mathcal{C}^\downarrow$ is, as commutative square in $\mathcal{C}$, a pullback square, then so too is $\mathfrak{T}_\bullet f$. This is equivalent to its being an indexed endofunctor in the sense of indexed categories, also to $(\mathfrak{T}_\bullet, \operatorname{Id}_\mathcal{C})$ being a morphism of fibrations – though not of opfibrations.*

Note that the slice endofunctors $\mathfrak{L}_B$ and $\mathfrak{R}_B$ are not preserved by pullback: although $\mathfrak{L}_\bullet$ and $\mathfrak{R}_\bullet$ are bundle 2-endomorphisms, they are not indexed. Hence we cannot use the language of indexed categories to discuss the interaction between them and $\mathfrak{T}_\bullet$. Instead we use the codomain bifibration explicitly.

**Definition 28.** *Let $\mathfrak{T}_\bullet$ be a bundle 2-endofunctor for $\mathcal{C}$. For each natural number $n$, we define the 2-natural transformation $\psi_n \colon \mathfrak{T}_\bullet \mathfrak{K}_n \to \mathfrak{K}_n \mathfrak{T}_\bullet$ recursively as follows. $\psi_0 \colon \mathfrak{T}_\bullet \to \mathfrak{T}_\bullet$ is the identity, and $\psi_{n+1}$ is (using Corollary 21 to deduce that $d_0 \mathfrak{T}_\bullet$ is prone) the unique 2-natural*

*transformation such that*

$$
\boxed{\psi_{n+1}} \;\; \vdots \;\; = \;\; \vdots \;\; \boxed{d_0} \;\; and \;\; \vdots \;\; = \;\; \vdots \;\; .
$$

For the rest of this section, we shall take it that we are given a representable category $\mathcal{C}$ and an indexed bundle 2-endofunctor $\mathfrak{T}_\bullet$ for it.

**Proposition 29.** *Each $\psi_n$ is a 2-natural isomorphism.*

*Proof.* Because $\mathfrak{T}_\bullet$ preserves proneness, we know that $\mathfrak{T}_\bullet d_0$ satisfies the conditions of Lemma 12 and hence is prone. This allows us to define $\psi_n^{-1}$ with diagrams similar to those of Definition 28. In the first equation each diagram is reflected left to right, while in the second they are upside down. One can then prove it is the inverse of $\psi_n$. $\qquad\square$

**Lemma 30.**

$$
\overset{(a)}{=} \qquad , \qquad \overset{(b)}{=} \qquad .
$$

*Proof.* By Corollary 21, taking $d_0\colon \mathfrak{K}_1 \to \mathfrak{K}_0 = \mathrm{Id}_{\mathcal{C}^\downarrow}$, it suffices to check for each equation that it holds when composed (i) with $\mathrm{cod}$ on the left, and (ii) with $d_0\mathfrak{T}_\bullet$ at the bottom. (i) is clear from the facts that each $\mathrm{cod}\,\psi_n$ is an identity, and $\mathrm{cod}\,\mathfrak{T}_\bullet = \mathrm{cod}$. It remains to check (ii) for each equation.

For (a) we have

$$
\mathrm{LHS} \;=\; \quad = \quad = \quad = \quad
$$

Here the second and third equations use Lemma 26 (c), and the fourth uses Definition 28.

For (b) we have

$$
= \quad = \quad = \quad = \quad
$$

For the first and third equations we have used Lemma 26 (a), while for the second and fourth we have used Definition 28. $\qquad\square$

**Definition 31.** *Using Lemma 24 with $\mathfrak{T}_\bullet$ for $\mathfrak{F}$, and using invertibility of $\psi_1$, we define the 2-natural transformation $\Psi_\bullet\colon \mathfrak{L}_\bullet\mathfrak{T}_\bullet \to \mathfrak{T}_\bullet\mathfrak{L}_\bullet$ as the unique such over* cod *satisfying*



In diagrammatic form, this works as follows. From Proposition 20, we get two pullback squares



Applying $\mathfrak{T}_\bullet$ to the first, we obtain that $\mathfrak{L}_B\mathfrak{T}_B E \cong \mathfrak{T}_{\Phi B}\mathfrak{L}_B E$. Composing this with

$$\underline{\mathfrak{T}_\bullet}\left(\begin{array}{c} \mathfrak{L}_B E = \mathfrak{L}_B E \\ \mathfrak{L}_B p \downarrow \qquad \downarrow d_1 \\ \Phi B \xrightarrow{\ d_1\ } B \end{array}\right) \tag{6-1}$$

gives us a 1-cell $\mathfrak{T}_{d_1}1\colon \mathfrak{L}_B\mathfrak{T}_B E \to \mathfrak{T}_B\mathfrak{L}_B E$, which will be the component at $(E,p)$ of our 2-natural transformation $\Psi_\bullet$.
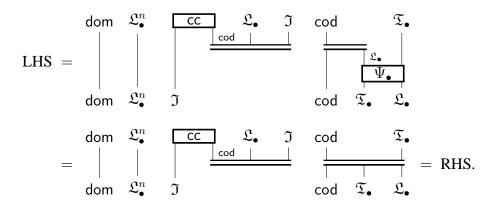
**Lemma 32.**



*Proof.* Note that the case $n = 0$ is simply Definition 31. Using Corollary 21, it suffices to prove the equation when composed (i) on the left with cod, and (ii) at the bottom with $d_0\mathfrak{T}_\bullet\mathfrak{L}_\bullet$.

For (i), applying $\mathrm{cod}$, and bearing in mind that each $\mathrm{cod}\,\psi_n$ and $\mathrm{cod}\,\Psi_\bullet$ is an identity, we get – modulo some identity 2-cells at top and bottom –

$$\mathrm{LHS} \;=\; \cdots \;=\; \mathrm{RHS}.$$

For (ii), we use induction on $n$. The base case, $n = 0$, has already been covered, so we assume $n > 0$. Now using Definitions 28 and 31, and Lemma 23, and induction in the fourth equation, we have –
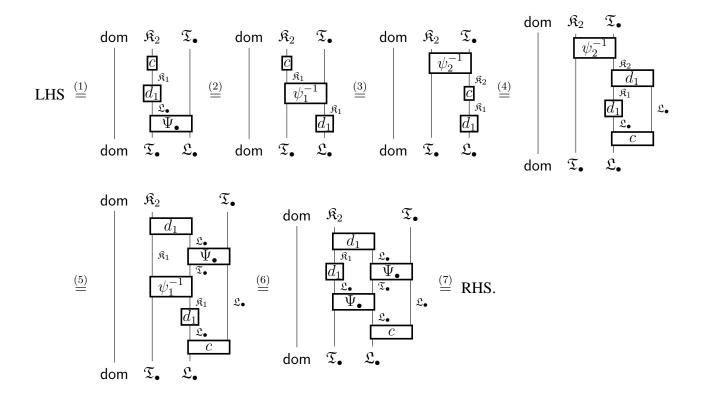
$$\mathrm{LHS} \;=\; \cdots \;=\; \cdots \;=\; \cdots$$

$$=\; \cdots \;=\; \cdots \;=\; \mathrm{RHS}.$$

$\square$

**Lemma 33.**

$$\Psi_\bullet = d_1 \, \mathfrak{K}_1 \,\Psi_\bullet = \psi_1^{-1}\, \mathfrak{K}_1 \, d_1 = d_1 \, \mathfrak{K}_1 =$$

*Proof.* The first and last equations use Lemma 26, and the other two use the definition of $\Psi_\bullet$ and Lemma 30.  □

**Lemma 34.**

$$c \,\Psi_\bullet = \Psi_\bullet \, c$$

*Proof.* We must prove the equation when composed with dom and with cod on the left. For cod this is immediate, since everything is over cod. For dom, after composing top and bottom with appropriate identity 2-cells, we calculate as follows. Here, equations (1) and (7) use the fact that dom $d_1$ is an identity; equations (2) and (6) use Definition 31; and equations (3)-(5) use Lemmas 30, 26 (b) and 32 respectively.

$$\text{LHS} \overset{(1)}{=} \quad \overset{(2)}{=} \quad \overset{(3)}{=} \quad \overset{(4)}{=}$$

$$\overset{(5)}{=} \quad \overset{(6)}{=} \quad \overset{(7)}{=} \text{RHS.}$$

$\square$

**Proposition 35.** *Let $\mathfrak{T}_\bullet$ be an indexed bundle 2-endofunctor for $\mathcal{C}$, and let $\mathfrak{L}_\bullet$ be the monad from Definition 13. Then $\Psi_\bullet$ is a 2-transition from $(\mathcal{C}^\downarrow, \mathfrak{L}_\bullet)$ to itself along $\mathfrak{T}_\bullet$.*

*Proof.* We have already shown that $\Psi_\bullet$ is 2-natural, and the two equations needed to make a 2-transition are those of Lemmas 33 and 34. $\square$

**Proposition 36.** *The 2-functor $\mathfrak{T}_\bullet$ lifts to an endofunctor $\widehat{\mathfrak{T}}_\bullet$ on the category $(\mathcal{C}^\downarrow)^{\mathfrak{L}_\bullet}$ of $\mathfrak{L}_\bullet$-pseudoalgebras $(E, c, \zeta, \theta)$. $\widehat{\mathfrak{T}}_\bullet$ preserves normality ($\zeta = 1$), and also the property that $\underline{c} = 1_B$.*

*Proof.* Given Proposition 35, the pseudoalgebra lifting and preservation of normality now follow from Proposition 9, while the final part follows from the fact that both $\mathfrak{T}_\bullet$ and $\Psi_\bullet$ are over cod. $\square$

**Theorem 37.** *Let $\mathcal{C}$ be a representable 2-category, and $\mathfrak{T}_\bullet$ an indexed bundle 2-endofunctor for $\mathcal{C}$. Then $\mathfrak{T}_\bullet$ preserves pseudofibrations, fibrations, pseudo-opfibrations and opfibrations in $\mathcal{C}$.*

*Proof.* For (pseudo-)opfibrations, we combine Proposition 36 with Proposition 17.

Working in $\mathcal{C}^{co}$ shows that the same development holds for the monad $\mathfrak{R}_\bullet$. This gives the result for (pseudo)fibrations. $\square$

## 7. Conclusions

As mentioned in the introduction, the starting point for this work was rather specific. Recent topos-theoretic approaches to quantum foundations can be understood (Fauser et al. 2012) as constructing "spectral bundles" in the category of locales. There are two technically distinct approaches, "presheaf" and "copresheaf", exemplified by (Döring & Isham 2011) and (Heunen, Landsman & Spitters 2009) respectively. In the presheaf approach, the spectral bundle is a local homeomorphism and hence an opfibration: it has fibre maps covariant with respect to specialization in the base. In the copresheaf approach, the spectral bundle is fibrewise compact regular (i.e. it corresponds to a compact regular locale in the topos of sheaves over the base) and hence a fibration: it has contravariant fibre maps.

We should like to apply constructions to these bundles, for example the valuation locale construction in order to gain access to probabilistic features of quantum physics, and it is natural to wish these constructions to work fibrewise on bundles. This naturally calls for constructions on locales that are geometric in the sense of being preserved under pullback, since fibres are pullbacks.

In general, geometric constructions will not preserve the bundle properties of being a local homeomorphism or of fibrewise compact regularity. However, it would still be useful to know that they preserve fibrations and opfibrations, and that is what this paper proves. Thus it seems that contextual physics (in which everything is considered to be fibred over a base space of contexts) would fall naturally into two kinds, opfibrational and fibrational, both closed under geometric constructions. The choice is determined by the choice for the spectral bundles, opfibrational for the presheaf approach, fibrational for copresheaves. (There is still debate over which is better. Our own opinion is that Gelfand-Naimark duality suggests that the spectral bundle should be fibrewise compact regular, thus leading to fibrations.)

Along the way the work took on other important ideas. The first was that "geometric" just means indexed, in the sense of indexed categories, and could alternatively be formulated using the codomain bifibration. The paper (Vickers & Townsend 2012) provides a way to guarantee the coherence needed for this, applicable to standard examples of geometric constructions on locales.

The new definition of geometric as indexed also becomes applicable to 2-categories much more general than $\mathbf{Loc}$, and this prompted a big generalization of the present work. In the more general setting it seemed very natural to use Street's characterization of fibrations and opfibrations in terms of pseudoalgebras, and lifting functors to algebra categories.

Throughout we have used tangle diagrams as a 2-dimensional calculus for 2-categories – in fact, we have even been able to use them in the 3-category of 2-categories, by separating out two levels of 2-dimensionality. We have found them more conclusive than the usual diagrams.

We have proved our results for a rather particular choice of laxities. We work in 2-categories, with 2-monads (hence all strict), but pseudoalgebras – to match Street's criterion. Also, the geometric endofunctors $\mathfrak{T}_\bullet$ are 2-functors. It may be that other combinations might be useful, or other indexed categories.

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