

Localic completion of generalized metric spaces

II: Powerlocales

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Abstract

The work investigates the powerlocales (lower, upper, Vietoris) of localic completions of generalized metric spaces. The main result is that all three are localic completions of generalized metric powerspaces, on the Kuratowski finite powerset.

Applications: (1) A localic completion is always open, and is compact iff its generalized metric space is totally bounded. (2) The Heine-Borel Theorem is proved in a strong form with continuous maps to the powerlocales of \mathbb{R} , $(x, y) \mapsto$ the closed interval $[x, y]$. (3) Every localic completion is a triquotient surjective image of a locale of Cauchy sequences.

The work is constructive in the topos-valid sense.

Keywords: locale, constructive, topos, metric, Heine-Borel.

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1 Introduction

A companion paper [19] describes the “localic completion” \overline{X} of a metric space X , generalized in the sense of Lawvere [10]. (The metric is required to have zero self-distance and the triangle inequality, but little else.) \overline{X} is a locale whose points are Cauchy filters of formal open balls in X . These are classically equivalent to ordinary spatial constructions, and in particular, to the standard construction in the symmetric case; however, we take the localic approach to topology as the one appropriate in constructive mathematics. This gives greater algebraic content to much of the working, though this can largely be wrapped up and hidden in the applications.

The purpose of the present paper is to investigate the powerlocales (lower, upper and Vietoris) of \overline{X} , locales whose points are equivalent to certain sublocales of \overline{X} and which are thus analogous to hyperspaces. We describe how the powerlocales of \overline{X} may themselves also be described as localic completions,

with respect to three different generalized metrics on the (Kuratowski) finite powerset $\mathcal{F}X$ of X .

The importance of this lies in the fact that some properties of locales can be expressed as structure existing in the powerlocales. For instance, a locale is compact iff its upper powerlocale is colocal (has, in a certain universal sense, a top point). This then makes it easy to characterize compactness of \overline{X} in terms of a total boundedness property on the generalized metric space X . A similar argument with the lower powerlocale yields the result that all completions \overline{X} are open as locales.

Two other applications are given.

The first uses all three powerlocales of the real line, but in particular the Vietoris powerlocale $V\mathbb{R}$ (or, rather, its positive version $V^+\mathbb{R}$ that “excludes the empty set”), to analyse the Heine-Borel theorem. It describes a “Heine-Borel” map HB_C from the sublocale \leq of \mathbb{R}^2 to $V^+\mathbb{R}$ that maps (x, y) to the closed interval $[x, y]$, and thus shows not only that $[x, y]$ is compact but also that it depends continuously on its endpoints. In addition we describe maps \sup and \inf from $V^+\mathbb{R}$ to \mathbb{R} . Thus if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a map then the supremum of f on $[x, y]$ can be described as $\sup \circ (V^+ f) \circ HB_C(x, y)$ and hence depends continuously on x and y .

The second application describes a locale $\text{Cauchy}_f(X)$ whose points are Cauchy sequences from X and uses powerlocale techniques to show that a certain map $\text{Cauchy}_f(X) \rightarrow \overline{X}$ is a localic surjection (in fact, a triquotient in the sense of [11]). This then reconstructs locally to some extent the classical idea that the completion of a metric space is a quotient of a set of Cauchy sequences.

The paper is constructive throughout in the sense of topos validity. In fact, most of the reasoning is stronger in that it conforms with *geometric* logic — the constructions are preserved by inverse image functors of geometric morphisms. This constructivism is adhered to not merely for philosophical reasons; it is expositionally essential in that it makes it possible to reason with locales in terms of their points. This is set out more fully in [19], as well as other papers such as [16], [18] and [17]. The essential reason is that to find enough points of a locale, one has to be prepared to move around to different set theories (and use *generalized* points), and it is the geometric constructivism that allows this to happen. Once that is accepted then it is possible validly to use an expositional language that treats locales and toposes as collections of points, and maps (continuous maps between locales or geometric morphisms between toposes) as transformations of points.

The overall achievement is to apply algebraic methods to topological investigations (presupposing that if we are to work constructively then the correct formalization of topology is to use locales). The definitions of the powerlocales are in terms of algebra — the frames (lattices of opens) for the powerlocales are defined as freely generated over the frames of the original locales, subject to preserving certain algebraic structure on them. Moreover, the proofs of the main results rely heavily on manipulating presentations by generators and relations. However, these algebraic manipulations also have a logical content, specifically with respect to geometric logic, and that can then be related through to the

topology.

2 Generalized metric completion

We summarize the definitions that appear in [19]. (The terse language here is explained in [19]. A phrase such as “the locale whose points are ...” bundles up a number of topos-theoretic concepts: first, that “...” describes a geometric theory; second, that its classifying topos is localic; and third, that we are defining the corresponding locale.)

Q_+ is the set of positive rationals.

$\widehat{[0, \infty]}$ is the locale whose points are “upper reals”, i.e. rounded upper sets of positive rationals. Classically, the upper reals are equivalent to ordinary reals, i.e. Dedekind sections. Constructively, however, there may be upper reals for which there is no corresponding lower set of rationals to make a Dedekind section.

The continuous arithmetic structure on $\widehat{[0, \infty]}$ includes order (numeric order is the reverse of specialization order), addition and multiplication, finitary max and min, and infinitary inf. It does not include any kind of subtraction (which would have to be contravariant in one argument, hence non-continuous because of the relationship between the numeric and specialization orders), or infinitary sup.

Definition 2.1 *A generalized metric space (or gms) is a set X equipped with a distance map $X(-, -) : X^2 \rightarrow \widehat{[0, \infty]}$ satisfying*

$$\begin{aligned} X(x, x) &= 0 && \text{(zero self-distance)} \\ X(x, z) &\leq X(x, y) + X(y, z) && \text{(triangle inequality)} \end{aligned}$$

If X and Y are two gms's, then a homomorphism from X to Y is a non-expansive function, i.e. a function $f : X \rightarrow Y$ such that $Y(f(x_1), f(x_2)) \leq X(x_1, x_2)$.

(This notion of homomorphism is in fact the natural one pertaining to gms's as models of a geometric theory. This is explained more fully in [19].)

Note that by *map* we mean continuous map, in this case between locales. So we are treating the set X^2 as the corresponding discrete locale.

Definition 2.2 *If X is a generalized metric space then we introduce the symbol “ $B_\delta(x)$ ”, a “formal open ball”, as alternative notation for the pair (x, δ) ($x \in X$, $\delta \in Q_+$). We write*

$$B_\varepsilon(y) \subset B_\delta(x) \text{ if } X(x, y) + \varepsilon < \delta$$

(more properly, if $\varepsilon < \delta$ and $X(x, y) < \delta - \varepsilon$) and say in that case that $B_\varepsilon(y)$ refines $B_\delta(x)$.

Definition 2.3 Let X be a generalized metric space. A subset F of $X \times Q_+$ is a Cauchy filter if –

1. F is upper with respect to \subset , i.e. if $B_\varepsilon(y) \subset B_\delta(x)$ and $B_\varepsilon(y) \in F$ then $B_\delta(x) \in F$.
2. If $B_\delta(x) \in F$ and $B_{\delta'}(x') \in F$ then there is some $B_\varepsilon(y) \in F$ with $B_\varepsilon(y) \subset B_\delta(x)$ and $B_\varepsilon(y) \subset B_{\delta'}(x')$.
3. For every $\delta \in Q_+$ there is some $x \in X$ such that $B_\delta(x) \in F$.

Definition 2.4 Let X be a generalized metric space. Then its localic completion \overline{X} is the locale whose points are the Cauchy filters of formal open balls.

Proposition 2.5 Let $\phi : X \rightarrow Y$ be a homomorphism of gms's. Then ϕ lifts to a map $\overline{\phi} : \overline{X} \rightarrow \overline{Y}$ defined by

$$B_\varepsilon(y) \in \overline{\phi}(F) \text{ iff } \exists B_\delta(x) \in F. B_\varepsilon(y) \supset B_\delta(\phi(x)).$$

Proof. [19]. $\overline{\phi}$ is also given a universal characterization there, but for the present work we shall not need that. ■

Proposition 2.6 Let $\phi, \psi : X \rightarrow Y$ be homomorphisms of gms's such that for all $x \in X$, $Y(\phi(x), \psi(x)) = 0$. Then $\overline{\phi} \sqsubseteq \overline{\psi}$.

Proof. Suppose F is a Cauchy filter for X and $B_\varepsilon(y) \in \overline{\phi}(F)$ with $B_\varepsilon(y) \supset B_\delta(\phi(x))$, $B_\delta(x) \in F$. Then

$$Y(y, \psi(x)) + \delta \leq Y(y, \phi(x)) + Y(\phi(x), \psi(x)) + \delta = Y(y, \phi(x)) + \delta < \varepsilon$$

so $B_\varepsilon(y) \supset B_\delta(\psi(x))$ and $B_\varepsilon(y) \in \overline{\psi}(F)$. ■

Definition 2.7 Let X be a generalized metric space. A subset F of $X \times Q_+$ is a left X -module if it is a rounded upper set of formal open balls –

1. F is upper with respect to \subset (as in Definition 2.3).
2. F is rounded, i.e. if $B_\delta(x) \in F$ then there is some $B_\varepsilon(y) \in F$ with $B_\varepsilon(y) \subset B_\delta(x)$.

We write $X\text{-Mod}$ for the locale whose points are the left X -modules.

(The term “module” is explained in [19] in terms of Lawvere’s use of enriched categories. In the present paper those ideas are less relevant but we keep the term.)

It is not hard to see that the frame of opens for \overline{X} can be presented by

$$\begin{aligned} \Omega\overline{X} &= \text{Fr}\langle B_\delta(x) \ (x \in X, \delta \in Q_+) \mid \\ &\quad B_\delta(x) \wedge B_{\delta'}(x') = \bigvee \{B_\varepsilon(y) \mid B_\varepsilon(y) \subset B_\delta(x) \text{ and } B_\varepsilon(y) \subset B_{\delta'}(x')\} \\ &\quad (x, x' \in X, \delta, \delta' \in Q_+) \\ \text{true} &= \bigvee_{x \in X} B_\delta(x) \ (\delta \in Q_+) \end{aligned}$$

We write $[\text{gms}]$ for the topos whose points are generalized metric spaces (i.e. the classifying topos for the geometric theory of generalized metric spaces). Between toposes, by “map” we mean geometric morphism.

3 Background on powerlocales

We summarize some facts about powerlocales.

Definition 3.1 *As in [9], a suplattice is a complete join semilattice. A suplattice homomorphism preserves all joins.*

As in [8] (and following Banaschewski), a preframe is a poset with finite meets and also directed joins, over which the binary meet distributes. A preframe homomorphism preserves the finite meets and directed joins.

Definition 3.2 *If Y is a locale, then the lower and upper powerlocales $P_L Y$ and $P_U Y$ are defined by letting $\Omega P_L Y$ and $\Omega P_U Y$ be the frames generated freely over ΩY qua suplattice and qua preframe respectively, with generators written as $\Diamond a$ and $\Box a$ ($a \in \Omega Y$).*

[5] The Vietoris powerlocale VY is the sublocale of $P_L Y \times P_U Y$ presented by relations $\Diamond a \wedge \Box b \leq \Diamond(a \wedge b)$ and $\Box(a \vee b) \leq \Box a \vee \Diamond b$.

All three are the functor parts of monads on the category **Loc** of locales. Following [14], we write these as $(P_L, \downarrow, \sqcup)$, (P_U, \uparrow, \sqcap) and $(V, \{-\}, \cup)$. We also overload the up and down arrows by writing $\downarrow: VY \rightarrow P_L Y$ and $\uparrow: VY \rightarrow P_U Y$ for the projection maps restricted to VY .

The notation here, as well as the name “powerlocale”, is explained by the fact that their points can be considered as sublocales. The results, all constructively valid, are surveyed in [15].

Theorem 3.3 *Given a global point $U: 1 \rightarrow P_L Y$, let P be the lax pullback*

$$\begin{array}{ccc} P & \longrightarrow & 1 \\ \downarrow & \sqsubseteq & \downarrow U \\ Y & \longrightarrow & P_L Y \\ & \downarrow & \end{array}$$

Then $P \rightarrow Y$ is a weakly closed sublocale with open domain, and this gives a bijection between such sublocales of Y and the global points of $P_L Y$.

Proof. This is essentially the result of [1] showing a bijection between those sublocales and suplattice homomorphisms from ΩY to Ω . ■

A *weakly closed* sublocale is one presented by relations $a \leq \Omega!(p)$ where $a \in \Omega Y$, $p \in \Omega = \mathcal{P}1$ and $!: Y \rightarrow 1$ is the unique map. (It is closed if p is **false**.) “*With open domain*” says that P is open as a locale, i.e. $P \rightarrow 1$ is an open map.

Classically, the weakly closed sublocales with open domain are exactly the closed sublocales and for present purposes that gives an adequate picture of the

points of $P_L Y$. Note in particular that weakly closed sublocales, like closed sublocales, are lower closed in the specialization order. This is obvious from the lax pullback construction.

Generalized points $U : Z \rightarrow P_L Y$ correspond to certain sublocales of $Y \times Z$ (see [15]).

This now explains the notation “ \downarrow ”, for if U is the composite map $\downarrow x = 1 \xrightarrow{x} Y \xrightarrow{\downarrow} P_L Y$, then P is the sublocale of points less than x in the specialization order.

The specialization order amongst points of $P_L Y$ corresponds to the inclusion order amongst the corresponding sublocales.

Now for the upper powerlocale.

Theorem 3.4 *Given a global point $U : 1 \rightarrow P_U Y$, let P be the lax pullback*

$$\begin{array}{ccc} P & \longrightarrow & 1 \\ \downarrow & \sqsubseteq & \downarrow U \\ Y & \longrightarrow & P_U Y \\ & \uparrow & \end{array}$$

Then $P \rightarrow Y$ is a compact fitted sublocale, and this gives a bijection between such sublocales of Y and the global points of $P_U Y$.

Proof. This is essentially Johnstone’s localic version [5] of the Hofmann-Mislove theorem. It shows that there is a bijection between compact fitted sublocales of Y and Scott open filters of the frame ΩY . ■

A *fitted* sublocale is a meet of open sublocales. Any fitted sublocale is upper closed in the specialization order. $\uparrow x$ corresponds to the sublocale of points bigger than x in the specialization order.

Note that the order in the lax pullback is the opposite to that for $P_L Y$. This has the consequence that specialization order in $P_U Y$ corresponds to the *opposite* of the inclusion order between sublocales. For instance, the empty sublocale is a top point of $P_U Y$.

Now for the Vietoris powerlocale.

Theorem 3.5 *Given a global point $U : 1 \rightarrow VY$, let P be the meet of the sublocales of Y corresponding to $\downarrow U$ and $\uparrow U$ (points of $P_L Y$ and $P_U Y$). Then P is a weakly semifitted sublocale with compact open domain, and this gives a bijection between such sublocales of Y and the global points of VY .*

Proof. See [15]; the result derives ultimately from [5]. ■

By a *weakly semifitted* sublocale is meant the meet of a weakly closed sublocale and a fitted sublocale.

One application of powerlocales is that openness and compactness of a locale are reflected in the structure of powerlocales.

We say that a locale X is *local* if the unique map $! : X \rightarrow 1$ has a left adjoint, a global point $\perp : 1 \rightarrow X$ such that $!; \perp \sqsubseteq \text{Id}_X$. This can be expressed by saying that \perp is less than the generic point, in other words that \perp is less

than all generalized points. Hence \perp is a bottom point of X in a strong sense. (This is a special case of the concept of *local topos*, which has been studied in [7]; see [6].) To prove that a locale is local, we shall normally give a geometric definition of the bottom point and show that it is less than every point. The geometricity allows us to deduce that the bottom point is less than the *generic* point.

Dually, a locale X is *colocal* iff the unique map $!$ has a *right* adjoint $\top : 1 \rightarrow X$ (a top point).

Theorem 3.6 [14]

1. A locale Y is open iff $P_L Y$ is colocal. Its top point then corresponds to Y as a sublocale of itself.
2. A locale Y is compact iff $P_U Y$ is local. Its bottom point then corresponds to Y as a sublocale of itself.

3.1 Powerlocales of continuous dcpos

We shall find it useful to refer to the localic theory of continuous dcpos (directed complete posets). By this we understand, in point-free terms, that its frame is constructively completely distributive (because these frames are the Stone duals of continuous dcpos). We use the characterization of [13] using continuous information systems (or *infosyses*), i.e. set D (of *tokens*) equipped with idempotent (transitive and interpolative) relation \prec .

Definition 3.7 A continuous information system (or infosys) is a set D (of tokens) equipped with an idempotent (transitive and interpolative) relation \prec .

An ideal of (D, \prec) is a directed lower subset $I \subseteq D$. In detail, it satisfies the following.

1. I is a lower set.
2. I is inhabited (nullary directedness).
3. If $s_1, s_2 \in I$ then there is some $s \in I$ with $s_1 \prec s$ and $s_2 \prec s$ (binary directedness).

A consequence of (3) is that I is rounded, i.e. if $s' \in I$ then there is some $s \in I$ with $s' \prec s$.

Theorem 3.8 A locale X is a continuous dcpo iff there is a continuous information system (D, \prec) such that X classifies the (geometric) theory $\text{Idl}(D)$ of ideals of D .

The frame $\Omega \text{Idl}(D)$ is the Scott topology on the set of points, ordered by subset inclusion. It can be presented as

$$\begin{aligned} \text{Fr}\langle \uparrow s \mid (s \in D) \mid \mathbf{true} \leq \bigvee_{s \in D} \uparrow s \\ \uparrow s \wedge \uparrow t \leq \bigvee \{ \uparrow u \mid s \prec u, t \prec u \} \rangle. \end{aligned}$$

From this it is clear that the opens $\uparrow s$ form a base.

Proof. See [13]. ■

Notation: If $A \subseteq D$ then we write $\prec A$ for $\{s \in D \mid \exists t \in A. s \prec t\}$. We also write $\prec s$ for $\prec \{s\}$.

In the case of continuous dcpos, the three powerlocale constructions on the ideal completions can be described in terms of the information systems [13]. If D is a continuous information system, then the powerlocales $P_L \text{Idl}(D)$, $P_U \text{Idl}(D)$ and $V \text{Idl}(D)$ are also continuous dcpos. They all take their tokens from the Kuratowski finite powerset $\mathcal{F}D$, but with three different orders: respectively,

- the *lower* order $S \prec_L T$ iff $\forall s \in S. \exists t \in T. s \prec t$
- the *upper* order $S \prec_U T$ iff $\forall t \in T. \exists s \in S. s \prec t$
- the *convex* order $S \prec_C T$ iff $S \prec_L T$ and $S \prec_U T$

To avoid confusion when we work on balls with \supset as our ordering, note that \succ_L is *not* the relational converse of \prec_L – in fact, that relational converse is \succ_U .

In due course we shall show that a parallel idea works for gms completions, defining distance functions on the finite powerset of a gms. It is clearly reminiscent of the Vietoris metric and its asymmetric parts, but the information system flavour shows up in the fact that we define it only on finite subsets.

Now some lemmas.

Lemma 3.9 *Let (D, \prec) be a continuous information system, and let $S \in \mathcal{F}D$.*

1. *If I is an ideal of $(\mathcal{F}D, \prec_L)$, then I is in the open $\bigwedge_{s \in S} \Diamond \uparrow s$ for $P_L \text{Idl}(D)$ iff $S \prec_L T$ for some $T \in I$.*
2. *If I is an ideal of $(\mathcal{F}D, \prec_U)$, then I is in the open $\Box(\bigvee_{s \in S} \uparrow s)$ for $P_U \text{Idl}(D)$ iff $S \prec_U T$ for some $T \in I$.*
3. *If I is an ideal of $(\mathcal{F}D, \prec_C)$, then the same conditions hold for opens of $V \text{Idl}(D)$.*

Proof. The proof of [13, Theorem 4.3] deals explicitly with the Vietoris powerlocale. From it we see that $\Diamond \uparrow s$ corresponds to $\bigvee \{\uparrow T \mid T \cap \succ s \neq \emptyset\}$, which equals $\bigvee \{\uparrow T \mid \{s\} \prec_L T\}$. Now I is in the open $\uparrow T$ iff $T \in I$, and so we can deduce (in the context of part 3) that I is in $\Diamond \uparrow s$ iff $\{s\} \prec_L T$ for some $T \in I$. The first condition readily follows. Similarly, $\Box(\bigvee_{s \in S} \uparrow s)$ corresponds to $\bigvee \{\uparrow T \mid T \subseteq \succ s\}$, which equals $\bigvee \{\uparrow T \mid S \prec_U T\}$.

[13] leaves to the reader the easier cases of the lower and upper powerlocales, but they yield the same conditions (for parts 1 and 2). ■

Lemma 3.10 *Let (D, \prec) be a continuous information system.*

1. *The map $\downarrow: V \text{Idl}(D) \rightarrow P_L \text{Idl}(D)$ maps each ideal I of $(\mathcal{F}D, \prec_C)$ to $\prec_L I$.*

2. The map $\uparrow: \text{VIdl}(D) \rightarrow \text{P}_U \text{Idl}(D)$ maps each ideal I of $(\mathcal{F}D, \prec_C)$ to $\prec_U I$.

Proof. 1. Suppose J is an ideal of $(\mathcal{F}D, \prec_L)$. Using Lemma 3.9 we have $S \in J$ iff J is in $\bigwedge_{s \in S} \diamond \uparrow s$. It follows that $S \in \downarrow I$ iff $S \prec_L T$ for some $T \in I$, i.e. $S \in \prec_L I$.

2. This is similar. ■

The following lemma is the analogue for continuous information systems of the fact that non-expansive functions between gms's lift to maps between their completions.

In [19] the gms result is proved in a rather deep form, saying that a certain localic geometric morphism (the completion of the generic gms) is an opfibration. In fact, an analogous opfibration result holds here. If $[\text{Infosys}]$ is the classifying topos for the theory of idempotent relations, then the ideal completion of the generic information system gives a localic map $[\text{Infosys}][\text{ideal}] \rightarrow [\text{Infosys}]$. The underlying result is that this localic map is an opfibration.

Proposition 3.11 *Let (D, \prec) and (E, \prec) be two continuous information systems, and let $f : D \rightarrow E$ preserve \prec . (Thus f is just a homomorphism of information systems.) Then f lifts to a map $f' : \text{Idl } D \rightarrow \text{Idl } E$,*

$$f'(I) = \prec \{f(s) \mid s \in I\}.$$

Proof. It is straightforward to check that if I is an ideal of D then $f'(I)$ as defined above is an ideal of E . ■

3.2 Synthetic reasoning with powerlocales

In [14] are developed some synthetic methods for reasoning with powerlocales, which allow points of $\text{P}_L Y$ or $\text{P}_U Y$ to be treated like collections of points of Y . We briefly recall some notation from there:

- $! : Y \rightarrow 1$ is the unique map.
- If x and U are points of Y and $\text{P}_L Y$, then we write $x \in U$ iff $\downarrow x \sqsubseteq U$.
- If x and U are points of Y and $\text{P}_U Y$, then we write $x \in U$ iff $\downarrow x \sqsupseteq U$.
- $\times : \text{P}_* Y \times \text{P}_* Z \rightarrow \text{P}_*(Y \times Z)$ (where $*$ is either L or U) is the “Cartesian product map”, $(x, y) \in U \times V$ iff $x \in U$ and $y \in V$.

Note also that $\text{P}_L 1 \cong \text{P}_U 1 \cong \mathbb{S}$, the Sierpinski locale, and $\downarrow : 1 \rightarrow \text{P}_L 1$ and $\uparrow : 1 \rightarrow \text{P}_U 1$ are, respectively, the top and bottom points \top and \perp of \mathbb{S} .

The basic technique arising from [14] is that if U and V are points of $\text{P}_L Y$, then $U \sqsubseteq V$ iff every $x \in U$ is also in V . (The techniques for the upper powerlocale are formally dual, with $U \sqsupseteq V$ iff every $x \in U$ is also in V .) This is in fact an instance of our geometric reasoning, for the generic “ $x \in U$ ” is just the generic point of the corresponding lax pullback (Theorem 3.3 and the remarks after it).

We also have to consider powerlocale points of the form $P_L f(U)$ where $f : Y \rightarrow Z$ is a map. To show $P_L f(U) \subseteq V$ it is equivalent to show that for every $x \in U$ we have $f(x) \in V$; however, the reverse direction \supseteq is trickier. We need that if $y \in V$ then $y \in P_L f(U)$, i.e. $\downarrow y \subseteq P_L f(U)$. From the basic definition of the specialization order \sqsubseteq , this amounts to showing that if y is in $b \in \Omega Z$ (it suffices to take b from a basis) then $P_L f(U)$ is in $\Diamond b$, i.e. U is in $\Diamond \Omega f(b)$. Notice that in classical point-set topology, the points of $P_L Y$ are the closed subsets of Y , and U is in $\Diamond a$ iff U meets a : so U is in $\Diamond \Omega f(b)$ iff U meets $f^{-1}(b)$, i.e. iff $f(U)$ meets b . Interpreted classically, therefore, the reasoning shows that y is in $P_L f(U)$ iff every open neighbourhood of y meets $f(U)$, and this is exactly what would be called for if $P_L f(U)$ were the closure of the direct image of U . It is remarkable that the constructively valid synthetic reasoning recreates a classical argument, even though the classical justification of the argument fails quite comprehensively.

4 The main results

4.1 The ball domain

Before dealing with the powerlocales, we first investigate a *ball domain* construction that turns out to be technically useful. It relates the gms powerlocale constructions to continuous dcpos. The ball domain is also related to the continuous dcpo of “formal balls” used in Edalat and Heckmann [2], though their spatial construction is constructively inequivalent to ours.

The definition of Cauchy filter (Definition 2.3) falls naturally into two parts. The first is that it is a *filter* of formal balls, and then the second, the Cauchy property, is the full strength of condition (3). Our ball domain is got by considering arbitrary filters, whether Cauchy or not, as ideals with respect to \supset , and thereby gaining access to the results in Subsection 3.1.

The order \subset defined on the formal balls (elements of $X \times Q_+$) is transitive and interpolative, and so the same goes for its relational coverse \supset .

Definition 4.1 *The ball domain $\text{Ball}(X)$ of a gms X is the ideal completion of $(X \times Q_+, \supset)$.*

In practice it is more natural to view the points of the ball domain as *filters* of $X \times Q_+$ with respect to \subset .

Proposition 4.2 *A point of $\text{Ball}(X)$ is a subset M of $X \times Q_+$ satisfying the conditions (1) and (2) of Definition 2.3, and in addition that for some δ there is x with $B_\delta(x) \in M$.*

Proof. Condition (1) says that M is rounded lower with respect to \supset , (2) is binary directedness and the new condition is nullary directedness, i.e. inhabitedness. ■

Hence \overline{X} embeds as a sublocale of the continuous dcpo $\text{Ball}(X)$, and in turn $\text{Ball}(X)$ embeds as a sublocale of $X\text{-Mod}$. The points of $X\text{-Mod}$ are the

rounded upper sets of balls, those of $\text{Ball}(X)$ are the filters, and those of \overline{X} are the Cauchy filters. We can be more precise about the second embedding, using powerlocales.

Proposition 4.3 *Let X be a gms. Then $X\text{-Mod}$ is homeomorphic to $P_L(\text{Ball}(X))$.*

Proof. In [13] it is shown that for any idempotent order (D, \prec) , the opens of $\text{Idl}(D)$ are the rounded upper subsets of D and that these are equivalent to points of the Hoffmann-Lawson dual (which corresponds to taking the opposite of the idempotent order) of the upper powerlocale $P_U \text{Idl}(D)$. Now the points of $X\text{-Mod}$, the rounded upper sets of formal open balls, are the opens of the dual of $\text{Ball}(X)$, and hence points of the dual of the upper powerlocale of the dual of $\text{Ball}(X)$. But this is homeomorphic to $P_L(\text{Ball}(X))$. ■

To express this more concretely, a point of $X\text{-Mod}$, a rounded upper subset of $X \times Q_+$ with respect to \subset , gives an ideal \mathcal{FI} of $\mathcal{F}(X \times Q_+)$ with respect to \supset_L . In the inverse direction, J gives rise to $\{s \mid \{s\} \in J\}$.

4.2 The lower powerlocale, P_L

Definition 4.4 *Let X be a gms. We define its lower powerspace, $\mathcal{F}_L X$, by taking the elements to be the finite subsets of X , with distance $\mathcal{F}_L X(S, T) = \max_{x \in S} \min_{y \in T} X(x, y)$.*

This defines a map $\mathcal{F}_L : [\text{gms}] \rightarrow [\text{gms}]$.

The fact that \mathcal{F}_L is a map (a geometric morphism) depends on the geometricity of the definition. In slightly more detail,

$$\mathcal{F}_L X(S, T) < q \text{ iff } \forall x \in S. \exists y \in T. X(x, y) < q,$$

which is geometric.

We can conveniently summarize properties of \mathcal{F}_L in the following result.

Proposition 4.5 *\mathcal{F}_L is the object part of a monad on $[\text{gms}]$ (in other words, it gives a monad in the 2-category **Top**).*

Proof. For any gms X we have homomorphisms $\{-\} : X \rightarrow \mathcal{F}_L X$, $x \mapsto \{x\}$, and $\bigcup : \mathcal{F}_L^2 X \rightarrow \mathcal{F}_L X$, $\mathcal{U} \mapsto \bigcup \mathcal{U}$. In fact, $\{-\}$ is an isometry. To show that \bigcup is a homomorphism (non-expansive),

$$\begin{aligned} \mathcal{F}_L^2 X(\mathcal{U}, \mathcal{V}) < q &\iff \forall U \in \mathcal{U}. \exists V \in \mathcal{V}. \mathcal{F}_L X(U, V) < q \\ &\iff \forall U \in \mathcal{U}. \exists V \in \mathcal{V}. \forall u \in U. \exists v \in V. X(u, v) < q \\ &\implies \forall U \in \mathcal{U}. \forall u \in U. \exists V \in \mathcal{V}. \exists v \in V. X(u, v) < q \\ &\iff \forall u \in \bigcup \mathcal{U}. \exists v \in \bigcup \mathcal{V}. X(u, v) < q \\ &\iff \mathcal{F}_L X(\bigcup \mathcal{U}, \bigcup \mathcal{V}) < q \end{aligned}$$

The rest follows from the known properties of \mathcal{F} as a monad on **Set**. ■

We shall show that $\overline{\mathcal{F}_L X}$ is homeomorphic to the lower powerlocale $P_L \overline{X}$, and we work by embedding $\overline{\mathcal{F}_L X}$ and $P_L \overline{X}$ in two continuous dcpos, namely $\text{Ball}(\mathcal{F}_L X)$ and $P_L \text{Ball}(X)$ respectively.

A point of $\text{Ball}(\mathcal{F}_L X)$ is an ideal of $\mathcal{F}X \times Q_+$ with respect to the \supset corresponding to \mathcal{F}_L , while a point of $P_L \text{Ball}(X)$ is an ideal of $\mathcal{F}(X \times Q_+)$ with respect to \supset_L . Let us define $\phi : \mathcal{F}X \times Q_+ \rightarrow \mathcal{F}(X \times Q_+)$ by $\phi(B_\delta(S)) = \{B_\delta(s) \mid s \in S\}$. We have

$$\begin{aligned} B_\delta(S) \supset B_\varepsilon(T) &\iff \mathcal{F}_L X(S, T) + \varepsilon < \delta \\ &\iff \varepsilon < \delta \text{ and } \forall s \in S. \exists t \in T. X(s, t) + \varepsilon < \delta \\ &\iff \varepsilon < \delta \text{ and } \forall s \in S. \exists t \in T. B_\delta(s) \supset B_\varepsilon(t) \\ &\iff \varepsilon < \delta \text{ and } \phi(B_\delta(S)) \supset_L \phi(B_\varepsilon(T)). \end{aligned}$$

(The explicit condition “ $\varepsilon < \delta$ ” is needed only to cover the case $S = \emptyset$. If $S \neq \emptyset$ then it is already implied by $\phi(B_\delta(S)) \supset_L \phi(B_\varepsilon(T))$.)

It follows by Proposition 3.11 that we get a continuous map $\phi' : \text{Ball}(\mathcal{F}_L X) \rightarrow P_L \text{Ball}(X)$, mapping I to $\supset_L \{\phi(B_\delta(S)) \mid B_\delta(S) \in I\}$. We can simplify this description by using the homeomorphism of Proposition 4.3, and in these terms we have

$$\phi'(I) = \supset \{B_\delta(s) \mid B_\delta(S) \in I, s \in S\}.$$

Lemma 4.6 $B_\delta(s) \in \phi'(I)$ iff there is some $B_\delta(S) \in I$ with $s \in S$.

Proof. \Rightarrow : Suppose $B_\delta(s) \supset B_\varepsilon(t)$ with $t \in T$ and $B_\varepsilon(T) \in I$. Then $B_\delta(\{s\} \cup T) \supset B_\varepsilon(T)$.

\Leftarrow : Given $B_\delta(S) \in I$, there is some $\delta' < \delta$ with $B_{\delta'}(S) \in I$, and then $B_\delta(s) \supset B_{\delta'}(s)$. ■

ϕ' is not itself a homeomorphism, but we shall show it restricts to a homeomorphism between $\overline{\mathcal{F}_L X}$ and $P_L \overline{X}$.

We must identify the points of $X\text{-Mod} \cong P_L \text{Ball}(X)$ that lie in $P_L \overline{X}$.

Lemma 4.7 Let J be a point of $X\text{-Mod}$, expressed as a subset of $X \times Q_+$. Then the following are equivalent:

1. J is in the image of $P_L \overline{X}$.
2. Balls in J have arbitrarily small refinements: in other words, if $\alpha > 0$ and $B_\delta(x) \in J$, then there is some $B_\varepsilon(y) \in J$ with $\varepsilon < \alpha$ and $B_\delta(x) \supset B_\varepsilon(y)$.

Proof. Let $Y \rightarrow Z$ be an arbitrary locale embedding, with ΩY presented over ΩZ by relations $a \leq b$ for $(a, b) \in R \subseteq \Omega Z \times \Omega Z$. By a routine application of the coverage theorem (see [15]), we have

$$\Omega Y \cong \text{SupLat} \langle \Omega Z \text{ (qua } \text{SupLat}) \mid a \wedge c \leq b \wedge c \text{ } ((a, b) \in R, c \in \Omega Z) \rangle$$

and it follows that

$$\begin{aligned}\Omega P_L Y &= \text{Fr}\langle \Omega Y \text{ (qua SupLat)} \rangle \\ &\cong \text{Fr}\langle \Omega Z \text{ (qua SupLat)} \mid a \wedge c \leq b \wedge c \ ((a, b) \in R, c \in \Omega Z) \rangle \\ &\cong \text{Fr}\langle \Omega P_L Z \text{ (qua Fr)} \mid \Diamond(a \wedge c) \leq \Diamond(b \wedge c) \ ((a, b) \in R, c \in \Omega Z) \rangle\end{aligned}$$

In our present case we have $Y = \overline{X}$, $Z = \text{Ball}(X)$, with $\Omega \overline{X}$ presented over $\Omega \text{Ball}(X)$ by relations $\mathbf{true} \leq \bigvee_y B_\alpha(y)$ ($\alpha > 0$). Hence, using the fact that the $B_\delta(x)$'s are a base for $\text{Ball}(X)$, we find that $\Omega P_L \overline{X}$ is presented over $\Omega P_L \text{Ball}(X)$ by relations

$$\begin{aligned}\Diamond B_\delta(x) &\leq \bigvee_y \Diamond(B_\delta(x) \wedge B_\alpha(y)) \\ &= \bigvee \{ \Diamond B_{\varepsilon'}(y') \mid B_\delta(x) \supset B_{\varepsilon'}(y') \text{ and } \varepsilon' < \alpha \}\end{aligned}$$

Equivalence of (1) with (2) now follows, because (using [13]) J satisfies $\Diamond B_\delta(x)$ iff $B_\delta(x) \in J$. ■

Notice that although the statement of the lemma is geometric, its proof is not – it uses frames and suplattices, which are not geometric constructions. However, the proof is topos-valid and so holds at every stage of definition. Implicitly, it also uses the fact that the powerlocale constructions themselves are geometric [17].

Theorem 4.8 *If X is a gms then $\overline{\mathcal{F}_L X}$ is homeomorphic to the lower powerlocale $P_L \overline{X}$.*

Proof. Suppose we are given I in $\text{Ball}(\mathcal{F}_L X)$. We see that if I is in $\overline{\mathcal{F}_L X}$ (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (2) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_L X}$ to $P_L \overline{X}$. Its inverse ψ is given by

$$\psi(J) = \{B_\delta(S) \mid \forall s \in S. B_\delta(s) \in J\}$$

To show that this is directed, first note that for balls of $\mathcal{F}_L X$ we have

$$\begin{aligned}B_\delta(S) \supset B_\varepsilon(T) &\Leftrightarrow \varepsilon < \delta \text{ and } \forall s \in S. B_\delta(\{s\}) \supset B_\varepsilon(T) \\ &\Leftrightarrow \varepsilon < \delta \text{ and } \forall s \in S. \exists t \in T. B_\delta(\{s\}) \supset B_\varepsilon(\{t\}).\end{aligned}$$

Allowing for the case $S = \emptyset$, we see it suffices to find arbitrarily small common refinements in $\psi(J)$ for families $B_{\delta_i}(S_i)$ where each S_i is a singleton. So suppose $B_{\delta_i}(s_i) \in J$ ($1 \leq i \leq n$) and $\alpha \in Q_+$. For each i we can find $\delta'_i < \delta_i$ with $B_{\delta'_i}(s_i) \in J$. Let $\varepsilon = \min(\alpha, \min_i(\delta_i - \delta'_i))$. Then for each i we can find $B_{\varepsilon_i}(t_i) \in J$ with $B_{\varepsilon_i}(t_i) \subset B_{\delta'_i}(s_i)$ and $\varepsilon_i \leq \varepsilon$. Then $B_\varepsilon(t_i) \in J$ and

$$X(s_i, t_i) + \varepsilon < X(s_i, t_i) + \varepsilon_i + \varepsilon < \delta'_i + \varepsilon \leq \delta_i,$$

so $B_\varepsilon(t_i) \subset B_{\delta_i}(s_i)$. Setting $T = \{t_i \mid 1 \leq i \leq n\}$, we have $B_\delta(\{s_i\}) \supset B_\varepsilon(T) \in \psi(J)$.

The proof that ψ is indeed the inverse of the restriction of ϕ' is largely routine. The least trivial part is to show that $\psi(\phi'(I)) \subseteq I$. Suppose $B_\delta(S) \in \psi(\phi'(I))$. For every $s \in S$ there is some $B_\delta(T) \in I$ with $s \in T$. We can now find $B_\alpha(U) \in I$ with $\alpha < \delta$ such that $B_\delta(T) \supset B_\alpha(U)$ for each of these $B_\delta(T)$ s. Then $B_\delta(S) \supset B_\alpha(U)$, so $B_\delta(S) \in I$. ■

The lower powerlocale $P_L Y$ is always local, its bottom point corresponding to the empty sublocale of Y . For many purposes it is desirable to exclude this and work with the open sublocale $P_L^+ Y$ (or $\Diamond \mathbf{true}$). We show that this corresponds to excluding the empty set from our finite powerspace (recall that for finite sets, emptiness is decidable). Notice that including the empty set had inevitably taken us beyond finitary metrics, for $\mathcal{F}_L X(T, \emptyset) = \infty$ if $T \neq \emptyset$. ($\mathcal{F}_L X(\emptyset, T) = 0$ always.)

Proposition 4.9 *The homeomorphism of Theorem 4.8 restricts to a homeomorphism between $P_L^+ \overline{X}$ and $\mathcal{F}_L^+ X$, where the elements of the space $\mathcal{F}_L^+ X$ are the finite non-empty subsets of X , and its metric is as before.*

Proof. Suppose $I \subseteq \mathcal{F}X \times Q_+$ is a point of $\overline{\mathcal{F}_L X}$. For every ε we have $B_\varepsilon(\emptyset) \supset B_{\varepsilon/2}(T) \in I$ for some T . But $\{B_\varepsilon(\emptyset) \mid \varepsilon \in Q_+\}$ is already a point of $\overline{\mathcal{F}_L X}$ and hence must be the bottom point, so $P_L^+ \overline{X}$ corresponds to those I that contain some $B_\varepsilon(T)$ with $T \neq \emptyset$. But once we have some such $B_\varepsilon(T)$ then we have arbitrarily small ones, for if $B_\alpha(S)$ refines both $B_\varepsilon(T)$ and $B_\delta(\emptyset)$ then $S \neq \emptyset$ and $\alpha < \delta$. The result now follows. ■

Corollary 4.10 *If X is a gms then \overline{X} is open (i.e. as a locale).*

Proof. If $\delta > \varepsilon$ then $B_\delta(S) \supset B_\varepsilon(S \cup T)$. It follows that $\mathcal{F}X \times Q_+$ is a point of $\overline{\mathcal{F}_L X}$, and hence must be the top point. We can now apply Theorem 3.6. ■

4.3 The upper powerlocale, P_U

Definition 4.11 *Let X be a gms. We define its upper powerspace, $\mathcal{F}_U X$, by taking the elements to be the finite subsets of X , with distance $\mathcal{F}_U X(S, T) = \max_{y \in T} \min_{x \in S} X(x, y)$.*

This defines a map $\mathcal{F}_U : [\mathbf{gms}] \rightarrow [\mathbf{gms}]$.

Proposition 4.12 *\mathcal{F}_U is the object part of a monad on $[\mathbf{gms}]$.*

Proof. For any gms X we have homomorphisms $\{-\} : X \rightarrow \mathcal{F}_U X$, $x \mapsto \{x\}$, and $\cup : \mathcal{F}_U^2 X \rightarrow \mathcal{F}_U X$, $\mathcal{U} \mapsto \bigcup \mathcal{U}$. Again, $\{-\}$ is an isometry. The proof that \bigcup is a homomorphism is similar to that for \mathcal{F}_L . ■

We shall prove that $\overline{\mathcal{F}_U X}$ is homeomorphic to the upper powerlocale $P_U \overline{X}$ using techniques somewhat similar to those for the lower powerlocale (though without the convenient embedding in $X\text{-Mod}$). We embed $\overline{\mathcal{F}_U X}$ in $\text{Ball}(\mathcal{F}_U X)$ and $P_U \overline{X}$ in $P_U \text{Ball}(X)$. The same function $\phi : \mathcal{F}X \times Q_+ \rightarrow \mathcal{F}(X \times Q_+)$ preserves order:

$$B_\delta(S) \supset B_\varepsilon(T) \text{ iff } \varepsilon < \delta \text{ and } \phi(B_\delta(S)) \supset_U \phi(B_\varepsilon(T))$$

(though this time \supset corresponds to the upper distance function, $\mathcal{F}_U X(S, T) + \varepsilon < \delta$). It follows that we get a continuous map $\phi' : \text{Ball}(\mathcal{F}_U X) \rightarrow \text{P}_U \text{Ball}(X)$,

$$\phi'(I) = \supset_U \{ \phi(B_\delta(S)) \mid B_\delta(S) \in I \},$$

which we show restricts to a homeomorphism between $\overline{\mathcal{F}_U X}$ and $\text{P}_U \overline{X}$. Again, the bulk of the work lies in identifying the points of $\text{P}_U \text{Ball}(X)$ that lie in $\text{P}_U \overline{X}$.

Lemma 4.13 *Let J be a point of $\text{P}_U \text{Ball}(X)$. Then the following are equivalent:*

1. J is in $\text{P}_U \overline{X}$.
2. J contains elements $\phi(B_\varepsilon(T))$ for arbitrarily small ε .
3. If $\alpha > 0$ and $U \in J$, then there is some $B_\varepsilon(T)$ with $\varepsilon < \alpha$ and $U \supset_U \phi(B_\varepsilon(T)) \in J$.

Proof. Let $Y \rightarrow Z$ be an arbitrary locale embedding, with ΩY presented over ΩZ by relations $a \leq b$ for $(a, b) \in R \subseteq \Omega Z \times \Omega Z$. By a routine application of the preframe coverage theorem [8], we have

$$\Omega Y \cong \text{PreFr}(\Omega Z \text{ (qua PreFr)} \mid a \vee c \leq b \vee c \text{ } ((a, b) \in R, c \in \Omega Z))$$

and it follows that

$$\begin{aligned} \Omega \text{P}_U Y &= \text{Fr}(\Omega Y \text{ (qua PreFr)}) \\ &\cong \text{Fr}(\Omega Z \text{ (qua PreFr)} \mid a \vee c \leq b \vee c \text{ } ((a, b) \in R, c \in \Omega Z)) \\ &\cong \text{Fr}(\Omega \text{P}_U Z \text{ (qua Fr)} \mid \Box(a \vee c) \leq \Box(b \vee c) \text{ } ((a, b) \in R, c \in \Omega Z)) \end{aligned}$$

In our present case we have $Y = \overline{X}$, $Z = \text{Ball}(X)$, with $\Omega \overline{X}$ presented over $\Omega \text{Ball}(X)$ by relations $\mathbf{true} \leq \bigvee_y B_\varepsilon(y)$ ($\varepsilon > 0$). The c 's appearing above make no difference ($\mathbf{true} \vee c = \mathbf{true}$), so we find that $\Omega \text{P}_U \overline{X}$ is presented over $\Omega \text{P}_U \text{Ball}(X)$ by relations

$$\mathbf{true} \leq \Box \bigvee_y B_\varepsilon(y) = \bigvee^\dagger \{ \Box \bigvee_{y \in T} B_\varepsilon(y) \mid T \in \mathcal{F}X \}$$

Equivalence of (1) with (2) now follows, because J satisfies $\Box \bigvee_{y \in T} B_\varepsilon(y)$ iff $\phi(B_\varepsilon(T)) \in J$.

(3) \Rightarrow (2) follows easily because J is inhabited (so we can find a U in it). For the converse, let J satisfy (2), and let $U \in J$, $\alpha > 0$. We can find $U \supset_U U' \in J$, and by pressing the finitely many strict inequalities involved we can find $\varepsilon > 0$ such that

$$\forall B_\gamma(y) \in U'. \exists B_\beta(x) \in U. B_\beta(x) \supset B_{\gamma+\varepsilon}(y)$$

In addition, we can require $\varepsilon < \alpha$. Choose S such that $\phi(B_\varepsilon(S)) \in J$, let V' be a common refinement of U' and $\phi(B_\varepsilon(S))$ in J , and let $T = \{z \mid \exists \delta. B_\delta(z) \in V'\}$. If $B_\delta(z) \in V'$ then $\delta < \varepsilon$, and it follows that $\phi(B_\varepsilon(T)) \in J$. If $B_\delta(z) \in V'$ then we can find $B_\gamma(y) \in U'$ and $B_\beta(x) \in U$ with $B_\gamma(y) \supset B_\delta(z)$ and $B_\beta(x) \supset B_{\gamma+\varepsilon}(y)$, and so $B_\beta(x) \supset B_{\delta+\varepsilon}(z) \supset B_\varepsilon(z)$. It follows that $U \supset_U \phi(B_\varepsilon(T))$. ■

Lemma 4.14 *If I is a point of $\overline{\mathcal{F}_U X}$ and $B_\varepsilon(\emptyset) \in I$ for some ε , then I contains every ball $B_\delta(S)$.*

Proof. We can find some $B_\alpha(T) \in I$ with $B_\alpha(T) \subset B_\varepsilon(\emptyset)$ and $\alpha < \delta$, so $\mathcal{F}_U X(\emptyset, T) + \alpha < \varepsilon$. It follows that $T = \emptyset$, for otherwise $\mathcal{F}_U X(\emptyset, T) = \infty$. Then $B_\delta(S) \supset B_\alpha(\emptyset)$. ■

Theorem 4.15 *$\overline{\mathcal{F}_U X}$ is homeomorphic to the upper powerlocale $P_U \overline{X}$.*

Proof. Suppose we are given I in $\text{Ball}(\mathcal{F}_U X)$. If I is in $\overline{\mathcal{F}_U X}$, (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (2) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_U X}$ to $P_U \overline{X}$.

Its inverse is given by $J \mapsto \phi^{-1}(J)$. A little care is needed in showing $\phi^{-1}(\phi'(I)) \subseteq I$. If $B_\delta(S) \in \phi^{-1}(\phi'(I))$ then $\phi(B_\delta(S)) \supset_U \phi(B_\varepsilon(T))$ for some $B_\varepsilon(T) \in I$. If $T \neq \emptyset$ then $B_\delta(S) \supset B_\varepsilon(T)$, so $B_\delta(S) \in I$. But if $T = \emptyset$ then by Lemma 4.14 $B_\delta(S) \in I$. ■

Just as for the lower powerlocale, we can “exclude the empty set”, which in the upper powerlocale corresponds to a top point. Excluding this gives us a closed sublocale $P_U^+ Y$ (the complement of $\square\text{false}$). Again, including the empty set had taken us beyond finitary metrics: $\mathcal{F}_U X(\emptyset, T) = \infty$ if $T \neq \emptyset$.

Proposition 4.16 *The homeomorphism of Theorem 4.15 restricts to a homeomorphism between $P_U^+ \overline{X}$ and $\overline{\mathcal{F}_U^+ X}$, where the elements of the space $\mathcal{F}_U^+ X$ are the finite non-empty subsets of X , and its metric is as before.*

Proof. Identifying points of $\overline{\mathcal{F}_U X}$ with certain subsets I of $\mathcal{F}X \times Q_+$, it is clear that the top point is the whole of $\mathcal{F}X \times Q_+$. By Lemma 4.14 we see that the points of $P_U^+ \overline{X}$ are those I containing no $B_\varepsilon(\emptyset)$, and the result follows. ■

As an application, we can prove that a gms completion \overline{X} is compact iff X is totally bounded. (More information on totally bounded quasimetric spaces can be found in [12].) The usual definition of this is that for every $\delta > 0$ there is a finite subset $S \subseteq X$ that is a δ -cover in the sense that for every $x \in X$ there is some $s \in S$ with $X(s, x) < \delta$. The universal quantification here ($\forall x \in X$) is intuitionistic but not geometric, so we give a geometric definition in which the relationship between δ and S is given as a relation Cov in the theory. Hence total boundedness is not (geometrically) a *property* of a gms but additional *structure*. We shall, however, present it in such a way that it is unique when it exists.

Definition 4.17 *A totally bounded gms X is one equipped with a relation $\text{Cov} \subseteq \mathcal{F}X \times Q_+$ satisfying the following axioms:*

$$\forall x \in X. \forall S \in \mathcal{F}X. \forall \delta \in Q_+. (\text{Cov}(S, \delta) \rightarrow \exists s \in S. X(s, x) < \delta) \quad (\text{TB1})$$

$$\forall \delta \in Q_+. \exists S \in \mathcal{F}X. \text{Cov}(S, \delta) \quad (\text{TB2})$$

$$\forall S \in \mathcal{F}X. \forall \delta \in Q_+. (\text{Cov}(S, \delta) \rightarrow \exists \delta' \in Q_+. (\delta' < \delta \wedge \text{Cov}(S, \delta'))) \quad (\text{TB3})$$

$$\begin{aligned} \forall S, T \in \mathcal{F}X. \forall \delta, \varepsilon \in Q_+. (\text{Cov}(T, \varepsilon) \wedge \mathcal{F}_U X(S, T) < \delta \\ \rightarrow \text{Cov}(S, \delta + \varepsilon)) \end{aligned} \quad (\text{TB4})$$

Note that from TB1 we can deduce that if $\text{Cov}(S, \delta)$ then $\mathcal{F}_U X(S, T) < \delta$ for every $T \in \mathcal{F}X$.

Note also that if Cov is viewed (in the obvious way) as a set of formal open balls for $\mathcal{F}_U X$, then conditions TB2, TB3 and TB4 are equivalent to saying that Cov is a rounded upper set containing balls of arbitrarily small radius (the Cauchy property).

Proposition 4.18 *Let X be a totally bounded gms. Then $\text{Cov}(S, \delta)$ iff $\exists \delta' < \delta$ such that S is a δ' -cover (i.e. $\forall x \in X. \exists s \in S. X(s, x) < \delta'$).*

Proof. \Rightarrow : Combine TB3 with TB1.

\Leftarrow : By TB2, we can find T with $\text{Cov}(T, \delta - \delta')$, and by hypothesis $\mathcal{F}_U X(S, T) < \delta'$. Then $\text{Cov}(S, \delta)$ by TB4. ■

It follows that on any given gms X , if Cov can be defined at all then it is unique.

The axiomatization is strong in order to characterize Cov uniquely. However, it is useful to know that we can get by with weaker structure.

Proposition 4.19 *Let X be a gms and let $\text{Cov}_0 \subseteq \mathcal{F}X \times Q_+$ satisfy axioms TB1 and TB2 for Cov . Then X can be given the structure of total boundedness.*

Proof. Define $\text{Cov}(S, \delta)$ if for some $\delta' < \delta$ and some T we have $\mathcal{F}_U X(S, T) < \delta'$ and $\text{Cov}_0(T, \delta - \delta')$. We prove the four axioms.

TB1: Suppose $\text{Cov}(S, \delta)$ with T and δ' as above, and suppose $x \in X$. There is some $t \in T$ with $X(t, x) < \delta - \delta'$, and then some $s \in S$ with $X(s, t) < \delta'$. Then $X(s, x) < \delta$.

TB2: Choose T with $\text{Cov}_0(T, \delta/2)$. We have $\mathcal{F}_U X(T, T) = 0 < \delta/2$, and it follows that $\text{Cov}(T, \delta)$.

TB3: Given $\text{Cov}(S, \delta)$, in the part of the definition that says $\mathcal{F}_U X(S, T) < \delta'$ we can reduce δ' to some δ'' and thereby reduce δ to $\delta - (\delta' - \delta'')$.

TB4: Given the hypotheses of TB4, we have some $\varepsilon' < \varepsilon$ and U such that $\text{Cov}_0(U, \varepsilon - \varepsilon')$ and $\mathcal{F}_U X(T, U) < \varepsilon'$. Then $\mathcal{F}_U X(S, U) < \delta + \varepsilon'$, so $\text{Cov}(S, \delta + \varepsilon)$. ■

Lemma 4.20 *Let X be a totally bounded gms. Then $\overline{\mathcal{F}_U X}$ is local, with Cov its least point.*

Proof. Since Cov is a subset of $\mathcal{F}X \times Q_+$, we can trivially reinterpret it as a set of formal open balls for $\mathcal{F}X$. Our task then is to show that it is a Cauchy filter, and that it is least amongst all Cauchy filters. The first and third conditions in Definition 2.3 follow directly from TB4 and TB2. For the second condition, suppose $\text{Cov}(S_i, \delta_i)$ ($i = 1, 2$). By TB3 we can find $\delta'_i < \delta_i$ and $\text{Cov}(S_i, \delta'_i)$. Let $\varepsilon = \min_i(\delta_i - \delta'_i)$ and choose T with $\text{Cov}(T, \varepsilon)$. Then it follows that $\mathcal{F}_U X(S_i, T) + \varepsilon < \delta_i$, which is what was needed.

Now let F be any point of $\overline{\mathcal{F}_U X}$. We want to show $\text{Cov} \subseteq F$. Suppose $\text{Cov}(S, \delta)$, and find $\delta' < \delta$ such that $\text{Cov}(S, \delta')$. By the Cauchy property for F ,

there is some T with $B_{\delta-\delta'}(T) \in F$. Then $\mathcal{F}_U X(S, T) < \delta'$ and so $B_{\delta-\delta'}(T) \subset B_\delta(S)$ and $B_\delta(S) \in F$. ■

Note the key role of geometricity here. When we let F be “any” point of $\overline{\mathcal{F}_U X}$, we allow arbitrary generalized points, not just global points. Essentially we show that Cov is less than the generic point of $\overline{\mathcal{F}_U X}$. Then geometricity allows us to transfer the argument to the classifying topos for $\overline{\mathcal{F}_U X}$.

Lemma 4.21 *Let X be a gms and suppose $\overline{\mathcal{F}_U X}$ is local. Then X is totally bounded.*

Proof. Let K be the least point of $\overline{\mathcal{F}_U X}$. We define $\text{Cov}(S, \delta)$ if $B_\delta(S) \in K$. Axioms TB2, TB3 and TB4 follow easily from the fact that K is a Cauchy filter. To prove TB1, suppose $B_\delta(S) \in K$ and we are given $x \in X$. There is a Cauchy filter for X , called $\mathcal{Y}(x)$ in the notation of [19], defined as $\{B_\varepsilon(y) \mid X(y, x) < \varepsilon\}$. The corresponding point $\uparrow \mathcal{Y}(x)$ of $\text{P}_U \overline{X} \cong \overline{\mathcal{F}_U X}$ is $\{B_\varepsilon(T) \mid \mathcal{F}_U X(T, \{x\}) < \varepsilon\}$. We have $K \sqsubseteq \uparrow \mathcal{Y}(x)$, and it follows that $\mathcal{F}_U X(S, \{x\}) < \delta$. Hence there is some $s \in S$ with $X(s, x) < \delta$. ■

Theorem 4.22 *Let X be a gms. Then \overline{X} is compact iff X is totally bounded.*

Proof. Combine Theorem 3.6 with the previous two lemmas. ■

4.4 The Vietoris powerlocale, \mathbf{V}

Definition 4.23 *Let X be a gms. We define its convex powerspace, $\mathcal{F}_C X$, by taking the elements to be the finite subsets of X , with distance $\mathcal{F}_C X(S, T) = \max(\mathcal{F}_L X(S, T), \mathcal{F}_U X(S, T))$.*

This defines a map $\mathcal{F}_C : [\text{gms}] \rightarrow [\text{gms}]$.

Proposition 4.24 *\mathcal{F}_C is the object part of a monad on $[\text{gms}]$.*

Proof. Just as before with $\{-\}$ and \cup . ■

Note that if X is symmetric, then so is $\mathcal{F}_C X$: for $\mathcal{F}_L X(S, T) = \mathcal{F}_U X(T, S)$.

We shall prove that $\overline{\mathcal{F}_C X}$ is homeomorphic to the Vietoris powerlocale $\mathbf{V} \overline{X}$ by embedding $\overline{\mathcal{F}_C X}$ in $\text{Ball}(\mathcal{F}_C X)$ and $\mathbf{V} \overline{X}$ in $\mathbf{V} \text{Ball}(X)$. The same function $\phi : \mathcal{F} X \times Q_+ \rightarrow \mathcal{F}(X \times Q_+)$ preserves order: we have

$$B_\delta(S) \supset B_\varepsilon(T) \text{ iff } \varepsilon < \delta \text{ and } \phi(B_\delta(S)) \supset_C \phi(B_\varepsilon(T))$$

(but this time \supset corresponds to the convex distance function). It follows that we get a continuous map $\phi' : \text{Ball}(\mathcal{F}_C X) \rightarrow \mathbf{V} \text{Ball}(X)$,

$$\phi'(I) = \sup_C \{\phi(B_\delta(S)) \mid B_\delta(S) \in I\},$$

which we show restricts to a homeomorphism between $\overline{\mathcal{F}_C X}$ and $\mathbf{V} \overline{X}$. Again, the bulk of the work lies in identifying the points of $\mathbf{V} \text{Ball}(X)$ that lie in $\mathbf{V} \overline{X}$.

Lemma 4.25 *Let J be a point of $\mathbf{V} \text{Ball}(X)$. Then the following are equivalent:*

1. J is in $\mathbf{V}\overline{X}$.
2. J contains elements $\phi(B_\varepsilon(T))$ for arbitrarily small ε .
3. If $\alpha > 0$ and $U \in J$, then there is some $B_\varepsilon(T)$ with $\varepsilon < \alpha$ and $U \supset_C \phi(B_\varepsilon(T)) \in J$.

Proof. Again, let $Y \rightarrow Z$ be an arbitrary locale embedding, with ΩY presented over ΩZ by relations $a \leq b$ for $(a, b) \in R \subseteq \Omega Z \times \Omega Z$. Combining the calculations of Theorems 4.8 and 4.15, we get

$$\begin{aligned} \Omega \vee Y &= \text{Fr}(\Omega Y \text{ (qua SupLat)}, \Omega Y \text{ (qua PreFr)}) \mid \\ &\quad \Box d \wedge \Diamond e \leq \Diamond(d \wedge e), \\ &\quad \Box(d \vee e) \leq \Box d \vee \Box e \\ &\cong \text{Fr}(\Omega \vee Z \text{ (qua Fr)} \mid \Diamond(a \wedge c) \leq \Diamond(b \wedge c), \Box(a \vee c) \leq \Box(b \vee c) \\ &\quad ((a, b) \in R, c \in \Omega Z)) \end{aligned}$$

In our present case, these relations reduce to

$$\begin{aligned} \Diamond B_\delta(x) &\leq \bigvee \{ \Diamond B_{\varepsilon'}(y') \mid B_\delta(x) \supset B_{\varepsilon'}(y') \text{ and } \varepsilon' < \varepsilon \} \\ \mathbf{true} &\leq \bigvee^\uparrow \{ \Box \bigvee_{y \in T} B_\varepsilon(y) \mid T \in \mathcal{FX} \} \end{aligned}$$

However, given the second, we have the first (in fact they are equivalent):
for

$$\begin{aligned} \Diamond B_\delta(x) &\leq \bigvee^\uparrow \{ \Diamond B_\delta(x) \wedge \Box \bigvee_{y \in T} B_\varepsilon(y) \mid T \in \mathcal{FX} \} \\ &\leq \bigvee^\uparrow \{ \Diamond(B_\delta(x) \wedge \bigvee_{y \in T} B_\varepsilon(y)) \mid T \in \mathcal{FX} \} \\ &= \bigvee_{y \in X} \Diamond(B_\delta(x) \wedge B_\varepsilon(y)) \leq \text{RHS of first} \end{aligned}$$

From Lemma 3.9 we see that J satisfies $\Box \bigvee_{y \in T} B_\varepsilon(y)$ iff $\phi(B_\varepsilon(T)) \supset_U U$ for some $U \in J$, and then $\phi(B_\varepsilon(T')) \supset_C U$ for some finite $T' \subseteq T$. The conditions therefore reduce to (2).

(3) \Rightarrow (2) is easy. For (2) \Rightarrow (3), suppose $U \in J$. Choose U' with $U \supset_C U' \in J$, and $\eta > 0$ such that $\eta < \alpha$, and

$$\begin{aligned} \forall B_\delta(x) \in U. \exists B_\varepsilon(y) \in U'. B_\delta(x) \supset B_{\varepsilon+\eta}(y), \\ \forall B_\varepsilon(y) \in U'. \exists B_\delta(x) \in U. B_\delta(x) \supset B_{\varepsilon+\eta}(y). \end{aligned}$$

Find S such that $\phi(B_\eta(S)) \in J$, and let V be a common refinement of U' and $\phi(B_\eta(S))$ in J . Then, much as in Lemma 4.13, we see that if $T = \{z \mid \exists b. B_\beta(z) \in V\}$ then $U \supset_C \phi(B_\eta(T)) \supset_C V$. ■

Lemma 4.26 *If I is a point of $\overline{\mathcal{F}_C X}$ and $B_\varepsilon(\emptyset) \in I$ for some ε , then $I = \{B_\delta(\emptyset) \mid \delta \in Q_+\}$.*

Proof. If $B_\delta(S) \supset B_\varepsilon(T)$ then $\mathcal{F}_C X(S, T)$ is finite, and it follows that S and T are either both empty or both non-empty. Hence for every $B_\alpha(S) \in I$ we have $S = \emptyset$. Then by the Cauchy property, I contains every $B_\delta(\emptyset)$. ■

Theorem 4.27 *$\overline{\mathcal{F}_C X}$ is homeomorphic to the Vietoris powerlocale $V \overline{X}$.*

Proof. The proof is similar to that for Theorem 4.15. Suppose we are given I in $\text{Ball}(\mathcal{F}_C X)$. If I is in $\overline{\mathcal{F}_C X}$ (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (2) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_C X}$ to $V \overline{X}$. Its inverse is given by $J \mapsto \phi^{-1}(J)$. (Lemma 4.26 is needed in proving $\phi^{-1}(\phi(I)) \subseteq I$.) ■

Again we can “exclude the empty set”, which in the Vietoris powerlocale VY is neither bottom nor top, but is isolated (a clopen point). Excluding it gives us a clopen sublocale $V^+ Y$ ($\Diamond \mathbf{true}$ is now the complement of $\Box \mathbf{false}$).

Proposition 4.28 *The homeomorphism of Theorem 4.27 restricts to a homeomorphism between $V^+ \overline{X}$ and $\overline{\mathcal{F}_C^+ X}$, where the elements of the space $\mathcal{F}_C^+ X$ are the finite non-empty subsets of X , and its metric is as before.*

Proof. By Lemma 4.26, each point of $\overline{\mathcal{F}_C X}$ either is $\{B_\varepsilon(\emptyset) \mid \varepsilon \in Q_+\}$ (which corresponds to $\Box \mathbf{false}$) or contains only balls $B_\varepsilon(T)$ with T non-empty (corresponding to $\Diamond \mathbf{true}$). The result follows. ■

We shall later need the following observation.

Proposition 4.29 *For any gms X , the identity function on $\mathcal{F}X$ gives non-expansive maps $\mathcal{F}_C X \rightarrow \mathcal{F}_L X$ and $\mathcal{F}_C X \rightarrow \mathcal{F}_U X$. These lift to the maps $\downarrow: V \overline{X} \rightarrow P_L \overline{X}$ and $\uparrow: V \overline{X} \rightarrow P_U \overline{X}$.*

Proof. The first assertion is obvious.

For the rest, consider first \downarrow . We must show that the outer square commutes in this diagram.

$$\begin{array}{ccccc}
 \overline{\mathcal{F}_C X} & & \xrightarrow{\cong} & & V \overline{X} \\
 & \searrow & & \swarrow & \\
 & \text{Ball}(\mathcal{F}_C X) & \xrightarrow{\phi'} & & V(\text{Ball } X) \\
 \text{Id} \downarrow & \downarrow \text{Ball(Id)} & & \downarrow \downarrow & \downarrow \downarrow \\
 & \text{Ball}(\mathcal{F}_L X) & \xrightarrow{\phi'} & & P_L(\text{Ball } X) \\
 & \nearrow & & \nwarrow & \\
 \overline{\mathcal{F}_L X} & & \xrightarrow{\cong} & & P_L \overline{X}
 \end{array}$$

All the diagonal morphisms are inclusions, and so it suffices to show that the outer square commutes when postcomposed with $P_L \overline{X} \hookrightarrow P_L(\text{Ball } X)$; but for this it suffices to show that all the inner quadrilaterals commute. For the top

and bottom trapezia, this follows from the construction of the isomorphisms in Theorems 4.27 and 4.8. For the left hand trapezium, it is seen by comparing the calculations for $\overline{\text{Id}}$ and $\text{Ball}(\text{Id})$, using Propositions 2.5 and 3.11. For the right-hand trapezium, it follows from naturality of \downarrow . Finally, for the inner square it follows from Lemma 3.10 and Proposition 3.11.

The proof for \uparrow is similar. ■

5 The Heine-Borel Theorem

As an application of the powerlocale methods, we get a new constructive proof of the localic Heine-Borel theorem [3]. Incidentally, this is an illustration of the disadvantages of trying to use point-set topology constructively. In any Grothendieck topos we can construct (non-geometrically) a Dedekind real number object R , the set of points of the locale \mathbb{R} ; then we can construct (again non-geometrically) a spatial locale of “spatial reals” whose frame is the subframe of $\mathcal{P}R$ given by the usual topology. But then the subspace $[0, 1]$ is not in general compact [4].

Recall from [19] that the real line \mathbb{R} (the locale whose points are the Dedekind sections of \mathbb{Q}) is homeomorphic to the localic completion of \mathbb{Q} with its usual metric, $\mathbb{Q}(q, r) = |q - r|$.

It is not hard to show that $[0, 1]$ is homeomorphic to $\overline{(0, 1) \cap \mathbb{Q}}$, after which its compactness follows from Theorem 4.22, and a similar technique works for other closed intervals. In fact in the case where $x < y$ we have $[x, y] \cong [0, 1]$. However, we can go further to show not only that the closed interval $[x, y]$ is compact, but also that it varies continuously with x and y : there is a corresponding “Heine-Borel” map HB_U from $\mathbb{R} \times \mathbb{R}$ to the upper powerlocale $P_U \mathbb{R}$.

We shall also strengthen the result by factoring the map HB_U via the Vietoris powerlocale. However, there is a problem here regarding the inverse image of the empty sublocale \emptyset : $[x, y]$ is empty iff $x > y$, and as a sublocale of $\mathbb{R} \times \mathbb{R}$, the relation $>$ is open but not closed. The upper powerlocale is compatible with this, for \emptyset is the top point and $\{\emptyset\}$ is open (it’s the open $\square \mathbf{false}$). In the lower powerlocale, on the other hand, \emptyset is the bottom point and $\{\emptyset\}$ is closed (the complement of $\diamond \mathbf{true}$). In the Vietoris powerlocale \emptyset is an isolated point. It follows that for neither of these can we map the whole of $\mathbb{R} \times \mathbb{R}$ to the powerlocale taking (x, y) to the sublocale $[x, y]$. We show instead that we have continuous maps

$$\begin{aligned} \text{HB}_L : \leq &\rightarrow P_L^+ \mathbb{R} \\ \text{HB}_C : \leq &\rightarrow V^+ \mathbb{R} \end{aligned}$$

where “ \leq ” here denotes the corresponding closed sublocale of $\mathbb{R} \times \mathbb{R}$.

The virtue of working with $V \mathbb{R}$ is that its points contain more information than those of $P_U \mathbb{R}$, and so more can be done with them. For instance, their sups and infs can be computed – we have maps sup and $\text{inf} : V^+ \mathbb{R} \rightarrow \mathbb{R}$ (lifting the non-expansive functions max and $\text{min} : \mathcal{F}_C^+ \mathbb{Q} \rightarrow \mathbb{Q}$), whereas we do not have

corresponding maps on $P_U^+ \mathbb{R}$. Hence for any map $f : \mathbb{R} \rightarrow \mathbb{R}$ we can compute its sup on $[x, y]$ as $\sup \circ V^+ f \circ \text{HB}_C(x, y)$.

5.1 Bounds of Vietoris points

In this subsection we examine the the maps \sup and \inf from $V^+ \mathbb{R}$ to \mathbb{R} . (We have to use $V^+ \mathbb{R}$ rather than $V \mathbb{R}$, because the sup and inf of the empty set would have to be infinite.) Defining the maps is an easy application of Proposition 4.28.

Proposition 5.1 *The functions \max and $\min : \mathcal{F}_C^+ \mathbb{Q} \rightarrow \mathbb{Q}$ are non-expansive.*

Proof. We prove the result for \max . For \min it is dual, by order reversal on \mathbb{Q} .

Let S and T be in $\mathcal{F}_C^+ \mathbb{Q}$, let $s_{\max} = \max(S)$, $t_{\max} = \max(T)$, and let $q = \mathcal{F}_C^+ \mathbb{Q}(S, T)$. Since $t_{\max} \in T$ we have that t_{\max} is within q of some $s \in S$ and then $t_{\max} \leq s + q \leq s_{\max} + q$. Similarly, $s_{\max} \leq t_{\max} + q$ and so $|t_{\max} - s_{\max}| \leq q$. ■

Definition 5.2 *We define $\sup : V^+ \mathbb{R} \rightarrow \mathbb{R}$ as $\overline{\max} : V^+ \mathbb{R} \cong \overline{\mathcal{F}_C^+ \mathbb{Q}} \rightarrow \overline{\mathbb{Q}} \cong \mathbb{R}$. Similarly, \inf is $\overline{\min}$.*

Our goal now is to show how, if U is a point of $V^+ \mathbb{R}$, $\sup(U)$ genuinely is the supremum of U . (We focus on \sup from now on; the results for \inf are entirely dual.) We show in fact that it is the greatest element of U : $\sup(U) \in U$, and if $x \in U$ (using the notation of the synthetic reasoning, as in Subsection 3.2) then $x \leq \sup(U)$.

Lemma 5.3 *If U is a point of $V^+ \mathbb{R}$, then $\sup(U) \in U$.*

Proof. We must show $\downarrow \sup(U) \sqsubseteq \downarrow U$ and $\uparrow \sup(U) \sqsupseteq \uparrow U$. Now $\sup; \downarrow : V^+ \mathbb{R} \rightarrow \mathbb{R} \rightarrow P_L \mathbb{R}$ lifts $\max; \{-\} : \mathcal{F}_C^+ \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow \mathcal{F}_L \mathbb{Q}$ and by Proposition 4.29 $\downarrow : V^+ \mathbb{R} \rightarrow P_L \mathbb{R}$ lifts $\text{Id} : \mathcal{F}_C^+ \mathbb{Q} \rightarrow \mathcal{F}_L \mathbb{Q}$. Hence to prove $\downarrow \circ \sup \sqsubseteq \downarrow$ it suffices by Proposition 2.6 to show that $\mathcal{F}_L^+ \mathbb{Q}(\{\max(S)\}, S) = 0$ for every $S \in \mathcal{F}^+ \mathbb{Q}$. This is obvious, because $\max(S) \in S$. The other half, $\uparrow \circ \sup \sqsupseteq \uparrow$, is dual. ■

Lemma 5.4 *If x is a point of \mathbb{R} then $\sup(\{x\}) = x$.*

Proof. $\{-\}; \sup : \mathbb{R} \rightarrow V^+ \mathbb{R} \rightarrow \mathbb{R}$ lifts $\{-\}; \max : \mathbb{Q} \rightarrow \mathcal{F}_C^+ \mathbb{Q} \rightarrow \mathbb{Q}$, which is the identity. ■

We now proceed to show that if $U \subseteq V$ are points of $V^+ \mathbb{R}$, then $\sup(U) \leq \sup(V)$. However, we first investigate a locale that will be useful.

Definition 5.5 *Let $\overrightarrow{\mathbb{Q}}$ be the gms whose elements are the rationals, but whose metric is defined as truncated minus,*

$$\overrightarrow{\mathbb{Q}}(x, y) = x \dot{-} y = \max(0, x - y).$$

It is shown in [19] that $\overrightarrow{\mathbb{Q}}$ is homeomorphic to the ideal completion of $(\mathbb{Q}, <)$, in other words its points are equivalent to rounded lower inhabited subsets of \mathbb{Q} . We shall write this locale as $\overrightarrow{(-\infty, \infty]}$.

Lemma 5.6 *In $\overrightarrow{\mathbb{Q}}$ we have $B_\varepsilon(y) \subset B_\delta(x)$ iff $\varepsilon < \delta$ and $x - \delta < y - \varepsilon$.*

Proof.

$$\begin{aligned} B_\varepsilon(y) \subset B_\delta(x) &\Leftrightarrow (x \dot{-} y) + \varepsilon < \delta \\ &\Leftrightarrow \varepsilon < \delta \text{ and } (x \dot{-} y) < \delta - \varepsilon \\ &\Leftrightarrow \varepsilon < \delta \text{ and } x < y + \delta - \varepsilon \end{aligned}$$

■

Lemma 5.7 *$\text{Id} : \mathbb{Q} \rightarrow \overrightarrow{\mathbb{Q}}$ lifts to the map $\downarrow : \mathbb{R} \rightarrow \overrightarrow{(-\infty, \infty]}$ that takes each Dedekind section (L, R) to L .*

Proof. It is shown in [19] that (L, R) as a Cauchy filter is

$$\{B_\delta(x) \mid x - \delta \in L, x + \delta \in R\}.$$

This maps to

$$\{B_\varepsilon(y) \mid \exists x, \delta. (x - \delta \in L \text{ and } x + \delta \in R \text{ and } \delta < \varepsilon \text{ and } y - \varepsilon < x - \delta)\}$$

and this in turn corresponds, as an ideal of $(\mathbb{Q}, <)$, to

$$\{y - \varepsilon \mid \exists x, \delta. (x - \delta \in L \text{ and } x + \delta \in R \text{ and } \delta < \varepsilon \text{ and } y - \varepsilon < x - \delta)\}.$$

But this is just L again. To show that it contains L , suppose $z \in L$, and by roundedness find $z < z' \in L$. Now find $w \in R$ and let $x = (z' + w)/2$, $\delta = (w - z')/2 = x - z'$. Then $x - \delta = z' \in L$ and $x + \delta = w \in R$. Let $y = x$, $\varepsilon = y - z > \delta$. Then $y - \varepsilon = z < z' = x - \delta$. ■

Lemma 5.8 *If $U \subseteq V$ are points of $V^+ \mathbb{R}$, then $\sup(U) \leq \sup(V)$.*

Proof. The function $\max : \mathcal{F}_L^+ \mathbb{Q} \rightarrow \overrightarrow{\mathbb{Q}}$ is non-expansive. For suppose $S, T \in \mathcal{F}^+ \mathbb{Q}$, $s_{\max} = \max(S)$, $t_{\max} = \max(T)$ and $q = \mathcal{F}_L^+ \mathbb{Q}(S, T) \geq 0$. Since $s_{\max} \in S$ we can find $t \in T$ within q of s_{\max} , and then $s_{\max} \leq t + q \leq t_{\max} + q$ and $s_{\max} \dot{-} t_{\max} \leq q$.

Using Proposition 4.29 and Lemma 5.7, the commutative square of non-expansive functions

$$\begin{array}{ccccc} & \mathcal{F}_C^+ \mathbb{Q} & \xrightarrow{\max} & \mathbb{Q} & \\ \text{Id} \downarrow & & & \downarrow & \text{Id} \\ & \mathcal{F}_L^+ \mathbb{Q} & \xrightarrow{\max} & \overrightarrow{\mathbb{Q}} & \end{array}$$

lifts to

$$\begin{array}{ccccc}
V^+ \mathbb{R} & \xrightarrow{\sup} & \mathbb{R} & & \\
\downarrow & \downarrow & \downarrow & \xrightarrow{\quad} & \downarrow \\
P_L^+ \mathbb{R} & \xrightarrow{\overline{\max}} & (-\infty, \infty] & &
\end{array}$$

Since $U \subseteq V$ we have $\downarrow U \subseteq \downarrow V$, and it follows that $\downarrow \sup(U) \subseteq \downarrow \sup(V)$. This says that $\sup(V)$ has the larger left half in its Dedekind section, in other words that $\sup(U) \leq \sup(V)$. ■

We can similarly show that $\inf(U) \geq \inf(V)$, but this time the proof has to use the upper powerlocale.

The proofs show something of the reason why we need points of the Vietoris powerlocale if we are to calculate \sup and \inf . Given a point of the lower powerlocale, we can approximate its \sup from below but not from above. This gives a point of $(-\infty, \infty]$, which in fact is what $\overline{\max}$ calculates in the above proof. Similarly, we can approximate its \inf from above but not below, getting a point in the dual $[-\infty, \infty)$ (completing the dual metric $\overleftarrow{\mathbb{Q}}(x, y) = y - x$). The same applies to points of the upper powerlocale, but the other way round. To get \sup and \inf as full Dedekind sections, approximated from both below and above, we need to start with a point of the Vietoris powerlocale.

Putting all these together, we obtain

Theorem 5.9 *There are maps $\sup, \inf : V^+ \mathbb{R} \rightarrow \mathbb{R}$ such that $\sup(U)$ is the greatest point in U and $\inf(U)$ is the least.*

5.2 The Heine-Borel maps

To find the closed interval $[x, y]$ as a point of a powerlocale, our working in effect requires us to define the “upper” and “lower” distances from S to $[x, y]$ for every finite $S \subseteq \mathbb{Q}$. In classical terms these would appear as $\sup_{z \in [x, y]} \min_{s \in S} d(s, z)$ and $\max_{s \in S} \inf_{z \in [x, y]} d(s, z)$. The upper distance is less than q iff every z in $[x, y]$ is within q of some s in S , in other words $\{B_q(s) \mid s \in S\}$ covers $[x, y]$. The lower distance is less than q iff every s in S is in the interval $(x - q, y + q)$. However, we must express these geometrically.

Definition 5.10 *Let x and y be Dedekind reals, and let S be a finite subset of $\mathbb{Q} \times \mathbb{Q}_+$. We say that $\{B_\varepsilon(s) \mid (s, \varepsilon) \in S\}$ covers $[x, y]$ iff either $x > y$ or there is some non-empty finite sequence (s_i, ε_i) ($1 \leq i \leq n$) in S such that*

$$\begin{aligned}
s_1 - \varepsilon_1 &< x \\
s_{i+1} - \varepsilon_{i+1} &< s_i + \varepsilon_i \quad (1 \leq i < n) \\
y &< s_n + \varepsilon_n
\end{aligned}$$

It is simpler for us not to assume that $s_1 \leq s_2 \leq \dots \leq s_n$. It could be that the ball $B_{\varepsilon_{i+1}}(s_{i+1})$ is a long way to the left of $B_{\varepsilon_i}(s_i)$ with a big gap in between. However, it turns out that doesn't matter.

Note that we can not assume in this second case that $x \leq y$. Although $<$ and \geq are complementary sublocales of $\mathbb{R} \times \mathbb{R}$, this does not entitle us to assume for generic reals x and y that $x < y$ is either true or impossible.

Lemma 5.11 *Suppose $\{B_\varepsilon(s) \mid (s, \varepsilon) \in S\}$ covers $[x, y]$ with a sequence (s_i, ε_i) as described. Suppose also $x \leq z \leq y$. Then for some i we have $s_i - \varepsilon_i < z < s_i + \varepsilon_i$.*

Proof. The sequence also covers $[x, z]$, for $z \leq y < s_n + \varepsilon_n$ implies $z < s_n + \varepsilon_n$. (Proof: choose a rational q such that $y < q < s_n + \varepsilon_n$. Then we have either $z < s_n + \varepsilon_n$, as desired, or $q < z$. But this second alternative implies $y < z$, which is impossible.) Hence without loss of generality we can assume that $z = y$. We use induction on n . If $n = 1$ then we have $s_1 - \varepsilon_1 < x \leq y < s_1 + \varepsilon_1$ and so $s_1 - \varepsilon_1 < y < s_1 + \varepsilon_1$. Now suppose $n > 1$. Since $s_n - \varepsilon_n < s_{n-1} + \varepsilon_{n-1}$, we have either $s_n - \varepsilon_n < y$, in which case we can take $i = n$, or $y < s_{n-1} + \varepsilon_{n-1}$, in which case we can use induction. ■

It is clear that if we have a cover and we enlarge all the balls then we still have a cover; and also that we are able to shrink the balls slightly and still have a cover.

The following technical construction will be useful. It simply divides a rational interval into k equal contiguous subintervals $(s_i - \mu/2, s_i + \mu/2)$.

Definition 5.12 *Let $a < b$ be rationals and $k \geq 1$ a natural number. Then the k -fold subdivision of (a, b) is the sequence $(s_i)_{1 \leq i \leq k}$ defined as follows. Let μ , the mesh of the subdivision, be $(b - a)/k$. Then*

$$s_i = a + (2i - 1)\mu/2$$

Simple calculations show

$$\begin{aligned} s_1 - \mu/2 &= a \\ s_{i+1} - \mu/2 &= s_i + \mu/2 \quad (1 \leq i < k) \\ b &= s_k + \mu/2 \end{aligned}$$

We are now able to define our distances from S to $[x, y]$.

Definition 5.13 *Let x and y be Dedekind reals. Then we define subsets $\text{HB}_U(x, y)$, $\text{HB}_L(x, y)$ and $\text{HB}_C(x, y)$ of $\mathcal{FQ} \times Q_+$ by*

$$\begin{aligned} B_\delta(S) &\in \text{HB}_U(x, y) \text{ iff } \{B_\delta(s) \mid s \in S\} \text{ covers } [x, y] \\ B_\delta(S) &\in \text{HB}_L(x, y) \text{ iff } \forall s \in S. (x < s + \delta \wedge s - \delta < y) \\ \text{HB}_C(x, y) &= \text{HB}_U(x, y) \cap \text{HB}_L(x, y) \end{aligned}$$

We must show that these define Cauchy filters on the powerspaces, but first we prove a series of lemmas.

Lemma 5.14 *If $B_\delta(S) \in \text{HB}_*(x, y)$ (where $*$ is U , L or C), then there is some $\delta' < \delta$ with $B_{\delta'}(S) \in \text{HB}_*(x, y)$.*

Proof. Obvious. ■

Lemma 5.15 Suppose $x \leq y$ are reals, $S, T \in \mathcal{F}^+\mathbb{Q}$, $B_\delta(S) \in \text{HB}_U(x, y)$ and $B_\varepsilon(T) \in \text{HB}_L(x, y)$. Then

$$\mathcal{F}_U\mathbb{Q}(S, T) = \mathcal{F}_L\mathbb{Q}(T, S) < \delta + \varepsilon.$$

Proof. The first equality is immediate from the symmetry of our gms structure on \mathbb{Q} . If $t \in T$ then t is in the open interval $(x - \varepsilon, y + \varepsilon)$. However, the closed interval $[x - \varepsilon, y + \varepsilon]$ is covered by $\{B_{\delta+\varepsilon}(s) \mid s \in S\}$, and so by Lemma 5.11 there is some $s \in S$ with $s - \delta - \varepsilon < t < s + \delta + \varepsilon$. Hence $\mathcal{F}_U\mathbb{Q}(S, T) < \delta + \varepsilon$. ■

Lemma 5.16 Let x and y be Dedekind reals with $x \leq y$ and let $\varepsilon \in Q_+$. Then there is some $U \in \mathcal{F}\mathbb{Q}$ such that $B_\varepsilon(U) \in \text{HB}_C(x, y)$.

Proof. Choose rationals a and b with $a < x < a + \varepsilon$ and $b - \varepsilon < y < b$. Now let $(u_i)_{1 \leq i \leq k}$ be a k -fold subdivision of (a, b) with mesh strictly less than 2ε , and let $U = \{u_i \mid 1 \leq i \leq k\}$. ■

Lemma 5.17 Suppose $x \leq y$ are reals, and suppose (with $*$ being either U , L or C) we have

$$B_{\delta_\lambda}(S_\lambda) \in \text{HB}_*(x, y) \quad (\lambda = 1, 2).$$

Then there is some $B_\varepsilon(T) \in \text{HB}_C(x, y)$ such that $\mathcal{F}_*\mathbb{Q}(S_\lambda, T) + \varepsilon < \delta_\lambda$.

Proof. By Lemma 5.14 we can find ε such that $B_{\delta_\lambda - 2\varepsilon}(S_\lambda) \in \text{HB}_*(x, y)$ for $\lambda = 1, 2$. By Lemma 5.16 we can find T with $B_\varepsilon(T) \in \text{HB}_C(x, y)$. Now by Lemma 5.15 if $*$ is U or L we deduce

$$\mathcal{F}_*\mathbb{Q}(S_\lambda, T) < \delta_\lambda - 2\varepsilon + \varepsilon = \delta_\lambda - \varepsilon.$$

The case when $*$ is C follows. ■

Proposition 5.18 The definitions above define maps

1. $\text{HB}_U: \mathbb{R} \times \mathbb{R} \rightarrow \overline{\mathcal{F}_U\mathbb{Q}} \cong P_U\mathbb{R}$,
2. $\text{HB}_U: \leq \rightarrow \overline{\mathcal{F}_U^+\mathbb{Q}} \cong P_U^+\mathbb{R}$,
3. $\text{HB}_L: \leq \rightarrow \overline{\mathcal{F}_L^+\mathbb{Q}} \cong P_L^+\mathbb{R}$,
4. $\text{HB}_C: \leq \rightarrow \overline{\mathcal{F}_C^+\mathbb{Q}} \cong V^+\mathbb{R}$.

Proof. In each part, we need to prove the three properties of Cauchy filters (Definition 2.3), namely upper closedness, binary filteredness and the Cauchy property. In parts 2, 3 and 4 (with $x \leq y$) we see that the last two properties follow from Lemmas 5.17 and 5.16, so it remains only to consider upper closedness. Part 1 is slightly more intricate because we cannot assume $x \leq y$, but it does subsume most of part 2.

1. Suppose $B_\varepsilon(T) \in \text{HB}_U(x, y)$ and $\mathcal{F}_U\mathbb{Q}(S, T) + \varepsilon < \delta$. We want $\{B_\delta(s) \mid s \in S\}$ to cover $[x, y]$. If $x > y$ there is no problem. In the other case, we cover $[x, y]$ using a sequence (t_i, ε) , $t_i \in T$. For each t_i there is some $s_i \in S$ within $\delta - \varepsilon$ of it. Then each ball $B_\delta(s_i)$ is bigger than $B_\varepsilon(t_i)$, and taken together they cover $[x, y]$.

Now suppose $\varepsilon \in Q_+$. Find rationals a and b with $a < x$ and $b > y$. Without loss of generality we can suppose $a < b$. By Lemma 5.16 we can then find U with $B_\varepsilon(U) \in \text{HB}_C(a, b) \subseteq \text{HB}_U(a, b) \subseteq \text{HB}_U(x, y)$.

Finally, suppose $B_{\delta_\lambda}(S_\lambda) \in \text{HB}_U(x, y)$ ($\lambda = 1, 2$). We can find $\varepsilon' \in Q_+$ with $B_{\delta_\lambda - 2\varepsilon'}(S_\lambda) \in \text{HB}_U(x, y)$. We shall now find $B_\varepsilon(T)$ such that $B_\varepsilon(T) \in \text{HB}_U(x, y)$ and $\mathcal{F}_U\mathbb{Q}(S_\lambda, T) + \varepsilon < \delta_\lambda$.

If $x > y$ then we can choose $T = \emptyset$ and $\varepsilon = \varepsilon'$. In the other case, we have covering sequences $((s_i^\lambda, \delta_\lambda - 2\varepsilon'))_{1 \leq i \leq m_\lambda}$ taken from S_λ . Let

$$a = \max_\lambda (s_1^\lambda - \delta_\lambda + 2\varepsilon') < x$$

$$b = \min_\lambda (s_{m_\lambda}^\lambda + \delta_\lambda - 2\varepsilon') > y.$$

Note that $B_{\delta_\lambda - \varepsilon'}(S_\lambda) \in \text{HB}_U(a, b)$. If $b < a$ then $y < x$, so (since the order is decidable on \mathbb{Q}) we can assume $a \leq b$, and then by Lemma 5.17 we can find

$$B_\varepsilon(T) \in \text{HB}_C(a, b) \subseteq \text{HB}_U(a, b) \subseteq \text{HB}_U(x, y)$$

such that

$$\mathcal{F}_U\mathbb{Q}(S_\lambda, T) + \varepsilon < \delta_\lambda - \varepsilon' < \delta_\lambda.$$

2. It suffices to show that if $x \leq y$ then $\text{HB}_U(x, y)$ does not contain $B_\delta(\emptyset)$ for any δ . But if $B_\delta(\emptyset) \in \text{HB}_U(x, y)$ then it is immediate from the definition that $x > y$.

3. Suppose $x \leq y$ and $B_\varepsilon(T) \in \text{HB}_L(x, y)$ and $\mathcal{F}_L\mathbb{Q}(S, T) + \varepsilon < \delta$. If $s \in S$ then s is within $\delta - \varepsilon$ of some $t \in T$, and we have $x < t + \varepsilon$ and $t - \varepsilon < y$. It follows that $x < s + \delta$ and $s - \delta < y$, so $B_\delta(S) \in \text{HB}_L(x, y)$.

4. Suppose $B_\varepsilon(T) \in \text{HB}_C(x, y)$ and $\mathcal{F}_C\mathbb{Q}(S, T) + \varepsilon < \delta$. By the previous parts we know that $B_\delta(S)$ is in both $\text{HB}_U(x, y)$ and $\text{HB}_L(x, y)$. ■

Lemma 5.19 *Let x and y be reals with $x \leq y$. Then*

1. $\downarrow \text{HB}_C(x, y) = \text{HB}_L(x, y)$ and
2. $\uparrow \text{HB}_C(x, y) = \text{HB}_U(x, y)$.

Proof. We use Proposition 4.29, and in fact the same technique works for both parts.

We must show that, if $*$ stands for U or L, then

$$B_\delta(S) \in \text{HB}_*(x, y) \text{ iff } \exists B_\varepsilon(T) \in \text{HB}_C(x, y). \mathcal{F}_*(S, T) + \varepsilon < \delta.$$

The non-trivial direction is \Rightarrow , but it follows from Lemma 5.17 by taking both $B_{\delta_\lambda}(S_\lambda)$'s to be $B_\delta(S)$. ■

Theorem 5.20 1. $\text{HB}_U: \mathbb{R} \times \mathbb{R} \rightarrow \text{P}_U \mathbb{R}$ is a map such that $\text{HB}_U(x, y)$ corresponds to the sublocale $[x, y]$ (the closed interval).

2. $\text{HB}_L: \leq \rightarrow \text{P}_L^+ \mathbb{R}$ is a map such that $\text{HB}_L(x, y)$ corresponds to the sublocale $[x, y]$.

3. $\text{HB}_C: \leq \rightarrow \text{V}^+ \mathbb{R}$ is a map such that $\text{HB}_C(x, y)$ corresponds to the sublocale $[x, y]$.

Proof. 1. We show that a point z of \mathbb{R} has $\uparrow z \sqsupseteq \text{HB}_U(x, y)$ iff $x \leq z \leq y$. We have that $\uparrow z \sqsupseteq \text{HB}_U(x, y)$ iff whenever $\{B_\varepsilon(s) \mid s \in S\}$ covers $[x, y]$ then z is in $B_\varepsilon(s)$ for some s in S . That this is implied by $x \leq z \leq y$ has already been proved in Lemma 5.11. For the converse we wish to show $x \leq z$ (i.e. that (x, z) is in the closed complement of the open sublocale $>$ of \mathbb{R}^2), so suppose $z < q < x$ for some rational q . Choosing also a rational $r > y$ such that $r > q$, the ball $B_{(r-q)/2}((q+r)/2)$ covers $[x, y]$ but does not contain z . Similarly, $z \leq y$.

2. We show that a point z of \mathbb{R} has $\downarrow z \sqsubseteq \text{HB}_L(x, y)$ iff $x \leq z \leq y$. We have that $\downarrow z \sqsubseteq \text{HB}_L(x, y)$ iff whenever $s - \varepsilon < z < s + \varepsilon$ then $x < s + \varepsilon$ and $s - \varepsilon < y$. This is obviously implied by $x \leq z \leq y$. For the converse, if $z < x$ then we can find rationals q and r with $q < z < r < x$ and then by taking $s = q$ and $\varepsilon = r - q$ we get a contradiction. Hence $x \leq z$, and similarly $z \leq y$.

3. Combining Theorem 3.5 with Lemma 5.19, we now see that as a sublocale, $\text{HB}_C(x, y)$ corresponds to the sublocale meet of $\text{HB}_U(x, y)$ and $\text{HB}_L(x, y)$, so it is just $[x, y]$ again. ■

Corollary 5.21 (Heine-Borel Theorem) If x and y are reals, then the closed interval $[x, y]$ is compact.

Proof. This follows already from part (1) of the theorem, since the points of $\text{P}_U \mathbb{R}$ are equivalent to compact fitted sublocales of \mathbb{R} [15]. ■

6 Cauchy sequences

Classically, one is used to constructing metric completions by taking equivalence classes of Cauchy sequences. Locally this does not work, for there is no convenient way to form quotient locales. Nonetheless, we shall show in this section that we can define a localic analogue of the set of (certain) Cauchy sequences, and there is a (localic) surjection from it to the completion. What is more, this surjection is *triquotient* in the sense of [11], which gives it good properties. For instance, triquotients are stable under pullback and have effective descent. Our surjection is a rare natural example of a triquotient map that is neither open nor proper, and to prove its triquotiency we use the lower powerlocale and the main result Theorem 4.8. The section also illustrates the synthetic reasoning for powerlocales of [14], another manifestation of the geometric reasoning.

Throughout this section we fix a gms X .

Smyth has given a definition of Cauchy sequence appropriate to quasimetric spaces. (This is the original sequential form of the definition of Cauchy net that was given in [19].)

Definition 6.1 [12] A sequence $(x_n)_{n \in \mathbb{N}}$ in X is forward Cauchy iff for every rational $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that whenever $N \leq m \leq n$ we have $X(x_m, x_n) < \varepsilon$.

(It is backward Cauchy iff it is forward Cauchy in X^{op} , i.e. in the above definition we have $X(x_n, x_m)$ instead of $X(x_m, x_n)$.)

For a geometric theory of Cauchy, the modulus of convergence (the dependence of N on ε) has to be supplied explicitly as part of the structure. We shall make do with a fixed, canonical rate of convergence:

A forward Cauchy sequence (x_n) has canonical convergence iff whenever $N \leq m \leq n$ we have $X(x_m, x_n) < 2^{-N}$; equivalently we can say $X(x_m, x_{m+k}) < 2^{-m}$ for all m, k .

We write $\text{Cauchy}_f(X)$ for the locale classifying the geometric theory of forward Cauchy sequences in X with canonical convergence. Henceforth, we shall tacitly assume that all forward Cauchy sequences mentioned have canonical convergence.

Proposition 6.2 There is a map $\lim : \text{Cauchy}_f(X) \rightarrow \overline{X}$ such that if (x_n) is forward Cauchy, then $\lim x_n$ is defined by

$$B_\varepsilon(y) \in \lim x_n \text{ if } \exists n. B_\varepsilon(y) \supset B_{2^{-n}}(x_n).$$

Proof. All is clear except the filter property. Suppose $B_{\varepsilon_\lambda}(y_\lambda) \in \lim x_n$ ($\lambda = 1, 2$), with $X(y_\lambda, x_{n_\lambda}) + 2^{-n_\lambda} < \varepsilon_\lambda$. We can now find η such that $X(y_\lambda, x_{n_\lambda}) + 2^{-n_\lambda} + \eta < \varepsilon_\lambda$, and $m > \max(n_1, n_2)$ such that $2^{-m} < \eta$. Then $B_\eta(x_m) \in \lim x_n$, and $B_\eta(x_m) \subset B_{\varepsilon_\lambda}(y_\lambda)$, for

$$\begin{aligned} X(y_\lambda, x_m) + \eta &\leq X(y_\lambda, x_{n_\lambda}) + X(x_{n_\lambda}, x_m) + \eta \\ &< X(y_\lambda, x_{n_\lambda}) + 2^{-n_\lambda} + \eta < \varepsilon_\lambda. \end{aligned}$$

■

In the language of “flat modules” [19], one would say that $(\lim x_n)(x) = \inf_n (X(x, x_n) + 2^{-n})$. This would more normally be constructed as a $\lim \inf, \sup_N \inf_{n \geq N} X(x, x_n)$. This is constructively difficult in $\overline{[0, \infty]}$ (we don’t have sups), but they are classically equal. To show $\inf_n (X(x, x_n) + 2^{-n}) \geq \sup_N \inf_{n \geq N} X(x, x_n)$, we must show for every m and N that

$$X(x, x_m) + 2^{-m} \geq \inf_{n \geq N} X(x, x_n)$$

If $m \geq N$, then take $n = m$. If $m < N$, then

$$\text{RHS} \leq X(x, x_N) \leq X(x, x_m) + X(x_m, x_N) \leq \text{LHS}.$$

For the reverse inequality, we must show that for every $\varepsilon > 0$ there is some m such that

$$X(x, x_m) + 2^{-m} < \sup_N \inf_{n \geq N} X(x, x_n) + \varepsilon$$

Choose N such that $2^{-N} < \varepsilon/2$, and then choose $m \geq N$ such that $X(x, x_m) < \inf_{n \geq N} X(x, x_n) + \varepsilon/2$. We then have that $X(x, x_m) + 2^{-m} < \inf_{n \geq N} X(x, x_n) + \varepsilon$.

Our aim now is to show that \lim is a surjection. However, surjective maps in generality are not well behaved – not preserved under pullback, for instance – and we prefer to show that \lim is in a better class of surjections. One would normally hope to show that a surjection is either open or proper, but in general \lim is neither.

Example 6.3 Consider $X = \mathbb{Q} \cap (-2, 2)$. First, it is not too hard to show that its completion is homeomorphic to $[-2, 2]$. \lim is neither open nor proper.

Proof. Consider the open sublocale of sequences (x_i) comprising those for which $x_0 = 0$. If such a sequence has limit x , then x is in $[-1, 1]$ – for

$$X(0, x) = \inf_i (X(0, x_i) + 2^{-i}) = \inf_i (X(x_0, x_i) + 2^{-i}) \leq \inf_i (1 + 2^{-i}) = 1$$

But – classically at least – every real in the interval $[-1, 1]$ is the limit of such a sequence starting at 0, so $[-1, 1]$ is the direct image under \lim of an open. It follows that \lim is not an open map.

But neither is it proper, for inverse image under proper maps preserves compactness. $[-2, 2]$ is compact, but $\text{Cauchy}_f(X)$ is not – it is covered by the open sets $(x_0 = q)$ for q in X , but there is no finite subcover. (This argument was shown me by Till Plewe.) ■

Nonetheless, we shall show that \lim is triquotient. This class of localic surjections was proposed by Plewe [11], who has proved that it is pullback stable, that it includes both open surjections and proper surjections, that triquotient maps have effective descent, and that any triquotient map is the coequalizer of its kernel pair. From this last property we see in effect that the completion is got from the locale of Cauchy sequences by factoring out an equivalence relation, though a direct construction this way would be problematic.

Definition 6.4 [11] A map $f : Y \rightarrow Z$ is triquotient iff there is a function $f_{\#} : \Omega Y \rightarrow \Omega Z$ (a triquotient assignment) such that –

1. $f_{\#}$ preserves directed joins
2. $f_{\#}(a \wedge \Omega f(b)) = f_{\#}(a) \wedge b$ ($a \in \Omega Y$, $b \in \Omega Z$)
3. $f_{\#}(a \vee \Omega f(b)) = f_{\#}(a) \vee b$ ($a \in \Omega Y$, $b \in \Omega Z$)

The usual special cases are open surjections ($f_{\#}$ is left adjoint to Ωf) and proper surjections ($f_{\#}$ right adjoint to Ωf). In any case, we see that a triquotient assignment $f_{\#}$ preserves **false** and **true** (put $b = \mathbf{false}$ in (2), **true** in (3)) and $f_{\#} \circ \Omega f(b) = b$ (put $a = \mathbf{true}$ in (2)), showing that f is a surjection. In our case, where f is \lim , we shall have an $f_{\#}$ that preserves all joins, and we see that a join-preserving function $f_{\#}$ is a triquotient assignment for f iff it preserves **true** and satisfies condition (2), the Frobenius identity for \wedge . Note that a function

$f_{\#}$ preserving all joins is equivalent to a map from Z to the lower powerlocale $P_L Y$.

In the following lemma we translate this sufficient condition into localic form so that we can apply the synthetic methods of [14] (see Subsection 3.2 here).

Lemma 6.5 *Let $f : Y \rightarrow Z$ be a map of locales, and let $g : Z \rightarrow P_L Y$. Then f is triquotient (with triquotient assignment $f_{\#} = \Diamond; \Omega g : \Omega Y \rightarrow \Omega P_L Y \rightarrow \Omega Z$) if*

1. $g; P_L ! = !; \downarrow : Z \rightarrow P_L 1$
2. $g; P_L \langle \text{Id}_Y, f \rangle = \langle g, \text{Id}_Z \rangle; (\text{Id} \times \downarrow); \times : Z \rightarrow P_L(Y \times Z)$

Proof. $f_{\#}$ is a suplattice homomorphism so by the above discussion it suffices to prove that it preserves 1 and that the Frobenius identity for \wedge holds.

First, we apply the two sides of (1) to $\Diamond \mathbf{true}$ in $P_L 1$:

$$\begin{aligned} \Omega(g; P_L !)(\Diamond \mathbf{true}) &= \Omega g(\Diamond \mathbf{true}) = f_{\#} \mathbf{true} \\ \Omega(!; \downarrow)(\Diamond \mathbf{true}) &= \mathbf{true} \end{aligned}$$

Next, we apply the two sides of (2) to $\Diamond(a \otimes b)$ in $P_L(Y \times Z)$:

$$\begin{aligned} \Omega(g; P_L \langle \text{Id}_Y, f \rangle)(\Diamond(a \otimes b)) &= \Omega g(\Diamond \langle \text{Id}_Y, f \rangle(a \otimes b)) = f_{\#}(a \wedge \Omega f(b)) \\ \Omega(\langle g, \text{Id}_Z \rangle; (\text{Id} \times \downarrow); \times)(\Diamond(a \otimes b)) &= \Omega \langle g, \text{Id}_Z \rangle(\Omega(\text{Id} \times \downarrow)(\Diamond(a \otimes b))) \\ &= \Omega \langle g, \text{Id}_Z \rangle(\Diamond(a \otimes b)) = f_{\#} a \wedge b \end{aligned}$$

■

Proposition 6.6 *The points of $P_L(\text{Cauchy}_f(X))$ are the lower closed subsets U of $\mathcal{F}(\mathbb{N} \times X)$ such that if $S \in U$ then –*

1. *if S contains (n, x) and (n, y) then $x = y$ (in other words, S is a finite partial function from \mathbb{N} to X);*
2. *if $n \in \mathbb{N}$ then $S \cup \{(n, z)\} \in U$ for some z ;*
3. *if S contains (n, x) and $(n + k, y)$ then $X(x, y) < 2^{-n}$.*

Note that (1) relies on the decidability of equality in \mathbb{N} . Geometrically, it is

$$\forall (m, x), (n, y) \in S. (m \neq n \vee x = y).$$

Proof. By the Suplattice Coverage Theorem. If we required U to be an ideal (closed under \cup), then we should just be describing the points of $\text{Cauchy}_f(X)$: if the point is a sequence (x_i) , then U is the set of finite subsets of the set $\{(i, x_i) \mid i \in \mathbb{N}\}$. Dropping the closure under \cup gives points of the lower powerlocale. ■

Proposition 6.7 *We can define a map $g : \overline{X} \rightarrow P_L(\text{Cauchy}_f(X))$ by $S \in g(F)$ iff –*

1. $\forall(i, x) \in S. B_{2^{-i}}(x) \in F$
2. $\forall(i, x), (j, y) \in S. (x = y \vee (i < j \wedge X(x, y) < 2^{-i}) \vee (j < i \wedge X(y, x) < 2^{-j}))$

Proof. (Note that $g(F)$ is a geometrically defined subset of $\mathcal{F}(\mathbb{N} \times X)$. This exploits the fact that universal quantification bounded over finite sets is geometric. Condition (2) rewrites the axioms (1) and (3) of Proposition 6.6 as a geometric formula.) The only difficult part is axiom (2) in Proposition 6.6. Suppose $S \in g(F)$ and $n \in \mathbb{N}$. We have $\forall(i, x) \in S. (n \leq i \vee i < n)$, and from the finiteness of S it follows that either $\exists(i, x) \in S. n \leq i$ or $\forall(i, x) \in S. i < n$.

In the first case, suppose we have $(k, x) \in S$ with $n \leq k$; let k be the least such: so $\forall(i, y) \in S. (i < n \vee k \leq i)$. Then $S \cup \{(n, x)\} \in g(F)$.

The second case is when $\forall(i, x) \in S. i < n$. Suppose we enumerate the elements of S as $\{(n_\lambda, x_\lambda) \mid 1 \leq \lambda \leq m-1\}$, and choose x_m such that $B_{2^{-n}}(x_m) \in F$. Let us write $n_m = n$, and $\varepsilon_\lambda = 2^{-n_\lambda}$ ($1 \leq \lambda \leq m$), so that $B_{\varepsilon_\lambda}(x_\lambda) \in F$. Then we can find $B_\varepsilon(x) \in F$ such that $B_\varepsilon(x) \subset B_{\varepsilon_\lambda}(x_\lambda)$ for every λ . It follows that $X(x_\lambda, x) < 2^{-n_\lambda}$ ($1 \leq \lambda \leq m-1$) and $B_{2^{-n}}(x) \in F$, so that $S \cup \{(n, x)\} \in g(F)$. ■

Theorem 6.8 *lim is triquotient.*

Proof. We use Lemma 6.5, with g as defined in Proposition 6.7. Let F be a Cauchy filter of balls. First we must show that $P_L!(g(F)) = \downarrow !$ in P_L . The troublesome direction is \sqsubseteq : we must show that $! \in P_L!(g(F))$, and for this we must show that $g(F)$ contains a non-empty set. We can find x such that $B_1(x) \in F$, and then $\{(0, x)\} \in g(F)$.

Next, we must show $P_L(\text{Id}, \text{lim}) \circ g(F) = g(F) \times \downarrow F$ in $P_L(\text{Cauchy}_f(X) \times \overline{X})$. For \sqsubseteq , following the remarks in Subsection 3.2, it suffices to show that if $(x_i) \in g(F)$ then $\langle \text{Id}, \text{lim} \rangle((x_i)) \in g(F) \times \downarrow F$, i.e. $\text{lim}(x_i) \sqsubseteq F$. If $\text{lim}(x_i)$ has $B_\varepsilon(y)$ then $B_\varepsilon(y) \supset B_{2^{-n}}(x_n)$ for some n . But from $(x_i) \in g(F)$ we know that F has $B_{2^{-n}}(x_n)$ and hence also $B_\varepsilon(y)$.

For the reverse, \supseteq , we show that if $(x_i) \in g(F)$ then $\langle (x_i), F \rangle \in P_L(\text{Id}, \text{lim}) \circ g(F)$. A basic open neighbourhood of $\langle (x_i), F \rangle$ will be described by some $S \in \mathcal{F}(\mathbb{N} \times X)$, being a finite subsequence of (x_i) , and $B_\varepsilon(y) \in F$. Its inverse image under $\langle \text{Id}, \text{lim} \rangle$ is then

$$\bigvee \{ \bigwedge (S \cup \{(n, z)\}) \mid B_{2^{-n}}(z) \subset B_\varepsilon(y) \}.$$

It follows that we must find (n, z) so that the finite sequence $S \cup \{(n, z)\}$ has the Cauchy property of Proposition 6.7 (2), with $B_\varepsilon(y) \supset B_{2^{-n}}(z) \in F$. Choose n so that $n > i$ for every (i, x_i) in S , and then choose $B_\delta(z) \in F$ such that

$$\begin{aligned} B_\delta(z) &\subset B_{2^{-i}}(x_i) \text{ (for every } (i, x_i) \in S), \\ B_\delta(z) &\subset B_\varepsilon(y), \\ \delta &< 2^{-n}. \end{aligned}$$

This (n, z) does what we want. ■

7 Conclusions

The localic completion of [19] provides a means of making constructive localic analogues of complete metric spaces, and even of completions of generalizations in which most of the metric axioms are dropped. The present paper studies the powerlocale constructions, lower, upper and Vietoris (which correspond to hyperspaces), as well as their positive sublocales (“excluding the empty set”). The main result is that all three constructions on the completions can be performed on the uncompleted space in a strongly constructive way. (In fact the construction, using finite powersets, is geometric, preserved under inverse image functors of geometric morphisms.) The construction defines distances between finite subsets, and is reminiscent of the Hausdorff metric but elementary in nature. The effect is that the main result provides means for calculating with the powerlocales.

The powerlocales have fundamental uses in locale theory, and the paper describes some examples in which the main results are applied. The first set are general results concerning compactness and openness of localic completions. The second set is a study of the particular example of the real line as completion of the rationals, and gives a constructive localic account of the Heine-Borel theorem in which it is shown how the closed interval $[x, y]$ is a point of the Vietoris powerlocale and depends continuously on its endpoints x and y . The third example relates the localic completion (which is described using Cauchy filters of formal open balls) to Cauchy sequences. Although locally we cannot describe the completion as a space of Cauchy sequences modulo an equivalence relation, nonetheless we can describe a locale of Cauchy sequences and a localic surjection from it to the localic completion. We use the powerlocale calculations in a proof that this map is a surjection, and indeed one of a good class — Plewe’s triquotients.

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