Theorems and Definitions

1. Prime Number: A positive $n \in \mathbb{Z}$ such that $n > 1$ is prime if its only divisors are 1 and n.
2. Fundamental Theorem of Arithmetic: Suppose $n \in \mathbb{Z} > 1$. Then
   (a) $\exists \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N}$ and primes $p_1, p_2, \ldots, p_n$ such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, or $n = \prod_{i=1}^{n} p_i^{\alpha_i}$.
   (b) The expression $\prod_{i=1}^{n} p_i^{\alpha_i}$ is unique up to permutation of the primes.
   (c) Lemma: Suppose p is prime and $a, b \in \mathbb{Z}$.
      i. If $p \nmid a$, then p and a are relatively prime.
      ii. If $p \mid ab$, then $p \mid a$ xor $p \mid b$.
   (d) Corollary: If p is prime and divides $a_1, a_2, \ldots, a_n$, then $p|a_i$ for some $i \in \{1, 2, \ldots, n\}$.
   (e) Corollary: If m is not a perfect square, then $\sqrt{m} \notin \mathbb{Q}$.
3. Exact Division: If $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$, then p^α exactly divides n, noted as $p^\alpha || n$.
4. Theorem: Suppose $a_1, a_2, \ldots, a_k$ are pairwise relatively prime and are greater than 0. If $a_1 a_2 \ldots a_k = x^m$ for some $x \in \mathbb{N}$ and $m \geq 2$, then $\exists x_i \in \mathbb{N}$ such that $a_i = x_i^m$ for all i.
5. Theorem: There are infinitely many primes.
6. Prime Number Theorem: Let $\pi(x)$ denote the number of primes such that $p_i \leq x$. We have that
   (a) $\lim_{x \to \infty} \pi(x) = \infty$
   (b) $\pi(x) < x$ for $x \geq 2$.

   We can also define the logarithmic integral as $\mathcal{L}_i(x) = \int_2^x \frac{1}{\log t}\, dt$. Then

$$\lim_{x \to \infty} \frac{\mathcal{L}_i}{\pi(x)} = \lim_{x \to \infty} \frac{\frac{x}{\log x}}{\pi(x)} = 1$$