We can define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

We know this function to be dicontinuous under the usual absolute value. In $\mathbb{R}$ under the usual absolute value, we define a function h to be continuous at a point a in the domain of h by

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |h(x) - h(a)| < \epsilon.$$

We can reformulate this definition in the p-adic sense by

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a|_p < \delta \implies |h(x) - h(a)|_p < \epsilon.$$

If we refomulate our original function, then we can show that under the p-adic absolute value, the function is continuous at every point. We will define $g : \mathbb{Q}_p \to \mathbb{Q}_p$ as

$$g(x) = \begin{cases} 0, & x \in \mathbb{Z} \\ 1, & x \in \mathbb{Q}_p \setminus \mathbb{Z} \end{cases}$$

as an analog to the funtion f we defined earlier. We will show that this function is indeed continuous.

Proof: Let $\epsilon > 0$ be given. If g is continuous at a point c, then $\exists \delta > 0$ such that $\forall \epsilon > 0$, $|x - c|_p < \delta \implies |g(x) - g(c)|_p < \epsilon$. We can determine our δ as follows:

If $a \in \mathcal{O}_p$, then

$$|g(x) - g(a)|_p = |g(x) - 0|_p = |g(x)|_p = p^{-v_p(g(x))} \leq 1.$$

If $a \in \mathbb{Q}_p \setminus \mathcal{O}_p$, then

$$|g(x) - g(a)|_p = |g(x) - 1|_p \leq \max\{|g(x)|_p, |-1|_p\} \leq \max\{1, 1\} = 1.$$

We now let $\delta = 1$. We need to consider the cases where $a \in \mathcal{O}_p$ and $a \in \mathbb{Q} \setminus \mathcal{O}_p$. If $a \in \mathcal{O}_p$ then we know $|g(x) - g(a)|_p \leq 1$. We need to show that there is nothing in the disk of radius 1 centerd 0 except for 0 and 1 (i.e. there is nothing between 0 and 1).