Exam Review

1. Find the general solutions to the following ordinary differential equations.

   (a) $y'(t) + 4y(t) = 1 + 2t$

   Homogeneous Solution:

   $$\begin{gathered} y' + 4y = 0 \\ r + 4 = 0 \\ r = -4 \\ y_g(t) = ce^{-4t} \end{gathered}$$

   Particular Solution:

   $$\begin{gathered} y' + 4y = 1 + 2t \\ y = At + B \qquad y' = A \\ A + 4At + 4B = 1 + 2t \\ 4A = 2 \qquad A + 4B = 1 \\ A = \frac{1}{2} \qquad B = \frac{1}{8} \\ y_p(t) = \frac{1}{2}t + \frac{1}{8} \end{gathered}$$

   $$y(t) = ce^{-4t} + \frac{1}{2}t + \frac{1}{8}$$

   (b) $y''(t) + y(t) = \cos \omega t$

   Homogeneous Solution:

   $$\begin{gathered} y''(t) + y(t) = 0 \\ r^2 + 1 = 0 \\ r = \pm\, i \\ y_g(t) = c_1 \cos t + c_2 \sin t \end{gathered}$$

   Particular Solution:

2. Here is another Question

3. Solve the initial boundary value problem.

   $$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = f(x),\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases} \qquad f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1 - x & \frac{1}{2} \le x \le 1 \end{cases}$$

   $$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x) \\ u(x,0) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\pi b_n \sin(n\pi x) = 0 \end{aligned}$$

$$
\begin{aligned}
a_n &= 2\left(\int_0^1 f(x)\sin(n\pi x)\,dx\right) \\
&= 2\left(\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{1/2}^1 (1-x)\sin(n\pi x)\,dx\right) \\
&= 2\left(\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{1/2}^1 \sin(n\pi x)\,dx - \int_{1/2}^1 x\sin(n\pi x)\,dx\right) \\
&= 2\left(\left[\frac{\sin(n\pi x)}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right]\Bigg|_0^{1/2} + \left[\frac{-\cos(n\pi x)}{n\pi}\right]\Bigg|_{1/2}^1 + \left[\frac{x\cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{(n\pi)^2}\right]\Bigg|_{1/2}^1\right) \\
&= 2\left(\frac{\sin\frac{n\pi}{2}}{(n\pi)^2} - \frac{\cos\frac{n\pi}{2}}{2n\pi} + \frac{\cos\frac{n\pi}{2}}{n\pi} - \frac{\cos n\pi}{n\pi} - \frac{\cos\frac{n\pi}{2}}{2n\pi} + \frac{\sin\frac{n\pi}{2}}{(n\pi)^2} + \frac{\cos n\pi}{n\pi} - \frac{\sin n\pi}{(n\pi)^2}\right) \\
&= \frac{4\sin\frac{n\pi}{2}}{(n\pi)^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{\frac{1}{\pi}\int_0^1 0\,dx}{n\pi} \\
&= 0
\end{aligned}
$$

$$
u(x,t) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos(2k\pi t + \pi t)\sin(2k\pi t + \pi t)
$$

4. Solve the following partial differential equation.

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = 0 \\ u(0,t) = A\cos t,\ u(1,t) = 0 \end{cases}$$

$$\begin{aligned} \delta(x,t) &= A\cos t - xA\cos t \\ \gamma(x,t) &= u(x,t) - \delta(x,t) \\ &= u - A\cos t + xA\cos t \\ \gamma_t(x,t) &= u_t + A\sin t - xA\sin t \end{aligned}$$