Threshold Phenomenon in a Nonlinear Heat Equation

1 Introduction

This paper concerns the behavior of the following non-linear heat equation given by

$$u_t = u_{xx} + u^3,\ 0 < x < \pi,\ t > 0 \tag{1}$$
$$u(0,t) = u(\pi,t) = 0,\ t > 0 \tag{2}$$
$$u(x,0) = u_0(x),\ 0 < x < \pi. \tag{3}$$

This system corresponds to several physical situation, including exothermic reactions between chemicals. We know that this equation has a solution $u(x,t)$, but it may help us to consider the *steady state* equation corresponding to this system. The steady state arises from holding time constant; thus, our differential equation becomes dependent only on x.

2 Steady State Analysis

The steady state equation of (1) is given by

$$u_{xx} + u^3 = 0 \tag{4}$$
$$u(0) = u(\pi) = 0. \tag{5}$$

We want to find a solution $u_s(x)$ that satisfies the following properties

1. If $u_0(x) < u_s(x)$, then $u(x,t) \to u_s(x)$ as $t \to \infty$.

   $$\lim_{t\to\infty} u(x,t) = u_s(x).$$

   In this case, the solution to (1) would independent of time. If this property is fulfilled, then there is a *global solution* $u(x,t)$ to the boundary value problem described in (1).

2. If $u_0(x) > u_s(x)$, then $u(x,t)$ blows up in a finite time. This means that the solution tends towards infinity in a finite time T.

   $$\lim_{t\to T} u(x,t) = \infty.$$

   This case if often indicative of a singularity in the solution $u(x,t)$. The solution u of (1) would not be global in this case.

We can now attempt to solve this steady state problem.

2.1 Solution to the Steady State Problem

We are attempting to solve the ordinary differential equation

$$u'' = -u^3.$$

Multiplying both sides of the equation gives us

$$\begin{aligned} u''u' &= -u'u^3 \\ \int u''u' &= \int -u'u^3 \\ \frac{1}{2}(u')^2 &= -\frac{1}{4}u^4 + c \\ 2(u')^2 &= -u^4 + c_1. \end{aligned}$$

This first order equation does not lend itself to separation of variables as readily as we would like (although it is possible if we want to deal with imaginary numbers). Instead, we will use a technique called the *shooting method* in order to determine a solution. The shooting method provides us with a way to relax boundary conditions into initial conditions. Recall that (5) provides us with boundary conditions that arose from the steady state of (1). We will leave the first boundary condition alone and change the second boundary condition so that $2u'(0)^2 = a^4$. This will allow us to factor the resulting term:

$$\begin{aligned} u'(x)^2 &= \frac{1}{2}(a^4 - u^4) \\ \frac{du}{dx} &= \sqrt{\frac{1}{2}(a^2 + u^2)(a^2 - u^2)} \\ \frac{dx}{du} &= \frac{1}{\sqrt{\frac{1}{2}(a^2 + u^2)(a^2 - u^2)}}. \end{aligned}$$

Integrating both sides and substituting s for u gives us

$$x = \int_0^u \frac{1}{\sqrt{\frac{1}{2}(a^2 + s^2)(a^2 - s^2)}}\, ds\,.$$

We can now make another substitution for s in terms of t. Letting

$$s = \sqrt{\frac{a^2 t^2}{2(1 - t^2/2)}}$$

yields

$$a^2 + s^2 = \frac{2a^2}{2(1 - t^2/2)} \qquad \text{and} \qquad a^2 - s^2 = \frac{a^2(1 - t^2)}{(1 - t^2/2)}.$$

We also have

$$ds = \frac{a(t-1)(t-2)}{2\sqrt{2}(1 - t^2/2)^{3/2}}\, dt,$$

which gives us the new integral

$$ax = \int_0^{\sqrt{\frac{2u}{a^2 + u^2}}} \frac{1}{\sqrt{(1 - t^2)(1 - t^2)}}.$$

We may notice that this integral is one of the ways to define $sn(x, k)$. Rewriting it in terms of our Jacobi elliptic function gives us

$$x = \int_0^{sn(x,k)} \frac{1}{\sqrt{(1 - t^2)(1 - t^2/2)}}.$$