Problems Involving Legendre Polynomials

1. Show $\int_{-1}^{1} x^2 p_l(x)\,dx = 0$ for $l \geq 3$.

   We know that the general form for a Legendre polynomial $p_l(x)$ is given by Rodrigues' Formula to be
   $$p_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left[(x^2-1)^l\right].$$
   We can use integration by parts to perform the integration:
   $$\begin{array}{ccc} (+) & x^2 & \frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right] \\ (-) & 2x & \frac{d^{l-2}}{dx^{l-2}}\left[(x^2-1)^l\right] \\ (+) & 2 & \frac{d^{l-3}}{dx^{l-3}}\left[(x^2-1)^l\right]. \end{array}$$
   We now have that
   $$\int_{-1}^{1} x^2 p_l(x)\,dx = x^2\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right] - 2x\frac{d^{l-2}}{dx^{l-2}}\left[(x^2-1)^l\right] + 2\frac{d^{l-3}}{dx^{l-3}}\left[(x^2-1)^l\right].$$
   It is clear that the $l \geq 3$ requirement is necessary, as any $l < 3$ would lead to a negative order derivative. We now need to show that this value is equal to 0; to do so, we will start by substituting $Q(x) = (x^2-1)^l$. We now have
   $$\begin{aligned} &x^2\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right] - 2x\frac{d^{l-2}}{dx^{l-2}}\left[(x^2-1)^l\right] + 2\frac{d^{l-3}}{dx^{l-3}}\left[(x^2-1)^l\right] \\ &= x^2\frac{d^{l-1}}{dx^{l-1}}Q(x) - 2x\frac{d^{l-2}}{dx^{l-2}}Q(x) + 2\frac{d^{l-3}}{dx^{l-3}}Q(x) \\ &= x^2Q^{(l-1)}(x) - 2xQ^{(l-2)}(x) + 2Q^{(l-3)}(x). \end{aligned}$$
   Let's consider the case where $l = 3$ to show that this integral is indeed 0. Substituting 3 for l yields
   $$x^2Q^{''}(x) - 2xQ^{'}(x) + 2Q(x).$$
   Through some careful integration, we find that
   $$\begin{aligned} Q &= (x^2-1) \\ Q^{'} &= 6x(x^2-1)^2 \\ Q^{''} &= 24x^2(x^2-1) + 6(x^2-1)^2. \end{aligned}$$
   Notice that in every term of each polynomial there is an $(x^2-1)^n$ term, where $n \in \mathbb{N}$. If we factor this term, we find that we have $(x+1)^n(x-1)^n$. The original integral was over the interval $(-1,1)$, which yields a 0 in each term when we evaluate the anti-derivative at those bounds. Hence,
   $$\begin{aligned} x^2Q^{''}(x) - 2xQ^{'}(x) + 2Q(x)\Big|_{-1}^{1} &= x^2(0) - 2x(0) + 2(0) \\ &= 0. \end{aligned}$$

We have shown that for $l = 3$, the value of the integral is 0. However, we need to show that $\forall l \geq 3$, the value of the integral is 0. We can use the Leibniz Rule to express the l^{th} derivative in terms of a summation.

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x).$$

We are trying to prove that $x^2Q^{(l-1)}(x) - 2xQ^{(l-2)}(x) + 2Q^{(l-3)}(x) = 0$ for all $l \geq 3$. We know $Q = (x^2-1)^l$, so we can take one derivative of Q, which gives us $2xl(x^2-1)^{l-1}$. Let $f(x) = 2xl$ and $g(x) = (x^2-1)^{l-1}$. Our goal is to find the l^{th} derivative of Q. By the Leibniz Rule we have

$$(fg)^{(l)} = \sum_{k=0}^{l} \binom{l}{k} f^{(l-k)}g^{(k)}.$$

Consider the case where we take another derivative. We know that we will have a $(x^2-1)^{(l-2)}$ term. Again, we can take another derivative, which would provide us with $(x^2-1)^{(l-3)}$. We know that this term will continue to appear until $k > l$. However, the Leibniz Rule will stop when $k = l$. Therefore, every product $f^{(l-k)}g^{(k)}$ will have a $(x^2-1)^n$ term where n is the $l-k$ power. We can also rewrite $(x^2-1)^n$ as $(x-1)^n(x+1)^n$, so we now know that every term in the summation will have an $(x-1)^n(x+1)^n$ term. Recall that we are evaluating this term from -1 to 1, so when we ultimately substitute into the equation, we will either have $(x-1)^n = 0$ or $(x+1)^n = 0$. Therefore, we can conclude that every term in the summation will be 0. This implies that the l^{th} derivative of Q is 0 for any $l \geq 3$. Therefore $x^2Q^{(l-1)}(x) - 2xQ^{(l-2)}(x) + 2Q^{(l-3)}(x) = 0$.

2. We want to find a_l in the series

$$f(x) = \sum_{l=0}^{\infty} a_l p_l(x)$$

where

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & -1 < x \leq 1 \end{cases} .$$

We know that we can express a_l as

$$a_l = \frac{\int_{-1}^{1} f(x) p_l(x)\, dx}{\int_{-1}^{1} p_l^2(x)\, dx}.$$

Since f is piecewise defined to be 0 on $-1 < x \leq 1$, we are only dealing with

$$a_l = \frac{\int_{0}^{1} x p_l(x)\, dx}{\int_{-1}^{1} p_l^2(x)\, dx}.$$

We will begin by analyzing the numerator. We are solving

$$\int_0^1 x p_l(x)\, dx\,.$$

Using integration by parts gives us

$$\begin{aligned}
&x \frac{1}{2^l l!} \frac{d^{l-1}}{dx^{l-1}} \left[(x^2-1)^l\right] - \frac{1}{2^l l!} \frac{d^{l-2}}{dx^{l-2}} \left[(x^2-1)^l\right] \Big|_{x=0}^{x=1} \\
&= x p_{l-1}(x) - p_{l-2}(x) \Big|_{x=0}^{x=1} \\
&= [1 \cdot p_{l-1}(1) - p_{l-2}(1)] - [0 - p_{l-2}(0)] \\
&= p_{l-2}(0).
\end{aligned}$$

We can express $p_l(0)$ as

$$p_l(0) = \begin{cases} \frac{(-1)^{l/2}}{2^l} \binom{l}{l/2} & x \in \mathbb{E} \\ 0 & x \in \mathbb{O} \end{cases} .$$

We will consider even l here, since all odd l will produce a 0. We can now express a_l as

$$\begin{aligned}
a_l &= \frac{(2l+1) p_{l-2}(0)}{2} \\
&= \frac{4(2l+1)}{l!}.
\end{aligned}$$

The Fourier series is now

$$\sum \frac{4(2l+1)}{l!} p_l(x)$$

for even l.