Steven Rosendahl
Proofs Exam 3

1. Prove that for all positive integers N,

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3n+9}$$

Solution: We first need to show a base case. If we let $n = 1$, we have

$$\begin{aligned}
\sum_{k=1}^{1} \frac{1}{(k+2)(k+3)} &= \frac{1}{3(1)+9} \\
\frac{1}{(1+2)(1+3)} &= \frac{1}{12} \\
\frac{1}{12} &= \frac{1}{12}.
\end{aligned}$$

Now we can form our induction hypothesis, given by

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3n+9}.$$

Now we need to prove:

$$\sum_{k=1}^{n+1} \frac{1}{(k+2)(k+3)} = \frac{n+1}{3(n+1)+9}$$

$$\begin{aligned}
\therefore \sum_{k=1}^{n+1} \frac{1}{(k+2)(k+3)} &= \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} + \sum_{k=n+1}^{n+1} \frac{1}{(k+2)(k+3)} \\
\text{By the induction hypothesis, } &= \frac{n}{3n+9} + \frac{1}{(n+3)(n+4)} \\
&= \frac{n(n+4)+3}{3(n+3)(n+4)} \\
&= \frac{n^2+4n+3}{3(n+3)(n+4)} \\
&= \frac{(n+3)(n+1)}{3(n+3)(n+4)} \\
&= \frac{n+1}{3(n+4)} \\
&= \frac{n+1}{3(n+1+3)} \\
&= \frac{n+1}{3(n+1)+9}
\end{aligned}$$

$\triangle$

2. Let R be a relation on $\{1,2,3,4,5,6\}$ be defined by xRy if and only if $x+y \leq 8$. Determine which of the following properties R has: reflexive, symmetric, transitive. Give a proof for each property it satisfies, and give a counterexample for each of those properties it does not have.

   Reflexive: Let $x \in \{1,2,3,4,5,6\}$. Then xRx means that $x+x \leq 8$. However $6+6 = 12 \nleq 8$. Therefore, R is not reflexive.

   Symmetric: Let $x, y \in \{1,2,3,4,5,6\}$ such that xRy. Then $x+y \leq 8$. Since addition is commutative, $x+y = y+x$. Then $x+y = y+x \leq 8$. Therefore yRx, which means R is symmetric.

   Transitive: Let $a, b, c \in \{1,2,3,4,5,6\}$ such that aRb and bRc. Then $a+b \leq 8$, and $b+c \leq 8$. However, if we let $a=5$, $b=2$, and $c=6$, then aRb, since $5+2 = 7 \leq 8$, and bRc, since $2+6 = 8 \leq 8$, but $a\not{R}c$, since $5+6 \nleq 8$.

   $\triangle$

3. Assume relation R on $A = \{1,2,3,4\}$ is an equivalence relation where

$$R = \{(1,1),(2,2),(\ \ ,\ \ ),(4,4),(1,2),(2,4),(\ \ ,\ \ ),(\ \ ,\ \ ),(4,2),(4,1)\}.$$

   Note that R has exactly 3 blank entries.

   (a) Fill in the 3 missing entries of R.

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,4),(2,1),(1,4),(4,2),(4,1)\}.$$

   (b) Clearly write all of the equivalence classes of R.

$$[1] = [2] = [3] = \{1,2,4\}$$
$$[3] = \{3\}$$

   (c) Write the partition associated with R.

$$P = \{[1],[3]\} = \{[2],[3]\} = \{[4],[3]\}$$

4. Let R be a relation from $A = \{1,2,3,4,5\}$ to $B = \{a,b,c,d\}$ given by

$$R = \{(1,d),(2,c),(3,a),(4,d),(5,b)\}.$$

   (a) Why is R a function from A to B?

      R is a function since every element in A maps to an element in B, and no one element in A maps to two elements in B at the same time.

   (b) Is R an injection from A to B? Why?

      R is not an injection since both 1 and 4 map to d.

   (c) Is R a surjection from A to B? Why?

      R is a surjection since every value in the codomain B has a pre-image in A.

5. (a) Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x + 1$ is an injection.

   Solution: Let $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$. Then $2x + 1 = 2y + 1$, or $2x = 2y$. Therefore, $x = y$, which means f is injective.

   $\triangle$

   (b) Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 7x - 1$ is an surjection.

   Solution: Let $y \in \mathbb{R}$. Then $y = 7x - 1$, or $x = \frac{y+1}{7}$. We know this is in $\mathbb{R}$ since it is either in $\mathbb{Q}, \mathbb{Z},$ or $\mathbb{R}$. Substituting x into f gives us

$$\begin{aligned} f\left(\frac{y+1}{7}\right) &= 7\left(\frac{y+1}{7}\right) - 1 \\ &= y + 1 - 1 \\ &= y \end{aligned}$$

   Therefore, f is surjective.

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6. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{3\}$ be defined by $f(x) = \frac{3x}{x-1}$.

   (a) Show $f(x)$ is a bijection.

   Injection: Let $x, y \in \mathbb{R} - \{1\}$ such that $f(x) = f(y)$. Then

$$\begin{aligned} \frac{3x}{x-1} &= \frac{3y}{y-1} \\ (3x)(y-1) &= (3y)(x-1) \\ 3xy - 3x &= 3xy - 3y \\ -3x &= -3y \\ x &= y. \end{aligned}$$

   Therefore, f is an injection.

   Surjection: Let $y \in \mathbb{R} - \{3\}$. Then $y = \frac{3x}{x-1}$, or $x = -\frac{y}{3-y}$, which is an element of $\mathbb{R} - \{3\}$. Substituting back into f gives us

$$\begin{aligned} f\left(-\frac{y}{3-y}\right) &= \frac{3\left(-\frac{y}{3-y}\right)}{\left(-\frac{y}{3-y}\right) - 1} \\ &= \frac{-3y}{-y-(3-y)} \\ &= \frac{-3y}{-y-3+y} \\ &= \frac{-3y}{-3} \\ &= y. \end{aligned}$$

   Therefore f is a surjection.

   Therefore f is a bijection.
   $\triangle$

   (b) Find the inverse of $f(x)$.

Let $g(x) = -\frac{x}{3-x}$. Then

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f\left(-\frac{x}{3-x}\right) \\ &= \frac{3\left(-\frac{x}{3-x}\right)}{\left(-\frac{x}{3-x}\right)-1} \\ &= \frac{-3x}{-x-(3-x)} \\ &= \frac{-3x}{-x-3+x} \\ &= \frac{-3x}{-3} \\ &= x \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(\frac{3x}{x-1}) \\ &= -\frac{\frac{3x}{x-1}}{3-\frac{3x}{x-1}} \\ &= \frac{-3x}{3(x-1)-3x} \\ &= \frac{-3x}{3x-3-3x} \\ &= \frac{-3x}{-3} \\ &= x \end{aligned}$$

Therefore, $g(x) = f^{-1}(x)$.

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