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Homework 4

1. Recall that $f(x) = x^2 + 1$ has two zeros $\alpha, \beta \in \mathbb{Z}_5$ satisfying $|\alpha - 2|_5 \leq 1/5$ and $|\beta - 3|_5 \leq 1/5$. Find the first three terms b_0, b_1, and b_2 of the *p-adic* expansion of β.

   Since $b = 3$ satisfies $f(x) = 0$, we know $b_0 = 3$. We can consider β of the form

   $$\beta = b_0 + 5b_1 + 5^2 b_2 + \ldots,$$

   where $0 \leq b_n \leq 5$. To find b_1, we look at $f(\beta) \equiv 0 \mod 5^2$. Then

   $$\begin{aligned} (3 + 5b_1)^2 + 1 &\equiv 0 \mod 25 \\ 9 + 5b_1 + 25b_2^2 + 1 &\equiv 0 \mod 25 \\ 10 + 5b_1 &\equiv 0 \mod 25 \\ 5b_1 &\equiv 15 \mod 25 \\ \implies b_1 &= 3. \end{aligned}$$

   Similarly, for b_2, we have

   $$\begin{aligned} (18 + 25b_2)^2 + 1 &\equiv 0 \mod 125 \\ 74 + 25b_2 + 1 &\equiv 0 \mod 125 \\ 25b_2 &\equiv 50 \mod 125 \\ 5b_2 &\equiv 10 \mod 25 \\ \implies b_2 &= 2. \end{aligned}$$

   Therefore $\beta \approx 3 + 5 \cdot 3 + 5^2 \cdot 2$.

2. User Hensel's Lemma to verify that $f(x) = x^3 + 1$ has a zero $\alpha \in \mathbb{Z}_7$. Find an integer n such that $|\alpha - n|_7 \leq 1/7$.

   If we let $\alpha = 3$, then $f(\alpha) = 27 + 1 \equiv 0 \mod 7$. We also have that $f'(\alpha) = 3(9) = 27 \not\equiv 0 \mod 7$, so Hensel's Lemma tells us that there is a zero of the function. Let $n10$. Then $|3 - 10|_7 = |-7|_7 = |7|_7 = 1/7 \leq 1/7$.

3. Show that Hensel's Lemma fails to apply to the polynomial $f(x) = x^3 + 1$ in $\mathbb{Z}_3$. Is this sufficient to conclude that f has no zeros in $\mathbb{Z}_3$?

   Hensel's Lemma requires that $f'(x) \not\equiv 0 \mod p$. In this case, $f'(x) = 3x^2$, and $3x^2 \equiv 0 \mod 3$, so we cannot use it. This is not enough to say that f has no zeros. Hensel's Lemma only tells us about the properties of a zero, but it does not guarantee the existence of a zero.