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Homework 2

1. If F is a finite field, prove that the only absolute value on F is the trivial absolute value.

   Proof: By definition, if we take any $x \in F$ such that $x \neq 0$, then there is some integer $m \in \mathbb{Z}$ such that $x^m = 1$, meaning that $|x| = 1$. If $x = 0$, then $|x| = 0$, so the absolute value is trivial.

2. Suppose p is prime. Prove that $|p^n|_p$ tends to 0 as $n \to \infty$.

   Proof: We know that $v_p(x) > 0$ for all $x \in \mathbb{Z}$, and that $v_p(x)$ counts the number of factors of p in x. If we consider p^n, there are n factors of p, meaning that $v_p(p^n) = n$. Then $|p^n|_p = p^{-v_p(p^n)} = p^{-n}$, so
   $$\lim_{n\to\infty} p^{-n} = 0.$$

3. Suppose that F is a field and $|\ |$ is a non-Archemedian absolute value on F. If $c \in F$ and r is a positive real number, we define the *open ball (or disk) centered at c of radius r* by
   $$B(c, r) = \{x \in F : |x - c| < r\}.$$
   If $d \in B(c, r)$, prove that $B(c, r) = B(d, r)$.

   Proof: Suppose $x \in B(d, r)$. Then
   $$\begin{aligned} |x - d| &= |x - d - c + c| \\ &= |x - c + c - d| \\ &\leq \max\{|x - c|, |c - d|\} \\ &= \max\{|x - c|, |d - c|\} \\ &\leq \max\{|x - c|, r\}. \end{aligned}$$
   Suppose $|x - d| \leq |x - c|$. Then the value $|x - d|$ is always within the disk $B(c, r)$.

   Suppose $x \in B(c, r)$. Then
   $$\begin{aligned} |x - c| &= |x - c - d + d| \\ &= |x - d + d - c| \\ &\leq \max\{|x - d|, |d - c|\} \\ &< \max\{|x - d|, r\}. \end{aligned}$$
   Suppose $\max\{|x - d|, r\} = |x - d|$. Then $|x - c|$ is always within the disk $B(d, r)$.