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Homework 5

1. Let $f(x,y) = x^2 + y^2$. Observe that $(x,y) = (0,0)$ defines a solution to the equation $f(x,y) = 0$. Do the remainder of this problem, we shall call $(0,0)$ the trivial solution to $f(x,y) = 0$.

   (a) Prove that $f(x,y) = 0$ has no non-trivial solutions with $x, y \in \mathbb{Q}$.

   (b) Prove that there exists a prime p such that $f(x,y) = 0$ has a nontrivial solution with $x, y \in \mathbb{Q}_p$.

   (c) Verify that f satisfies the Hasse Principle without referring to the Hasse-Minkowski Theorem.

2. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of points in $\mathbb{Z}_p$. Prove that the series

$$\sum_{n=0}^{\infty} a_n p^n$$

converges in $\mathbb{Q}_p$.

Proof: We know that a series will converge if and only if its sequence of partial sums converges. We can test for convergence by calculating

$$\lim_{n\to\infty} \left|a_{n+1}p^{n+1} - a_n p^n\right|_p .$$

We can say

$$\begin{aligned}\left|a_{n+1}p^{n+1} - a_n p^n\right|_p &\leq \max\left\{\left|a_{n+1}p^{n+1}\right|_p, \left|a_n p^n\right|_p\right\}\\ &= \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\}.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\} &= \max\{0,0\}\\ &= 0.\end{aligned}$$

Since the p-adic absolute value is greater than or equal to 0, we know

$$\lim_{n\to\infty} \left|a_{n+1}p^{n+1} - a_n p^n\right|_p = 0.$$

Therefore, the sequence of partial sums is Cauchy, and thus convergent, so the series converges.

3. Suppose that $k \in \mathbb{N}$.

   (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: If we can show that the sequence of partial sums does not converge, then the series will not converge. We want to evaluate (and eventually take the limit of)

$$\left|\frac{1}{(n+1)^k} - \frac{1}{n^k}\right|_p .$$

We know by definition that

$$\begin{aligned}\left|\frac{1}{(n+1)^k} - \frac{1}{n^k}\right|_p &= \left|\frac{n^k - (n+1)^k}{n^k(n+1)^k}\right|_p\\ &= p^{-\left[v_p(n^k-(n+1)^k) - v_p(n^k) - v_p((n+1)^k)\right]}.\end{aligned}$$

Through some tedious calculation, we can solve for the valuation of each term in the exponent.

$$\begin{aligned}
v_p\left(n^k - (n+1)^k\right) &= \frac{\log\left(n^k - (n+1)^k\right) - \log(m_0)}{\log(p)} \\
v_p(n^k) &= \frac{k\log(n) - \log(m_1)}{\log(p)} \\
v_p((n+1)^k) &= \frac{k\log(n+1) - \log(m_2)}{\log(p)},
\end{aligned}$$

where m_0, m_1, and m_2 do not divide p. Combining these valuations as required yields

$$-\left[v_p(n^k - (n+1)^k) - v_p(n^k) - v_p((n+1)^k)\right] = \frac{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)}{\log p} + c$$

where c is some constant number independant of n. We can now evaluate the real valued limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)}{\log p} + c = c + \frac{1}{\log p}\lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right).$$

The function on the inside of the limit is continuous for all $k \in \mathbb{N}$, so we can determine the limit as follows:

$$\begin{aligned}
\lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right) &= L \\
e^{\lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)} &= e^L \\
\lim_{n\to\infty} e^{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)} &= e^L \\
\lim_{n\to\infty} \left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right) &= e^L.
\end{aligned}$$

For the fraction inside the limit, notice that after multiplication, the highest order term in the numerator will be n^{2k}, while the highest order term in the denominator will be n^k, so the limit behaves like

$$\lim_{n\to\infty} \frac{n^{2k}}{n^k} = \lim_{n\to\infty} n^k = \infty = e^L.$$

Taking the logarithm of both sides changes nothing, nor does dividing by $\log p$ where $p \neq \infty$, or adding c. Hence, we have that

$$\lim_{n\to\infty} \left|\frac{1}{(n+1)^k} - \frac{1}{n^k}\right|_p = \infty,$$

so the sequence of partial sums does **not** converge, and therefore the series does not converge.

(b) Prove that the series

$$\sum_{n=1}^{\infty} n^k$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: We want to show that

$$\lim_{n\to\infty} \left|(n+1)^k - n^k\right|_p \neq 0.$$

Using the binomial theorem, we find that

$$\left|(n+1)^k - n^k\right|_p = \left|\sum_{j=0}^{k} \binom{k}{j} n^j - n^k\right|_p = \left|\sum_{j=0}^{k-1} \binom{k}{j} n^j\right|_p$$