# Formal Solutions To ODE's

## 1 Homogeneous Solutions

### 1.1 First Order

We have the following differential equation

$$y' + ay = 0.$$

Then

$$y = ce^{-at},$$

where $c$ is determined by the initial condition.

### 1.2 Second Order

We have the following differential equation

$$y'' + ay' + by = 0. \tag{1}$$

Then we solve the following equation for $r$,

$$r^2 + ar + b = 0.$$

If $r = \alpha \pm i\beta$, then the general solution is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

where $c_1$ and $c_2$ are determined by the initial conditions.

If Equation 1 has two real solutions, $r_1$ and $r_2$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

where $c_1$ and $c_2$ are determined by the initial conditions.

If Equation 1 has a real repeated root, $r$, then

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

where $c_1$ and $c_2$ are determined by the initial conditions.

## 2 Nonhomogeneous

Suppose we have the following differential equation.

$$ay'' + by' + cy = f(x) \tag{2}$$

We begin by finding the solution to the homogeneous equation,

$$ay'' + by' + cy = 0.$$

Call this solution the general solution or $y_g$. Note, do **NOT** apply any initial conditions for this solution.

We now have a table for $f(x)$ and the form of the particular solution.

| $f$ | $y_p$ |
|---|---|
| $k$ | $A$ |
| $x$ | $Ax + B$ |
| $x^2$ | $Ax^2 + Bx + C$ |
| $e^x$ | $Ae^x$ |
| $\cos(n\pi x)$ or $\sin(n\pi x)$ | $A\cos(n\pi x) + B\sin(n\pi x)$ |
| $x\cos(n\pi x)$ or $\sin(n\pi x)$ | $(Ax + B)\cos(n\pi x) + (Cx + D)\sin(n\pi x)$ |

If your $f$ is a linear combination of the above, then your $y_p$ is a linear combination of the applicable entries above. To find the $A, B, C, D$, plug $y_p$ back into Equation 2, and match terms with $f(x)$. The solution to Equation 2 is $y = y_g + y_p$. Now apply your initial conditions to determine your as of now undetermined coefficients.