Formal Solutions To ODE's

1 Homogeneous Solutions

1.1 First Order

We have the following differential equation
$$y' + ay = 0.$$
Then
$$y = ce^{-at},$$
where c is determined by the initial condition.

1.2 Second Order

We have the following differential equation
$$y'' + ay' + by = 0. \tag{1}$$
Then we solve the following equation for r,
$$r^2 + ar + b = 0.$$
If $r = \alpha \pm i\beta$, then the general solution is
$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$
where c_1 and c_2 are determined by the initial conditions.

If Equation 1 has two real solutions, r_1 and r_2, then
$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
where c_1 and c_2 are determined by the initial conditions.

If Equation 1 has a real repeated root, r, then
$$y = c_1 e^{rx} + c_2 x e^{rx}$$
where c_1 and c_2 are determined by the initial conditions.

2 Nonhomogeneous

Suppose we have the following differential equation.
$$ay'' + by' + cy = f(x) \tag{2}$$
We begin by finding the solution to the homogeneous equation,
$$ay'' + by' + cy = 0.$$
Call this solution the general solution or y_g. Note, do **NOT** apply any initial conditions for this solution.

We now have a table for $f(x)$ and the form of the particular solution.

f	y_p
k	A
x	$Ax + B$
x^2	$Ax^2 + Bx + C$
e^x	Ae^x
$\cos(n\pi x)$ or $\sin(n\pi x)$	$A\cos(n\pi x) + B\sin(n\pi x)$
$x\cos(n\pi x)$ or $\sin(n\pi x)$	$(Ax + B)\cos(n\pi x) + (Cx + D)\sin(n\pi x)$

If your f is a linear combination of the above, then your y_p is a linear combination of the applicable entries above. To find the A, B, C, D, plug y_p back into Equation 2, and match terms with $f(x)$. The solution to Equation 2 is $y = y_g + y_p$. Now apply your initial conditions to determine your as of now undetermined coefficients.