# Chapter 12: Inertial Reference Fames

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# Contents

# Chapter 12

# Inertial Reference Frames

*Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium" suggest that the phenomena of electromagnetism as well as mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, . . . , the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, . . . , namely that light is always propagated in empty space with a definite velocity c, which is independent of the state of motion of the emitting body* [1]

Albert Einstein

## 12.1 Introduction

In order to describe physical events that occur in space and time such as the motion of bodies, we introduced a coordinate system. Its spatial and temporal coordinates can now specify a *space-time event*. In particular, the position of a moving body can be described by space-time events specified by its space-time coordinates. Place an observer at the origin of coordinate system. The coordinate system with your observer acts as a *reference frame* for describing the position, velocity, and acceleration of bodies. The position vector of the body depends on the choice of origin (location of your observer) but the displacement, velocity, and acceleration vectors are independent of the location of the observer.

You can always choose a second reference frame that is moving with respect to the

[1]A. Einstein, Zur Elektrodynamik begetter Körper, (On the Electrodynamics of Moving Bodies), Ann. Physik, 17, 891 (1905); translated by W. Perrett and G.B. Jeffrey, 1923, in The Principle of Relativity, Dover, New York.

first reference frame. Then the position, velocity and acceleration of bodies as seen by the different observers do depend on the relative motion of the two reference frames. The relative motion can be described in terms of the relative position, velocity, and acceleration of the observer at the origin, $O$, in reference frame $S$ with respect to a second observer located at the origin, $O'$, in reference frame $S'$.

## 12.2 Galilean Coordinate Transformations

Let the vector $\vec{\mathbf{R}}$ point from the origin of frame $S$ to the origin of reference frame $S'$. Suppose an object is located at a point 1. Denote the position vector of the object with respect to origin of reference frame $S$ by $\vec{\mathbf{r}}$. Denote the position vector of the object with respect to origin of reference frame $S'$ by $\vec{\mathbf{r}}'$ (Figure **??** The position vectors are

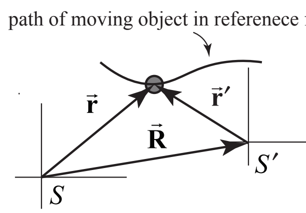


Figure 12.1: Two reference frames $S$ and $S'$.

related by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{R}}. \tag{12.1}$$

These coordinate transformations are called the *Galilean Coordinate Transformations*. They enable the observer in frame $S$ to predict the position vector in frame $S'$, based only on the position vector in frame $S$ and the relative position of the origins of the two frames.

The relative velocity between the two reference frames is given by the time derivative of the vector $\vec{\mathbf{R}}$,

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{r}}}{dt}. \tag{12.2}$$

### 12.2.1 Relatively Inertial Reference Frames and the Principle of Relativity

If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}. \tag{12.3}$$

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are called *relatively inertial reference frames*.

We can reinterpret Newton's First Law

> Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

as the *Principle of Relativity*:

> In relatively inertial reference frames, if there is no net force impressed on an object at rest in frame S, then there is also no net force impressed on the object in frame $S'$.

## 12.3 Law of Addition of Velocities

Suppose the object in Figure **??** is moving; then observers in different reference frames will measure different velocities. Denote the velocity of the object in frame $S$ by $\vec{\mathbf{v}} = \dfrac{d\vec{\mathbf{r}}}{dt}$, and the velocity of the object in frame $S'$ by $\vec{\mathbf{v}}' = \dfrac{d\vec{\mathbf{r}}'}{dt}$. Because the derivative of the position is velocity, the velocities of the object in two different reference frames are related according to

$$\frac{d\vec{\mathbf{r}}'}{dt} = \frac{d\vec{\mathbf{r}}}{dt} - \frac{d\vec{\mathbf{R}}}{dt}, \tag{12.4}$$

or in a more conventional form

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}}, \tag{12.5}$$

This is called the *Law of Addition of Velocities*.

## 12.4 Worked Examples

### 12.4.1 Relative Velocities of Two Moving Planes

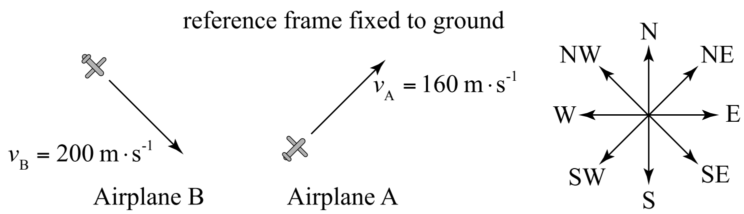


Figure 12.2: Motion of two planes.

An airplane A is traveling northeast with a speed of $v_A = 160\ \mathrm{m \cdot s^{-1}}$ . A second airplane B is traveling southeast with a speed of $v_B = 200\ \mathrm{m \cdot s^{-1}}$ . (Figure **??**).

(a) Choose a coordinate system and write down an expression for the velocity of each airplane as vectors, $\vec{\mathbf{v}}_{\mathrm{A}}$ and $\vec{\mathbf{v}}_{\mathrm{B}}$. Carefully use unit vectors to express your answer.

(b) Sketch the vectors $\vec{\mathbf{v}}_{\mathrm{A}}$ and $\vec{\mathbf{v}}_{\mathrm{B}}$ on your coordinate system.

(c) Find a vector expression that expresses the velocity of aircraft A as seen from an observer flying in aircraft B. Calculate this vector. What is its magnitude and direction? Sketch it on your coordinate system.

**Solution**: An observer at rest with respect to the ground defines a reference frame

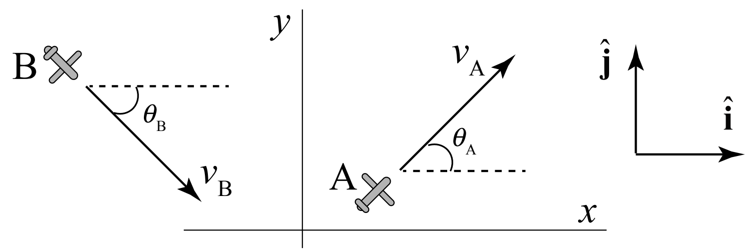


Figure 12.3: Coordinate System for motion of two planes.

$S$. Choose a coordinate system shown in Figure **??**. According to this observer, airplane A is moving with velocity $\vec{\mathbf{v}}_{\mathrm{A}} = v_{\mathrm{A}}\cos(\theta_{\mathrm{A}})\hat{\mathbf{i}} + v_{\mathrm{A}}\sin(\theta_{\mathrm{A}})\,\hat{\mathbf{j}}$, and airplane B is moving with velocity $\vec{\mathbf{v}}_{\mathrm{B}} = v_{\mathrm{B}}\,\cos(\theta_{\mathrm{B}})\hat{\mathbf{i}} + v_{\mathrm{B}}\sin(\theta_{\mathrm{B}})\,\hat{\mathbf{j}}$. According to the information given in the problem airplane A flies northeast so $\theta_{\mathrm{A}} = \pi/4$ and airplane B flies southeast east so $\theta_{\mathrm{B}} = -\pi/4$. Thus $\vec{\mathbf{v}}_{\mathrm{A}} = (80\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{i}} + (80\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_{\mathrm{B}} = (100\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{i}} - (100\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{j}}$

Consider a second observer moving along with airplane B, defining reference frame $S'$. What is the velocity of airplane A according to this observer moving in airplane B? The velocity of the observer moving along in airplane B with respect to an observer at rest on the ground is just the velocity of airplane B and is given by $\vec{\mathbf{V}} = \vec{\mathbf{v}}_{\mathrm{B}}$. Using the Law of Addition of Velocities, Equation (11.3.2), the velocity of airplane A with respect to an observer moving along with Airplane B is given by

$$\begin{aligned}
\vec{\mathbf{v}}'_{\mathrm{A}} &= \vec{\mathbf{v}}_{\mathrm{A}} - \vec{\mathbf{V}} = (v_{\mathrm{A}}\cos(\theta_{\mathrm{A}})\hat{\mathbf{i}} + v_{\mathrm{A}}\sin(\theta_{\mathrm{A}})\,\hat{\mathbf{j}}) - (v_{\mathrm{B}}\,\cos(\theta_{\mathrm{B}})\hat{\mathbf{i}} + v_{\mathrm{B}}\sin(\theta_{\mathrm{B}})\,\hat{\mathbf{j}}) \\
&= (v_{\mathrm{A}}\cos(\theta_{\mathrm{A}}) - v_{\mathrm{B}}\,\cos(\theta_{\mathrm{B}}))\hat{\mathbf{i}} + (v_{\mathrm{A}}\sin(\theta_{\mathrm{A}})\ - v_{\mathrm{B}}\sin(\theta_{\mathrm{B}}))\hat{\mathbf{j}} \\
&= ((80\sqrt{2}\ \mathrm{m\cdot s^{-1}}) - (100\sqrt{2}\ \mathrm{m\cdot s^{-1}}))\hat{\mathbf{i}} + ((80\sqrt{2}\ \mathrm{m\cdot s^{-1}})\ + (100\sqrt{2}\ \mathrm{m\cdot s^{-1}}))\hat{\mathbf{j}} \\
&= -(20\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{i}} + (180\sqrt{2}\ \mathrm{m\cdot s^{-1}})\hat{\mathbf{j}} \\
&= v'_{Ax}\hat{\mathbf{i}} + v'_{Ay}\hat{\mathbf{j}}
\end{aligned} \tag{12.6}$$

Figure **??** shows the velocity of airplane A with respect to airplane B in reference frame $S'$. The magnitude of velocity of airplane A as seen by an observer moving with

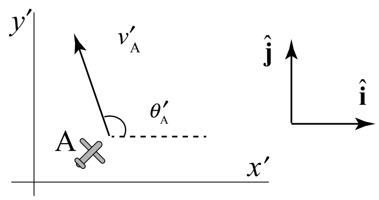


Figure 12.4: Airplane A as seen from observer in airplane B.

airplane B is given by

$$\left|\vec{\mathbf{v}}'_{\mathrm{A}}\right| = ({v'_{\mathrm{A},x}}^2 + {v'_{\mathrm{A},y}}^2)^{1/2} = ((-20\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})^2 + (180\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})^2)^{1/2} = 256\ \mathrm{m{\cdot}s}^{-1} \quad (12.7)$$

The angle of velocity of airplane A as seen by an observer moving with airplane B is given by,

$$\begin{aligned}\theta'_A &= \tan^{-1}(v'_{\mathrm{A},y}/v'_{\mathrm{A},x}) = \tan^{-1}((180\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})/(-20\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})) \\ &= \tan^{-1}(-9) = 180^\circ - 83.7^\circ = 96.3^\circ\end{aligned} \quad . \quad (12.8)$$

### 12.4.2 Relative Motion and Polar Coordinates

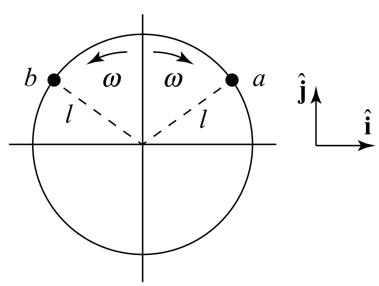


Figure 12.5: Particles $a$ and $b$ moving relative to each other

Particles $a$ and $b$ are moving in opposite directions around a circle with angular velocity of magnitude $\omega$, as shown in Figure **??**. At $t = 0$ they are both at the top of the circle with position vector point $\vec{\mathbf{r}} = l\hat{\mathbf{j}}$, where $l$ is the radius of the circle. Find the velocity of $a$ relative to $b$.

**Solution**:

Let $S$ denote the reference frame with the origin fixed at the center of the circle. Let $S'$ be a reference frame moving with particle $b$. Let the vector $\vec{\mathbf{R}} = l\ \hat{\mathbf{r}}_b$ point from the origin of frame $S$ to the origin of reference frame $S'$. The position vector of particle

$a$ relative to frame $S$ is given by $\vec{\mathbf{r}} = l\ \hat{\mathbf{r}}_a$. The position vector $\vec{\mathbf{r}}'$ of particle $a$ relative to reference frame $S'$ is then

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{R}} = l\ \hat{\mathbf{r}}_a - l\ \hat{\mathbf{r}}_b \tag{12.9}$$

We can decompose each of the unit vectors $\hat{\mathbf{r}}_a$ and $\hat{\mathbf{r}}_b$ with respect to the Cartesian unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (Figure **??**):

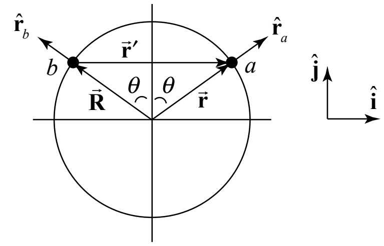


Figure 12.6: position functions in different reference frames $S$ and $S'$ for particle $a$.

$$\begin{aligned} \hat{\mathbf{r}}_b &= -\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}}, \\ \hat{\mathbf{r}}_a &= \sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}}. \end{aligned} \quad . \tag{12.10}$$

Then Equation **??** becomes

$$\vec{\mathbf{r}}' = l\ (\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}}) - l\ (-\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}}) = 2l\sin\theta\ \hat{\mathbf{i}} \tag{12.11}$$

The velocity vector of particle $a$ relative to reference frame $S'$ (i.e. with respect to particle $b$) is then found by differentiating Equation **??**:

$$\vec{\mathbf{v}}' = \frac{d}{dt}(2l\sin\theta)\ \hat{\mathbf{i}} = (2l\cos\theta)\frac{d\theta}{dt}\ \hat{\mathbf{i}} = 2\omega l\cos\theta\ \hat{\mathbf{i}} \tag{12.12}$$

### 12.4.3 Example: Recoil in Different Frames

A person of mass $m_1$ is standing on a cart of mass $m_2$. Assume that the cart is free to move on its wheels without friction. Relative to the cart, the person throws a ball of mass $m_3$ with a speed $v'_b$ at an angle of $\theta'$ with respect to the horizontal (Figure **??**).

(a) What is the final velocity of the ball as seen by an observer fixed to the ground?

(b) What is the final velocity of the cart as seen by an observer fixed to the ground?

(c) With respect to the horizontal, what angle the fixed observer see the ball leave the cart?

**Solution**:

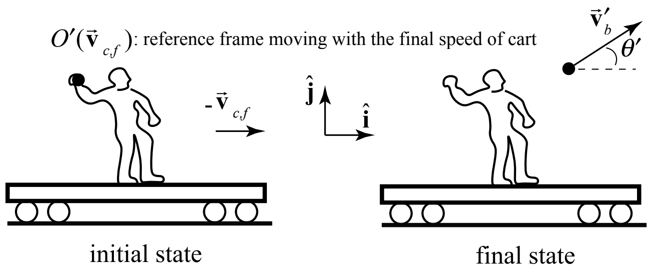


Figure 12.7: Recoil of a person on cart due to thrown ball.

Choose a reference frame fixed to the ground. The initial state: before the ball is thrown (cart, ball, throwing person stationary) and the final state: after the ball is thrown. We are assuming that there is no friction, and so there are no external forces acting in the horizontal direction. The initial $x$-component of the total momentum is zero,

$$p_{x,0}^{\text{total}} = 0. \tag{12.13}$$

The ball is thrown with a speed $v'_b$ and at an angle $\theta'$ with respect to the horizontal as measured by the person in the cart. Therefore the person in the cart throws the ball with velocity

$$\vec{\mathbf{v}}'_b = v'_b \cos\theta' \,\hat{\mathbf{i}} + v'_b \sin\theta' \,\hat{\mathbf{j}}. \tag{12.14}$$

Just after the ball leaves the person's hand, the cart is moving in the negative $x$-direction with velocity $\vec{\mathbf{v}}_{c,f} = -v_{c,f}\,\hat{\mathbf{i}}$. The $x$-component of the velocity of the ball as measured by an observer on the ground is given by

$$(v_{x,f})_{ball} = v'_b \cos\theta' - v_{c,f}\,. \tag{12.15}$$

The ball appears to have a smaller $x$-component of the velocity according to the observer on the ground. The velocity of the ball as measured by an observer on the ground is

$$\vec{\mathbf{v}}_{b,f} = v_b \cos\theta\hat{\mathbf{i}} + v_b \sin\theta\hat{\mathbf{j}} = (v'_b \cos\theta' - v_{c,f})\,\hat{\mathbf{i}} + v'_b \sin\theta' \,\hat{\mathbf{j}}. \tag{12.16}$$

The final momentum of the ball according to an observer on the ground is

$$\vec{\mathbf{p}}_{b,f} = m_3 \left[ (v'_b \cos\theta - v_{c,f})\,\hat{\mathbf{i}} + v'_b \sin\theta \,\hat{\mathbf{j}} \right]. \tag{12.17}$$

The momentum flow diagram is shown in Figure **??**. After the ball is thrown, the final momentum of the person, ball and cart is

$$\vec{\mathbf{p}}_f = -(m_2 + m_1)v_{c,f}\,\hat{\mathbf{i}} + m_3(v_b \cos\theta\hat{\mathbf{i}} + v_b \sin\theta)\hat{\mathbf{j}}. \tag{12.18}$$

as measured by the person on the ground, where $v_{f,c}$ is the speed of the person and cart. Because the $x$-component of the momentum of the system is constant, we have that

$$\begin{aligned} 0 &= (p_{x,f})_{cart} + (p_{x,f})_{ball} \\ &= -(m_2 + m_1)v_{c,f} + m_3(v'_b \cos\theta - v_{c,f}). \end{aligned} \tag{12.19}$$

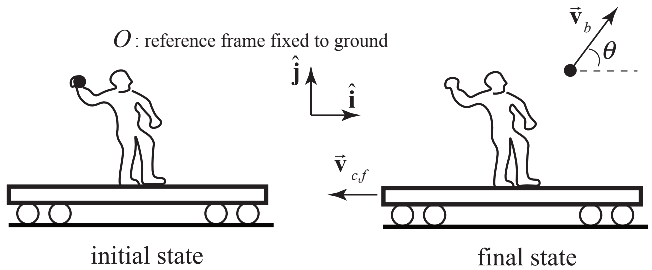


Figure 12.8: Momentum flow diagram for recoil.

We can solve Equation **??** for the final speed and velocity of the cart as measured by an observer on the ground,

$$v_{c,f} = \frac{m_3\, v'_b \cos\theta'}{m_2 + m_1 + m_3}. \tag{12.20}$$

Thus the final velocity of the cart is

$$\vec{\mathbf{v}}_{c,f} = -v_{c,f}\ \hat{\mathbf{i}} = -\frac{m_3\, v'_b \cos\theta'}{m_2 + m_1 + m_3}\hat{\mathbf{i}}. \tag{12.21}$$

Note that the $y$-component of the momentum is not constant because as the person is throwing the ball they are pushing off the cart and the normal force with the ground exceeds the gravitational force so the net external force in the $y$-direction is non-zero.

We can find the final velocity of the ball in the reference frame fixed to the ground by substituting Equation **??** into Equation **??** gives

$$\vec{\mathbf{v}}_{f,b} = \frac{m_1 + m_2}{m_1 + m_2 + m_3}(v'_b \cos\theta')\hat{\mathbf{i}} + (v'_b \sin\theta')\ \hat{\mathbf{j}}. \tag{12.22}$$

As a check, note that in the limit $m_3 << m_1 + m_2$, $\vec{\mathbf{v}}_{f,b}$ has speed $v'_b$ and is directed at an angle $\theta'$ above the horizontal; the fact that the much more massive person-cart combination is free to move doesn't affect the flight of the ball as seen by the fixed observer. Also note that in the unrealistic limit $m_3 >> m_1 + m_2$ the ball is moving at a speed much smaller than $v'_b$ as it leaves the cart.

c) The angle $\theta$ at which the ball is thrown as seen by the observer on the ground is given by

$$\begin{aligned}\theta = \tan^{-1}\frac{(v_{b,f})_y}{(v_{b,f})_x} &= \tan^{-1}\left(\frac{v'_b \sin\theta'}{[(m_1+m_2)/(m_1+m_2+m_3)]\, v'_b\cos\theta'}\right) \\ &= \tan^{-1}\left[\left(\frac{m_1+m_2+m_3}{m_1+m_2}\right)\tan\theta'\right].\end{aligned} \tag{12.23}$$

For arbitrary values for the masses, the above expression will not reduce to a simplified form. However, we can see that $\tan\theta > \tan\theta'$ for arbitrary masses, and that in the limit $m_3 << m_1 + m_2$, $\theta \to \theta'$ and in the unrealistic $m_3 >> m_1 + m_2$, $\theta \to \pi/2$. Can you explain this last odd prediction?