Partial Differential Equations

1 Bessel Functions

The 2D heat equation provided us a way to describe heat flow through a 2D surface, such as a thin rod. Now imagine that we have a surface that is shaped like a large cylinder. We've now moved into 3D space, and as such we will have to come up with a way to describe the heat transfer through the cylinder. Let's start by recalling Laplace's equation,

$$\nabla^2 u = 0,$$

which described a vector composed of the all the partial derivatives of u. We will need to keep this equation in mind as we concern ourselves with the 3D heat equation.

Our end goal will be to come up with a formal solution for the equation

$$u_t = k^2(u_{xx} + u_{yy}).$$

This is, of course, the heat equation expanded to a cylinder. Recall that when we dealt with Laplace's equation on a disk, we moved into polar coordinates. The 3D heat equation contains the laplacian in it, represented as

$$u_{xx} + u_{yy}.$$

Moving into polar coordinates will allow us to have rectangular bounds on our function. Transforming our heat equation yields

$$u_t = k^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}).$$

We now have a rectangular bound, which allows us to use the familiar separation of variables technique to solve the PDE. In this case, we want to separate the spacial variables (r, θ) from the time variable (t). We have a general solution of

$$u = V(r, \theta)T(t) = 0.$$

Setting them equal to a common value yields

$$\frac{T}{k^2 T} = \frac{V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta}}{V} = -\lambda^2.$$

As expected, we are left with 2 ODEs. The first ODE, concerning T, will be left alone for now. The remaining ODE can be solved by using separation of variables again. Doing so yields

$$V = R(r)\nu(\theta)$$

$$\begin{cases} \eta^{''} + \gamma\eta = 0 \\ r^2 R'' + rR' + (k^2 r^2 - \gamma^2)R = 0 \end{cases}.$$

Again, we will leave the η function for later, and instead turn our attention to the equation concerning R. This leads us to our discussion of Bessel Functions.