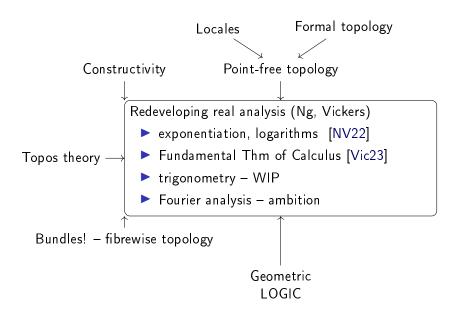
Real analysis via logic

Steve Vickers

School of Computer Science University of Birmingham

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- Topology and continuity as logical phenomena (geometric logic).
- Now applying logical approach to real analysis.
- Gives fresh perspective on the analysis.



Topology, continuity as logical phenomena

Usual approach: maths is discrete

Topological space (point-set) = set + extra structure

Continuous map = function respecting that structure

Geometric logic: "all" maths is continuous

Discrete maths of sets very restricted – eg $\mathbb Q$ is a set, real line $\mathbb R$ isn't. Can still access $\mathbb R$, by taking logical, *point-free* approach –

Topological space = logical theory (point = model of theory)

Continuity of map = definition respects logical constraints

Example: real line $\mathbb R$ as logical theory

Signature

For each rational $q \in \mathbb{Q}$: two propositional symbols $[\cdot < q]$, $[q < \cdot]$.

Axioms - eg

Model x is real number as Dedekind cut:

Specify truth values [x < q] and [q < x] for every q, ie which rationals are bigger than x, which are smaller.

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Defining maps

Think of maps $f: \mathbb{R} \to \mathbb{R}$ in a style of programming languages.

Declare formal parameter x.

Do some auxiliary calulations.

Define result f(x) as model:

- specify truth values [f(x) < q], [q < f(x)],
- prove that axioms hold.

eg absolute value $|\cdot| \colon \mathbb{R} \to \mathbb{R}$

Let
$$x:\mathbb{R}$$

$$[|x| < q] := [x < q] \land [-q < x]$$

$$[q < |x|] := [q < x] \lor [x < -q]$$
 ... and prove axioms

Inside the box, in scope of x, is a different mathematics!

1. Lots of non-standard truth values [x < q], [q < x] (for each rational q)

Each expresses where (ie for which models x) something is true.

2. Continuity = different logic

Continuity: inverse image of open is open

$$f^{-1}([x < q]) = [f(x) < q]$$
 is made from truth values of form $[x < r]$ and $[s < x]$ using \land and \bigvee . Similarly for $[q < f(x)]$.

We want continuity, therefore restrict mathematics inside box to limit how we construct f(x).

Geometricity

Pure logic

Restricted formulae: \bigvee , \wedge , =, \exists .

Axioms as sequents:

formula ⊢_{context} formula

Context = finite stock of free variables, with implicit \forall .

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Corresponding maths

Restricted maths of sets: Disjoint unions, quotients, finite products, equalizers, free algebras.

Function spaces Y^X , powersets $\mathcal{P}X$, the real line \mathbb{R} are not sets! They must be dealt with as spaces.

Infinite \bigvee can often be avoided by using \exists with an infinite set. eg

$$[\cdot < q] \vdash_{q:\mathbb{Q}} (\exists q' : \mathbb{Q}) (q' < q \land [\cdot < q']) ext{ for } [\cdot < q] \vdash \bigvee_{q' < q} [\cdot < q'].$$

Technicalities

- ► The "maths inside the box" is the geometric fragment of the internal mathematics of the classifying topos S[X].
 "Map" = geometric morphism.
- ightharpoonup "Classifying topos" is slippery constructively depends on choice of a base topos \mathcal{S} . To avoid that dependency, work without infinite disjunctions. [Vic17]

[Vic99] shows the technique in action in domain theory.

[Vic07] explains how standard topos results (eg [MLM92]) arrive at this point of view.

[Vic22] gives a more up-to-date discussion.

Logical manipulations I: Decomposing theories

Theory of Dedekind reals =

theory of *lower reals* to account for $[q<\cdot]$

- + theory of *upper reals* to account for $[\cdot < q]$
- + two axioms to relate them

Good strategy for point-free analysis (eg exp, log, integration)

- 1. Deal with lower and upper cases separately,
- 2. then combine them.

Can provide fresh insights – eg Ostrowski's Theorem in number theory (Ng [Ng22, NV]).

Logical manipulations II: Building up theories

```
Let x:\mathbb{R}
:
:
:
Space \mathbb{T}(x) := \cdots
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Declare formal parameter x.

Do some auxiliary calculations.

Define space $\mathbb{T}(x)$ as theory:

- define signature (as set)
- define axiom set (with appropriate structure).

Geometricity suggests $\mathbb{T}(x)$ depends continuously on x.

Logically – $(\mathbb{T}(x))_{x:\mathbb{R}}$ defines extension of theory of reals.

Models = pairs (x, y), $x:\mathbb{R}$, y model of $\mathbb{T}(x)$.

Dependent type theory: write $\sum_{x \in \mathbb{R}} \mathbb{T}(x)$.

Forgetful map $\sum_{x:\mathbb{R}} \mathbb{T}(x) \to \mathbb{R}$, $(x,y) \mapsto x$, makes a bundle over \mathbb{R} .

None of this works satisfactorily in point-set topology

Can't describe bundle as continuously indexed family of spaces.

Logical approach works better!

Logical manipulations III: Modal logic → hyperspaces

$Hyperspace^1 = space of subspaces$

eg Use \square modality to construct new theory, models are to be subspaces W.

 $\Box \phi$ assigned value true for those W in which every point has ϕ true in old theory.

Clearly $\Box(\phi \wedge \psi) \equiv \Box \phi \wedge \Box \psi$.

For \mathbb{R} , signature has elements of the form $\Box \bot$, $\Box [q < \cdot]$, $\Box [\cdot < r]$ and $\Box ([q < \cdot] \lor [\cdot < r])$.

Under suitable axioms, model = compact subspace of \mathbb{R} .

Application eg: Heine-Borel Theorem, closed intervals [x, y] are compact

Using hyperspace, can demonstrate that [x, y] depends continuously on x and y.

Other hyperspaces available; works for all spaces

¹Point-free hyperspaces aka *powerlocales*. For geometricity see [Vic04].

Conclusion

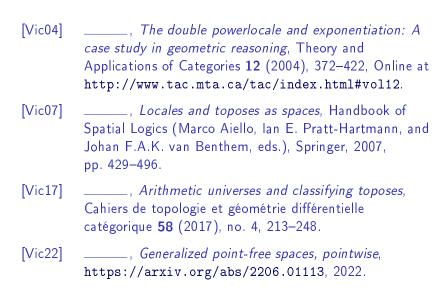
Logic makes topology work ... better than topology does!

See [NV22] "Point-free construction of real exponentiation" for introduction to putting this into practice.

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Bibliography III

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