Elliptic Functions

We will begin the discussion of Elliptic functions by analyzing the familiar trig functions, sin and cos. Consider the following system of ODEs:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$

This system has the solution $x(t) = \cos t$ and $y(t) = \sin t$. We will use this definition to formulate the Elliptic functions.

We define a system of ODE's by letting $k \in (0,1)$ and $t \in \mathbb{R}$ representative of time. We can define three functions, $sn(t,k), cn(t,k), dn(t,k)$, as solutions to the system

$$\begin{cases} \frac{dx}{dt} = yz \\ \frac{dy}{dt} = -zx \\ \frac{dz}{dt} = -k^2xy \end{cases} \implies \begin{cases} \frac{d}{dt}sn(t,k) = cn(t,k)dn(t,k) \\ \frac{d}{dt}cn(t,k) = -dn(t,k)sn(t,k) \\ \frac{d}{dt}dn(t,k) = -k^2sn(t,k)cn(t,k) \end{cases}$$

The functions are not arbitrary; they have special properties. The first one we notice is that

$$\lim_{k\to 0^+} sn(t,k) = \sin t \qquad \lim_{k\to 0^+} cn(t,k) = \cos t \qquad \lim_{k\to 0^+} dn(t,k) = 1.$$

In addition, we have that

$$\lim_{k\to 1^-} sn(t,k) = \tanh t \qquad \lim_{k\to 1^-} cn(t,k) = \operatorname{sech} t \qquad \lim_{k\to 1^-} dn(t,k) = \operatorname{sech} t.$$