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Homework 1

1. What are the possible remainders when a perfect square is divided by 3 or by 6?

Let $k \in \mathbb{Z}$ such that $a = k^2$. By the division algorithm, we have that

$$k = 3q_1 + r_1 \qquad \text{and} \qquad k = 6q_2 + r_2.$$

<u>case $k = 3q_1 + r_1$</u>: Squaring k yields

$$k^2 = 9q_1^2 + 6q_1 r_1 + r_1^2.$$

Since $k^2 = a$, we have that

$$\begin{aligned} a &= 3(3q_1^2 + 2q_1 r_1) + r_1^2 \\ &= 3q_0 + r_0, \text{ where } q_0 = (3q_1^2 + 2q_1 r_1) \text{ and } r_0 = r_1^2. \end{aligned}$$

We know by the division algorithm that $r_0 \in \{0, 1, 2\}$. If we let $r_0 = 0$ or $r_0 = 1$, then $r_1^2 < 4$ and still in $\{0, 1, 2\}$. Therefore, 0 and 1 are possible remainders. If we let $r_0 = 2$, then $r_1^2 = 4$, which is not less than 4. However, we have that

$$\begin{aligned} a &= 3q_0 + 2^2 \\ &= 3q_0 + 4 \\ &= 3q_0 + 3 + 1 \\ &= 3(q_0 + 1) + 1, \end{aligned}$$

Which implies that 1 is a valid remainder. Therefore, 0 and 1 are the only valid remainders.

<u>case $k = 6q_2 + r_2$</u>: If we square k, then we have

$$k^2 = 36q_2^2 + 12q_2 r_2 + r_2^2.$$

Since $a = k^2$, we have that

$$\begin{aligned} a &= 36q_2^2 + 12q_2 r_2 + r_2^2 \\ &= 6(6q_2^2 + 2q_2 r_2) + r_2^2 \\ &= 6q_0 + r_0, \text{ where } q_0 = 6q_2^2 + 2q_2 r_2 \text{ and } r_0 = r_2^2. \end{aligned}$$

We know that for $r_0 = 0, 1,$ and 2, $r_2^2 \leq 6$, so they are valid remainders. For $r_0 = 3, 4,$ and 5, we have

<u>$r_0 = 3$</u>

$$\begin{aligned} a &= 6q_0 + 9 \\ &= 6q_0 + 6 + 3 \\ &= 6(q_0 + 1) + 3 \end{aligned}$$

$\therefore$ 3 is a valid remainder.

<u>$r_0 = 4$</u>

$$\begin{aligned} a &= 6q_0 + 16 \\ &= 6q_0 + 12 + 4 \\ &= 6(q_0 + 2) + 4 \end{aligned}$$

$\therefore$ 4 is a valid remainder.

<u>$r_0 = 5$</u>

$$\begin{aligned} a &= 6q_0 + 25 \\ &= 6q_0 + 24 + 1 \\ &= 6(q_0 + 4) + 1 \end{aligned}$$

$\therefore$ 1 is a valid remainder.

Therefore, the valid remainders are $\{0, 1, 2, 3, 4\}$.

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2. Suppose $a, b, c, d \in \mathbb{Z}$ are such that $a|b$ and $c|d$. Prove that $ac|bd$.

Since $a|b$, we have that $b = an$ for some $n \in \mathbb{Z}$. We also have that $c|d$, so $d = cm$ for some $m \in \mathbb{Z}$. The product $bd = (an)(cm)$, which, after rearranging, yields $bd = (nm)(ac)$. We know that $nm \in \mathbb{Z}$, so we let $j = nm$. Then $bd = jac$, so $ac|bd$.

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3. Suppose $a, b, m \in \mathbb{Z}$ and $m \neq 0$. Prove that $a|b$ if and only if $ma|mb$.

   Suppose $a|b$. Then $b = na$ for some $n \in \mathbb{Z}$. Multiplying both sides of the equation yields $mb = mna$; therefore $ma|mb$.
   Suppose $ma|mb$, and $m \neq 0$. Then $mb = mna$ for some $n \in \mathbb{Z}$. We know that $\frac{m}{m} = 1$, so dividing both sides by m gives us $b = na$. Therefore $a|b$.

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4. Suppose $a, b \in \mathbb{Z}$ with $b \neq 0$ and $a|b$. Prove that $|a| \leq |b|$.

   Since $a|b$, we have that $b = an$ for some $n \in \mathbb{Z}$. We first need to show that $n \neq 0$. Since $b \neq 0$, $an \neq 0$. Therefore both a and n must not be equal to 0. If we apply the absolute value to b, we have that $|b| = |an|$. We know that $|an| = |a||n|$. Since $n \neq 0$, $|n| \geq 1$. We also know that $|a| \geq 1$, since $a \neq 0$. Therefore $|a||n| \geq |a|$, or $|an| \geq |a|$. Since $|b| = |an|$, we have that $|a| \leq |b|$.

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5. Suppose that $a, b, c, d \in \mathbb{Z}$ are such that $a|b$ and $c|d$. Does $a + c$ necessarily divide $b + d$?

   Assume $a + c$ divides $b + d$. Let $a = 1$ and $c = -1$. Both 1 and -1 divide all elements of $\mathbb{Z}$, but $1 + -1 = 0$, and 0 does not divide anything. Therefore by counterexample it does not.

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6. Suppose that $b \in \mathbb{Z}$.

   (a) If $4|b^2$, prove that $2|b$.

   Assume that $4|b^2$ but $2\nmid b$. Then $b = 2n + r$ for some n and r in $\mathbb{Z}$. We know that $r \neq 0$, and that $0 < r < 2$. Therefore $r = 1$. If we square both sides, we get

$$\begin{aligned} b^2 &= (2n + 1)^2 \\ &= 4n^2 + 4n + 1 \\ &= 4(n^2 + n) + 1. \end{aligned}$$

   Therefore $4\nmid b$, since it has a remainder of 1.