Chapter 4: One Dimensional Kinematics

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September 6, 2023

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Chapter 4

One Dimensional Kinematics

Where was the chap I saw in the picture somewhere? Ah yes, in the dead sea floating on his back, reading a book with a parasol open. Couldn't sink if you tried: so thick with salt. Because the weight of the water, no, the weight of the body in the water is equal to the weight of the what? Or is it the volume equal to the weight? It's a law something like that. Vance in High school cracking his fingerjoints, teaching. The college curriculum. Cracking curriculum. What is weight really when you say weight? Thirtytwo feet per second per second. Law of falling bodies: per second per second. They all fall to the ground. The earth. It's the force of gravity of the earth is the weight. [1]

James Joyce

4.1 Introduction to the Vector Description of Motion in Two Dimensions

We have introduced the concepts of position, velocity and acceleration to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in more than one dimension, we shall study the motion of a projectile in two-dimension moving under the action of uniform gravitation.
We extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, the position vector $\vec{\mathbf{r}}(t)$ with respect to some choice of origin for the object at time t is given by

[1] James Joyce, *Ulysses*, The Corrected Text edited by Hans Walter Gabler with Wolfhard Steppe and Claus Melchior, Random House, New York.

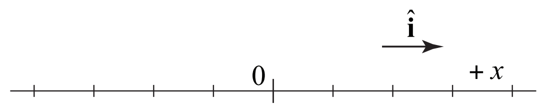

Figure 4.1: A one-dimensional Cartesian coordinate system.

4.2 Position, Time Interval, Displacement

4.2.1 Position

Consider a point-like object moving in one dimension. We denote the *position coordinate* of the object with respect to the choice of origin by $x(t)$. The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. The position of the object with respect to the origin has both direction and magnitude, and hence is a vector (Figure 4.2), which we shall denote as the *position vector* (or simply position) and write as

$$\vec{\mathbf{r}} = x(t)\hat{\mathbf{i}} \tag{4.1}$$

Denote the position coordinate at $t = 0$ by the symbol $x_0 \equiv x(t = 0)$. The SI unit for position is the meter [m].

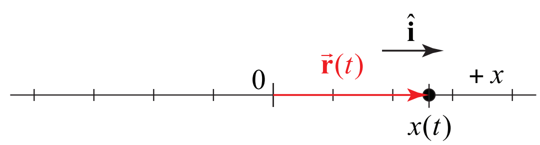

Figure 4.2: The position vector, with reference to a chosen origin.

4.2.2 Time Interval

Consider a closed interval of time $[t_1, t_2]$. We characterize this time interval by the difference in endpoints of the interval,

$$\Delta t = t_2 - t_1 \tag{4.2}$$

The SI units for time intervals are seconds [s].

4.2.3 Displacement

The *displacement* of a body during a time interval $[t_1, t_2]$ (Figure 4.3) is defined to be the change in the position of the body

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t_2) - \vec{\mathbf{r}}(t_1) = (x(t_2) - x(t_1))\hat{\mathbf{i}} = \Delta x\hat{\mathbf{i}}, \tag{4.3}$$

where $\Delta x = (x(t_2) - x(t_1))$ is the x-component of the displacement during the time interval Δt, which is a scalar quantity. The displacement $\Delta\vec{\mathbf{r}}$ is a vector quantity.

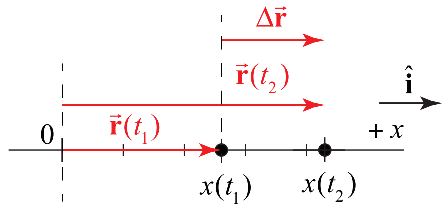

Figure 4.3: The displacement vector of an object over a time interval is the vector difference between the two position vectors.

4.3 Velocity

When describing the motion of objects, words like "speed" and "velocity" are used in natural language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

4.3.1 Average Velocity

The x-component of the average velocity, $v_{x,ave}$, for a time interval Δt is defined to be the displacement Δx during the time interval divided by the time interval Δt,

$$v_{x,ave} \equiv \frac{\Delta x}{\Delta t} \tag{4.4}$$

Because we are describing one-dimensional motion we shall drop the subscript x and denote $v_{ave} \equiv v_{x,ave}$.When we introduce two-dimensional motion we will distinguish the components of the velocity by subscripts. The average velocity vector is then

$$\vec{\mathbf{v}}_{ave} \equiv \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} = v_{ave}\hat{\mathbf{i}}, \tag{4.5}$$

The SI units for average velocity are meters per second $[\mathrm{m} \cdot \mathrm{s}^{-1}]$. The average velocity is not necessarily equal to the distance traveled divided by the time interval Δt. For example, during a time interval Δt, an object moves in the positive x-direction and then returns to its starting position, the displacement of the object is zero, but the distance traveled is non-zero.

4.3.2 Instantaneous Velocity

Consider a body moving in one direction. During the time interval Δt, the average velocity corresponds to the slope of the line connecting the points $(t, x(t)$ and $(t + \Delta t, x(t + \Delta t)$. The slope, the rise over the run, is the change in position divided by the change in time, and is given by

$$v_{ave} \equiv \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}. \tag{4.6}$$

As $\Delta t \to 0$, the slope of the lines connecting the points $(t, x(t)$ and $(t + \Delta t, x(t + \Delta t)$, approach slope of the tangent line to the graph of the function $x(t)$ at the time t (Figure 4.4). The limiting value of this sequence is defined to be the x-component of the

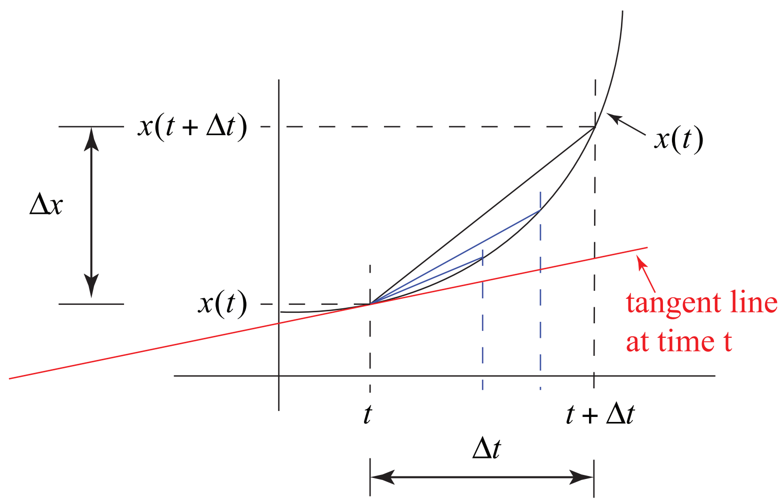

Figure 4.4: Plot of position vs. time showing the tangent line at time t.

instantaneous velocity at the time t:

$$v(t) \equiv \lim_{\Delta t \to 0} v_{ave} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \tag{4.7}$$

The instantaneous velocity vector is then

$$\vec{\mathbf{v}}(t) = v(t)\hat{\mathbf{i}} \tag{4.8}$$

The component of the velocity, $v(t)$, can be positive, zero, or negative, depending on whether the object is traveling in the positive x-direction, instantaneously at rest, or the negative x -direction.

4.3.3 Determining velocity from position

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + \frac{1}{2}bt^2, \tag{4.9}$$

where x_0 is the initial position of the object at $t = 0$. We can explicitly calculate the x-component of instantaneous velocity from Equation (4.5) by first calculating the displacement in the x-direction, $\Delta x = x(t+\Delta t) - x(t)$. We need to calculate the position at time $t + \Delta t$,

$$x(t+\Delta t) = x_0 + \frac{1}{2}b\,(t+\Delta t)^2 = x_0 + \frac{1}{2}b\,(t^2 + 2t\Delta t + \Delta t^2). \tag{4.10}$$

Then the x-component of instantaneous velocity is

$$\begin{aligned} v(t) &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\left(x_0 + \frac{1}{2}b\,(t^2 + 2t\,\Delta t + \Delta t^2)\right) - \left(x_0 + \frac{1}{2}b\,t^2\right)}{\Delta t}. \end{aligned} \tag{4.11}$$

This expression reduces to

$$v(t) = \lim_{\Delta t \to 0} \left(b\,t + \frac{1}{2}b\,\Delta t\right). \tag{4.12}$$

The first term is independent of the interval Δt and the second term vanishes because in the limit as $(1/2)b\Delta t \to 0$. Therefore the x-component of instantaneous velocity at time t is

$$v(t) = bt. \tag{4.13}$$

In Figure 4.5 we plot the instantaneous velocity, $v(t)$, as a function of time t.

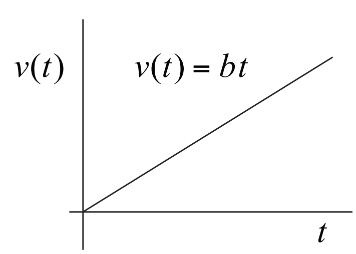

Figure 4.5: Plot of instantaneous velocity instantaneous velocity as a function of time.

4.3.4 Mean value theorem

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + v_0 t + \frac{1}{2}bt^2. \tag{4.14}$$

The graph of $x(t)$ vs. t is shown in Figure 4.6.

The x-component of the instantaneous velocity is

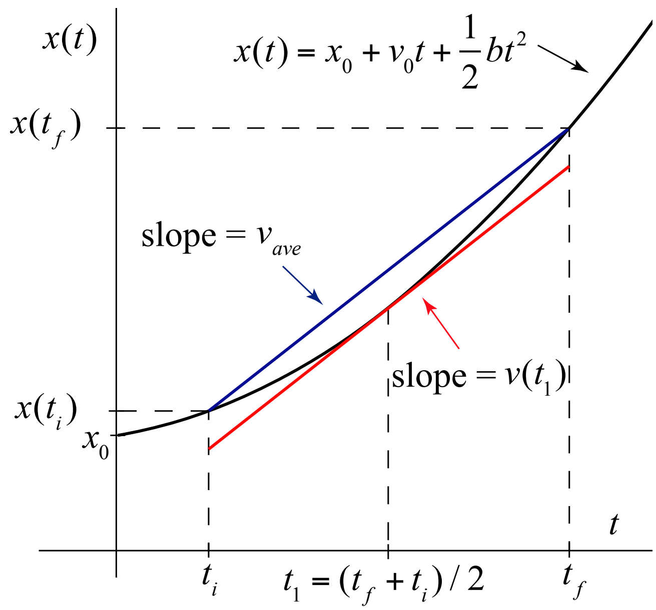

Figure 4.6: Intermediate value theorem

$$v(t) = \frac{dx(t)}{dt} = v_0 + bt. \tag{4.15}$$

For the time interval $[t_i, t_f]$, the displacement of the object is

$$x(t_f) - x(t_i) = \Delta x = v_0(t_f - t_i) + \frac{1}{2}b(t_f{}^2 - t_i{}^2) = v_0(t_f - t_i) + \frac{1}{2}b(t_f - t_i)(t_f + t_i). \tag{4.16}$$

Recall that the x-component of the average velocity is defined by the condition that

$$\Delta x = v_{ave}(t_f - t_i) \tag{4.17}$$

We can determine the average velocity by substituting Equation 4.17 into Equation 4.16 yielding

$$v_{ave} = v_0 + \frac{1}{2}b(t_f + t_i). \tag{4.18}$$

The *Mean Value Theorem* states that there exists an instant in time t_1, with $t_i < t_1 < t_f$, such that the$x$-component of the instantaneously velocity, $v(t_1)$, satisfies

$$\Delta x = v(t_1)(t_f - t_i). \tag{4.19}$$

Geometrically this means that the slope of the straight line (blue line in Figure 4.6 connecting the points $(t_i, x(t_i))$ to $(t_f, x(t_f))$ is equal to the slope of the tangent line (red line in Figure 4.6) to the graph of $x(t)$ vs. t at the point $(t_1, x(t_1))$ (Figure 4.6)),

$$v(t_1) = v_{ave}. \tag{4.20}$$

We know from Equation (4.3.13) that

$$v(t_1) = v_0 + bt_1. \tag{4.21}$$

We can solve for the time t_1 by substituting Equations. 4.21 and (4.18 into Equation (4.3.20) yielding

$$t_1 = (t_f + t_i)/2. \tag{4.22}$$

This intermediate value $v(t_1)$ is also equal to one-half the sum of the initial velocity and final velocity

$$v(t_1) = \frac{1}{2}(v(t_i) + v(t_f)) = \frac{1}{2}((v_0 + bt_i) + (v_0 + bt_f)) = v_0 + \frac{1}{2}b(t_f + t_i) = v_0 + bt_1 \tag{4.23}$$

For any time interval, the quantity $\frac{1}{2}(v(t_i) + v(t_f))$ is the *arithmetic* mean of the initial velocity and the final velocity (but unfortunately is also sometimes referred to as the average velocity). The average velocity, which we defined as $v_{ave} = (x_f - x_i)/\Delta t$, and the arithmetic mean, , are only equal in the special case when the velocity is a linear function in the variable t as in this example, (Equation 4.15)). We shall only use the term average velocity to mean displacement divided by the time interval.

4.4 Acceleration

We shall apply the same physical and mathematical procedure for defining acceleration, as the rate of change of velocity with respect to time. We first consider how the instantaneous velocity changes over a fixed time interval of time and then take the limit as the time interval approaches zero.

4.4.1 Average Acceleration

Average acceleration is the quantity that measures a change in velocity over a particular time interval. Suppose during a time interval $[t, t + \delta t]$ a body undergoes a change in velocity

$$\Delta \vec{\mathbf{v}} = \vec{\mathbf{v}}(t + \Delta t) - \vec{\mathbf{v}}(t). \tag{4.24}$$

Note: The change in the x -component of the velocity, Δv, for the time interval $[t, t+\delta t]$ is then $\Delta v = v(t+\Delta t) - v(t)$. The average acceleration for the time interval Δt is defined as

$$\vec{\mathbf{a}}_{ave} = a_{ave}\ \hat{\mathbf{i}} \equiv \frac{\Delta v}{\Delta t}\ \hat{\mathbf{i}} = \frac{(v(t + \Delta t) - v(t))}{\Delta t}\ \hat{\mathbf{i}}. \tag{4.25}$$

The SI units for average acceleration are meters per second squared, $[\mathrm{m}\cdot\mathrm{s}^{-2}]$.

4.4.2 Instantaneous acceleration

Consider the graph of the x-component of velocity, $v(t)$, (Figure 4.7). The average

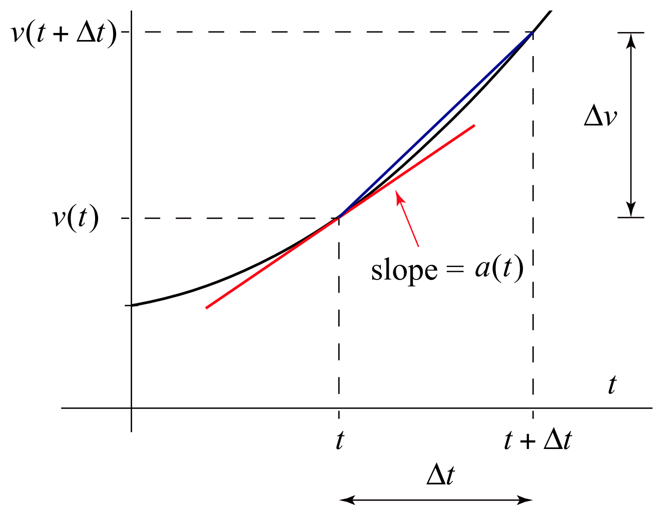

Figure 4.7: Graph of velocity vs. time showing the tangent line at time t.

acceleration for a fixed time interval $\Delta t$is the slope of the straight line connecting the two points $(t, v(t))$ and $(t+\Delta t, v(t+\delta t))$. In order to define the x-component of the instantaneous acceleration at time t, we employ the same limiting argument as we did when we defined the instantaneous velocity in terms of the slope of the tangent line.

The *x-component of the instantaneous acceleration* at time t is the slope of the tangent line at time t of the graph of the x-component of the velocity as a function of time,

$$a(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{(v(t+\Delta t) - v(t))}{\Delta t} \equiv \frac{dv}{dt}. \tag{4.26}$$

The instantaneous acceleration vector at time t is then

$$\vec{\mathbf{a}} = a(t)\hat{\mathbf{i}} \tag{4.27}$$

Because the velocity is the derivative of position with respect to time, the x-component of the acceleration is the second derivative of the position function,

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} \tag{4.28}$$

4.4.3 Determining acceleration from velocity

Suppose the position function for the body is given by $x(t) = x_0 + \frac{1}{2}bt^2$, and the x-component of the velocity is $v(t) = bt$. The x -component of the instantaneous acceleration is the first derivative (with respect to time) of the x-component of the velocity:

$$a(t) = \frac{dv(t)}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{b\,t + b\,\Delta t - b\,t}{\Delta t} = b. \tag{4.29}$$

Note that in Equation 4.29, the ratio $\delta v/\Delta t$ is independent of t, consistent with the constant slope as shown in Figure 4.5.

4.5 Constant Acceleration

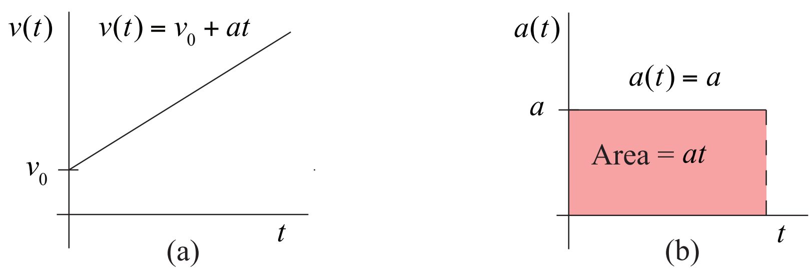

Figure 4.8: Constant acceleration: (a) velocity, (b) acceleration

When the x-component of the velocity is a linear function (Figure 4.8(a)), the average acceleration, $\Delta v/\Delta t$, is a constant and hence is equal to the instantaneous acceleration (Figure 4.8(b)). Let's consider a body undergoing constant acceleration for a time interval $[0, t]$, where $\Delta t = t$. Denote the x-component of the velocity at time t by $v_0 \equiv v(t = 0)$. Therefore the x-component of the acceleration is given by

$$a(t) = \frac{\Delta v}{\Delta t} = \frac{v(t) - v_0}{t}. \tag{4.30}$$

, Thus the x-component of the velocity is a linear function of time given by

$$v(t) = v_0 + at. \tag{4.31}$$

4.5.1 Velocity: area under the acceleration vs. time graph

In Figure 4.8(b), the area under the acceleration vs. time graph, for the time interval $[0, t]$, where $\Delta t = t$, is

$$\text{Area}(a(t); t) = a\,t. \tag{4.32}$$

From Equation 4.31, the area is the change in the x-component of the velocity for the interval $[0, t]$:

$$\text{Area}(a(t); t) = a\,t = v(t) - v_0 = \Delta v. \tag{4.33}$$

4.5.2 Area under the velocity vs. time graph

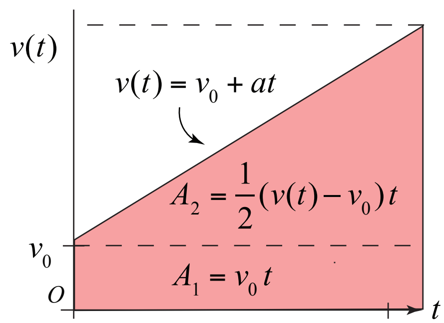

Figure 4.9: Graph of velocity as a function of time for constant a.

Figure 4.9 illustrates a graph of the x-component of the velocity vs. time for the case of constant acceleration (Equation 4.31).

The region under the velocity vs. time curve is a trapezoid, formed from a rectangle with area $A_1 = v_0 t$, and a triangle with area $A_2 = \frac{1}{2}(v(t) - v_0)t$. The total area of the trapezoid is given by

$$\text{Area}(v(t); t) = A_1 + A_2 = v_0\,t + \frac{1}{2}(v(t) - v_0)t. \tag{4.34}$$

Substituting for the velocity (Equation 4.31) yields

$$\text{Area}(v(t);, t) = v_0\,t + \frac{1}{2}a\,t^2. \tag{4.35}$$

Recall using the Mean Value Theorem , (Equation 4.18, where we have set $b = a$ and $\Delta t = t$),

$$v_{ave}t = (v_0 + \frac{1}{2}at)t = \Delta x = x(t) - x_0. \tag{4.36}$$

Therefore Equation 4.35 can be rewritten as

$$\text{Area}(v(t); t) = (v_0 + \frac{1}{2}a\,t)t = v_{ave}t = \Delta x = x(t) - x_0. \tag{4.37}$$

The displacement is equal to the area under the graph of the x-component of the velocity vs. time. The position as a function of time can now be found by rewriting Equation

4.37 as

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2. \tag{4.38}$$

Figure 4.10 shows a graph of this equation. Notice that at $t = 0$ the slope is non-zero, corresponding to the x-component of the initial velocity v_0.. subsectionAccelerating car

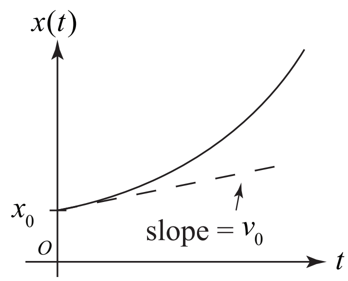

Figure 4.10: Graph of position vs. time for constant acceleration.

A car, starting at rest at $t = 0$, accelerates in a straight line for $100\mathrm{m}$ with an unknown constant acceleration. It reaches a speed of $20\mathrm{m} \cdot \mathrm{s}^{-1}$ and then continues at this speed for another $10\mathrm{s}$.

(a) Write down the equations for position and velocity of the car as a function of time.

(b) How long was the car accelerating?

(c) What was the magnitude of the acceleration?

(d) Plot speed vs. time, acceleration vs. time, and position vs. time for the entire motion.

(e) What was the average velocity for the entire trip?

Answer

(a) For the acceleration a, the position $x(t)$ and velocity $v(t)$ as a functions of time t for a car starting from rest are

$$\begin{aligned} x(t) &= (1/2)\, a\, t^2, \\ v_x(t) &= a\, t. \end{aligned} \tag{4.39}$$

b) Denote the time interval during which the car accelerated by t_1. We know that the position $x(t_1) = 100\mathrm{m}$ and $v(t_1) = 20\mathrm{m} \cdot \mathrm{s}^{-1}$. Note that we can eliminate the acceleration a between the Equations 4.39 to obtain

$$x(t) = (1/2)\, v(t)t. \tag{4.40}$$

We can solve this equation for time as a function of the distance and the final speed giving

$$t = 2\frac{x(t)}{v(t)} \tag{4.41}$$

We can now substitute our known values for the position $x(t_1) = 100\mathrm{m}$ and $v(t_1) = 20\mathrm{m} \cdot \mathrm{s}^{-1}$ and solve for the time interval that the car has accelerated

$$t_1 = 2\frac{x(t_1)}{v(t_1)} = 2\frac{100\ \mathrm{m}}{20\ \mathrm{m} \cdot \mathrm{s}^{-1}} = 10\,\mathrm{s} \tag{4.42}$$

c) We can substitute into either of the expressions in Equation (4.4.10); the second is slightly easier to use,

$$a = \frac{v(t_1)}{t_1} = \frac{20\ \mathrm{m} \cdot \mathrm{s}^{-1}}{10\,\mathrm{s}} = 2.0\,\mathrm{m} \cdot \mathrm{s}^{-2} \tag{4.43}$$

d) The x -component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are piece-wise functions given by

$$a(t) = \begin{cases} 2\ \mathrm{m} \cdot \mathrm{s}^{-2}; & 0 < t \leq 10\ \mathrm{s} \\ 0; & 10\ \mathrm{s} < t < 20\ \mathrm{s} \end{cases}$$

$$v(t) = \begin{cases} (2\ \mathrm{m} \cdot \mathrm{s}^{-2})\, t; & 0 < t \leq 10\ \mathrm{s} \\ 20\ \mathrm{m} \cdot \mathrm{s}^{-1}; & 10\ \mathrm{s} \leq t \leq 20\ \mathrm{s} \end{cases}$$

$$x(t) = \begin{cases} (1/2)\,(2\ \mathrm{m} \cdot \mathrm{s}^{-2})\, t^2; & 0 < t \leq 10\ \mathrm{s} \\ 100\ \mathrm{m} + (20\ \mathrm{m} \cdot \mathrm{s}^{-2})(\,t - 10\ \mathrm{s}); & 10\ \mathrm{s} \leq t \leq 20\ \mathrm{s} \end{cases}$$

The graphs of the x-component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are shown in Figure 4.11.

(e) After accelerating, the car travels for an additional ten seconds at constant speed and during this interval the car travels an additional distance $\Delta x = v(t_1)(10\ \mathrm{s}$ (note that this is twice the distance traveled during the $(10\ \mathrm{s}$ of acceleration), so the total distance traveled is $300\mathrm{m}$ and the total time is $(20\ \mathrm{s}$, for an average velocity of

$$v_{\mathrm{ave}} = \frac{300\,\mathrm{m}}{20\,\mathrm{s}} = 15\,\mathrm{m} \cdot \mathrm{s}^{-1} \tag{4.44}$$

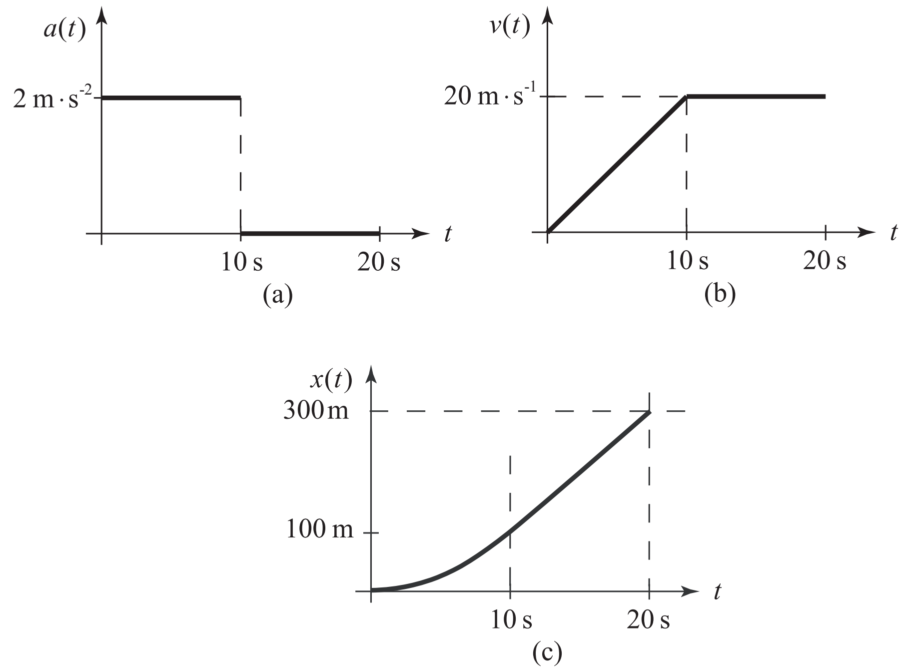

Figure 4.11: Graphs of the x-components of (a) acceleration, (b) velocity and (c) position as piece-wise functions

4.5.3 Car and bus

At the instant a traffic light turns green, a car starts from rest with a given constant acceleration, $3.0 \times 10^{-1}\ \mathrm{m \cdot s^{-2}}$. Just as the light turns green, a bus, traveling with a given constant speed, $1.6 \times 10^{1}\ \mathrm{m \cdot s^{-1}}$, passes the car. The car speeds up and passes the bus some time later. How far down the road has the car traveled, when the car passes the bus?

Answer There are two moving objects, bus and the car. Each object undergoes one stage of one-dimensional motion. We are given the acceleration of the car, the velocity of the bus, and infer that the position of the car and the bus are equal when the bus just passes the car. Figure 4.12 shows a qualitative sketch of the position of the car and bus as a function of time. Choose a coordinate system with the origin at the traffic light and the positive x-direction such that car and bus are traveling in the positive x-direction. Set time $t = 0$ as the instant the car and bus pass each other at the origin when the light turns green. Figure 4.13 shows the position of the car and bus at time t. Let $x_1(t)$ denote the position function of the car, and $x_2(t)$ the position function for the bus. The initial position and initial velocity of the car are both zero, $x_{1,0} = 0$ and $x_{2,0} = 0$, and the acceleration of the car is non-zero $a_1 \neq 0$. Therefore the position and velocity

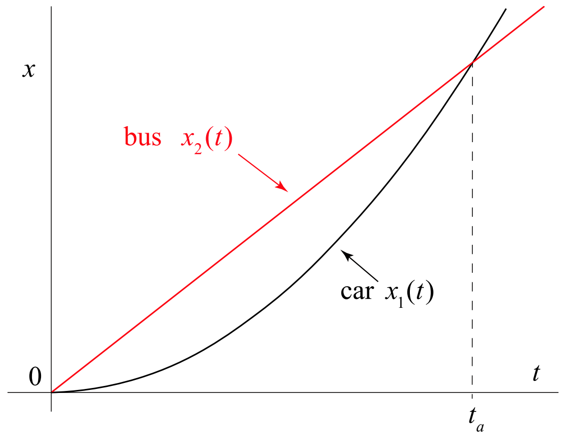

Figure 4.12: Position vs. time of the car and bus.

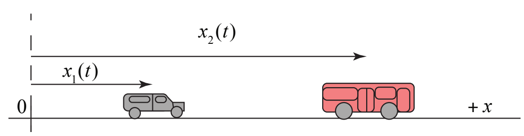

Figure 4.13: A coordinate system for car and bus.

functions of the car are given by

$$x_1(t) = \tfrac{1}{2}a_1t^2,$$
$$v_1(t) = a_1t.$$

The initial position of the bus is zero, $x_{2,0} = 0$, the initial velocity of the bus is non-zero, $v_{2,0} \neq 0$, and the acceleration of the bus is zero, $a_2 = 0$. Therefore the velocity is constant, $v_2(t) = v_{2,0}$, and the position function for the bus is given by $x_2(t) = v_{2,0}t$.

Let t_a correspond to the time that the car passes the bus. Then at that instant, the position functions of the bus and car are equal, $x_1(t_a) = x_2(t_a)$. We can use this condition to solve for t_a:

$$(1/2)a_1{t_a}^2 = v_{2,0}t_a.$$

Hence

$$t_a = \frac{2v_{2,0}}{a_1} = \frac{(2)(1.6\times 10^1\ \mathrm{m}\cdot\mathrm{s}^{-1})}{(3.0\,\mathrm{m}\cdot\mathrm{s}^{-2})} = 1.1\times 10^1\mathrm{s}.$$

Therefore the position of the car at time t_a is

$$x_1(t_a) = \frac{1}{2}a_1{t_a}^2 = \frac{2{v_{2,0}}^2}{a_1} = \frac{(2)(1.6\times 10^1\ \mathrm{m}\cdot\mathrm{s}^{-1})^2}{(3.0\ \mathrm{m}\cdot\mathrm{s}^{-2})} = 1.7\times 10^2\mathrm{m}.$$

4.6 One Dimensional Kinematics and Integration

When the acceleration $a(t)$ of an object is a non-constant function of time, we would like to determine the time dependence of the position function $x(t)$ and the x-component of the velocity $v(t)$. Because the acceleration is non-constant we no longer can use Equations 4.31 and 4.38. Instead we shall use integration techniques to determine these functions.

4.6.1 Change of velocity as the indefinite integral of acceleration

Consider a time interval $t_1 \leq t \leq t_2$. Recall that by definition the derivative of the velocity $v(t)$ is equal to the acceleration $a(t)$,

$$\frac{dv(t)}{dt} = a(t). \tag{4.45}$$

Integration is defined as the inverse operation of differentiation or the 'anti-derivative'. For our example, the function $v(t)$ is called the *indefinite integral* of $a(t)$ with respect to t, and is unique up to an additive constant C., which is denoted by

$$v(t) + C = \int a(t)dt. \tag{4.46}$$

The symbol $\int ...dt$ means "the integral, with respect to t, of . . . ", and is thought of as the inverse of the symbol $\frac{d}{dt}....$ Equivalently we can write the *differential* $dv(t) = a(t)dt$, called the *integrand*, and then Equation 4.46 can be written as

$$v(t) + C = \int dv(t). \tag{4.47}$$

which we interpret by saying that the integral of the differential of function is equal to the function plus a constant.

4.6.2 Example: Non-constant acceleration 1

Suppose an object at time $t = 0$ has initial non-zero velocity v_0 and acceleration $a(t) = bt^2$, where b is a constant. Then $dv(t) = bt^2dt$. The velocity is then

$$v(t) + C = \int d(bt^3/3) = bt^3/3.$$

At $t = 0$, $v_0 + C = 0$. Therefore $v_0 + C = -v_0$. and the velocity as a function of time is then

$$v(t) = v_0 + bt^3/3.$$

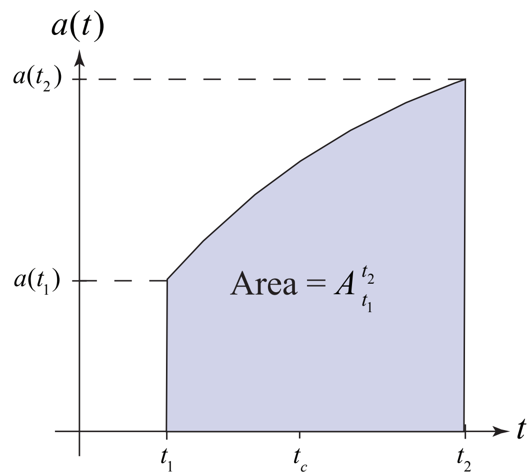

Figure 4.14: Area under the graph of acceleration over an interval $t_1 \leq t \leq t_2$.

4.6.3 Area as the indefinite integral of acceleration

Consider the graph of a positive-valued acceleration function $a(t)$ vs. t for the interval $t_1 \leq t \leq t_2$, shown in Figure 4.14. Denote the area under the graph of $a(t)$ by $A_{t_1}^{t_2}$. The Intermediate Value Theorem states that there is at least one time t_c such that the area $A_{t_1}^{t_2}$ is equal to

$$A_{t_1}^{t_2} = a(t_c)(t_2 - t_1). \tag{4.48}$$

In Figure 4.14b, the shaded regions above and below the curve have equal areas, and hence the area $A_{t_1}^{t_2}$ under the curve is equal to the area of the rectangle given by $a(t_c)(t_2 - t_1)$. We shall now show that the derivative of the area function is equal to the acceleration and therefore we can write the area function as an indefinite integral. From Figure 4.15, the area function satisfies the condition that

$$A_{t_1}^{t} + A_{t}^{t+\Delta t} = A_{t_1}^{t+\Delta t}. \tag{4.49}$$

Let a small increment of area be denoted by $\Delta A_{t_1}^{t} = A_{t_1}^{t+\Delta t} - A_{t_1}^{t} = A_{t}^{t+\Delta t}$. By the Intermediate Value Theorem

$$\Delta A_{t_1}^{t} = a(t_c)\Delta t. \tag{4.50}$$

where $t \leq t_c \leq t + \Delta t$. In the limit as $\Delta t \rightarrow 0$,

$$\frac{dA_{t_1}^{t}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta A_{t_1}^{t}}{\Delta t} = \lim_{t_c \rightarrow t} a(t_c) = a(t), \tag{4.51}$$

with the initial condition that when $t = t_1$, the area $A_{t_1}^{t_1}$ is zero. Because $v(t)$ is also an integral of $a(t)$, we have that

$$A_{t_1}^{t} = \int a(t)dt = v(t) + C. \tag{4.52}$$

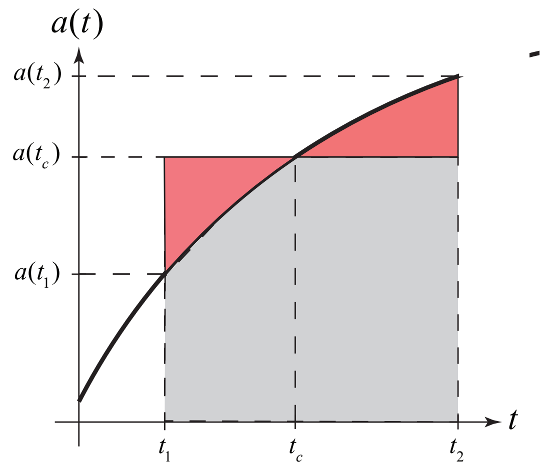

Figure 4.15: Intermediate value theorem. The shaded regions above and below the curve have equal areas.

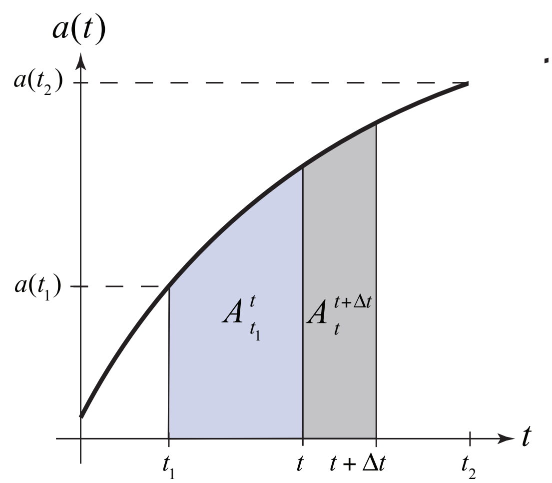

Figure 4.16: Area function is additive.

When we set $t = t_2$, Equation 4.52 becomes

$$A_{t_1}^{t_2} = v(t_2) - v(t_1) = \int a(t)dt. \tag{4.53}$$

The area under the graph of the positive-valued acceleration function for the interval $t_1 \le t \le t_2$ can be found by integrating $a(t)$.

4.6.4 Change of velocity as the definite integral of Acceleration

Let $a(t)$ be the acceleration function over the interval $t_i \le t \le t_f$. Recall that the velocity $v(t)$ is an integral of $a(t)$ because $dv(t)/dt = a(t)$. Divide the time interval $[t_i, t_f]$ into n equal time subintervals $\Delta t = (t_f - t_i)/n$. For each subinterval $[t_j, t_{j+1}]$, where the index $j = 1, 2, ..., n$, $t_1 = t_i$ and $t_{n+1} = t_f$, let t_{c_j} be a time such that $t_j \le t_{c_j} \le t_{j+1}$. Let

$$S_n = \sum_{j=1}^{j=n} a(t_{c_j})\ \Delta t. \tag{4.54}$$

S_n is the sum of the blue rectangles shown in Figure 4.17 for the case $n = 4$. The *Fundamental Theorem of Calculus* states that in the limit as $n \to \infty$, the sum is equal to the change in the velocity during the interval $[t_i, t_f]$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\ \Delta t = v(t_f) - v(t_i). \tag{4.55}$$

The limit of the sum in Equation 4.55 is a number, which we denote by the symbol

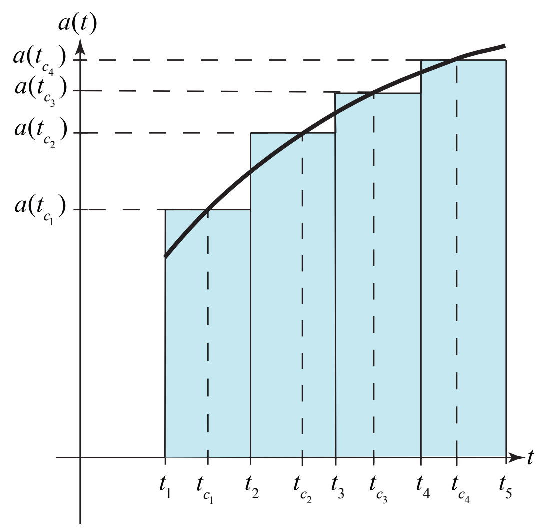

Figure 4.17: Graph of $a(t)$ vs.t with $n = 4$ rectangles.

$$\int_{t_i}^{t_f} a(t)\ dt \equiv \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\ \Delta t = v(t_f) - v(t_i). \tag{4.56}$$

and is called the *definite integral* of $a(t)$ from t_i to t_f. The times t_i and t_f are called the *limits of integration*, t_i the *lower limit* and t_f the *upper limit*. The definite integral is a linear map that takes a function $a(t)$ defined over the interval $[t_i, t_f]$ and gives a number. The map is linear because

$$\int_{t_i}^{t_f} (a_1(t) + a_2(t))\, dt = \int_{t_i}^{t_f} a_1(t)\, dt + \int_{t_i}^{t_f} a_2(t)\, dt. \tag{4.57}$$

Suppose the times t_{c_j}, $j = 1, 2, ..., n$, are selected such that each t_{c_j} satisfies the Intermediate Value Theorem,

$$\Delta v_j \equiv v(t_{j+1}) - v(t_j) = \frac{dv(t_{c_j})}{dt}\,\Delta t = a(t_{c_j})\Delta t. \tag{4.58}$$

where $a(t_{c_j}$ is the instantaneous acceleration at time t_{c_j}, (Figure 4.18). Then the sum

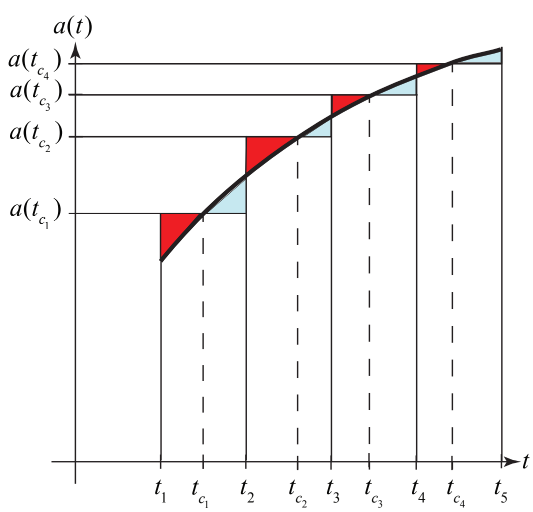

Figure 4.18: Graph of $a(t)$ vs.t where red areas are an overestimate and the blue areas are an underestimate of area.

of the changes in the velocity for the interval $[t_i, t_f]$ is

$$\sum_{j=1}^{j=n} \Delta v_j = (v(t_2) - v(t_1)) + (v(t_3) - v(t_2)) + \cdots + (v(t_{n+1}) - v(t_n)) = v(t_{n+1}) - v(t_1)$$
$$= v(t_f) - v(t_i). \tag{4.59}$$

where $v(t_f) = v(t_{n+1})$ and $v(t_i) = v(t_1)$. Substituting Equation 4.58 into Equation 4.59 yields the exact result that the change in the x-component of the velocity is give by this finite sum.

$$v(t_f) - v(t_i) = \sum_{j=1}^{j=n} \Delta v_j = \sum_{j=1}^{j=n} a(t_{c_j})\ \Delta t. \tag{4.60}$$

We do not specifically know the intermediate values $a(t_{c_j}$ and so Equation 4.58 is not useful as a calculating tool. The statement of the Fundamental Theorem of Calculus is that the limit as $n \to \infty$ of the sum in Equation **??** is independent of the choice of the set of t_{c_j}. Therefore the exact result in Equation 4.60 is the limit of the sum.

Thus we can evaluate the definite integral if we know any indefinite integral of the integrand $a(t)dt = dv(t)$.

Additionally, provided the acceleration function has only non-negative values, the limit is also equal to the area under the graph of $a(t)$ vs. t for the time interval, $[t_i, t_f]$:

$$A_{t_i}^{t_f} = \int_{t_i}^{t_f} a(t)\ dt. \tag{4.61}$$

In Figure 4.18, the red areas are an overestimate and the blue areas are an underestimate. As $n \to \infty$, the sum of the red areas and the sum of the blue areas both approach zero. If there are intervals in which $a(t)$ has negative values, then the summation is a sum of signed areas, positive area above the t-axis and negative area below the t-axis.

We can determine both the change in velocity for the time interval $[t_i, t_f]$ and the area under the graph of $a(t)$ vs. t by integration techniques instead of limiting arguments. We can turn the linear map into a function of time, instead of just giving a number, by setting $t_f = t$. In that case, Equation becomes

$$v(t) - v(t_i) = \int_{t'=t_i}^{t'=t} a(t')dt'. \tag{4.62}$$

Because the upper limit of the integral, $tf = t$, is now treated as a variable, we shall use the symbol t' as the integration variable instead of t.

4.6.5 Displacement as the definite integral of velocity

We can repeat the same argument for the definite integral of the x-component of the velocity $v(t)$ vs. time t. Because $x(t)$ is an integral of $v(t)$, the definite integral of $v(t)$ for the time interval $[t_i, t_f]$ is the displacement

$$x(t_f) - x(t_i) = \int_{t'=t_i}^{t'=t_f} v(t')dt'. \tag{4.63}$$

If we set $t_f = t$, then the definite integral gives us the position as a function of time

$$x(t) = x(t_i) + \int_{t'=t_i}^{t'=t} v(t')dt'. \tag{4.64}$$

Summarizing the results of these last two sections, for a given acceleration $a(t)$, we can use integration techniques, to determine the change in velocity and change in position for an interval $[t_i, t]$, and given initial conditions (x_i, v_i), we can determine the x-component of the velocity $v(t)$ and the position $x(t)$ as functions of time.

4.6.6 Example: Non-constant acceleration 2

Let's consider a case in which the acceleration, $a(t)$, is not constant in time,

$$a(t) = b_0 + b_1\,t + b_2\,t^2$$

The graph of the x-component of the acceleration vs. time is shown in Figure 4.19. Denote the initial velocity at $t = 0$ by v_0. Then, the change in the x-component of the

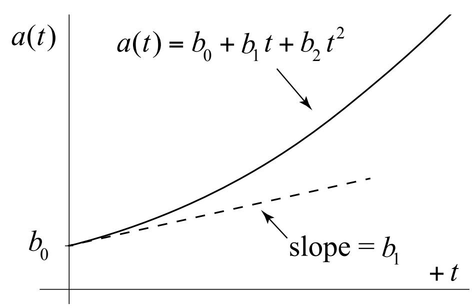

Figure 4.19: Non-constant acceleration vs. time graph

velocity as a function of time can be found by integration:

$$v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')\,dt' = \int_{t'=0}^{t'=t} (b_0 + b_1\,t' + b_2\,t'^2)\,dt' = b_0\,t + \frac{b_1\,t^2}{2} + \frac{b_2\,t^3}{3}.$$

The x-component of the velocity as a function in time is then

$$v(t) = v_0 + b_0\,t + \frac{b_1\,t^2}{2} + \frac{b_2\,t^3}{3}. \tag{4.65}$$

Denote the initial position at $t = 0$ by x_0. The displacement as a function of time is The x-component of the velocity as a function in time is then

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt'. \tag{4.66}$$

Use Equation 4.65 for the -component of the velocity in Equation 4.66 and then integrate to determine the displacement as a function of time:

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt'$$
$$= \int_{t'=0}^{t'=t} \left(v_0 + b_0\,t' + \tfrac{b_1\,t'^2}{2} + \tfrac{b_2\,t'^3}{3}\right) dt' = v_0\,t + \tfrac{b_0\,t^2}{2} + \tfrac{b_1\,t^3}{6} + \tfrac{b_2\,t^4}{12}.$$

Finally the position as a function of time is then

$$x(t) = x_0 + v_{x,0}\,t + \frac{b_0\,t^2}{2} + \frac{b_1\,t^3}{6} + \frac{b_2\,t^4}{12}. \tag{4.67}$$

4.6.7 Example: Bicycle and car

A car is driving through a green light at $t = 0$ located at $x = 0$ with an initial speed $v_{c,0} = 12\mathrm{m}\cdot\mathrm{s}^{-1}$. At time $t_1 = 1\mathrm{s}$, the car starts braking until it comes to rest at time t_2. The acceleration of the car as a function of time is given by the piecewise function

$$a_c(t) = \begin{cases} 0; & 0 \le t < t_1 = 1\ \mathrm{s} \\ b(t - t_1); & 1\ \mathrm{s} \le t \le t_2 \end{cases}$$

where $b = -6\mathrm{m}\cdot\mathrm{s}^{-3}$

(a) Find the x-component of the velocity and the position of the car as a function of time.

(b) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is $17\mathrm{m}$ behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Answer

Getting started We choose a coordinate system with the car and bicycle moving in the positive x-direction, and label the position of the car as a function of time by $x_c(t)$ and the position of the bicycle as a function of time by $x_b(t)$. We choose the origin such that at $t = 0$, the car is at the origin $x_{c,0} = 0$ and the bicycle is located at $x_{b,0} \neq 0$. The initial x-component of the velocity of the car is $v_{c,0}$. The initial x-component of the velocity of the bicycle is denoted by $v_{b,0}$ and is the quantity we are trying to find. We will substitute given values at the end of the problem. The coordinate system and coordinate functions are shown for the car and the bicycle in Figure 4.20

Strategy We first identify that the car has two stages of motion, and that at the end of the second stage, the bicycle and car intersect when the x-component of the velocity of the car is zero. We can express these two conditions as

$$x_b(t_2) = x_c(t_2) \tag{4.68}$$

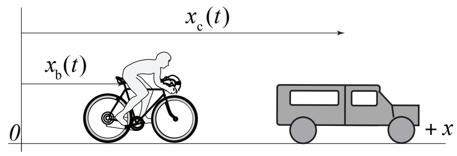

Figure 4.20: Coordinate system for car and bicycle at time t.

$$v_c(t_2) = 0 \tag{4.69}$$

Because we are given the piecewise function for the x-component of the acceleration of the car for the two stages of motion, the initial conditions, and the length of the first stage of motion, we can integrate the x-component of the acceleration to find the x-component of the velocity of the car. We can find t_2 by solving Equation 4.69. We then integrate the x-component of the velocity of the car to find the position of the car $x_c(t)$. We are also told that the bicycle is moving at a constant speed so we can easily find the position function $x_b(t)$, which will be a depend on the unknown initial speed $v_{b,0}$. We can then use Equation 4.68 to solve for $v_{b,0}$

a) In order to apply Equation 4.62, we shall treat each stage separately. For the time interval $0 \le t \le t_1$, the acceleration is zero so the x-component of the velocity is constant. For the second time interval $t_1 \le t \le t_2$, the definite integral becomes

$$v_c(t) - v_c(t_1) = \int_{t'=t_1}^{t'=t} b(t' - t_1)\, dt'$$

Because $v_c(t_1) = v_{c,0}$, the x-component of the velocity is then

$$v_c(t) = \begin{cases} v_{c,0}; & 0 \le t \le t_1 \\ v_{c,0} + \int\limits_{t'=t_1}^{t'=t} b(t' - t_1) dt'; & t_1 \le t \le t_2 \end{cases}$$

Integrate and substitute the two endpoints of the definite integral, yields

$$v_c(t) = \begin{cases} v_{c,0}; & 0 < t \le t_1 \\ v_{c,0} + \frac{1}{2} b(t - t_1)^2; & t_1 \le t < t_2 \end{cases}$$

In order to use Equation 4.64, we need to separate the definite integral into two integrals corresponding to the two stages of motion, using the correct expression for the velocity for each integral. The position function is then

$$x_c(t) = \begin{cases} x_{c0} + \int\limits_{t'=0}^{t'=t_1} v_{c,0} dt'; & 0 \le t \le t_1 \\ x_c(t_1) + \int\limits_{t'=t_1}^{t'=t} \left(v_{c,0} + \frac{1}{2} b(t' - t_1)^2\right) dt; & t_1 \le t \le t_2. \end{cases}$$

Upon integration we have

$$x_c(t) = \begin{cases} x_c(0) + v_{c,0}\,t; & 0 \le t \le t_1 \\ x_c(t_1) + \left(v_{c,0}(t' - t_1) + \frac{1}{6}b(t' - t_1)^3\right)\Big|_{t'=t_1}^{t'=t}; & t_1 \le t \le t_2 \end{cases}$$

We chose our coordinate system such that the initial position of the car was at the origin, $x_{c,0}$, therefore $x_c(t_1) = v_{c,0}t_1$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t) = \begin{cases} v_{c,0}t; & 0 \le t \le t_1 \\ v_{c,0}\,t_1 + v_{c,0}(t - t_1) + \frac{1}{6}b(t - t_1)^3; & t_1 \le t \le t_2. \end{cases}$$

(b) We are looking for the instant t_2 that the car has come to rest. So we use our expression for the x-component of the velocity during the interval $t_1 \le t \le t_2$. When $t = t_2$, $v_c(t_2) = 0$ hence:

$$0 = v_c(t_2) = v_{c,0} + \frac{1}{2}b(t_2 - t_1)^2$$

Solving for t_2 yields

$$t_2 = t_1 + \sqrt{-\frac{2v_{c,0}}{b}}$$

where we have taken the positive square root. Substitute the given values then yields

$$t_2 = 1\ \mathrm{s} + \sqrt{-\frac{2(12\ \mathrm{m\cdot s^{-1}})}{(-6\ \mathrm{m\cdot s^{-3}})}} = 3\ \mathrm{s}.$$

The position of the car at t_2 is then given by

$$\begin{aligned} x_c(t_2) &= v_{c,0}\,t_1 + v_{c,0}(t_2 - t_1) + \tfrac{1}{6}b(t_2 - t_1)^3 \\ x_c(t_2) &= v_{c,0}\,t_1 + v_{c,0}\sqrt{-2v_{c,0}/b} + \tfrac{1}{6}b(-2v_{c,0}/b)^{3/2} \\ x_c(t_2) &= v_{c,0}\,t_1 + \frac{2\sqrt{2}({v_{c,0}}^{3/2})}{3(-b)^{1/2}}, \end{aligned}$$

where we used the condition that $t_2 - t_1 = \sqrt{-\frac{2v_{c,0}}{b}}$. Substitute the given values then yields

$$x_c(t_2) = v_{c,0}\,t_1 + 2\frac{4\sqrt{2}(v_{c,0})^{3/2}}{3(-b)^{1/2}} = (12\ \mathrm{m\cdot s^{-1}})(1\ \mathrm{s}) + \frac{4\sqrt{2}((12\ \mathrm{m\cdot s^{-1}})^{3/2}}{3((6\ \mathrm{m\cdot s^{-3}}))^{1/2}} = 28\ \mathrm{m}.$$

Because the bicycle is traveling at a constant speed with an initial position $x_{b,0} = -17\mathrm{m}$, the position of the bicycle is given by $x_b(t) = 17\mathrm{m} + \mathrm{v_b t}$. The bicycle and car intersect at time $t_2 = 3\mathrm{s}$, where $x_b(t_2) = x_c(t_2)$. Therefore $-17\mathrm{m} + \mathrm{v_b}(3\mathrm{s}) = 28\mathrm{m}$. So the speed of the bicycle is $v_b = 15\mathrm{m\cdot s^{-1}}$.