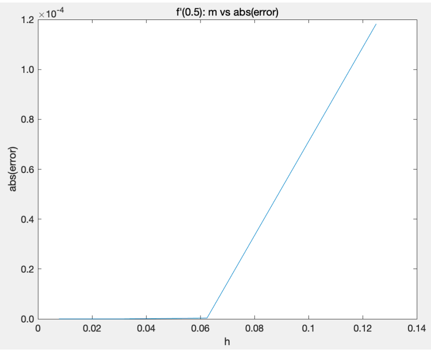
Part A: "natural cubic spline to approximate $f(x) = e^{-x}$ where $x = 0, 0.25, 0.5, 0.75, 1.0$."
The natural cubic spline yielded is:

$$S(x) = \begin{cases} 1 - 0.928453x + 0.698502x^3 & : x \in [0,0.25] \\ 0.778801 - 0.797484(x-0.25) + 0.523877(x-0.25)^2 - 0.361048(x-0.25)^3 & : x \in [0.25,0.5] \\ 0.606531 - 0.603242(x-0.5) + 0.253090(x-0.5)^2 - 0.053015(x-0.5)^3 & : x \in [0.5,0.75] \\ 0.472367 - 0.466757(x-0.75) + 0.292851(x-0.75)^2 - 0.390468(x-0.75)^3 & : x \in [0.75,1] \end{cases}$$

Using the derivative of the spline on $x \in [0.25,0.5]$: $S'(0.5) = -0.797384 + 1.047754(0.25) - 1.083144(0.25)^2 \approx -0.603142$ and $f'(0.5) = \frac{d}{dx}e^{-x}|_{x=0.5} = -e^{-x}|_{x=0.5} \approx -0.606531$. Then the absolute error for $f'(0.5)$: $|-0.606531 - (-0.603142)| = 0.003389$.
Using the second derivative of the spline on $x \in [0.25,0.5]$: $S''(0.5) = f''(0.5) = 1.047754 - 2.166288(0.25) \approx 0.506182$ and $f''(0.5) = \frac{d^2}{dx^2}e^{-x}|_{x=0.5} = e^{-x}|_{x=0.5} \approx 0.606531$. Then the absolute error for $f''(0.5)$: $|0.606531 - 0.506182| = 0.100349$.
Thus, the natural spline gives good approximation for the first derivative, whereas the second derivative does not. It is because the construction of the spline does not take the derivatives of any order of the function we seek to approximate into consideration.

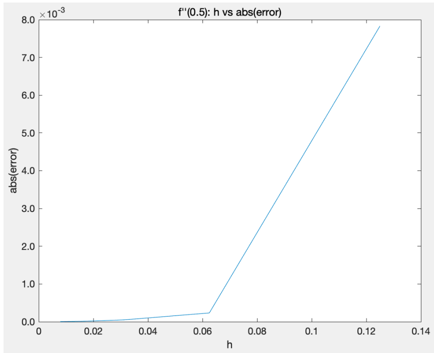
Part B: "Repeat Part A over the interval [0, 1] as using natural cubic spline with equal node spacing $h = 2^{-m}, m = 3,4,5,6,7$."



Plot_1. The absolute error for $f'(0.5)$ as a function of h as $h = 2^{-m}, m = 3,4,5,6,7$

Table_1

i	$p \Longrightarrow f'(0.5)$	$p \Longrightarrow f''(0.5)$
1	8.3825	5.0662
2	6.7832	2.2436
3	4.0017	2.0000
4	3.9998	2.0000



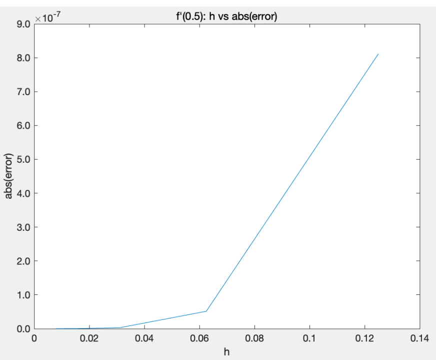
Plot_2. The absolute error for $f''(0.5)$ as a function of h as $h = 2^{-m}, m = 3,4,5,6,7$

From Plot_1 and Plot_2 of the absolute error for $f'(0.5)$ and $f''(0.5)$ as a function of *h*, we can do the ratio test to calculate p on the 5 errors. We have $e(h) = O(h^p)$ and divide this error into $\frac{e(h_i)}{e(h_{i+1})}$, where $1 \le i \le 4$. For h_1, we get $O(h_1^p) = O((2^{-3})^p)$ and divide this into $\frac{e(h_1)}{e(h_2)}$ which gives us $O(2^p)$. Thus, we could obtain the value of p by logging the ratio. We can take the other errors as shown in the Table_1 by finding whether the ratio converges to a limit or not since we have to look at the asymptotic behavior of the ratio for the Big-Oh. Thus, p converges to 4 and 2 for $f'(0.5)$ and $f''(0.5)$, respectively.

Part C: The clamped cubic spline yielded is:

$$S(x) = \begin{cases} 1 - 0.9999999977x + 0.49754996x^2 - 0.1469497x^3 & : x \in [0,0.25] \\ 0.778801 - 0.7787781(x-0.25) + 0.3873376(x-0.25)^2 - 0.1141889(x-0.25)^3 & : x \in [0.25,0.5] \\ 0.606531 - 0.6065197(x-0.5) + 0.3016959(x-0.5)^2 - 0.0889714(x-0.5)^3 & : x \in [0.5,0.75] \\ 0.472367 - 0.4723538(x-0.75) + 0.23496737(x-0.75)^2 - 0.06938272(x-0.75)^3 & : x \in [0.75,1] \end{cases}$$

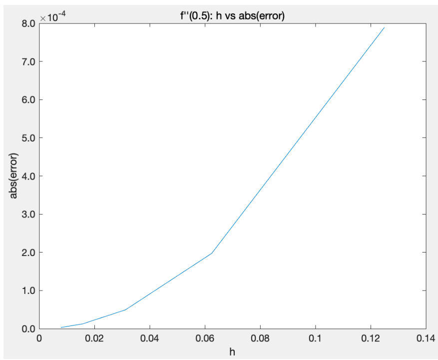
Using the derivative of the spline on $x \in [0.25,0.5]$: $S'(0.5) = -0.7787781 + 0.7746752(0.25) - 0.3425667(0.25)^2 \approx -0.6065197$ and $f'(0.5) = \frac{d}{dx}e^{-x}|_{x=0.5} = -e^{-x}|_{x=0.5} \approx -0.606531$. Then the absolute error for $f'(0.5)$: $|-0.606531 - (-0.6065197)| = 0.0000113$.
Using the second derivative of the spline on $x \in [0.25,0.5]$: $S''(0.5) = 0.7746752 - 0.6851334(0.25) \approx 0.60339185$ and $f''(0.5) = \frac{d^2}{dx^2}e^{-x}|_{x=0.5} = e^{-x}|_{x=0.5} \approx 0.606531$. Then the absolute error for $f''(0.5)$: $|0.606531 - 0.60339185| = 0.00313915$.



Plot_3. The absolute error for $f'(0.5)$ as a function of h as $h = 2^{-m}, m = 3,4,5,6,7$

Table_2

i	$p \Longrightarrow f'(0.5)$	$p \Longrightarrow f''(0.5)$
1	3.9812	1.9992
2	3.9994	1.9999
3	3.9999	2.0000
4	3.9998	2.0000



Plot_4. The absolute error for $f''(0.5)$ as a function of h as $h = 2^{-m}, m = 3,4,5,6,7$

From Plot_3 and Plot_4 of the absolute error for $f'(0.5)$ and $f''(0.5)$ as a function of *h*, as described in Part B, Table_2 illustrates p values for both $f'(0.5)$ and $f''(0.5)$ of $O(h^p)$ as required. Thus, p converges to 4 and 2 for $f'(0.5)$ and $f''(0.5)$, respectively.

Part D: The clamped spline is vastly superior since the value of p for $f'(0.5)$ and $f''(0.5)$ converges much faster as shown in the Table_2 comparing with the Table_1, which shows the p value for natural spline. It is because the boundary conditions for the clamped spline are exact as shown in the above plots, whereas for the natural spline, we are essentially assuming that.