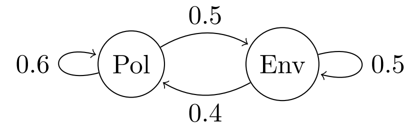
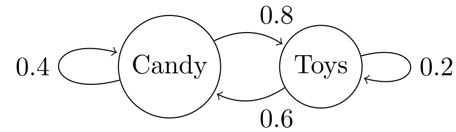
1. Two competing companies, Pollution Products and Environmental Hazards, simultaneously introduce new enzyme laundry detergents. Market tests indicate that during a year, Pollution keeps 60% of its customers and loses 40% of its customers to Environmental. On the other hand, Environmental keeps half of its customers and loses the other half to Pollution. Set up this process as a Markov chain. Determine the transition matrix and sketch a state diagram.

	Pollution	Environmental
Pollution	.6	.4
Environmental	.5	.5



2. Abigail spends her entire weekly allowance on either candy or toys. If she buys candy 1 week, she is 60% sure to buy toys the next week. The probability that she buys toys in two successive weeks is 1/5. Set up this process as a Markov chain. Determine the transition matrix and sketch a state diagram.

	Candy	Toys
Candy	.4	.6
Toys	.8	.2



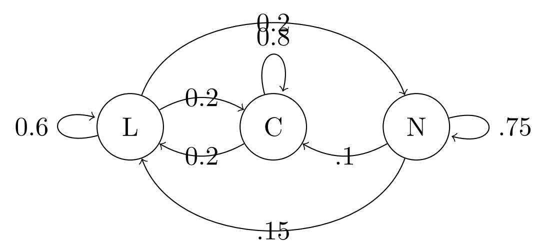
3. A political scientist in Canada discovered that of the children of Conservatives, 80% vote Conservative and the rest vote Labor; of the sons and daughters of Labor supporters, 60% vote Labor, 20% vote Conservative, and 20% vote for the New Democratic Party (NDP); and of the offspring of NDP followers, 75% vote NDP, 15% vote Labor, and 10% vote Conservative.

   (a) What is the probability that the grandchild of a Conservative will vote for the NDP?

   The probability that a grandchild of a conservative will vote for the NDP is (.2)(.2)

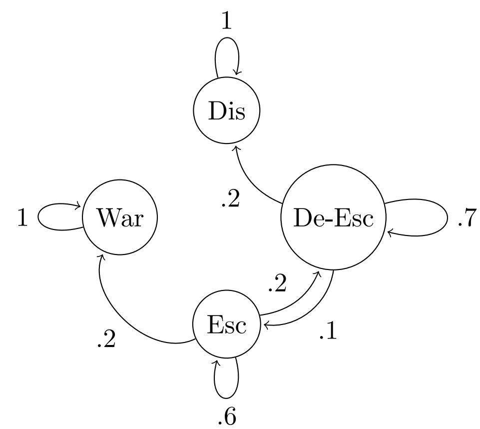
   (b) Set up this process as a Markov chain, with steps corresponding to successive generations. Determine the transition matrix and sketch the state diagram.

	L	C	N
L	.6	.2	.2
C	.2	.8	0
N	.15	.1	.75



4. A secret CIA report gives the following analysis of the arms race between India and Pakistan: There are four possible states: War, Total Disarmament, Escalating Arms Race, and De-escalating Arms Race. It is not possible to change the situation if War or Total Disarmament is occurring this year. If there is an Escalating Arms Race this year, the probability of continued escalation next year is .6, of de-escalation next year is .2, and of War next year is .2. If there is a De-escalating Arms Race this year, the probability of continued de-escalation next year is .7, of escalation next year is .1, and of total disarmament next year is .2. Set up this process as a Markov chain. Determine the transition matrix and sketch the state diagram

	War	Disarmed	Escalating	De-Escalating
War	1	0	0	0
Disarmed	0	1	0	0
Escalating	0.2	0	0.6	0.2
De-Escalating	0	0.2	0.1	0.7



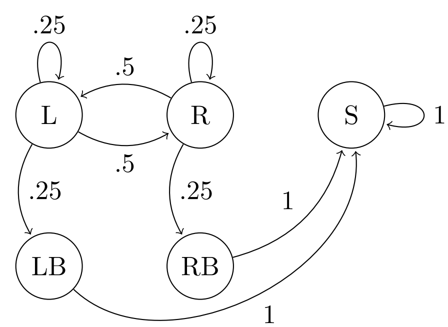
5. The National League and American League used to alternate as hosts for the opening game of each years World Series. Show that this process can be set up as a Markov process with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since the venues are alternating, there is a 100% chance that the venue will change, and a 0% chance it will stay the same.

6. A particle moves along a line from an initial position 2 feet to the right of the origin. Each minute it moves one foot to the right with probability 1/2 or 1 foot to the left. There are barriers at the origin and 4 feet to the right of the origin; if the particle hits a barrier, it remains there. Show that this process can be set up as a Markov process with five states. Determine the transition matrix and draw the state diagram.

	L	R	S	RB	LB
L	.25	.5	0	0	.25
R	.5	.25	0	.25	0
S	0	0	1	0	0
RB	0	0	1	0	0
LB	0	0	1	0	0



7. The random walk of Exercise 6 is modified so that if the particle reaches the barrier at the origin, it must move one foot to the right in the next minute, while if it hits the other barrier, it must move one foot to the left in the next minute. Determine the transition matrix for the associated Markov chain.

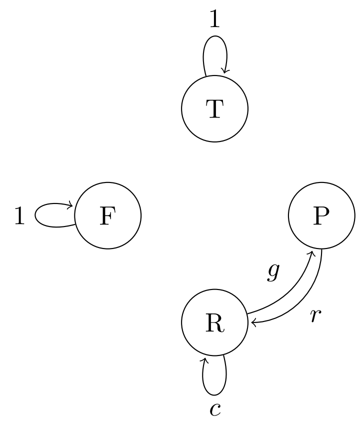
	L	R	S	RB	LB
L	.25	.5	0	0	.25
R	.5	.25	0	.25	0
S	0	0	1	0	0
RB	1	0	0	0	0
LB	0	1	0	0	0

8. Find the probability that a woman whose birthweight was average has a granddaughter with an average birthweight.

	Average	Not Average
Average	.5	.5
Not Average	.5	.5

The probability that a granddaughter will have average weight is $(.5)(.5) + (.5)(.5) = .5$.

9. Sketch the state diagram for the Lower Pine Cone College example.



11. Using probabilistic considerations only, show that the square of a stochastic matrix is also a stochastic matrix.

    If it was not a stochastic matrix, then there would be no way to represent the next state in the system. The total probability of all the transitions must sum to 1, since there is a 100% chance of a transition occurring in every step of the chain.

12. A stochastic matrix is doubly stochastic if the sum of the entries in each column is 1. If A and B are doubly stochastic square matrices, is A^2 doubly stochastic? Is AB?

    Yes, the square of a doubly stochastic matrix is still a doubly stochastic matrix. AB is also still doubly stochastic.

13. Write out inductive proofs for Theorems 2 and 3.

    Theorem 2: From the text, we know

$$p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots p_r^{(0)}) = (p_1, p_2, \ldots p_r) = p^0 \cdot p.$$

We can assume that $p^{(n)} = p^n$, and we want to show that $p^{(n+1)} = p^{n+1}$. We know that

$$\begin{aligned} p^{(n+1)} &= (p_1^{(n+1)}, p_2^{(n+1)}, \ldots p_r^{(n+1)}) \\ &= (p_1^{(n)} p_1^{(1)}, p_2^{(n)} p_2^{(1)}, \ldots p_r^{(n)} p_r^{(1)}) \\ &= p^{n+1}. \end{aligned}$$

Theorem 3: We know that $p^{(n)} = p^{(0)} p^n$. We want to show $p^{(n+1)} = p^{(0)} p^{n+1}$. From theorem 2, we have

$$\begin{aligned} p^{(n+1)} &= (p_1^{(n+1)}, p_2^{(n+1)}, \ldots p_r^{(n+1)}) \\ &= (p_1^{(n)} p_1^{(0)}, p_2^{(n)} p_2^{(0)}, \ldots p_r^{(n)} p_r^{(0)}) \\ &= p^{n+1} p^{(0)}. \end{aligned}$$

14. Use matrix multiplication to solve Exercise 8.

$$\begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix} \cdot \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix} = \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$$

Therefore, the probability is .5.

15. Abigail bought a toy with her allowance this week (see Exercise 2). Find the probability that she will buy a toy 4 weeks from now.

The probability that she will buy a toy can be found by raising the matrix to the fourth power.

$$\begin{pmatrix} .4 & .6 \\ .8 & .2 \end{pmatrix}^4 = \begin{pmatrix} .5824 & .4176 \\ .5568 & .4432 \end{pmatrix}.$$

Therefore, the probability is 0.4432.

16. Find the distribution of birth weights (Example 1) after one generation if the initial probability distribution is $(.4, .3, .3)$.

$$(.4, .3, .3) \cdot \begin{pmatrix} .5 & .45 & .05 \\ .25 & .5 & .25 \\ .3 & .3 & .4 \end{pmatrix} = (.365, .42, .215).$$

17. Suppose the distribution of birth weights of a generation of daughters is $p^{(1)} = 7(0.31, 0.45, 0.24)$. Can you find the distribution of birth weights of the mothers?

We need to solve the system

$$(x, y, z) \begin{pmatrix} .5 & .45 & .05 \\ .25 & .5 & .25 \\ .3 & .3 & .4 \end{pmatrix} = 7(.31, .45, .24).$$

Doing so yields

$$(x, y, z) = (1.4, 4.2, 1.4).$$

18. Use the state diagram in Fig. 11.3 to find the probability that the process reaches state S_3 in two steps if it starts in state S_1.

To move from S_1 to S_2 has a probability of a, and from S_2 to S_3 has a probability $(1 - b)$, so the total probability is $a(1 - b)$.

19. Let P be the transition matrix of Eq. (4). Compute P^2. What can you say about the first column of P^3 and P^n?

$$P^2 = \begin{pmatrix} 0 & a^2 & (1-a)^2 \\ 0 & b^2 & (1-b)^2 \\ 0 & c^2 & (1-c)^2 \end{pmatrix}.$$

The first column of P^n will always be 0.

21. The matrices determined in Exercises 1-7 are all transition matrices for Markov processes. Which ones are regular?

Exercises 1 and 2 are regular matrices, since they have all positive entries.

22. Find, if possible, a fixed-point vector for each of the transition matrices of Exercises 1-7.

(a) $\mathbf{w} = \left\{\frac{5}{9}, \frac{4}{9}\right\}$

(b) $\mathbf{w} = \left\{\frac{4}{7}, \frac{3}{7}\right\}$

(c) $\mathbf{w} = \left\{\frac{23}{70}, \frac{43}{70}, \frac{2}{35}\right\}$

(d) This is not a regular matrix, so no $\mathbf{w}$ exists.

(e) This is not a regular matrix, so no $\mathbf{w}$ exists.

(f) This is not a regular matrix, so no $\mathbf{w}$ exists.

(g) This is not a regular matrix, so no $\mathbf{w}$ exists.

23. Does a matrix of the form $A = \begin{pmatrix} a & b \\ -b & d \end{pmatrix}$ have a fixed-point vector?

If it has a fixed-point vector, then that vector is given by

$$\mathbf{w} = \left\{ \frac{c}{b+c}, \frac{b}{b+c} \right\} = \left\{ \frac{-b}{b-b}, \frac{b}{b-b} \right\} = \left\{ \frac{-b}{0}, \frac{b}{0} \right\}.$$

Therefore, there is no fixed-point vector for the matrix.

32. Which of the transition matrices of Exercises 1-7 represent absorbing Markov processes?

Exercises 4 and 6 represent absorbing Markov processes, since they have nodes which are not escapable.