Methods For Solving ODE's

1 First Order Equations

1.1 Separable Equations

$$\frac{dy}{dx} = g(x)h(y)$$
$$\frac{dy}{h(y)} = g(x)\,dx$$
$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

1.2 The Integrating Factor

$$\frac{dy}{dx} + \rho(x)y(x) = q(x)$$
$$\frac{dy}{dx}e^{-\int \rho(x)\,dx} + \rho(x)y(x)e^{-\int \rho(x)\,dx} = q(x)e^{-\int \rho(x)\,dx}$$
$$\frac{d}{dx}\left(y(x)e^{-\int \rho(x)\,dx}\right) = q(x)e^{-\int \rho(x)\,dx}$$
$$\int \frac{d}{dx}\left(y(x)e^{-\int \rho(x)\,dx}\right) = \int q(x)e^{-\int \rho(x)\,dx}$$

1.3 Exact Differential Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \begin{cases} M = \frac{\partial F}{\partial x} \\ N = \frac{\partial F}{\partial y} \end{cases}$$
$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx$$

A function $g(y)$ will be left over; use N to solve for it by $N = \dfrac{\partial F}{\partial y}$

2 Second Order Equations

2.1 The Wronskian and Linear Independence

1. If $\mathcal{W}(x, y)(x_0)$ for some x_0 in an interval $\mathcal{I}$ is ***not*** equal to 0, then the two functions are linearly independent.

2. If $f(x)$ and $g(x)$ are linearly dependent on $\mathcal{I}$, the $\mathcal{W}(f, g)(x) = 0$ for all x in $\mathcal{I}$.

3. If f and g are functions of the same variable, then

$$\mathcal{W}(f, g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

4. Given that y_1 and y_2 are two solutions to $y'' + p(x)y' + q(x)y = 0$,

$$\mathcal{W}(y_1, y_2)(x) = ce^{-\int p(x)\,dx}.$$

2.2 Algebraic Methods

Homogeneous

$$ay'' + by' + cy = 0 \qquad \text{becomes} \qquad ar^2 + br + c = 0$$

The roots of r determine the formal solution to the ODE

Two distinct roots in $\mathbb{R}$	Two identical roots in $\mathbb{R}$	Two conjugate roots in $\mathbb{C}$
$y_1(x) = e^{r_1 x}$	$y_1(x) = e^{rx}$	$r_1 = \alpha + i\beta$
$y_2(x) = e^{r_2 x}$	$y_2(x) = xe^{rx}$	$r_2 = \alpha - i\beta$
$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$y(x) = c_1 e^{rx} + c_2 x e^{rx}$	$y(x) = e^{\alpha x}(c_1 \cos(\beta x) - c_2 \sin(\beta x))$

2.3 The Method of Undetermined Coefficients

Given $y'' + p(x)y' + q(x)y = g(x)$, we know that $y(x)$ is the sum of the homogeneous solution and the non-homogeneous solution of the ODE.