Elliptic Functions

1 Introduction

The Jacobi Elliptic functions come from the result of integrating certain algebraic functions. The functions arose due to the lack on any elementary antiderivative to

$$u = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The Jacobi Elliptic functions provide a way to express the solution to this integral via the properties

$$\begin{aligned} sn(u) &= sn\left(\int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}\right) = \sin\phi \\ cn(u) &= cn\left(\int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}\right) = \cos\phi \\ dn(u) &= dn\left(\int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}\right) = \sqrt{1 - k^2 \sin^2 \phi}. \end{aligned}$$

The elliptic functions end up being applicable to more than just finding the antiderivative of u. We can express the solutions to several differential equations in terms of these functions, much like we can with sin and cos. For example, the system

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -zx \\ \dot{z} = -k^2 xy \end{cases} \qquad \text{where} \qquad \begin{cases} x(0) = 0 \\ y(0) = 1 \\ z(0) = 1 \end{cases}$$

can be solved by these elliptic functions, when subject to the initial conditions. More interestingly, we can find the solutions for several ODEs via the elliptic functions.

1.1 Applications to ODEs

Consider the second order ODE

$$\begin{cases} \ddot{x} = (1 - x^2)(1 - k^2 x^2) \\ \dot{x}(0) = 1 \\ x(0) = 0 \end{cases}.$$

Recall that we defined the Jacobi Elliptic functions as solutions to a prior system of ODEs. We had the relationship that $\dot{x} = yz$, where y and z were functions of t. Taking another derivative yields $\ddot{x} = \dot{y}z + y\dot{z}$.