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Homework 3

1. Suppose F is a field with a non-Archemedian absolute value $|\cdot|$ and $\mathcal{O}_F$ is the ring of integers of F. Prove that a point $x \in \mathcal{O}_F$ is a unit if and only if $|x| = 1$.

   Suppose $x \in \mathcal{O}_F$ is a unit and consider $x^{-1} \in \mathcal{O}_F$ as the inverse of x. We know $x^{-1} \in \mathcal{O}_F$ since x is a unit in $\mathcal{O}_F$. Since $x \in \mathcal{O}_F$, we know

   $$\begin{aligned} |x| &\leq 1 \\ |x^{-1}||x| \leq |x^{-1}| &\leq 1 \qquad \text{(since } x^{-1} \in \mathcal{O}_F\text{)} \\ |x^{-1}x| \leq |x^{-1}| &\leq 1 \\ |1| \leq |x^{-1}| &\leq 1 \\ 1 \leq |x^{-1}| &\leq 1. \end{aligned}$$

   The only choice we have for x^{-1} is 1, so

   $$\begin{aligned} x^{-1} &= 1 \\ xx^{-1} &= x \\ x &= 1 \\ |x| &= 1. \end{aligned}$$

   Suppose $|x| = 1$. We want to show that $x^{-1} \in \mathcal{O}_F$. Consider $|x^{-1}|$ where $x^{-1} \in F$. Then

   $$\begin{aligned} |x| &= 1 \\ |x||x^{-1}| &= |x^{-1}| \\ |xx^{-1}| &= |x^{-1}| \\ |1| &= |x^{-1}| \\ 1 &= |x^{-1}|. \end{aligned}$$

   Therefore $|x^{-1}| = 1$, so $x^{-1} \in \mathcal{O}_F$ and therefore x is a unit.

2. If n is a non-negative integer a p is prime, define $\mathcal{I}_n = \{x \in \mathcal{O}_p : |x|_p \leq p^{-n}\}$ where $\mathcal{O}_p = \{x \in \mathbb{Q} : |x|_p \leq 1\}$.

   (a) If $x, y \in \mathcal{I}_n$ prove that $x + y \in \mathcal{I}_n$.

   Suppose $x, y \in \mathcal{I}_n$, and consider $|x + y|_p$. Then

   $$\begin{aligned} |x + y|_p &\leq \max\{|x|_p, |y|_p\} \\ &\leq \max\{p^{-n}, p^{-n}\} \\ &= p^{-n} \end{aligned}$$

   Therefore $|x + y|_p \leq p^{-n}$, so $x + y \in \mathcal{I}_n$.

   (b) If $x \in \mathcal{I}_n$ and $r \in \mathcal{O}_p$ prove that $rx \in \mathcal{I}_n$.

   Let $|\cdot|$ be the p-adic absolute value and consider $|rx| = |r| \cdot |x|$. Since $|x| \leq p^{-n}$ and $|r| \leq 1$, we have $|r| \cdot |x| \leq 1 \cdot p^{-n} = p^{-n}$, so $|rx| \leq p^{-n}$ which implies that $rx \in \mathcal{I}_n$.

   (c) What is another way to describe the set $\mathbb{Z} \cap \mathcal{I}_n$?

   We know by definition of the p-adic absolute value that for all $x \in \mathbb{Z}$, $|x|_p \leq 1$. This implies $\mathbb{Z} \subset \mathcal{O}_p$. We want to find all $x \in \mathbb{Z}$ where $|x|_p \leq p^{-n}$ for some $n \in \mathbb{N}$. We cannot consider any x where x is a multiple of p, since those values would be in $\mathbb{Q}$. This leaves only 1 (when x does not share any factors of p) or 0 (when $x = 0$). For any value p, $0 < p^{-n} < 1$, so the only value we have is $x = 0$. Therefore, $\mathbb{Z} \cap \mathcal{I}_n = \{0\}$.