

## 1 Logarithmic Equations

It can often be helpful to express a logarithm in terms of base  $e$ . The following relationship can help us get to the natural log,  $\ln$ .

$$\log_n x = \frac{\ln x}{\ln n}$$

We can also use the properties of logarithmic functions to simplify exponential functions.

$$n^x = e^{\ln x}$$

Applying calculus to logarithmic and exponential functions can yield interesting results.

$$\frac{d}{dx}(\log_n(x)) = \frac{1}{x \ln(n)}$$

$$\int n^x dx = \frac{n^x}{\ln(n)}$$

## 2 Inverse Trigonometry Functions

The derivatives of inverse trig functions are useful for integration. We have the following derivatives:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec}(x)) = \frac{1}{|x|\sqrt{1-x^2}}$$

## 3 Integration Techniques

The main focus of Calculus II is integration. In Calculus I, we learned how to integrate by using a  $u$  substitution, but that technique is often not powerful enough to handle more complex integrals. Luckily, we have several different methods that allow us to integrate.

### 3.1 Integration By Parts

$$\int u dv = uv - \int v du$$

This technique is useful for integrating two functions,  $f(x)$  and  $g(x)$ . Suppose we want to find the integral  $\int xe^{x^2} dx$ , or  $\int x \sin \pi x dx$ , or even  $\int (3t+5) \ln(t/5) dt$ . All of these examples are set up to be solved by integration by parts. We want to split the integrand into two separate functions. We then differentiate one ( $u$ ), and integrate the other ( $dv$ ). Applying our formula will often simplify the original integral into something that we know how to easily integrate. Typically, when we choose our  $u$  and  $dv$ , we want to choose the function whose integral we know for  $dv$ , and the other one for  $u$  (remember, it is often easier to take a derivative than integrate).

### 3.2 Trig Combination

Ultimately, this technique will end up using integration by parts in most cases. There are a few trig identities that we need to know first:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

There are some common patterns that often appear in integrals that we can solve using this method.

1. If we see a  $\cos x$  to an odd power, then we should let  $u = \sin x$ .
2. If we see a  $\sin x$  to an odd power, then we should let  $u = \cos x$ .
3. If we see a  $\sec x$  to an odd power, then we should let  $u = \tan x$ .
4. If we have a  $\tan x$  to an odd power, and we have  $\sec x$  present, then we should let  $u = \sec x$ .
5. If we have  $\sec x$  to an odd power and  $\tan x$  to an even power, then convert the  $\tan x$  to  $\sec x$ .
6. If we **only** have  $\sec x$  to an odd power, then we need to use integration by parts.
7. If we have a  $\tan x$  and do **not** have a  $\sec x$  present, then we need to convert the  $\tan x$  into  $\sec x$ .
8. If we see a trig function to an even power, then we need to use the half angle formulas first.

### 3.3 Trig Substitution

This technique for integration is applicable mostly to integrals that contain functions that look like the derivatives of inverse trig functions. We have a few special cases that tell us what substitution to make.

$$\sqrt{a^2 - u^2} \rightarrow 1 - \sin^2 \theta = \cos^2 \theta \rightarrow u = a \sin \theta$$

$$\sqrt{a^2 + u^2} \rightarrow 1 + \tan^2 \theta = \sec^2 \theta \rightarrow u = a \tan \theta$$

$$\sqrt{u^2 + a^2} \rightarrow \sec^2 \theta - 1 = \tan^2 \theta \rightarrow u = a \sec \theta$$

### 3.4 Partial Fractions

We want to use this technique to solve fraction of functions:

$$\int \frac{\mathcal{P}(x)}{\mathcal{Q}(x)} dx \text{ Where the degree of } \mathcal{P} \text{ is less than the degree of } \mathcal{Q}$$

To use this method, we need to

1. Factor the denominator
2. Determine the decomposition
3. Determine the unknown values of the numerator

## 4 Series and Sequences

### 4.1 Telescoping Series

A telescoping series takes the form of

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

The series will expand in such a way that all the terms will cancel except for the first and last terms. This allows us to say that the series will converge at

$$\lim_{n \rightarrow \infty} 1 - \frac{1}{n+1}$$

### 4.2 Geometric Series

A geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1}$$

where  $a \neq 0$  and  $r$  is constant. We know that

$$|r| \geq 1 \implies \text{Divergence} \qquad |r| < 1 \implies \text{Convergence}$$

If the sequence converges, it converges to

$$\frac{s_0}{1-r}$$

where  $s_0$  is the first term in the series.

### 4.3 P Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If  $p \leq 1$ , then the series diverges. If  $p > 1$ , then the series converges.

## 4.4 Tests of Convergence

### 4.4.1 The Divergence Test

If

$$\lim_{n \rightarrow \infty} a_n \neq 0$$

then the series  $\sum a_n$  diverges. Note that this test **only** determines when the series diverges. It says nothing about the series converging.

### 4.4.2 Comparison Test

Let  $\sum a_n$  and  $\sum b_n$  be positive termed series.

- If  $b_n$  converges,  $a_n < b_n \implies a_n$  converges.
- If  $b_n$  diverges,  $a_n > b_n \implies a_n$  diverges.

### 4.4.3 Limit Comparison Test

Let  $\sum a_n$  and  $\sum b_n$  be positive termed series. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \mathcal{L}$$

then

1. If  $\mathcal{L} = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
2. If  $\mathcal{L} \rightarrow \infty$ , and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.
3. If  $\mathcal{L} > 0$ , then both series either both converge or both diverge.

### 4.4.4 Ratio and Root Test

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \mathcal{L} \qquad \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \mathcal{L}$$

1.  $\mathcal{L} < 1 \implies a_n$  converges
2.  $\mathcal{L} > 1 \implies a_n$  diverges
3.  $\mathcal{L} = 1$  is inconclusive

### 4.4.5 Alternating Series Test

The alternating series

$$\sum (-1)^{n+1} a_n$$

converges if

1.  $a_{n+1} \leq a_n$
2.  $\lim_{n \rightarrow \infty} a_n = 0$

### 4.4.6 Integral Test

Given the series

$$\sum_{n=j}^{\infty} a_n$$

let  $f(n) = a_n$ . The series will converge if and only if

$$\int_j^{\infty} f(n) dn$$

exists.

## 4.5 Types of Convergence

### 4.5.1 Absolute Convergence

A series converges absolutely if both  $\sum a_n$  and  $\sum |a_n|$  converge.

### 4.5.2 Conditional Convergence

A series converges conditionally if  $\sum a_n$  converges and  $\sum |a_n|$  diverges.

## 4.6 Special Series

### 4.6.1 Power Series

A power series is a special type of p-series that takes the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

All power series converge for  $x = x_0$ . With power series, we often want to know the radius of convergence of the series. The radius of convergence,  $\mathcal{R}$ , can be determined by finding the distance that  $x$  can move from  $x_0$  while the series still converges. The interval  $\mathcal{I} = [x_0 - \mathcal{R}, x_0 + \mathcal{R}]$  is called the interval of convergence.

### 4.6.2 Taylor Series

The Taylor series representation of a function can be given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

where  $f^{(n)}$  is the  $n^{\text{th}}$  derivative of  $f$ . Taylor series provide an approximation of a function.

### 4.6.3 Binomial Series

A binomial series is given by

$$\sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k$$