Equality of Absolute Values

Lemma 1. *Let $|\cdot|_1$ and $|\cdot|_2$ be absolute values on a field $\mathbb{F}$. Then the following are equivalent:*

1. *$|\cdot|_1$ and $|\cdot|_2$ are equivalent (i.e. their topologies are the same).*
2. *For any $x \in \mathbb{F}$, we have $|x|_1 < 1$ if and only if $|x|_2 < 1$.*
3. *There is an $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{F}$, $|x|_1 = |x|_2^\alpha$.*

Proof. We will start by showing $1 \implies 2$. Suppose $|\cdot|_1$ and $|\cdot|_2$ are equivalent. Since the topologies are the same, we have that any sequence a_n that converges under $|\cdot|_1$ will also converge under $|\cdot|_2$. If we consider any $x \in \mathbb{F}$, then we have

$$\lim_{n\to\infty} x^n = 0$$

exactly when $x < 1$, so $|x|_1 < 1$ if and only if $|x|_2 < 1$.

We must now show that $2 \implies 3$. Suppose that $x \in \mathbb{F}$ and that $|x|_1 < 1$ if and only if $|x|_2 < 1$. Our goal is to find an α such that for all $x \in \mathbb{F}$, $|x|_1 = |x|_2^\alpha$. We can find one such α rather easily. Consider an arbitrary $y \in \mathbb{F}$ where $|y|_1 < 1|$. Then

$$\begin{aligned}
|y|_1 &= |y|_2^\alpha \\
\log|y|_1 &= \log|y|_2^\alpha \\
\log|y|_1 &= \alpha \log|y|_2 \\
\alpha &= \frac{\log|y|_1}{\log|y|_2}.
\end{aligned}$$

We have asserted that such an α exists for y, but we want to show that this α does indeed work for any $x \in \mathbb{F}$ (ideally that α does not depend on our choice of x). We will choose another element of $\mathbb{F} \setminus \{0\}$, $z \neq x$. We know that if $|x|_1 = |y|_1$ then $|x|_2 = |y|_2$ since x/y or y/x would have $|\cdot|_2 < 1$ which would violate the assumption. $\square$