Locators point-free

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Abstract

Locators for Dedekind reals can be dealt with in point-free topology, giving a space $\mathbb{R}^{\mathfrak{L}}$ of reals equipped with locators. The projection $\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$ is an open surjection, and hence a coequalizer of its kernel pair. Consequently maps on reals can be defined by their action on reals with locators.

1 Introduction

A Dedekind real number x has the *locatedness* property that for any rationals q and r with $q < r$, we have either $q < x$ or $x < r$. Of course, for many pairs $q < r$ we shall have both, $q < x < r$, and this raises the possibility that computations might in some sense be non-deterministic: control could go in either of two directions depending on which of the two cases is chosen. Booij [Boo20] introduced the notion of *locators* for Dedekind real numbers to embody that choice. A locator for x is structure determining that choice for each pair $q < r$.

Booij investigated locators in the setting of Univalent Type Theory (UTT), by replacing the logical disjunction $(q < x) \vee (x < r)$ (property) by its interpretation $(q < x) + (x < r)$ (structure) in propositions as types. He showed that reals equipped with locators have computational advantages over bare reals.

The purpose of this note is to show that locators are also useful in point-free topology. Along with the space $\mathbb{R}$ of Dedekind reals there is also a space $\mathbb{R}^{\mathfrak{L}}$ of reals equipped with locators, and a forgetful map $\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$. We expect many of the analytic applications of locators developed in UTT to carry through to point-free spaces. However, our central result has some radically new consequences that were not possible in UTT. We show that pr_1 is an open surjection, from which it also follows that it is the coequalizer (in the category of point-free spaces) of its kernel pair: hence maps from $\mathbb{R}$ can be defined as maps from $\mathbb{R}^{\mathfrak{L}}$ that respect an intensional equality, the kernel pair.

This is made possible by the nature of point-free surjections, and the fact that there is no need to consider discontinuous maps. Thus although (just as in UTT) we cannot say that for every real there exists a locator, the surjection gives us the conservativity principles needed for results about reals with locators to descend to bare reals.

1.1 Terminological note

A *space* is a point-free space: in other words, its points are described (often implicitly) as the models of a geometric theory. In principle this may be a generalized space, but in this note they are all ungeneralized – the geometric theory is essentially propositional. Familiar ways to present propositional geometric theories include locales and formal topologies.

A *map* is a map between spaces. It can be described as a geometric transformation of points of the domain to points of the codomain, and is necessarily continuous. For locales these are the usual locale morphisms.

A map can alternatively be constructed as a *bundle*, by a geometric construction of spaces (the fibres) out of codomain points.

We say a bundle is "fibrewise" $\mathcal{X}$ if the generic fibre is $\mathcal{X}$. However, that only makes sense if the property $\mathcal{X}$ is preserved by pullback of the bundles: for then the fibre over a (generalized) point is just the pullback along the point. Note that this also allows $\mathcal{X}$, as property of bundles, to be viewed as a property of dependent types, since pullback corresponds to substitution.

A *set* X is a discrete space. In terms of locales, this means that the frame is isomorphic to the powerobject $\mathcal{P}X$ for some object X in some chosen elementary topos "of sets". As shown in [JT84, section V.5], X is discrete iff the unique map $X \to 1$ and the diagonal $X \to X \times X$ are both open maps.

A *function* is a map between sets. Note that the space of functions between two sets is not itself a set – the topology is not discrete.

A *truth value* is a subset of 1.[1]

2 $\mathbb{R}^{\mathfrak{L}}$ as pullback

We first describe how the space $\mathbb{R}^{\mathfrak{L}}$ of Dedekind reals equipped with locators, and the map $\mathrm{pr}_1 : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$, can be described as a pullback.

Let us write $Q_<$ for the strict numerical $<$ on the rationals $\mathbb{Q}$: the set of pairs (q, r) with $q < r$.

[1] This is incidental, but note that sub*spaces* of 1 are not the same as sub*sets*. The subsets ϕ are the open subspaces. When you bring in their closed complements $\neg\phi = \emptyset^{\phi}$, you get a Boolean algebra of subspaces even constructively. Excluded middle holds in the sense that the join (as subspace) of ϕ and $\neg\phi$ is 1, and $\neg\neg\phi = \phi$. See [MV12] for the description of $\neg\phi$ as a *Stone* space, not a discrete space. The Heyting logic arises only when you force the closed subspace $\neg\phi$ to be a subset by taking its set of points. From this point of view, much of intuitionistic logic is a study of the pathologies that arise when you neglect the topology on function spaces.

If x is a Dedekind real, then for each pair $q < r$ in $Q_<$ we have a pair of truth values $\phi_l = (q < x)$ and $\phi_r = (x < r)$ whose disjunction is $\top$. To provide a locator for x at $q < r$ is to choose an element of $(q < x) + (x < r)$

Definition 2.1. We write Λ for the space of pairs (ϕ_l, ϕ_r) of truth values, with $\phi_l \vee \phi_r = \top$.

We write $2\mathbb{S}$ for the space in which each point is a pair (ϕ_l, ϕ_r) together with a point of $\phi_l + \phi_r$.

The map $p\colon 2\mathbb{S} \to \Lambda$ forgets the point of $\phi_l + \phi_r$.

Remark 2.2.

1. The shape of "Λ" shows the layout of the three classical points $(\mathtt{t}, \mathtt{f}), (\mathtt{f}, \mathtt{t}), (\mathtt{t}, \mathtt{t})$.

2. We shall show later that $2\mathbb{S} \cong \mathbb{S} + \mathbb{S}$, where $\mathbb{S}$ is Sierpinski, the space of truth values.

3. Because each fibre of p was defined as a *set* (the set of elements of $\phi_l + \phi_r$), it is immediate that p is fibrewise discrete, in other words a local homeomorphism. Moreover, it is a surjection, because the unique function $\phi_l + \phi_r \to 1$ is epi (because its image $\phi_l \vee \phi_r$ is 1).

Hence for each real x and for each $q < r$ in $Q_<$, we have a point $(q < x, x < r)$ of Λ, thus giving a map $\mathbb{R} \times Q_< \to \Lambda$, and to provide a locator for x at $q < r$ is to choose an element of the fibre of p over $(q < x, x < r)$.

We now exploit the fact that $Q_<$, a set, is locally compact, and hence exponentiable. This gives us a map $\mathbb{R} \to \Lambda^{Q_<}$. Now to provide x with locator is simply to choose, for every $q < r$, an element of $(q < x) + (x < r)$. There is no interaction between the choices for different pairs $q < r$, so we just seek a map from $Q_<$ to $2\mathbb{S}$ lying over the map from $Q_<$ to Λ corresponding to x.

Definition 2.3. The space $\mathbb{R}^{\mathfrak{L}}$ of reals equipped with locators, and the projection pr_1, are given by the following pullback diagram.

$$\begin{tikzcd}
\mathbb{R}^{\mathfrak{L}} \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & (2\mathbb{S})^{Q_<} \arrow[d, "p^{Q_<}"] \\
\mathbb{R} \arrow[r, "x \mapsto (q<r) \mapsto (q<x, x<r)"'] & \Lambda^{Q_<}
\end{tikzcd}$$

To prove further properties of pr_1 we shall need more detail about $p^{Q_<}$ in order to prove some of its pullback-stable properties. In particular, we shall prove that it is an open surjection.

3 The map $p\colon 2\mathbb{S} \to \Lambda$

Proposition 3.1. *Λ is isomorphic to the ideal completion* $\mathrm{Idl}(\mathsf{K}\,\Lambda)$ *where* $\mathsf{K}\,\Lambda$ *is the following 3-element poset.*

$$\mathsf{K}\,\Lambda = \begin{tikzcd} & \top = (\mathtt{t},\mathtt{t}) \arrow[dl, dash] \arrow[dr, dash] & \\ \bot_l = (\mathtt{t},\mathtt{f}) & & \bot_r = (\mathtt{f},\mathtt{t}) \end{tikzcd}$$

Proof. Given an ideal I, take $\phi_l = \bot_l \in I$ and $\phi_r = \bot_r \in I$. Conversely, given ϕ_l and ϕ_r, take $I = \{\bot_l \mid \phi_l\} \cup \{\bot_r \mid \phi_r\} \cup \{\top \mid \phi_l \wedge \phi_r\}$. □

Thus Λ is a (point-free) algebraic dcpo, and $\mathsf{K}\,\Lambda$ is its compact base.

Better known is the result that $\mathbb{S}$ is the ideal completion of the 2-element poset $\mathtt{I}$ of classical truth values $\mathtt{f} \leq \mathtt{t}$.

In the following, we write the four elements of the poset $2\mathtt{I}$ as $\bot_l \leq \top_l$ and $\bot_r \leq \top_r$.

Proposition 3.2. *The following are isomorphic.*

1. $2\mathbb{S}$
2. *The space of quadruples of truthvalues $(\phi'_l, \phi_l, \phi'_r, \phi_r)$ such that ϕ'_l and ϕ'_r are Boolean complements, $\phi'_l \leq \phi_l$ and $\phi'_r \leq \phi_r$.*
3. $\mathrm{Idl}(2\mathtt{I})$
4. $\mathbb{S} + \mathbb{S}$

Proof. (1) ≅ (2): Given ϕ_l and ϕ_r, suppose $\alpha \in \phi_l + \phi_r$ and let α' be its image in 2. Then we define $\phi'_l = (\alpha' = 0)$ and $\phi'_r = (\alpha' = 1)$. The process is invertible.

(2) ≅ (3): Given an ideal I of $2\mathtt{I}$, the quadruple of truth values is

$$(\phi'_l = \bot_l \in I, \phi = \bot_l \in I \vee \top_r \in I, \phi'_r = \bot_r \in I, \phi_r = \bot_r \in I \vee \top_l \in I).$$

Conversely, given $(\phi'_l, \phi_l, \phi'_r, \phi_r)$, the ideal is

$$I = \{\bot_l \mid \phi'_l\} \cup \{\bot_r \mid \phi'_r\} \cup \{\top_l \mid \phi'_l \wedge \phi_r\} \cup \{\top_r \mid \phi'_r \wedge \phi_l\}.$$

The two transformations are mutually inverse.

(3) ≅ (4): The geometric account in [Vic99, section 4.2] generalizes to abitrary algebraic dcpos. □

There is an obvious monotone function from $2\mathtt{I}$ to $\mathsf{K}\,\Lambda$, mapping both $\top_l$ and $\top_r$ (in $2\mathbb{S}$) to $\top$ (in Λ) and p is its lift to the ideal completions. In the bundle picture we can see how the fibre over $\top$ (the case in Λ where both ϕ_l and ϕ_r are true) has two elements, for the two choices in the locator.

$$\begin{tikzcd}
 & & \top_l, \top_r \arrow[dl, dash] \arrow[d, mapsto] \arrow[dr, dash] & \\
2\mathbb{S} \arrow[d, "p"] & \bot_l \arrow[d, mapsto] & \top \arrow[dl, dash] \arrow[dr, dash] & \bot_r \arrow[d, mapsto] \\
\Lambda & \bot_l & & \bot_r
\end{tikzcd}$$

Using the representation in part (2) of the Proposition, p maps $(\phi'_l, \phi_l, \phi'_r, \phi_r)$ to (ϕ_l, ϕ_r).

4 Fibrewise overt is open

It is at least part of the folklore[2] that a map is open iff, as a bundle, it is fibrewise overt. We gather together a proof of this, using the lower hyperspace (powerlocale) P_L.

Proposition 4.1. *The following are equivalent for a space (ungeneralized) X.*

1. *X is overt.*
2. *The unique map $!\colon \mathsf{P}_L X \to 1$ has a right adjoint. (This says in a strong way that $\mathsf{P}_L X$ has a top point, which is the whole subspace X as a point of $\mathsf{P}_L X$.)*
3. *$\mathsf{P}_L X$ has a global point $\top$ such that $\eta \sqsubseteq !; \top$. (η here is the unit of the monad P_L, and $\sqsubseteq$ is the specialization order.)*

Proof. (1) $\Leftrightarrow$ (2) comes from [Vic12, theorem 39].

(2) $\Leftrightarrow$ (3): the adjoint is $\top\colon 1 \to \mathsf{P}_L X$ such that $\mathsf{Id}_{\mathsf{P}_L X} \sqsubseteq !; \top$. This is equivalent to (3) by [Vic95, proposition 4.4]. $\square$

This characterization justifies our use of the phrase "fibrewise overt", because it expresses overtness in terms of a construction, P_L, that is geometric [Vic04]. For bundles Y over X, we write $(\mathsf{P}_L/X)\, Y$ for the fibrewise hyperspace.

Openness of a map $f\colon Y \to X$ is defined in [JT84] using a suplattice homomorphism $\exists_f \colon \Omega Y \to \Omega X$, left adjoint to the inverse image function Ωf and with a Frobenius condition. This then corresponds to a map from X to $\mathsf{P}_L Y$.

Proposition 4.2. P_L *preserves coreflexive equalizers.*

Proof. (For the time being, we give a proof using frames.)

[2] Published results? Paul Taylor?

Suppose we have a coreflexive equalizer as follows, with $fh = gh = \mathsf{Id}_Y$:

$$\begin{tikzcd} X \arrow[r, "e"] & Y \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & Z \arrow[l, "h"'] \end{tikzcd} . \tag{1}$$

Then

$$\Omega X \cong \mathbf{Fr}\langle \Omega Y \text{ (qua Fr) } \mid f^* W = g^* W \quad (W \in \Omega Z)\rangle. \tag{2}$$

In order to calculate $\Omega\,\mathsf{P}_L\,X$, we must convert this frame presentation into a suplattice presentation, which we do using the coverage theorem – see [Vic04] for more details. Now

$$\begin{aligned}\Omega X \cong \mathbf{Fr}\langle \Omega Y \text{ (qua } \wedge\text{-semilattice)} \mid &\text{ joins preserved,} \\ & f^* W = g^* W \quad (W \in \Omega Z)\rangle \\ \cong \mathbf{SupLat}\langle \Omega Y \text{ (qua poset)} \mid &\text{ joins preserved,} \\ & f^* W = g^* W \quad (W \in \Omega Z)\rangle,\end{aligned}$$

where the final isomorphism follows from the coverage theorem *provided* that the relations are meet stable. For the preservations of joins this follows from frame distributivity in ΩY, while for the other relations it follows from coreflexivity using $f^* W \wedge V = f^*(W \wedge h^* V)$ and so on. Hence,

$$\begin{aligned}\Omega\,\mathsf{P}_L\,X &\cong \mathbf{Fr}\langle \Omega Y \text{ (qua suplattice)} \mid f^* W = g^* W \quad (W \in \Omega Z)\rangle \\ &\cong \mathbf{Fr}\langle \Omega\,\mathsf{P}_L\,Y \text{ (qua frame)} \mid (\mathsf{P}_L\,f)^* W = (\mathsf{P}_L\,g)^* W \quad (W \in \Omega\,\mathsf{P}_L\,Z)\rangle.\end{aligned}$$

Hence the equalizer is preserved by P_L. □

Now suppose $f\colon Y \to X$ is a map, and let us work in the context of $x : X$. Topos-theoretically, this means we are working internally in sheaves over X, with x the generic point. As long as our work is geometric, it can be transported to arbitrary points x.

We have a coreflexive equalizer

$$\begin{tikzcd} Y_x \arrow[r, "i_x"] & Y \arrow[r, shift left=2, "y \mapsto \langle x, y\rangle"] \arrow[r, shift right=2, "\langle p, \mathsf{Id}_y\rangle"'] & X \times Y \arrow[l, "\pi_2"'] \end{tikzcd} ,$$

where $i_x\colon Y_x \to Y$ is the inclusion, and hence by Proposition 4.2 a coreflexive equalizer

$$\begin{tikzcd} \mathsf{P}_L\,Y_x \arrow[r, "\mathsf{P}_L\,i_x"] & \mathsf{P}_L\,Y \arrow[r, shift left, "\mathsf{P}_L(y \mapsto \langle x, y\rangle)"] \arrow[r, shift right, "\mathsf{P}_L\langle p, \mathsf{Id}_y\rangle"'] & \mathsf{P}_L(X \times Y) \end{tikzcd} . \tag{3}$$

We now apply $\Sigma_{x:X}$ to bundle these fibres together, converting an internal space in the sheaf category $\mathcal{S}\,X$ into an external bundle over X, and getting an equalizer

$$\begin{tikzcd} (\mathsf{P}_L/X)\,Y \arrow[r] & X \times \mathsf{P}_L\,Y \arrow[r, shift left, "\langle \pi_1, \tau\rangle"] \arrow[r, shift right, "X \times \mathsf{P}_L\langle f, \mathsf{Id}_Y\rangle"'] & X \times \mathsf{P}_L(X \times Y) \end{tikzcd} .$$

where τ is the strength. We can simplify this as an equalizer

$$(\mathsf{P}_L/X)\,Y \longrightarrow X \times \mathsf{P}_L\,Y \underset{\pi_2\,\mathsf{P}_L\langle f, \mathsf{Id}_Y\rangle}{\overset{\tau}{\rightrightarrows}} \mathsf{P}_L(X \times Y)\ . \tag{4}$$

Theorem 4.3. *Let $p\colon Y \to X$ be a map. Then p is open iff it is fibrewise overt. Furthermore, it is an open surjection iff it is fibrewise positive overt.*

Proof. We first analyse fibrewise overtness. Fixing $x : X$, by Proposition 4.1 we seek a map $\top_x\colon 1 \to \mathsf{P}_L\,Y_x$, which, using the equalizer (3), is equivalent to a map $\top'_x\colon 1 \to \mathsf{P}_L\,Y$ such that

$$\top'_x; \mathsf{P}_L(y \mapsto \langle x, y\rangle) = \top'_x; \mathsf{P}_L(y \mapsto \langle py, y\rangle)$$

In addition we must have $\eta_x \sqsubseteq\ !; \top_x\colon Y_x \to \mathsf{P}_L\,Y_x$. This is equivalent to $\eta_x; \mathsf{P}_L\,i_x \sqsubseteq\ !; \top_x; \mathsf{P}_L\,i_x$, ie

$$i_x; \eta \sqsubseteq i_x; !; \top'_x.$$

Bundling the fibres together (apply $\Sigma_{x:X}$), we find that p is fibrewise overt iff we have a map $\top'\colon X \to \mathsf{P}_L\,Y$ such that the following diagram commutes, and the inequality holds.

$$\begin{tikzcd} X \arrow[r, "\top'"] \arrow[d, "\langle X, \top'\rangle"'] & \mathsf{P}_L\,Y \arrow[d, "\mathsf{P}_L\langle p, Y\rangle"] \\ X \times \mathsf{P}_L\,Y \arrow[r, "\tau"'] & \mathsf{P}_L(X \times Y) \end{tikzcd}\ , \tag{5}$$

$$\eta_Y \sqsubseteq p; \top'. \tag{6}$$

We now turn to openness. By [Vic95, theorem 4.7], p is open iff we have a map $p^{-1}\colon\ \mathsf{P}_L\,X \to \mathsf{P}_L\,Y$, right adjoint to $\mathsf{P}_L\,p$, such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\langle \eta, \eta; p^{-1}\rangle"'] & \mathsf{P}_L\,X \arrow[r, "p^{-1}"] & \mathsf{P}_L\,Y \arrow[d, "\mathsf{P}_L\langle p, Y\rangle"] \\ \mathsf{P}_L\,X \times \mathsf{P}_L\,Y \arrow[rr, "\times"'] & & \mathsf{P}_L(X \times Y) \end{tikzcd} \tag{7}$$

Because P_L is a KZ-monad, with multiplication μ right adjoint to $\mathsf{P}_L\,\eta$ (as well as left adjoint to $\eta\,\mathsf{P}_L$), we find that if p^{-1} is right adjoint to $\mathsf{P}_L\,p$ then $\mu; p^{-1}$ and $\mathsf{P}_L\,p^{-1}; \mu$ are both right adjoint to $\mathsf{P}_L(p; \eta) = \mathsf{P}_L(\eta; \mathsf{P}_L\,p)$ and hence equal. Thus p^{-1} is a P_L-algebra homomorphism and hence determined by a map $\top' = \eta; p^{-1}$ with $p^{-1} = \mathsf{P}_L\,\top'; \mu$. Factoring $\langle \eta, \eta; p^{-1}\rangle = \langle X, \eta; p^{-1}\rangle; (\eta \times \mathsf{P}_L\,Y)$, and noting that $(\eta \times \mathsf{P}_L\,Y); \times = \tau$, diagram (7) can be rewritten as diagram (5).

For any $\top'\colon X \to \mathsf{P}_L\,Y$, if we write p^{-1} for $\mathsf{P}_L\,\top'; \mu$ then (5) commutes iff (7) commutes. Also, we have (6) iff $\mathsf{Id} \sqsubseteq \mathsf{P}_L\,p; p^{-1}$, which is one half of the adjunction $\mathsf{P}_L\,p \dashv p^{-1}$.

We complete this part of the proof by showing that if (5) commutes then $p^{-1}; \mathsf{P}_L\, p \sqsubseteq \mathsf{Id}$. By composing (5) with $\mathsf{P}_L\, \pi_1$ at bottom right, we get the following diagram, which all commutes except for the indicated 2-cell.

$$\begin{tikzcd}
X \arrow[rr, "\top'"] \arrow[d, "\langle X, \top' \rangle"] \arrow[ddd, bend right=60, equal] & & \mathsf{P}_L\, Y \arrow[d, "\mathsf{P}_L \langle p, Y \rangle"] \arrow[ddd, bend left=60, "\mathsf{P}_L\, p"] \\
X \times \mathsf{P}_L\, Y \arrow[rr, "\tau"] \arrow[d, "X \times \mathsf{P}_L !"] & & \mathsf{P}_L(X \times Y) \arrow[d, "\mathsf{P}_L(X \times !)"] \\
X \times \mathsf{P}_L\, 1 \arrow[rr, "\tau"] \arrow[d, "\pi_1"] & & \mathsf{P}_L(X \times 1) \arrow[d, "\cong"] \\
X \arrow[rr, "\eta"'] \arrow[urr, phantom, "\sqsupseteq"] & & \mathsf{P}_L\, X
\end{tikzcd} \tag{8}$$

We deduce

$$\mathsf{P}_L \top'; \overset{2}{\mathsf{P}}_L\, p \sqsubseteq \mathsf{P}_L\, \eta_X \sqsubseteq \eta_{\mathsf{P}_L\, X}$$

and so

$$p^{-1}; \mathsf{P}_L\, p = \mathsf{P}_L \top'; \mu; \mathsf{P}_L\, p = \mathsf{P}_L \top'; \overset{2}{\mathsf{P}}_L\, p; \mu \sqsubseteq \mathsf{Id}.$$

This completes the proof that p is open iff it is fibrewise overt. It remains to show that surjectivity corresponds to fibrewise positivity.

$\mathsf{P}_L\, X$ has an open subspace $\mathsf{P}_L^+\, X$ whose points are the *positive* overt, weakly closed subspaces of X: they are the ones satisfying the open $\Diamond\top$, given by the map $\mathsf{P}_L!\colon \mathsf{P}_L\, X \to \mathsf{P}_L\, 1 \cong \mathbb{S}$. Hence an overt space is positive if its top point $\top$ of $\mathsf{P}_L\, X$ is in $\mathsf{P}_L^+\, X$, hence maps to $\top$ in $\mathbb{S}$. For a fibrewise overt space $p\colon Y \to X$, this comes down to this commutative square.

$$\begin{tikzcd}
X \arrow[r, "\top'"] \arrow[d, "!"'] & \mathsf{P}_L\, Y \arrow[d, "\mathsf{P}_L !"] \\
1 \arrow[r, "\top"'] & \mathbb{S}
\end{tikzcd} \tag{9}$$

On the other hand, an open map p is surjective iff its inverse image function Ωp is one-one; and because it has a left adjoint $\exists_p$ this happens iff $\Omega p; \exists_p = \mathsf{Id}$. Under the duality between P_L-Kleisli maps and suplattice homomorphisms of frames, this corresponds to the following commutative triangle.

$$\begin{tikzcd}
X \arrow[r, "\top'"] \arrow[dr, "\eta"'] & \mathsf{P}_L\, Y \arrow[d, "\mathsf{P}_L\, p"] \\
& \mathsf{P}_L\, X
\end{tikzcd} \tag{10}$$

Given (10), we can postcompose it with $\mathsf{P}_L!\colon \mathsf{P}_L\, X \to \mathbb{S}$, and use $\eta; \mathsf{P}_L! = !; \eta_1 = !; \top$ to get (9). Now suppose (9). Combining it with diagram (8), and using isomorphisms $\mathsf{P}_L\, 1 \cong \mathbb{S}$ and $X \times 1 \cong X$, we get a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[rr, "\top'"] \arrow[d, "\langle X, \top' \rangle"] \arrow[dd, bend right=60, "\langle X, !; \top \rangle"'] & & \mathsf{P}_L\, Y \arrow[d, "\mathsf{P}_L\, \langle p, Y \rangle"'] \arrow[dd, bend left=60, "\mathsf{P}_L\, p"] \\
X \times \mathsf{P}_L\, Y \arrow[rr, "\tau"] \arrow[d, "X \times \mathsf{P}_L !"] & & \mathsf{P}_L(X \times Y) \arrow[d, "\mathsf{P}_L(X \times !)"'] \\
X \times \mathbb{S} \arrow[rr, "\tau"] & & \mathsf{P}_L\, X
\end{tikzcd}$$

From $\tau(x, \top) = \eta(x)$ we get (10). □

5 $p^{Q_<}$ is an open surjection

We now apply Theorem 4.3 to the map $p^{Q_<}$, where p is as in Section 3.

Proposition 5.1. *Let X be a set with decidable equality. Then $p^X : (2\mathbb{S})^X \to \Lambda^X$ is an open surjection.*[3]

Proof. A map is open iff it is fibrewise overt: that is to say, the fibre over the generic point is overt. A point of Λ^X is a pair of subsets X_l and X_r of X such that $X_l \cup X_r = X$.

Let (X_l, X_r) be a point of Λ^X.[4] Then a point of the fibre is a complementary pair (X'_l, X'_r) of subsets of X such that $X'_l \subseteq X_l$ and $X'_r \subseteq X_r$. A subbase for the topology (hence a signature for the propositional geometric theory) is provided by $X+X$, with x_l (repectively x_r) comprising those points (X'_l, X'_r) with $x_l \in X'_l$ (respectively $x_r \in X'_r$). We then need relations as follows:

$$\begin{aligned}
\top &\vdash x_l \vee x_r \\
x_l \wedge x_r &\vdash \bot \\
x_l &\vdash \bigvee \{\top \mid x \in X_l\} \\
x_r &\vdash \bigvee \{\top \mid x \in X_r\}
\end{aligned}$$

However, for proving overtness it is more convenient to have a base of the topology, by taking finite meets of the subbasic opens. Writing (S_l, S_r) for $\bigwedge_{x \in S_l} x_l \wedge \bigwedge_{x \in S_r} x_r$ we define

$$B = \{(S_l, S_r) \in \mathcal{F}X_l \times \mathcal{F}X_r \mid (\forall x \in S_l)(\forall y \in S_r) x \neq y\}.^{5}$$

[3] I suspect this can be proved for more general p.

[4] That declaration in effect changes our context to the category of sheaves over Λ^X, with (X_l, X_r) the generic point. From now on it is fixed.

[5] It is necessary for B, the signature of the geometric theory, to be a *set*. First, $\mathcal{F}$, the Kuratowski finite powerset (free semilattice), is a geometric construction of sets. Second, the formula $(\forall x \in S_l)(\forall y \in S_r) x \neq y$ is a geometric formula. This is because the $\forall$s are finitely bounded, and X has decidable equality. See [Vic99] for more details.

We order B by the componentwise superset order:

$$(S_l, S_r) \leq (T_l, T_r) \text{ if } S_l \supseteq T_l \text{ and } S_r \supseteq T_r.$$

Then B can be made a flat site (presentation for an inductively generated formal topology) for our space by adding cover relations for the axioms $\top \vdash x_l \vee x_r$. These will say that (S_l, S_r) in B is covered by pairs in B that add x either left or right, or

$$(S_l, S_r) \lhd (\{(S_l \cup \{x\}, S_r)\} \cap B) \cup (\{(S_l, S_r \cup \{x\})\} \cap B).$$

To prove overtness we must provide a positivity predicate, which in this case is that all the basics are positive, and prove that it splits covers: in other words, the covering set above is always inhabited. Suppose we are given (S_l, S_r) and x. If $x \in S_l$ or $x \in S_r$, then (S_l, S_r) is already in the covering set. Suppose not (ie $(\forall y \in S_l \cup S_r) y \neq x$). Then we have $x \in X_l \vee x \in X_r$, and we get that one (or both) of $(S_l \cup \{x\}, S_r)$, $(S_l, S_r \cup \{x\})$ is in B. This completes the proof that p is open.

It is a surjection because all basics are positive, and that includes $(\emptyset, \emptyset)$, which is the whole space. □

Corollary 5.2. $\mathrm{pr}_1 \colon \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$ *is an open surjection, and hence the coequalizer of its kernel pair.*

Proof. [JT84, section V.4] tells us that open surjections are coequalizers of their kernel pairs and preserved under pullback.[6]

The proof in [JT84] is valid in elementary toposes using frames, but is not geometric. I conjecture that the results can be proved in a more purely geometric way from the "fibrewise overt" account, using the geometricity of presentations and the lower powerlocale [Vic04]. □

Thus, in defining maps out of $\mathbb{R}$, it is valid to define the map for reals equipped with locators, and prove that it acts equally for different locators on the same real – as if $\mathbb{R}$ were defined as a quotient of $\mathbb{R}^{\mathfrak{L}}$. This looks as if it is assuming that all Dedekind reals have locators, but it is actually more subtle than that. It is a conservativity principle that says some reasoning about reals with locators descends to give results about reals.

This also tells us something about locator lifts. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ lifts to $\overline{f} \colon \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}}$. Then we have conditions under which f can be recovered from $\overline{f}$.

6 Conclusions

The fibrewise reasoning used here is essentially that of dependent type theory, with geometricity (constructions preserved under pullback) corresponding to

[6] [JT84] also gives there the example of the space of epimorphisms from $\mathbb{N}$ to an inhabited set X. That function space maps surjectively to 1 even though there might not be any actual epimorphism from $\mathbb{N}$ to X. This, and the use made of it in the rest of the paper, shows something of the power of point-free surjections.

the ability to substitute specific points (terms) for generic ones (variables). It seems likely that much of the type-theoretic reasoning in UTT of [Boo20] could be transferred to our point-free setting, albeit with care needed for the lack of arbitrary exponentials.

Where UTT and point-free topology diverge is in topological issues, and in particular the peculiar advantages of point-free surjections. It seems that UTT does not have any straightforward ability to deal with notions such as overtness of types and openness of maps. Of course, it shold be possible in UTT to recreate point-free structures such as frames or formal topologies, but then the benefits of pointwise reasoning for point-free spaces would have to be painfully reconstructed.

The lesson to be drawn is that there is a trade-off between, on the one hand, arbitrary function spaces, and, on the other, using geometric reasoning to get a dependent type theory of point-free spaces.

Our main result, Corollary 5.2, is not an isolated result.

In the compact case, [Vic17] makes essential use of a proper surjection from Cantor space 2^ω to $[-1,1]$ that takes an infinite sequence $(s_i)_{i=1}^\infty$ of signs $\pm$ to $\sum_{i=1}^\infty s_i/2^i$.

For completions of generalized metric spaces [Vic05], an earlier Departmental Report [Vic98, proposition 7.2] showed that a limit map $\lim\colon \mathrm{Cauchy}_f X \to \overline{X}$ is a triquotient map in the sense of Plewe [Ple97]. Given a generalized metric space X, Cauchy_f is a space of forward Cauchy sequences with a canonical rate of convergence, and $\overline{X}$ is the localic completion. In the case where X is the rationals $\mathbb{Q}$ with the usual metric, $\overline{X}$ is $\mathbb{R}$.

Triquotients include both open surjections an dproper surjections, and Plewe proved that they coequalize their kernel pairs. Thus in all these examples we get the possibility of defining maps from the reals in terms of some extra structure (which need not always exist) and then showing that the result is independent of the extra structure.

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