1. If F is a field in which 0 has a mutiplicative inverse, show $|F| = 1$.

   Proof: Suppose $|F| > 1$, and let $x \in F$ such that $x = 0^{-1}$. Then $0x = 1 = 0$. We know that under multiplication, any element in F can be expressed as $1 \times y\ \forall y \in F$, and since $1 = 0$, $1 \times y = 0 \times y = 0$. Therefore the only element in F is 0, and $|F| = 1$

2. Suppose that $n > 1$ is an integer and let $\mathbb{Z}_n$ be equipped with addition and multiplication modulo n. Prove that $\mathbb{Z}_n$ is a field if and only if n is prime.

   In order to be a field, $\mathbb{Z}_n$ must fulfill the following axioms:

   (A1) Addition is commutative on F.
   (A2) Addition is associative on F.
   (A3) There is a uniqe additive identity, called 0.
   (A4) There is an additive inverse $-a$ for all $a \in F$.
   (M1) Multiplication is commutative on F.
   (M2) Multiplication is associative on F.
   (M3) There is a unique multiplicative identity called 1.
   (M4) There is an multiplicative inverse element a^{-1} for every $a \in F$.
   (D) For all $x, y, z \in F$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
   (ZO) The additive and multiplicative identity are distinct.

   Proof: Suppose n is prime. Then every element in $\mathbb{Z}_n$ is a unit, so $\mathbb{Z}_n$ is equipped with a multiplicative inverse. Since $\mathbb{Z}_n$ has addition and multiplication, we know it is commutative and associateve for both those operations. We know that $\mathbb{Z}_n$ contains 0, so it has the additive inverse. Addition and multiplication form the distributive law, so $\mathbb{Z}_n$ is equipped with the distributive law. Since $n > 1$, $\mathbb{Z}_n$ $0 \neq 1$. Therefore, by defintion, $\mathbb{Z}_n$ is a field since it fulfills all the field axioms.

   Suppose $\mathbb{Z}_n$ is a field with n not prime. Since n is not prime, then we can find an element $x \in \mathbb{Z}_n$ such that $gcd(n, x) \neq 1$. Therefore there is an element in $\mathbb{Z}_n$ that is not a unit, so $\mathbb{Z}_n$ is not a field, since there is an element without an inverse.

3. Suppose that F is a finite field and $x \in F \setminus \{0\}$. Prove that there exists $n \in \mathbb{N}$ such that $x^n = 1$.