

# Methods For Solving ODE's

## 1 First Order Equations

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### 1.1 Separable Equations

$$\begin{aligned}\frac{dy}{dx} &= g(x)h(y) \\ \frac{dy}{h(y)} &= g(x) dx \\ \int \frac{dy}{h(y)} &= \int g(x) dx\end{aligned}$$

### 1.2 The Integrating Factor

$$\begin{aligned}\frac{dy}{dx} + \rho(x)y(x) &= q(x) \\ \frac{dy}{dx} e^{-\int \rho(x) dx} + \rho(x)y(x)e^{-\int \rho(x) dx} &= q(x)e^{-\int \rho(x) dx} \\ \frac{d}{dx} \left( y(x)e^{-\int \rho(x) dx} \right) &= q(x)e^{-\int \rho(x) dx} \\ \int \frac{d}{dx} \left( y(x)e^{-\int \rho(x) dx} \right) &= \int q(x)e^{-\int \rho(x) dx}\end{aligned}$$

### 1.3 Exact Differential Equations

$$\begin{aligned}M(x, y) dx + N(x, y) dy &= 0 \\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} &\implies \begin{cases} M = \frac{\partial F}{\partial x} \\ N = \frac{\partial F}{\partial y} \end{cases} \\ F(x, y) &= \int \frac{\partial F}{\partial x} dx\end{aligned}$$

A function  $g(y)$  will be left over; use  $N$  to solve for it by  $N = \frac{\partial F}{\partial y}$

## 2 Second Order Equations

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### 2.1 The Wronskian and Linear Independence

1. If  $\mathcal{W}(x, y)(x_0)$  for some  $x_0$  in an interval  $\mathcal{I}$  is **not** equal to 0, then the two functions are linearly independent.
2. If  $f(x)$  and  $g(x)$  are linearly dependent on  $\mathcal{I}$ , the  $\mathcal{W}(f, g)(x) = 0$  for all  $x$  in  $\mathcal{I}$ .
3. If  $f$  and  $g$  are functions of the same variable, then

$$\mathcal{W}(f, g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

4. Given that  $y_1$  and  $y_2$  are two solutions to  $y'' + p(x)y' + q(x)y = 0$ ,

$$\mathcal{W}(y_1, y_2)(x) = ce^{-\int p(x) dx}.$$

### 2.2 Algebraic Methods

#### Homogeneous

$$ay'' + by' + cy = 0 \quad \text{becomes} \quad ar^2 + br + c = 0$$

The roots of  $r$  determine the formal solution to the ODE

Two distinct roots in  $\mathbb{R}$

$$\begin{aligned} y_1(x) &= e^{r_1 x} \\ y_2(x) &= e^{r_2 x} \\ y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \end{aligned}$$

Two identical roots in  $\mathbb{R}$

$$\begin{aligned} y_1(x) &= e^{rx} \\ y_2(x) &= xe^{rx} \\ y(x) &= c_1 e^{rx} + c_2 x e^{rx} \end{aligned}$$

Two conjugate roots in  $\mathbb{C}$

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \\ y(x) &= e^{\alpha x} (c_1 \cos(\beta x) - c_2 \sin(\beta x)) \end{aligned}$$

### 2.3 The Method of Undetermined Coefficients

Given  $y'' + p(x)y' + q(x)y = g(x)$ , we know that  $y(x)$  is the sum of the homogeneous solution and the non-homogeneous solution of the ODE.