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Homework 3

1. Suppose F is a field with a non-Archemedian absolute value $|\cdot|$ and $\mathcal{O}_F$ is the ring of integers of F. Prove that a point $x \in \mathcal{O}_F$ is a unit if and only if $|x| = 1$.

   Suppose $x \in \mathcal{O}_F$ is a unit and consider $x^{-1} \in \mathcal{O}_F$ as the inverse of x. Since $x \in \mathcal{O}_F$, we know

$$
\begin{aligned}
|x| &\le 1 \\
|x^{-1}||x| &\le |x^{-1}| \le 1 \qquad \text{(since } x^{-1} \in \mathcal{O}_F\text{)} \\
|x^{-1}x| &\le |x^{-1}| \le 1 \\
|1| &\le |x^{-1}| \le 1 \\
1 &\le |x^{-1}| \le 1.
\end{aligned}
$$

   The only choice we have for x^{-1} is 1, so

$$
\begin{aligned}
x^{-1} &= 1 \\
xx^{-1} &= x \\
x &= 1 \\
|x| &= 1.
\end{aligned}
$$

   Suppose $|x| = 1$.

2. If n is a non-negative integer a p is prime, define $\mathcal{I}_n = \{x \in \mathcal{O}_p : |x|_p \le p^{-n}\}$ where $\mathcal{O}_p = \{x \in \mathbb{Q} : |x|_p \le 1\}$.

   (a) If $x, y \in \mathcal{I}_n$ prove that $x + y \in \mathcal{I}_n$.

   Suppose $x, y \in \mathcal{I}_n$, and consider $|x+y|_p$. Then

$$
\begin{aligned}
|x+y|_p &\le \max\{|x|_p, |y|_p\} \\
&\le \max\{p^{-n}, p^{-n}\} \\
&= p^{-n}
\end{aligned}
$$

   Therefore $|x+y|_p \le p^{-n}$, so $x + y \in \mathcal{I}_n$.

   (b) If $x \in \mathcal{I}_n$ and $r \in \mathcal{O}_p$ prove that $rx \in \mathcal{I}_n$.

   Let $|\cdot|$ be the p-adic absolute value and consider $|rx| = |r| \cdot |x|$. Since $|x| \le p^{-n}$ and $|r| \le 1$, we have $|r| \cdot |x| \le 1 \cdot p^{-n} = p^{-n}$, so $|rx| \le p^{-n}$ which implies that $rx \in \mathcal{I}_n$.

   (c) What is another way to describe the set $\mathbb{Z} \cap \mathcal{I}_n$?