Cartesian bicategories and their Karoubi envelopes

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Abstract

If $\mathcal{B}$ is a cocartesian bicategory, then so is its Karoubi envelope $\mathsf{Kar}(\mathcal{B})$. If moreover $\mathcal{B}$ is compact closed, with the adjoints of maps given by their duals (which is the case for a bicategory of relations), then the same holds for $\mathsf{Kar}(\mathcal{B})$.

We define additional structure and properties for a cocartesian category $\mathcal{B}$ to be a "bicategory of entailments". Such a $\mathcal{B}$ is compact closed, with adjoints of maps given by their duals.

The entailment category Ent of Vickers is a bicategory of entailments.

1 Introduction

Amongst the cartesian bicategories of Carboni and Walters [CW87], particularly well behaved ones are the *bicategories of relations,* the paradigm example being Rel. For them, Frobenius identities lead to their being dagger closed. It is a relatively simple observation (our Section 4) that the property of being a compact closed, cartesian bicategory, which brings the ability to use methods of string diagrams, is inherited by the Karoubi envelope. In the case of Rel, we get [Vic93] the category of continuous dcpos and certain "non-deterministic maps", the Kleisli morphisms for the lower powerdomain monad.

Motivated by [Vic93], and drawing on the "multilingual sequent calculus" of [JKM99], the paper [Vic04] described the category of stably compact spaces and certain non-deterministic maps between them (this time the Kleisli morphisms for the upper hyperspace or powerlocale) as the Karoubi envelope of a 2-category Ent. Manipulation of Ent is complicated, but [Vic04] noted that there seemed to be structural features that could be captured with string diagrams in a simple way. The second strand of the present paper is to show how these again come out of the structure of a compact closed, cartesian bicategory.

Ent is not a bicategory of relations. Nonetheless, it has "outer" Frobenius equations that give it dagger-closed structure. In Definition 25 we describe the abstract structure of *bicategory of entailments* to capture this, and in Theorems 12 and 33 we show that Ent has that structure. It follows that Ent and its Karoubi envelope are both compact closed.

The working of the paper is constructive in a broad range of senses and applies to spaces understood in a point-free way: in other words, the points are described not "point-set", as elements of a set, but as models of a geometric theory by one means or another. However, in classical mathematics stably compact point-free spaces are spatial, and so the results are valid classically for stably compact point-set spaces.

2 Background: Cocartesian bicategories

We summarize some of [CW87], and use string diagrams. For the sake of our new example Ent of entailment systems, we have chosen to work with *co*cartesian bicategories. This means that our 2-cells are in the opposite direction to those of [CW87]. We believe this gives the most help to one's intuitions.

Definition 1. Let $\mathcal{B}$ be a poset-enriched symmetric monoidal category, with tensor $\otimes$ and unit $\mathbb{I}$. Then we say $\mathcal{B}$ is a *cocartesian bicategory* if it is equipped with the following structure.

(Δ) Each object X is a commutative comonoid by morphisms $\Delta_X \colon X \to X \otimes X$ and $t_X \colon X \to \mathbb{I}$.

(U) Every morphism $r \colon X \to Y$ is a lax comonoid homomorphism, lax in the direction that

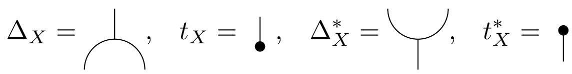
$$\begin{tikzcd} X \arrow[rr, "r"] \arrow[dr, "t_X"'] & \leq & Y \arrow[dl, "t_Y"] \\ & \mathbb{I} & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "r"] \arrow[d, "\Delta_X"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "\Delta_Y"] \\ X \otimes X \arrow[r, "r \otimes r"'] & Y \otimes Y \end{tikzcd} .$$

(M) Δ_X and t_X have left adjoints Δ_X^* and t_X^*.

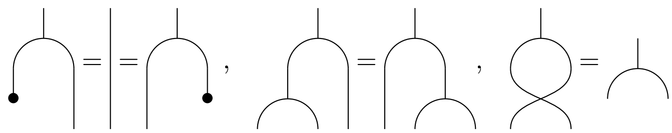
A morphism in a cocartesian bicategory $\mathcal{B}$ is a *map* if it has a left adjoint, and we write $\mathsf{Map}(\mathcal{B})$ for the category of objects and maps. Hence the definition says that each X is a commutative comonoid in $\mathsf{Map}(\mathcal{B})$. [CW87] show that in $\mathsf{Map}(\mathcal{B})$ the tensor product is cartesian, with the comonoid structure given by diagonals, and that maps are comonoid homomorphisms. Moreover, the comonoid structures are uniquely determined by the conditions.

Note that maps are not in general homomorphisms for the dual monoid structure given by Δ_X^* and t_X^*: to get the oplax homomorphism property for r, we need r to be a comap – ie to have a *right* adjoint.

We shall use string diagrams in the usual way, writing 1-cells vertically with domain at the top.

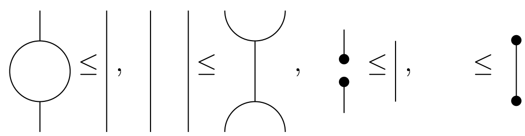


Thus counits, coassociativity and cocommutativity can be drawn as –



(1)

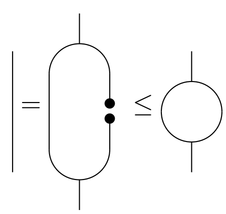
while the adjunctions are



(2)

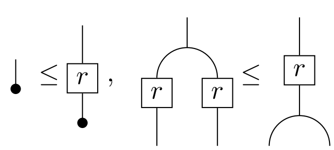
In the final condition, the blank on the left hand side is the identity on $\mathbb{I}$. This is in accordance with the usual convention that $\mathbb{I}$ appears as nothing, because tensoring with it has no effect, but it does point to a potential problem with our diagrammatic representation of $t_{\mathbb{I}}$ and $t_{\mathbb{I}}^*$. Fortunately [CW87] they are both equal to the identity on $\mathbb{I}$.

Remark 2. The first of the inequalities is actually an equality, for we have –



We do not have a corresponding equality involving t_X and t_X^*, as can be seen by considering the example of Rel (with reversed order).

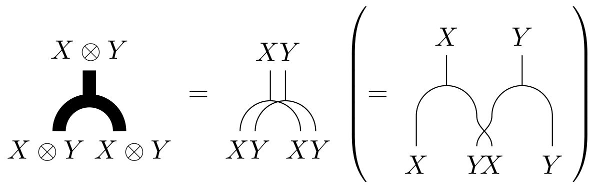
The lax homomorphism conditions are –



(3)

From uniqueness of the comonoid structure, it follows [CW87, Remark 1.3(ii)] that the comonoid structure on a tensor is got in the obvious way from other comonoid structures. For $\mathbb{I}$, $t_{\mathbb{I}}$ and $\Delta_{\mathbb{I}}$ are both (modulo the isomorphism $\mathbb{I} \otimes \mathbb{I} \cong \mathbb{I}$) the identity on $\mathbb{I}$.

For tensors $X \otimes Y$ it is expressed as string diagrams as follows.

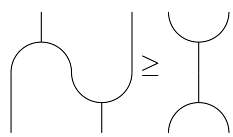


(4)

Here the thin versions of and are for objects X and Y, while the thick versions are for $X \otimes Y$. As indicated, the diagram with overlapping thin s is shorthand for a composition with a swap (the symmetry isomorphism). The corresponding equations will also hold for the adjoints and .

2.1 Bicategories of relations

An object X is *discrete* if it satisfies the *inner Frobenius equation*



(5)

(This is an equation because the other direction follows from coassociativity by using the adjunctions.)

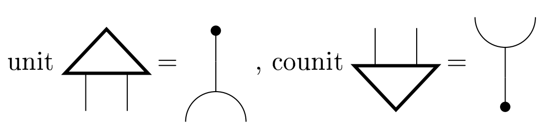
The phrase "inner Frobenius" is ours. This is because in Section 5.2 we shall use the same diagram for different operations, when we shall call it *outer* Frobenius.

The left-hand side has a mirror image, and the two are not equal in general. (A counterexample is in Section 3.2.) However, by commutativity each is got from the other by swapping arguments both top and bottom, and it follows that the Frobenius condition is equivalent to its mirror.

In the original paradigm example of cartesian bicategory, namely Rel, all objects are discrete, and such a cartesian bicategory is called a *bicategory of relations.*

Proposition 3. *Let $\mathcal{B}$ be a cocartesian bicategory. Then any discrete object X is its own dual X°, in the sense of compact closed categories.*

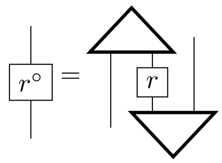
Proof. We define the unit and counit as



The yanking laws clearly follow by combining Frobenius with the (co)monoid laws. ☐

Corollary 4. *Any bicategory of relations is dagger closed, ie compact closed with each object self-dual.*

Recall that compact closed structure allows us to extend the duality on objects to one on morphisms. If $r\colon X \to Y$, then $r^\circ\colon Y^\circ \to X^\circ$ is defined by

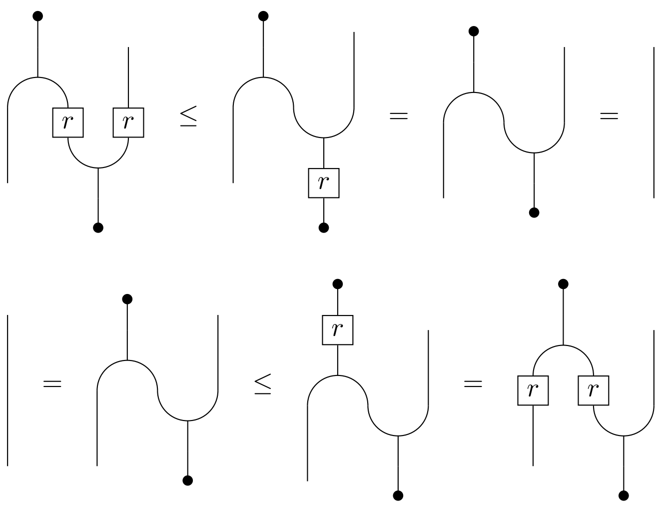


(6)

In the case of a bicategory of relations, dagger closed, the duality is an involution.

Proposition 5. *Let $\mathcal{B}$ be a cocartesian bicategory with inner Frobenius (ie $\mathcal{B}^{co}$ be a bicategory of relations). Then if r is a map, r° is its left adjoint.*

Proof.



$\square$

Definition 6. A *compact closed cocartesian bicategory* is a cocartesian bicategory equipped with compact closed structure such that $\curlyvee = \curlywedge^\circ$ and $\upharpoonleft = \downharpoonleft^\circ$.

It is *dagger closed* if the duality is the identity on objects.

Hence Proposition 5 shows that any bicategory of relations is the dual of a dagger closed cocartesian bicategory.

3 Entailment systems

3.1 Background

Motivated by Gentzen's sequent calculus, the "multilingual sequent calculus" of [JKM99] developed a notion of sequents "between logics" and applied it to a point-free treatment of stably compact spaces. In [Vic04] this was developed to obtain a category of stably compact spaces as Karoubi envelope of a category Ent with three equivalent manifestations: (1) a category of non-deterministic maps between powersets with their Scott topology; (2) (locale theoretically) the dual of the category of free frames and preframe homomorphisms; and (3) sets with certain relations between the their finite powersets (analogous to sequent calculi). (3) has a very simple description of the objects and morphisms, and of the duality that leads to de Groot duality on stably compact spaces. On the other hand, its "cut" composition is complicated and very difficult to work with – in effect it embodies a generalized distributive law. In fact, idempotents correspond to closure under the cut rule in the sequent calculus.

We recall the 2-category Ent of [Vic04], and establish some termoinology and notation in preparation for the definition.

If X is a set, then $\mathcal{F}X$ is the free semilattice over it, which serves as its Kuratowski finite powerset. We write $\between$ for the *overlap* relation on $\mathcal{F}X$, $U \between V$ if U and V have non-empty intersection. If $\mathcal{U} \in \mathcal{F}\mathcal{F}X$, then a *choice* of $\mathcal{U}$ is a finite total relation γ from $\mathcal{U}$ to $\bigcup \mathcal{U}$ such that if $U \; \gamma \; u$ then $u \in U$. (γ chooses at least one element of each U.) We write $\mathsf{Im}\gamma$ for the image of γ, and $\mathsf{Ch}\mathcal{U}$ for the set (which is Kuratowski finite) of choices of $\mathcal{U}$. If $\mathcal{U}, \mathcal{V} \in \mathcal{F}\mathcal{F}X$, then we say $\mathcal{U}$ is *diagonal* to $\mathcal{V}$, $\mathcal{U} \bowtie \mathcal{V}$, if for every $\gamma \in \mathsf{Ch}\mathcal{U}$ and $\delta \in \mathsf{Ch}\mathcal{V}$ we have $\mathsf{Im}\gamma \between \mathsf{Im}\delta$.

Definition 7. The 2-category Ent [Vic04] is defined as follows.

An object is a set.

A morphism $r\colon X \to Y$ is an up-closed relation between $\mathcal{F}X$ and $\mathcal{F}Y$. We extend this to a relation $\overline{r}$ between $\mathcal{F}\mathcal{F}X$ and $\mathcal{F}\mathcal{F}Y$, $\mathcal{U} \; \overline{r} \; \mathcal{V}$ if, for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$, we have $U \; r \; V$.

A 2-cell from R to S exists if $R \subseteq S$.

The identity morphism on X is the overlap relation.

If $r\colon X \to Y$ and $s\colon Y \to Z$, then the *cut composition* $r \dagger s\colon X \to Z$ is defined by $U \; (r \dagger s) \; W$ if there are $\mathcal{V}, \mathcal{V}' \in \mathcal{F}\mathcal{F}Y$ such that $\{U\} \; \overline{r} \; \mathcal{V} \bowtie \mathcal{V}' \; \overline{s} \; \{W\}$.

Ent is monoidal, with unit the empty set and tensor given by disjoint union. Note that $\mathcal{F}(X+Y) \cong \mathcal{F}X \times \mathcal{F}Y$. We shall commonly use this implicitly, writing finite subsets of $X+Y$ as pairs.

A tensor of morphisms is, perhaps surprisingly, a *disjunction:*

$$(U_1, U_2)(r \otimes s)(V_1, V_2) \text{ if } U_1 r V_1 \text{ or } U_2 s V_2.$$

There are two equivalent representations. One is frame theoretic: Ent^{op} is equivalent to the category of free frames and preframe homomorphisms. Then the maps in Ent correspond to preframe homomorphisms that have *right* adjoints, so they preserve all joins and must be frame homomorphisms. Dualizing again, the maps in Ent correspond to locale maps. Note, however, that they are not arbitrary locale maps. Their frame homomorphisms have right adjoints that preserve directed joins: so the locale maps are *perfect.*

The points of the locale with free frame over X are the subsets of X. A subbase of opens is give by $\{A \subseteq X \mid x \in A\}$, for $x \in X$. In fact this is just the Scott topology on the powerset $\mathcal{P}X$.

The tensor is product, using $\mathcal{P}(X+Y) \cong \mathcal{P}X \times \mathcal{P}Y$.

The cut composition in Ent can be difficult to work with, and that is our motivation for seeking to clarify the axioms that underly the diagrammatic reasoning. Nonetheless, there are at least two respects in which it has an advantage over the spatial or localic versions.

The first is that the duality of relational converse (which becomes de Groot duality for stably compact spaces) is extremely simple.

The second is more foundational. In predicative mathematics, frames and powersets are problematic in that they may fail to be admissible as sets. The manipulations in Ent, by contrast, are foundationally very robust. They are geometric, and even suitable for the form of geometric rreasoning based on arithmetic universes as in [Vic19].

3.2 Ent is a cocartesian bicategory

We define the cocartesian bicategory structure of Ent as follows.

$$\begin{aligned} \Delta_X &: U \curlywedge (V_1, V_2) \text{ if } U \between V_1 \cup V_2, \\ t_X &: U \downharpoonleft \emptyset \text{ never.} \end{aligned}$$

Their left adjoints $\curlyvee = \Delta_X^*$ and $\upharpoonleft = t_X^*$ are just the relational converses.

On the powersets $\mathcal{P}X$, $\curlywedge \colon \mathcal{P}X \to \mathcal{P}X \times \mathcal{P}X$ is a map, namely the diagonal. $\curlyvee \colon \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ is, as will become clear in Section 5.3, also a map. The left adjoint of the diagonal, it produces binary unions.

In calculations, one common technique is what we shall call *portwise cut*, by which we mean lemma 34 of [Vic04]. This applies in calculating a cut when one of the morphisms is a tensor product. It allows us in turn to fix one of those morphisms and cut just against the other.

Lemma 8.

1. *If $r \colon X + X \to Y$ then $U\ (\curlywedge \dagger r)\ W$ iff $(U, U)\ r\ W$.*
2. *If $r \colon \emptyset \to Y$ then $U\ (\downharpoonleft \dagger r)\ W$ iff $\emptyset\ r\ W$.*
3. *If $r \colon Y \to X$ and $U\ (r \dagger \curlywedge)\ (W_1, W_2)$ then $U\ r\ (W_1 \cup W_2)$.*
4. *If $r \colon Y \to X$ and $U\ (r \dagger \downharpoonleft)\ \emptyset$ then $U\ r\ \emptyset$.*

Proof. (1) For $(\Rightarrow)$, suppose $\{U\}\ \overline{\curlywedge}\ \mathcal{V} \bowtie \mathcal{V}'\ \overline{r}\ \{W\}$. For every $(V_1, V_2) \in \mathcal{V}$ we have either $U \between V_1$ or $U \between V_2$. Hence we can find $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and choices γ_i for $\mathcal{V}_i$ such that $\mathsf{Im}\gamma_i \subseteq U$. Combining them gives a choice of $\mathcal{V}$, for which we must have $(\mathsf{Im}\gamma_1, \mathsf{Im}\gamma_2)\ r\ W$, so $(U, U)\ r\ W$.

For the converse, if $(U, U)\ r\ W$ then

$$\{U\}\ \overline{\curlywedge}\ \{(\{u\}, \emptyset) \mid u \in U\} \cup \{(\emptyset, \{u\}) \mid u \in U\} \bowtie \{(U, U)\}\ \overline{r}\ \{W\}.$$

(2) is similar but easier.

(3) We have $\{U\}\ \overline{r}\ \mathcal{V} \bowtie \mathcal{V}'\ \overline{\curlywedge}\ \{(W_1, W_2)\}$. Then we can find $\mathcal{V}' = \mathcal{V}'_1 \cup \mathcal{V}'_2$ such that for every $V' \in \mathcal{V}'_i$ we have $V' \between W_i$; we then can find $\gamma_i \in \mathsf{Ch}\mathcal{V}'_i$ such that $\mathsf{Im}\gamma_i \subseteq W_i$. The conclusion follows.

(4) is similar. □

There is a dual version for $\curlyvee$.

Lemma 9. *$\curlywedge$ and $\downarrow$ are cocommutative comonoid structure (diagrams (1)).*

Proof. Cocommutativity is easy. One direction for the counit laws follows easily from Lemma 8, and the other direction also is easy.

For coassociativity, we first show that $U\ (\curlywedge \dagger (X \otimes \curlywedge))\ (W_1, W_2, W_3)$ iff $U \between W_1 \cup W_2 \cup W_3$. The "if" direction is easy, by picking an element of the intersection. For "only if", by Lemma 8 we have $(U, U)\ (X \otimes \curlywedge)\ (W_1, W_2, W_3)$, so either $U \between W_1$ or $U \curlywedge (W_2, W_3)$. Coassociativity follows using cocommutativity. □

Lemma 10. *Every morphism r is a lax comonoid homomorphism (diagrams (3)).*

Proof. The first condition is immediate, because its left-hand side is the empty relation. For the second, we can use Lemma 8 on the left-hand side to show $U\ (\curlywedge \dagger (r \otimes r))\ (W_1, W_2)$ iff $U\ r\ W_1$ or $U\ r\ W_2$. The right-hand side is $U\ (r \dagger \curlywedge)\ (W_1, W_2)$ iff $U\ r\ W_1 \cup W_2$, which clearly is implied by the left-hand side. □

Lemma 11. *$\uparrow \dashv \downarrow$ and $\curlyvee \dashv \curlywedge$ (diagrams (2)).*

Proof. First we prove the adjunctions (2). For $\curlyvee \dashv \curlywedge$, the first inequality (the counit of the adjunction), is immediate from Lemma 8. In fact we have an equality there: the counit is an isomorphism. If $U \between W$, then $\{U\}\ \overline{\curlywedge}\ \{(\{v\}, \emptyset)\} \bowtie \{(\{v\}, \emptyset)\}\ \overline{\curlyvee}\ \{W\}$.

For the unit, it is easy to show that $(U_1, U_2)\ (\curlyvee \dagger \curlywedge)\ (W_1, W_2)$ iff $U_1 \cup U_2 \between W_1 \cup W_2$, from which the unit follows immediately.

We now show $\uparrow \dashv \downarrow$. This time it is the unit that is an isomorphism, as both sides are the empty relation. (Note that $\{\emptyset\}\ \bar{\uparrow}\ \mathcal{V}$ iff $\mathcal{V}$ is empty, and then $\mathcal{V}$ does have a choice – itself empty.) For the counit, we have that the left hand side is the empty relation. □

Theorem 12. Ent *is a cocartesian bicategory.*

Proof. Following the lemmas, this is now done. □

Let us calculate the two sides of the Frobenius equation (5). The right-hand side was already calculated in the proof of Theorem 12. $(U_1, U_2)\ (\curlyvee \dagger \curlywedge)\ (W_1, W_2)$ iff $U_1 \cup U_2 \between W_1 \cup W_2$. In other words, $U_1 \between W_1$ or $U_1 \between W_2$ or $U_2 \between W_1$ or $U_2 \between W_2$.

If we write R for the left-hand side, we can calculate it using portwise cut and Lemma 8: $(U_1, U_2)\ R\ (W_1, W_2)$ iff $(U_1, U_1, U_2)\ (X \otimes \curlyvee)\ (W_1, W_2)$, ie $U_1 \between W_1$ or $U_1 \cup U_2 \between W_2$, ie $U_1 \between W_1$ or $U_1 \between W_2$ or $U_2 \between W_2$. It is easily shown that the converse also holds.

We thus have the $\leq$ direction of the Frobenius condition (which we knew already from cocommutativity of $\curlywedge$), but not $\geq$ – in the above calculation, with $X = \{x\}$, take $U_2 = W_1 = x$ and $U_1 = W_2 = \emptyset$.

Thus Ent is emphatically *not* a bicategory of relations. Nonetheless, it does have an involution (given by relational converse), and we shall see later that it does come from a Frobenius condition, just not the one involving $\curlywedge$ and $\curlyvee$.

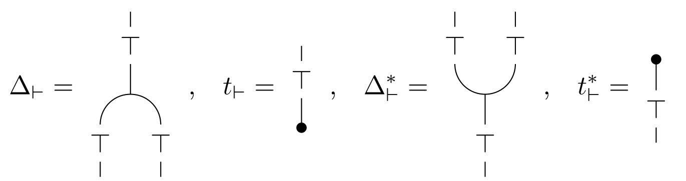
4 Karoubi envelopes of cocartesian bicategories

In this section we prove that the Karoubi envelope $\mathsf{Kar}(\mathcal{B})$ of a cocartesian bicategory $\mathcal{B}$ is also a cocartesian bicategory.

$\mathsf{Kar}(\mathcal{B})$ is clearly an order-enriched, symmetric monoidal category, inheriting its order and tensor from $\mathcal{B}$.

Definition 13. Let $\mathcal{B}$ be a cocoartesian bicategory, and $\vdash$ an idempotent 1-cell (an object of $\mathsf{Kar}(\mathcal{B})$).

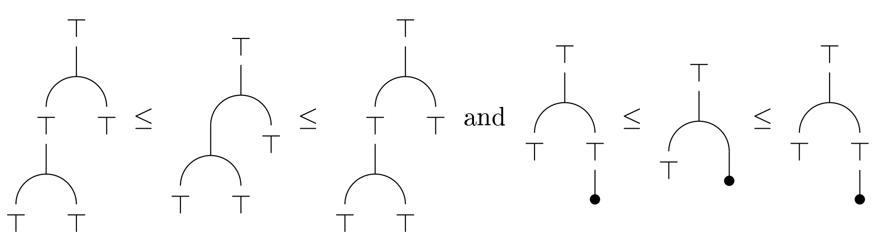
The cocommutative comonoid structure $\Delta_\vdash$ and $t_\vdash$, together with left adjoints $\Delta^*_\vdash$ and $t^*_\vdash$, are given by –



Lemma 14. *$\Delta_\vdash$ and $t_\vdash$ are cocommutative comonoid structure (diagrams (1)).*

Proof. Cocommutativity is clear.

By repeated use of the lax homomorphism properties (including the one for counits) we see that



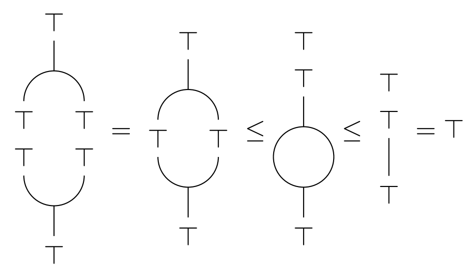
so that we get two equations as well as two more for the mirror images. These reduce the problem to the corresponding laws for $\curlywedge$. ☐

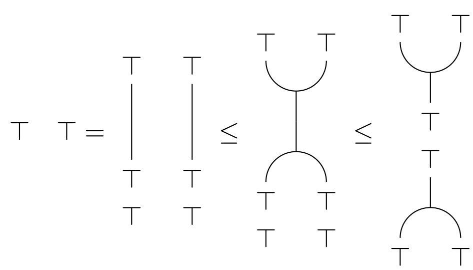
Lemma 15. *Every morphism r in $\mathsf{Kar}(\mathcal{B})$ is a lax comonoid homomorphism (diagrams (3)).*

Proof. This follows easily from the corresponding properties in $\mathcal{B}$, bearing in mind that that the morphisms in $\mathsf{Kar}(\mathcal{B})$ are those r for which $r = \vdash^\dagger r^\dagger \vdash$. ☐

Lemma 16. $\Delta_\vdash \dashv \Delta_\vdash^*$ *and* $t_\vdash \dashv t_\vdash^*$ *(diagrams* (2)*).*

Proof. For $\Delta_\vdash \dashv \Delta_\vdash^*$ we have the following. The other adjunction is similar but easier.





□

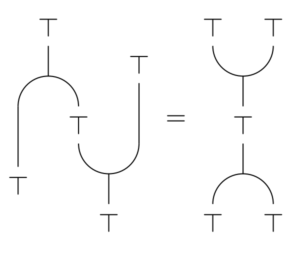
Theorem 17. *If $\mathcal{B}$ is a cocartesian bicategory then so is* $\mathsf{Kar}(\mathcal{B})$*.*

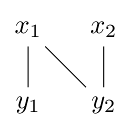
Proof. This has now been proved in the preceding lemmas. □

Example 18. The property of being a bicategory of relations (ie of having the Frobenius condition everywhere) is not inherited by Karoubi envelopes.

We can see this in Rel, which is a bicategory of relations. The objects of its Karoubi envelope are transitive, interpolative relations, and those include partial orders.

The Frobenius condition at $\vdash$ is on the left below. The significant direction is $\leq$, as Rel is cartesian, not cocartesian.



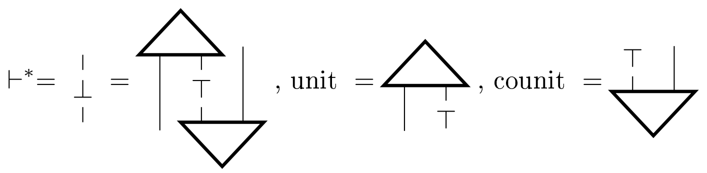


Consider the idempotent $\vdash$ given by $\geq$ in the Hasse diagram on the right above. By construction, (x_1, x_2) is related to (y_1, y_2) by the left-hand side of the Frobenius condition. However, to make them related by the right-hand side we should have to find an element that is both a lower bound of x_1 and x_2 and an upper bound of y_1 and y_2, and there is no such element.

Hence this object of the Karoubi envelope is not cartesian.

Proposition 19. *Let $\mathcal{B}$ be a compact closed category. Then so is* $\mathsf{Kar}(\mathcal{B})$.

Proof. Let $\vdash : X \to X$ be an idempotent in $\mathcal{B}$, with dual morphism (also idempotent) $\vdash^* : X^* \to X^*$. We define $\vdash^*$ to be the dual of $\vdash$ as objects of $\mathsf{Kar}(\mathcal{B})$, with unit and counit as follows.



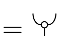
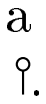
It is routine to prove, first, that the unit and counit are indeed morphisms in $\mathsf{Kar}(\mathcal{B})$ between $\mathbb{I}$ and $\vdash^* \otimes \vdash$ or $\vdash \otimes \vdash^*$, and, second, that the yanking equations hold. $\square$

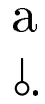
Corollary 20. *If $\mathcal{B}$ is a bicategory of relations, then* $\mathsf{Kar}(\mathcal{B})$ *is a compact closed cartesian bicategory.*

5 Bicategories of entailments

5.1 Lattice objects

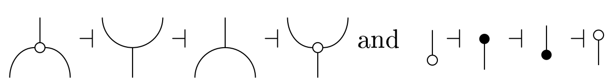
Definition 21. Let X be an object of a cocartesian bicategory $\mathcal{B}$.

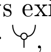
1. X is a *meet semilattice* if Δ_X and t_X have right adjoints, $\nabla_X =$  and $\tau_X =$ .

2. X is a *join semilattice* if Δ_X^* and t_X^* have left adjoints, $\nabla_X^* =$  and $\tau_X^* =$ .

3. X is a *lattice* if it is both a meet semilattice and a join semilattice.

Thus $\nabla_X^* \dashv \Delta_X^* \dashv \Delta_X \dashv \nabla_X$ and $\tau_X^* \dashv t_X^* \dashv t_X \dashv \tau_X$; diagrammatically,



These should be familar as semilattice structure coming as adjoints to diagonals. Note, however, the asymmetry from $\mathcal{B}$'s point of view. The join operator  always exists, but the effect of the definition is to make it a map. The meet operator , on the other hand, does not necessarily exist, but when it does it is automatically a map.

Proposition 22. *Let X be an object in a cocartesian bicategory $\mathcal{B}$.*

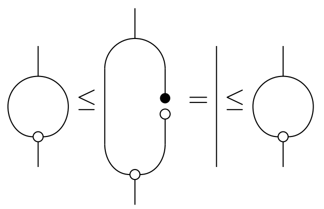
1. *If X is a join semilattice, then and make it an idempotent commutative monoid.*

2. *If X is a meet semilattice, then and make it an idempotent commutative monoid.*

In each case, idempotence is with respect to as diagonal. (Of course, it is *the diagonal in* $\mathsf{Map}(\mathcal{B})$*; but we need to be clear about what idempotence means for a monoid in* $\mathcal{B}$*.)*

Proof. (1) The commutative monoid structure comes from the cocommutative comonoid structure for Δ_X and t_X, as explained in [CW87]. Idempotence is Remark 2.

(2) The commutative monoid structure by by adjunctions from the comonoid structure in an entirely similar way. For idempotence we have



□

5.2 Distributivity and outer Frobenius

Lemma 23. *Let L be a lattice. Then L is distributive iff it satisfies $a \wedge (b \vee c) \leq (a \wedge b) \vee c$ for all $a, b, c \in L$.*

Proof. If L is distributive, then

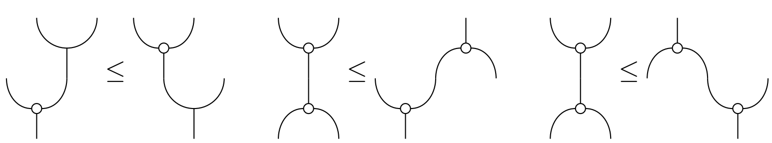
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \leq (a \wedge b) \vee c.$$

For the converse, we have

$$(a \vee c) \wedge (b \vee c) \leq ((a \vee c) \wedge b) \vee c = (b \wedge (a \vee c)) \vee c \leq ((b \wedge a) \vee c) \vee c = (a \wedge b) \vee c.$$

□

For a lattice X in a cocartesian bicategory, we should therefore define X to be distributive if it satisfies the left hand inequality below; then the adjunctions quickly show that that is equivalent to the middle inequality. Using commutativity, that is also equivalent to the right hand inequality.


(7)

Definition 24. Let X be a lattice in a cocartesian bicategory $\mathcal{B}$. Then X is a *distributive* lattice if it satisfies any one of the three inequalities above.

Clearly we are now approaching a Frobenius condition, as in Section **??**. Let us refer to equation (5), involving and , as the *inner* Frobenius condition, and the corresponding equation with and as the *outer* Frobenius equation.

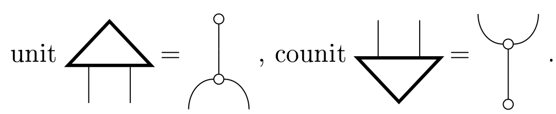
Recall that for inner Frobenius the $\leq$ direction of equation (5) was automatic, and corresponded by adjunctions to associativity of . For outer Frobenius we similarly get that the $\leq$ direction in the middle equation (7) corresponds by adjunctions to the distributivity condition, but that is no longer automatic. The other direction is also not automatic.

Definition 25. An object X of a cocartesian bicategory $\mathcal{B}$ is an *outer Frobenius object* if it is a distributive lattice for which equality holds in the middle (or right hand) part of equations (7).

$\mathcal{B}$ is a *bicategory of entailments* if every object is outer Frobenius.

Proposition 26. *Let $\mathcal{B}$ be a cocartesian bicategory. Then any outer Frobenius object X is its own dual X°, in the sense of compact closed categories.*

Proof. The proof is the same as for Proposition 3, except that the unit and counit are



□

Recall that in any cocartesian bicategory, every morphism r is a lax comonoid homomorphism for and , and a lax monoid homomorphism for and . The laxness is in the direction of diagram (3). If r is a map or comap (r has left or right adjoint) then it is a homomorphism for the comonoid or the monoid respectively.

Proposition 27. *Let $\mathcal{B}$ be a cocartesian bicategory in which every object is a lattice, and let r be a morphism.*

1. *If r is a map, then it is an oplax homomorphism for the monoids with and .*
2. *If r is a comap, then it is an oplax homomorphism for the comonoids with and .*
3. *If r is a map and so is its left adjoint r^*, then r is a homomorphism for the monoids with and .*

Proof. (1) r is a homomorphism (hence oplax) for and ; then use $\dashv$ etc.

(2) is dual.

(3) Since r^* is a map, it is an oplax homomorphism for and . Then by $r^* \dashv r$, r is a lax homomorphism and hence a homomorphism. □

Note that to get corresponding results for $\mathring{\curlywedge}$ and $\mathring{\downharpoonleft}$ we need r to be a comap.

Proposition 28. *Let $\mathcal{B}$ be a bicategory of entailments, and let r be both a map whose left adjoint r^* is also a map, and a comap. Then $r^* = r^\circ$.*

Proof. The argument to show $r^\circ \dashv r$ is the same diagram as in Proposition 5, replacing $\curlywedge$ by $\mathring{\curlywedge}$ and so on. However, some instances of $\leq$ and $=$ are exchanged. Whereas before r was a comonoid homomorphism, now (using Proposition 27) it is a monoid homomorphism. $\square$

The following result is our justification for using upside down symbols.

Proposition 29. *Let $\mathcal{B}$ be a bicategory of entailments. Then $\mathcal{B}$ is dagger closed, with $\curlyvee = \curlywedge^\circ$, $\upharpoonleft = \downharpoonleft^\circ$, $\mathring{\curlywedge} = \mathring{\curlyvee}^\circ$ and $\mathring{\downharpoonleft} = \mathring{\upharpoonleft}^\circ$.*

Proof. The first two follow from Proposition 28, because $\curlywedge$ and $\downharpoonleft$ satisfy the hypotheses.

Because $r \mapsto r^\circ$ is an order-preserving involution, if $r \dashv s$ then $s^\circ \dashv r^\circ$, and it follows that $\mathring{\curlyvee}^\circ \dashv \curlywedge^\circ = \curlyvee$, so $\mathring{\curlyvee}^\circ = \mathring{\curlywedge}$. The remaining equation is similar. $\square$

Proposition 30. *Let $\mathcal{B}$ be a bicategory of entailments. Then $\mathsf{Kar}(\mathcal{B})$ is a compact closed cocartesian bicategory.*

5.3 Ent is a bicategory of entailments

Definition 31. Let X be a set. We define $\nabla_X = \mathring{\curlyvee}\colon X + X \to X$ and $\tau_X = \mathring{\upharpoonleft}\colon \emptyset \to X$ in Ent by $(U_1, U_2)\, \mathring{\curlyvee}\, V$ if $U_1 \cap U_2 \between V$ and $\emptyset\, \mathring{\upharpoonleft}\, V$ if V non-empty.

$\nabla_X^* = \mathring{\curlywedge}\colon X \to X + X$ and $\tau_X^* = \mathring{\downharpoonleft}\colon X \to \emptyset$ are their relational converses.

Proposition 32. *Let X be a set.*

1. *$\mathring{\curlyvee}$ and $\mathring{\upharpoonleft}$ make X a meet semilattice in* Ent.
2. *$\mathring{\curlywedge}$ and $\mathring{\downharpoonleft}$ make X a join semilattice in* Ent.
3. *X is an outer Frobenius object in* Ent.

Proof. (1) First, $\curlywedge \dashv \mathring{\curlyvee}$. For the counit of the adjunction, suppose $(U_1, U_2)\ (\mathring{\curlyvee} \dagger \curlywedge)\ (V_1, V_2)$. Then by Lemma 8 we get $U_1 \cap U_2 \between V_1 \cup V_2$, so either $U_1 \between V_1$ or $U_2 \between V_2$, which gives us $(U_1, U_2)\mathsf{Id}(V_1, V_2)$.

For the unit, suppose $U \between W$, with $x \in U \cap W$. Then $\{U\}\ \overline{\curlywedge}\ \{\{(x,x)\}\} \bowtie \{\{(x,x)\}\}\ \overline{\mathring{\curlyvee}}\ \{W\}$.

Now we show $\downharpoonleft \dashv \mathring{\upharpoonleft}$. For the counit, we use Lemma 8 in a similar way.

For the unit, suppose $U \between W$ with $x \in U \cap W$. Then $\{U\}\ \overline{\downharpoonleft}\ \emptyset \bowtie \{\emptyset\}\ \overline{\mathring{\upharpoonleft}}\ \{W\}$.

(2) is immediate using relational converses.

(3) We show that for each side of the outer Frobenius equation, (U_1, U_2) is related to (W_1, W_2) iff $U_1 \cap U_2 \cap W_1 \cap W_2 \neq \emptyset$.

First, consider the left hand side, $\mathring{\curlyvee} \dagger \mathring{\curlywedge}$. If $\{(U_1, U_2)\}\ \overline{\mathring{\curlyvee}}\ \mathcal{V} \bowtie \mathcal{V}'\ \overline{\mathring{\curlywedge}}$ then for every $V \in \mathcal{V}$ we have $U_1 \cap U_2 \between V$, so there is some $\gamma \in \mathsf{Ch}\mathcal{V}$ with $\mathsf{Im}\gamma \subseteq U_1 \cap U_2$.

It follows that $U_1 \cap U_2 \between W_1 \cap W_2$. Conversely, if $x \in U_1 \cap U_2 \cap W_1 \cap W_2$ then the result follows from $(U_1, U_2) \curlyvee \{x\} \curlywedge (W_1, W_2)$.

Now consider the right hand side, $(X + \curlywedge) \dagger (\curlyvee + X)$. For this we use portwise cut. (U_1, U_2) is related to (W_1, W_2) iff there are $\mathcal{V} \bowtie \mathcal{V}'$ such that for all $V \in \mathcal{V}$ we have $U_2 \between V \cap W_2$ and for all $V' \in \mathcal{V}'$ we have $U_1 \cap V' \between W_1$. We find there is some choice γ for $\mathcal{V}$ such that $\mathsf{Im}\gamma \subseteq U_2 \cap W_2$, and so we get $U_1 \cap U_2 \cap W_2 \between W_1$. Again, the reverse inequality is easy. $\square$

We have now proved the main result of this section.

Theorem 33. Ent *is a bicategory of entailments.*

Corollary 34. *The category of stably compact locales and closed relations between them is a compact closed cocartesian bicategory.*

Proof. By [Vic04], this category is $\mathsf{Kar}(\mathsf{Ent})$. $\square$

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