9.3 Determining Optimal Mixed Strategies

As we have seen, each choice of strategies by R and C results in an expected value, representing the average payoff to R per play. In this section we shall give a method for choosing the best strategies. Let us begin by clarifying our notion of optimality.

DEFINITION To every choice of a strategy for R there is a best counterstrategy—that is, a strategy for C that results in the least expected value e. An **optimal mixed strategy for** R is one for which the expected value against C's best counterstrategy is as large as possible.

In a similar way we can define the optimal strategy for C.

DEFINITION To every choice of a strategy for C there is a best counterstrategy—that is, a strategy for R that results in the largest expected value e. An **optimal mixed strategy for** C is one for which the expected value against R's best counterstrategy is as small as possible.

It is most surprising that the optimal strategies for R and C in a non-strictly determined game may be determined using linear programming. To see how this is done, let us consider a particular problem.

EXAMPLE 1

Games as linear programs Suppose that a game has payoff matrix

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}.$$

Reduce the determination of an optimal strategy for R to a linear programming problem. *Note*: This game is not strictly determined. (Why?)

Solution Suppose that R plays the strategy $\begin{bmatrix} r_1 & r_2 \end{bmatrix}$. What is C's best counterstrategy? If C plays $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then the expected value of the game is

$$\begin{aligned}\begin{bmatrix} r_1 & r_2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 5r_1 + r_2 & 3r_1 + 4r_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (5r_1 + r_2)c_1 + (3r_1 + 4r_2)c_2 \end{bmatrix}.\end{aligned}$$

If C pursues his best counterstrategy, then he will try to minimize the expected value of the game. That is, C will try to minimize

$$(5r_1 + r_2)c_1 + (3r_1 + 4r_2)c_2.$$

Since $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 = 1$, this expression has as its minimum value the smaller of the terms $5r_1 + r_2$ or $3r_1 + 4r_2$. That is, if $5r_1 + r_2$ is the smaller, then C should choose the strategy $c_1 = 1$, $c_2 = 0$; whereas if $3r_1 + 4r_2$ is the smaller, then C should choose the strategy $c_1 = 0$, $c_2 = 1$. In any case the expected value of the game if C adopts his best counterstrategy is the smaller of $5r_1 + r_2$ and $3r_1 + 4r_2$. The goal of R is to maximize this expected value. In other words, the mathematical problem R faces is this:

$$\text{Maximize the minimum of } 5r_1 + r_2 \text{ and } 3r_1 + 4r_2,$$
$$\text{where} \quad r_1 \geq 0, \quad r_2 \geq 0, \quad r_1 + r_2 = 1.$$

Let v denote the minimum of $5r_1 + r_2$ and $3r_1 + 4r_2$. Clearly, $v > 0$. Then

$$\begin{aligned} 5r_1 + r_2 &\geq v \\ 3r_1 + 4r_2 &\geq v \end{aligned} \tag{1}$$

Maximizing v is the same as minimizing $1/v$. Moreover, the inequalities (1) may be rewritten in the form

$$\begin{aligned} 5\frac{r_1}{v} + \frac{r_2}{v} &\geq 1 \\ 3\frac{r_1}{v} + 4\frac{r_2}{v} &\geq 1 \end{aligned} \tag{2}$$

Moreover, since $r_1 \geq 0$, $r_2 \geq 0$, and $r_1 + r_2 = 1$, we see that

$$\frac{r_1}{v} \geq 0, \qquad \frac{r_2}{v} \geq 0, \qquad \frac{r_1}{v} + \frac{r_2}{v} = \frac{1}{v}. \tag{3}$$

This suggests that we introduce new variables:

$$y_1 = \frac{r_1}{v}, \qquad y_2 = \frac{r_2}{v}.$$

Then (3) and (2) may be rewritten as

$$\begin{aligned} y_1 + y_2 &= \frac{1}{v} \\ 5y_1 + y_2 &\geq 1 \\ 3y_1 + 4y_2 &\geq 1 \\ y_1 \geq 0, \quad y_2 &\geq 0 \end{aligned}$$

We wish to minimize $1/v$, so we may finally state our original question in terms of a linear programming problem: Minimize $y_1 + y_2$ subject to the constraints

$$\begin{cases} 5y_1 + y_2 \geq 1 \\ 3y_1 + 4y_2 \geq 1 \\ y_1 \geq 0, \quad y_2 \geq 0. \end{cases}$$

Now Try Exercise 1

In terms of the solution to this linear programming problem we may calculate R's optimal strategy as follows:

$$r_1 = vy_1 \qquad r_2 = vy_2, \qquad \text{where} \quad v = \frac{1}{y_1 + y_2}.$$

In the preceding derivation it was essential that the entries of the matrix were positive numbers, for this is how we derived that $v > 0$. The same reasoning used in Example 1 can be used in general to convert the determination of R's optimal strategy to a linear programming problem, *provided that the payoff matrix has positive entries*. If the payoff matrix does not have positive entries, then just add a large positive constant to each of the entries so as to give a matrix with positive entries. The new matrix will have the same optimal strategy as the original one. However, since all its entries are positive, we may use the previous reasoning to reduce determination of the optimal strategy to a linear programming problem.

Optimal Strategy for R Let the payoff matrix of a game be

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where all entries of the matrix are positive numbers. Let $y_1, y_2, \ldots, y_m$ be chosen so as to minimize

$$y_1 + y_2 + \cdots + y_m$$

subject to the constraints

$$\begin{cases} y_1 \geq 0,\ y_2 \geq 0,\ \ldots,\ y_m \geq 0 \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq 1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq 1 \\ \qquad \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq 1. \end{cases}$$

Let

$$v = \frac{1}{y_1 + y_2 + \cdots + y_m}.$$

Then an optimal strategy for R is $\begin{bmatrix} r_1 & r_2 & \cdots & r_m \end{bmatrix}$, where

$$r_1 = vy_1, \quad r_2 = vy_2, \quad \ldots, \quad r_m = vy_m.$$

Furthermore, if C adopts the best counterstrategy, then the expected value is v.

Note that the determination of $y_1, y_2, \ldots, y_m$ is a linear programming problem whose solution can be obtained using either the method of Chapter 3 (if $m = 2$) or the simplex method of Chapter 4 (any m). The next example illustrates the preceding result.

EXAMPLE 2

Finding the optimal strategy for R Suppose that a game has payoff matrix

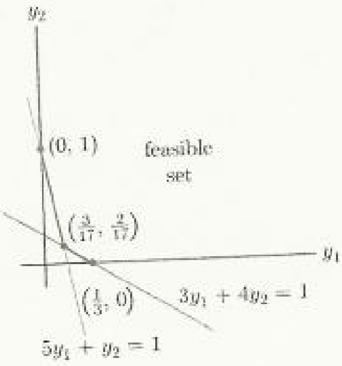
$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}.$$

(a) Determine an optimal strategy for R.

(b) Determine the expected payoff to R if C uses the best counterstrategy.

Solution (a) The associated linear programming problem asks us to minimize $y_1 + y_2$ subject to the constraints

$$\begin{cases} y_1 \geq 0, \quad y_2 \geq 0 \\ 5y_1 + \ \ y_2 \geq 1 \\ 3y_1 + 4y_2 \geq 1. \end{cases}$$



Vertex	$y_1 + y_2$
$(0,1)$	1
$\left(\frac{3}{17}, \frac{2}{17}\right)$	$\frac{5}{17}$
$\left(\frac{1}{3}, 0\right)$	$\frac{1}{3}$

Figure 1

In Fig. 1 we have sketched the feasible set for this problem and evaluated the objective function at each vertex.

The minimum value of $y_1 + y_2$ is $\frac{5}{17}$ and occurs when $y_1 = \frac{3}{17}$ and $y_2 = \frac{2}{17}$. Further,

$$v = \frac{1}{y_1 + y_2} = \frac{1}{\frac{5}{17}} = \frac{17}{5}$$

$$r_1 = vy_1 = \tfrac{17}{5} \cdot \tfrac{3}{17} = \tfrac{3}{5}$$

$$r_2 = vy_2 = \tfrac{17}{5} \cdot \tfrac{2}{17} = \tfrac{2}{5}$$

Thus the optimal strategy for R is $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$.

(b) The expected value against the best counterstrategy is $v = \frac{17}{5}$. ■

There is a similar linear programming technique for determining the optimal strategy for C.

Optimal Strategy for C Let the payoff matrix of a game be

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where all entries of the matrix are positive numbers. Let $z_1, z_2, \ldots, z_n$ be chosen so as to maximize

$$z_1 + z_2 + \cdots + z_n$$

subject to the constraints

$$\begin{cases} z_1 \geq 0, \quad z_2 \geq 0, \quad \ldots, \quad z_n \geq 0 \\ a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n \leq 1 \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n \leq 1 \\ \qquad \vdots \\ a_{m1}z_1 + a_{m2}z_2 + \cdots + a_{mn}z_n \leq 1. \end{cases}$$

Let $v = 1/(z_1 + z_2 + \cdots + z_n)$. Then an optimal strategy for C is

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where $c_1 = vz_1$, $c_2 = vz_2$, $\ldots$, $c_n = vz_n$.

MPLE 3 **Finding the optimal strategy for C** Determine the optimal strategy for C for the game with payoff matrix

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}.$$

Solution We must maximize $z_1 + z_2$ subject to the constraints

$$\begin{cases} z_1 \geq 0, \quad z_2 \geq 0 \\ 5z_1 + 3z_2 \leq 1 \\ z_1 + 4z_2 \leq 1. \end{cases}$$

The solution is as follows: The maximum value is $\frac{5}{17}$ and it occurs when $z_1 = \frac{1}{17}$, $z_2 = \frac{4}{17}$. Therefore, $v = \frac{17}{5}$, and the optimal strategy for C is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where

$$c_1 = vz_1 = \frac{17}{5} \cdot \frac{1}{17} = \frac{1}{5}$$

Now Try Exercise 3

$$c_2 = vz_2 = \frac{17}{5} \cdot \frac{4}{17} = \frac{4}{5}.$$

■

Notice that in both Examples 2 and 3 we obtained $v = \frac{17}{5}$. This was not just a coincidence. This phenomenon always occurs, and the number v is called the **value** of the game. An easy computation shows that for the matrix of Examples 2 and 3, when R and C each use their optimal strategies, the expected value is $v = \frac{17}{5}$. That is,

$$\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{17}{5} \end{bmatrix}.$$

Let us briefly reconsider the calculations of optimal strategies for R and C. We begin in every case with the matrix for the game, A. Let us assume that A is an $m \times n$ matrix and that each entry of A is a positive number. To find the optimal strategy for C, we find the matrix Z that maximizes the objective function EZ subject to the constraints $AZ \leq B$ and $Z \geq \mathbf{0}$, where

$$E = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{n \text{ entries}}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \quad \text{and} \quad B = \left.\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right\} m \text{ entries.}$$

The dual of the linear programming problem associated with finding an optimal strategy for C is to find the matrix Y that minimizes the objective function B^TY subject to the constraints $A^TY \geq E^T$ and $Y \geq \mathbf{0}$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

But this is exactly the problem of finding an optimal strategy for R.

The problem of finding the optimal strategy for R is a linear programming problem whose dual is the problem of finding an optimal strategy for C, and vice versa.

Our previous work with duality leads us to conclude the following:

1. If there exists an optimal strategy for R, then there exists an optimal strategy for C, and vice versa.
2. The minimum of the objective function $y_1 + y_2 + \cdots + y_m$ and the maximum of the objective function $z_1 + z_2 + \cdots + z_n$ are equal—since the linear programming problems are duals of each other. Hence, the value of the optimal strategy for C is the same as the value of the optimal strategy for R, that is,

$$\frac{1}{z_1 + z_2 + \cdots + z_n} = \frac{1}{y_1 + y_2 + \cdots + y_m}.$$

EXAMPLE 4

Optimal strategies when matrix entries are not positive Determine the optimal strategies for R and C for the game with payoff matrix

$$\begin{bmatrix} 1 & -1 \\ -3 & 0 \end{bmatrix}.$$

Solution We cannot apply our technique directly, since only one of the entries of the given matrix is a positive number. However, if we add 4 to each entry, then the new matrix will be

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix},$$

which does have all positive entries. These two payoff matrices have the same optimal strategies. The only difference is that the values in the new matrix are 4 more than that of the given matrix. Now, the optimal strategies and the value of a game with the new matrix were found in Examples 2 and 3 to be

$$\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}, \quad \text{and} \quad \frac{17}{5}.$$

Therefore, the optimal strategies for the given matrix are

$$\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix},$$

Now Try Exercise 5

and the value is $\frac{17}{5} - 4 = -\frac{3}{5}$. ■

EXAMPLE 5

Finding optimal strategies by the simplex method Use the simplex method and the resulting tableau to determine the optimal strategies for the game of Example 4.

Solution As in Example 4, add 4 to each entry to get a matrix with positive entries, and set up the tableau for finding the optimal strategy for C. (We choose this linear programming problem because it is a maximization problem.) The transformed matrix A is

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}.$$

To find the optimal strategy for C, we need to find the values of z_1 and z_2 that maximize $z_1 + z_2$ subject to the constraints

$$\begin{cases} 5z_1 + 3z_2 \leq 1 \\ z_1 + 4z_2 \leq 1 \\ z_1 \geq 0, \quad z_2 \geq 0. \end{cases}$$

We set up the tableau using slack variables t and u and display the initial and final tableaux:

$$\begin{array}{c} \\ t \\ u \\ M \end{array}\begin{array}{c} \begin{array}{cccccc} z_1 & z_2 & t & u & M & \end{array} \\ \left[\begin{array}{ccccc|c} 5 & 3 & 1 & 0 & 0 & 1 \\ 1 & 4 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} \\ z_1 \\ z_2 \\ M \end{array}\begin{array}{c} \begin{array}{cccccc} z_1 & z_2 & t & u & M & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{4}{17} & -\frac{3}{17} & 0 & \frac{1}{17} \\ 0 & 1 & -\frac{1}{17} & \frac{5}{17} & 0 & \frac{4}{17} \\ \hline 0 & 0 & \frac{3}{17} & \frac{2}{17} & 1 & \frac{5}{17} \end{array}\right] \end{array}.$$

Thus the solution is $z_1 = \frac{1}{17}$, $z_2 = \frac{4}{17}$ with $M = z_1 + z_2 = \frac{5}{17}$. Then $v = \frac{17}{5}$, and the optimal strategy for C is

$$\begin{bmatrix} vz_1 \\ vz_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}.$$

This agrees with previous solutions, and the value of this game is $\frac{17}{5}$, which is 4 more than the original game. Thus the value of the game is $\frac{17}{5} - 4 = -\frac{3}{5}$.

But the optimal strategy for R can be read from the final tableau since y_1 and y_2 are the values of the variables in the dual of the problem we solved. So $y_1 = t = \frac{3}{17}$, $y_2 = u = \frac{2}{17}$, and $M = \frac{5}{17}$. Then $v = \frac{17}{5}$ and the optimal strategy for R is $\begin{bmatrix} vy_1 & vy_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$. ■

Now Try Exercise 11

We actually have a very useful fact.

Fundamental Theorem of Game Theory Every two-person zero-sum game has a solution.

Verification of the Fundamental Theorem of Game Theory If the given two-person game has a saddle point, then the game is strictly determined, and optimal strategies for R and C are given by the position of the saddle point.

If the game is not strictly determined, then let us assume that the $m \times n$ payoff matrix A has only positive entries. We let B be an $m \times 1$ column matrix in which each entry is 1, and let E be a $1 \times n$ row matrix of 1's. Then

1. There is an optimal feasible solution to the problem:

$$\text{Maximize } M = EZ \text{ subject to } AZ \leq B \text{ and } Z \geq \mathbf{0}. \qquad \text{(P)}$$

2. There is an optimal feasible solution to the problem:

$$\text{Minimize } M = B^T Y \text{ subject to } A^T Y \geq E^T \text{ and } Y \geq \mathbf{0}. \qquad \text{(D)}$$

3. The solutions to (P) and (D) give a solution to the game.

To see that characteristics 1 and 2 hold, we note that there is a feasible solution for the inequalities of the primal problem (P). The $n \times 1$ matrix of zeros, $Z = \mathbf{0}$, satisfies $AZ \leq B$. Also, there is a feasible solution for the inequalities of the dual problem (D). This can be seen by noting that since every element of the matrix

A is positive, we can find an $m \times 1$ matrix $Y \geq \mathbf{0}$ with sufficiently large entries to guarantee that $A^TY \geq E^T$. Since the inequalities of both (P) and (D) have a feasible solution, the fundamental theorem of duality (Chapter 4) tells us that both (P) and (D) have optimal feasible solutions.

Let Z^* and Y^* be optimal feasible solutions of (P) and (D), respectively. Say that

$$Z^* = \begin{bmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{bmatrix} \quad \text{and} \quad Y^* = \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_m^* \end{bmatrix}.$$

The maximum for (P),

$$M = z_1^* + z_2^* + \cdots + z_n^*,$$

equals the minimum for (D),

$$M = y_1^* + y_2^* + \cdots + y_m^*,$$

and

$$AZ^* \leq B \quad \text{and} \quad A^TY^* \geq E^T.$$

Recall that B is an $m \times 1$ matrix of 1's and E^T is an $n \times 1$ matrix of 1's.

M must be strictly greater than zero since at least one of the y_i^* must be >0 in order for $A^TY \geq E^T$ to hold. Therefore, $1/M$ is defined. We let

$$C = \begin{bmatrix} \frac{1}{M} z_1^* \\ \frac{1}{M} z_2^* \\ \vdots \\ \frac{1}{M} z_n^* \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and

$$R = \begin{bmatrix} \frac{1}{M} y_1^* & \frac{1}{M} y_2^* & \cdots & \frac{1}{M} y_m^* \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & \cdots & r_m \end{bmatrix}.$$

Furthermore, C and R represent optimal strategies for players C and R, respectively. To verify that C and R are legitimate strategies, we note that since $M > 0$, $Z^* \geq \mathbf{0}$, and $Y^* \geq \mathbf{0}$, every entry in C and R is nonnegative. We only need to check that

$$c_1 + c_2 + \cdots + c_n = 1 \quad \text{and} \quad r_1 + r_2 + \cdots + r_m = 1.$$

This follows directly from the definitions of M, C, and R. ■

Practice Problems 9.3

1. Determine the optimal strategy for C for the game with payoff matrix

$$\begin{bmatrix} 2 & 14 \\ 6 & 12 \\ 8 & 6 \end{bmatrix}.$$

2. Determine by inspection the optimal strategies for C for the games whose payoff matrices are given.

(a) $\begin{bmatrix} 0 & 12 \\ 4 & 10 \\ 6 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 0 \\ 4 & 3 \\ 8 & -1 \end{bmatrix}$

ERCISES 9.3

1. Suppose that a game has payoff matrix

$$\begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}.$$

Reduce the determination of an optimal strategy for R into a linear programming problem. Just set up the problem showing the constraints and the objective function.

2. Suppose a game has a payoff matrix

$$\begin{bmatrix} 10 & 6 \\ 5 & 7 \end{bmatrix}.$$

Reduce the determination of an optimal strategy for R into a linear programming problem. Just set up the problem showing the constraints and the objective function.

n Exercises 3–8, determine optimal strategies for R and r C for the games whose payoff matrices are given.

3. $\begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -6 \\ -5 & 4 \end{bmatrix}$ 6. $\begin{bmatrix} 5 & 2 \\ 7 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$ 8. $\begin{bmatrix} 5 & -8 \\ 3 & 6 \end{bmatrix}$

In Exercises 9 and 10, determine optimal strategies for R for the games whose payoff matrices are given.

9. $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 6 \end{bmatrix}$ 10. $\begin{bmatrix} -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

In Exercises 11 and 12, determine optimal strategies for C for the games whose payoff matrices are given.

11. $\begin{bmatrix} -3 & 1 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$ 12. $\begin{bmatrix} 0 & 2 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$

13. Two players, Renée and Carlos, play a game with payoff matrix

$$\begin{bmatrix} 5 & -3 \\ -3 & 1 \end{bmatrix}.$$

Is the game strictly determined? Determine the optimal mixed strategy for each player. What is the value of the game? Explain.

14. Determine the optimal mixed strategy for each player, given the payoff matrix

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Give the value of the game.

15. **Smuggler's Strategy** A rumrunner attempts to smuggle rum into a country having two ports. Each day the coast guard is able to patrol only one of the ports. If the rumrunner enters via an unpatrolled port, he will be able to sell his rum for a profit of \$7000. If he enters the first port and it is patrolled that day, he is certain to be caught and will have his rum (worth \$1000) confiscated and be fined \$1000. If he enters the second port (which is big and crowded) and it is patrolled that day, he will have time to jettison his cargo and thereby escape a fine.
   (a) What is the optimal strategy for the rumrunner?
   (b) What is the optimal strategy for the coast guard?
   (c) How profitable is rumrunning? That is, what is the value of the game?

16. **Which Hand?** Ralph puts a coin in one of his hands and Carl tries to guess which hand holds the coin. If Carl guesses incorrectly, he must pay Ralph \$2. If Carl guesses correctly, then Ralph must pay him \$3 if the coin was in the left hand and \$1 if it was in the right.
   (a) What is the optimal strategy for Ralph?
   (b) What is the optimal strategy for Carl?
   (c) Whom does this game favor?

17. **Advertising Strategies** The Carter Company can choose between two advertising strategies (I and II). Its most important competitor, Rosedale Associates, has a choice of three advertising strategies (a, b, c). The estimated payoff to Rosedale Associates away from the Carter Company is given by the payoff matrix

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} -2 & 1 \\ 2 & -3 \\ 1 & -2 \end{bmatrix} \end{array},$$

where the entries represent thousands of dollars per week. Determine the optimal strategies for each company.

Solutions to Practice Problems 9.3

1. The associated linear programming problem is: Maximize $z_1 + z_2$ subject to the constraints

$$\begin{cases} z_1 \geq 0, \quad z_2 \geq 0. \\ 2z_1 + 14z_2 \leq 1 \\ 6z_1 + 12z_2 \leq 1 \\ 8z_1 + \ 6z_2 \leq 1. \end{cases}$$

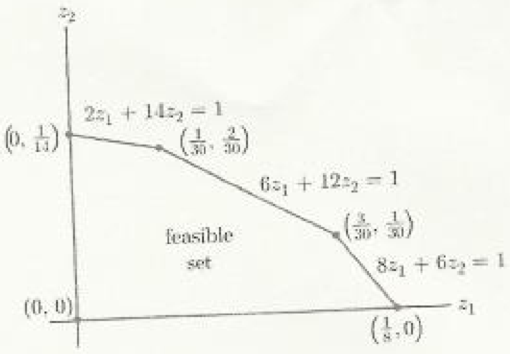
In Fig. 2 we have sketched the feasible set and evaluated the objective function at each vertex. The maximum value of $z_1 + z_2$ is $\frac{4}{30}$, which is achieved at the vertex $(\frac{3}{30}, \frac{1}{30})$. Therefore,

$$v = \frac{1}{z_1 + z_2} = \frac{1}{\frac{4}{30}} = \frac{30}{4}$$

$$c_1 = v \cdot z_1 = \frac{30}{4} \cdot \frac{3}{30} = \frac{3}{4}$$

$$c_2 = v \cdot z_2 = \frac{30}{4} \cdot \frac{1}{30} = \frac{1}{4}.$$

That is, the optimal strategy for C is $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$.



Vertex	$z_1 + z_2$
$(0, 0)$	0
$(0, \frac{1}{14})$	$\frac{1}{14}$
$(\frac{1}{30}, \frac{2}{30})$	$\frac{3}{30}$
$(\frac{3}{30}, \frac{1}{30})$	$\frac{4}{30}$
$(\frac{1}{8}, 0)$	$\frac{1}{8}$

Figure 2

2. (a) $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$. If we add 2 to each entry, we obtain the payoff matrix of Problem 1, so these two games have the same optimal strategies.

(b) Always play column 2. This game is strictly determined and has the entry 3 as saddle point. It is a good idea always to check for a saddle point before looking for a mixed strategy.

CHAPTER SUMMARY

1. A *zero-sum* game is one in which a payoff to one player results in a loss of the same amount to the other player.
2. The entry in the ith row and jth column of a payoff matrix gives the payoff to the row player (equivalently, the loss to the column player) when the row player chooses row i and the column player chooses column j.
3. A *pure strategy* is one in which the player consistently chooses the same row or column. Strategies involving varied moves are called *mixed strategies*.
4. The *optimal pure strategy* for the row player is to choose the row whose least element is maximal. The optimal pure strategy for the column player is to choose the column whose greatest element is minimal.
5. A *saddle point* is an entry in a payoff matrix that is simultaneously the least element of its row and the greatest element of its column. A game need not have a saddle point. If a game has more than one saddle point, then the saddle points are equal.
6. A game with a saddle point is called a *strictly determined game*. In a strictly determined game, the optimal pure strategy for each player is to choose a row or column containing a saddle point.
7. In a strictly determined game, if both players use optimal pure strategies, then the saddle point gives the payoff to the row player. The value of the saddle point is called the *value of the game*.
8. If a game is not strictly determined, then the players should use mixed strategies. A *mixed strategy* for the row player is a row matrix whose ith entry is the probability that the row player will choose row i on any repetition of the game. A mixed strategy for the column player is a column matrix whose jth entry is the probability that the column player will choose column j on any repetition of the game.