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Homework 5

1. Let $f(x,y) = x^2 + y^2$. Observe that $(x,y) = (0,0)$ defines a solution to the equation $f(x,y) = 0$. Do the remainder of this problem, we shall call $(0,0)$ the trivial solution to $f(x,y) = 0$.

   (a) Prove that $f(x,y) = 0$ has no non-trivial solutions with $x, y \in \mathbb{Q}$.

   (b) Prove that there exists a prime p such that $f(x,y) = 0$ has a nontrivial solution with $x, y \in \mathbb{Q}_p$.

   (c) Verify that f satisfies the Hasse Principle without referring to the Hasse-Minkowski Theorem.

2. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of points in $\mathbb{Z}_p$. Prove that the series

$$\sum_{n=0}^{\infty} a_n p^n$$

converges in $\mathbb{Q}_p$.

Proof: We know that a series will converge if and only if its sequence of partial sums converges. We can test for convergence by calculating

$$\lim_{n\to\infty} \left| a_{n+1}p^{n+1} - a_n p^n \right|_p .$$

We can say

$$\begin{aligned} \left| a_{n+1}p^{n+1} - a_n p^n \right|_p &= \max\left\{ \left| a_{n+1}p^{n+1} \right|_p, \left| a_n p^n \right|_p \right\} \\ &= \max\left\{ \frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}} \right\}. \\ \lim_{n\to\infty} \max\left\{ \frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}} \right\} &= \max\{0,0\} \\ &= 0. \end{aligned}$$

Therefore, the sequence of partial sums is Cauchy, and thus convergent, so the series converges.

3. Suppose that $k \in \mathbb{N}$.

   (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: Suppose otherwise. Then

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p = 0.$$

We can analyze the p-adic absolute value inside the limit:

$$\left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p \le \max\left\{ p^{-v_p\left(1/(n+1)^k\right)}, p^{-v_p\left(1/n^k\right)} \right\}$$

We can solve for the valuation in the first term:

$$v_p\left(\frac{1}{(n+1)^k}\right) = -\frac{k\log(n+1) + \log m}{\log p},$$

where $m \nmid p$. If we take the limit as $n \to \infty$ of this expression, we see that it tends towards negative infinity, so the p-adic absolute value tends towards infinity. Similarly, we can evaluate the limit of $|1/n^k|_p$, and we will see that it too tends towards infinity. Therefore,

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p \neq 0,$$

so the series does not converge since its sequence of partial sums does not converge.

(b) Prove that the series

$$\sum_{n=1}^{\infty} n^k$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: We want to show that

$$\lim_{n\to\infty} \left|(n+1)^k - n^k\right|_p \neq 0.$$