Partial Differential Equations

1 First Order Partial Differential Equations

Ex: Solve the following PDE

$$u_x + u_y + u = x + y$$

Solution: In this case, we have a non-zero term on the right hand side of the equation. We will use substitution to solve this. For the time being, we will let T be $x + y$. To find our S, we can use a similar method to the one that we used above. We know that the coefficients on the u_x and u_y are 1 in this case. Now we have:

$$\begin{aligned}\frac{dy}{dx} &= 1\\ dy &= dx\\ \int dy &= \int dx\\ x &= y + c\\ x - y &= c\end{aligned}$$

Now we can say that $S = x - y$; however, we need to check our choice for T to make sure this substitution will not just yield a trivial solution. We can do this by Next we have to actually use the substitution. We need to determine whether the coefficient matrix, or the Jacobian, of our substitutions has a determinant of zero. If it does, then we need to choose a different T.

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \neq 0.$$

Our choices of variables will work here, so now we need to express u_x and u_y in terms of S and T. The following equations come from the chain rule; when the S and T terms are differentiated, we get u_S and u_T left behind.

$$\begin{aligned} u_x &= u_S\frac{\partial S}{\partial x} + u_T\frac{\partial T}{\partial x} & \qquad u_y &= u_S\frac{\partial S}{\partial y} + u_T\frac{\partial T}{\partial y}\\ u_x &= -u_S + u_T & u_y &= u_S + u_T\end{aligned}$$

We can now plug in our values into the original equation. Note that we didn't do anything with u; it gets left alone. In addition, if we needed u_{xx} or u_{yy}, we can just differentiate again. Plugging in yields:

$$\begin{aligned}(-u_S + u_T) + (u_S + u_t) + u &= T\\ 2u_T + u &= T\end{aligned}$$

This is just a first order ODE. Since it's not homogeneous, we need to solve for the general solution, and then the particular solution. Let's start with the general.

$$\begin{aligned}2y^{'} + y &= 0\\ 2r + 1 &= 0\\ r &= -\frac{1}{2}\\ y &= ce^{\left(-\frac{1}{2}\right)x}\end{aligned}$$

In this case, we know that our x is really T. So we finally have $y = ce^{\left(-\frac{1}{2}\right)T}$. We will back substitute soon, but let's find the particular solution before we do that. Recall from ODE that we have several cases for the particular solution. In our case, we have a linear equation on the right hand side of our ODE. This lets us say:

$$\begin{aligned}y &= AT + B\\ y^{'} &= A\end{aligned}$$

Now we can substitute again into our non-homogeneous ODE as follows:

$$2y^{'} + y = T \equiv 2A + AT + B = T$$

And we have that $A = 1$, and $B = -2$. Now we have a final solution of:

$$y = ce^{\left(-\frac{1}{2}\right)T} + T - 2$$
$$y = ce^{\left(-\frac{1}{2}\right)(x+y)} + (x + y) - 2$$

We aren't quite done yet, as we have a solution to an ODE, not a PDE. It isn't too difficult to switch back; The c is really the only difference. In our ODE, the c is a constant term, but to our PDE, the c is a function. The c now becomes $f(s)$. We will go ahead and back substitute, which will give us $f(y-x)$. Now we have our final solution:

$$u(x, y) = f(y - x)e^{\left(-\frac{1}{2}\right)(x+y)} + (x + y) - 2$$

And we are done.

2 Classification of Partial Differential Equations

We mentioned earlier the topic of classification of PDE's. Since we only have this and Fourier Series to talk about, we will go ahead and knock out classification. There's some information that we need to go ahead and review here. This will be explained in the problem, but it's handy to have it for reference:

$$\begin{aligned}
b^2 - ac < 0 &\implies \text{Elliptic with complex roots} \\
b^2 - ac = 0 &\implies \text{Parabolic with the same root twice} \\
b^2 - ac > 0 &\implies \text{Hyperbolic with two distinct real roots}
\end{aligned}$$

Another helpful piece of information is what's known as the *classical trinity* of PDEs.

$$\begin{aligned}
u_t - ku_{xx} = 0 &\qquad \text{The heat equation (parabolic)} \\
u_{tt} - c^2 u_{xx} = 0 &\qquad \text{The wave equation (hyperbolic)} \\
u_{xx} + u_{yy} = 0 &\qquad \text{The Laplace equation (elliptic)}
\end{aligned}$$

Let's go ahead and look at an example.

Ex. Classify the following PDE:

$$yu_{xx} + u_{yy} = 0$$

Solution: The first thing we need to identify are the coefficients on the u_{xx}, u_{yy}, and u_{xy} or u_{yx} terms. Note that the u_{xy} and u_{yx} are interchangeable. We will assign these coefficients to a, b, and c corresponding to u_{xx}, u_{xy}, and u_{yy}, respectively. Looking at the PDE, we can see that

$$a = y \qquad b = 0 \qquad c = 1$$

Now we look at $b^2 - ac$. We are really dealing with $b = \frac{b_0}{2}$. In other words, we need to divide our b value by 2. In the case of the first problem, we are dealing with $b = 0$. Now we can classify our PDE as follows:

$$b^2 - ac = 0 - y < 0 \implies \text{Elliptic}$$

Note that we assume $y > 0$. So our PDE is elliptic. Now we need to make it look like it's corresponding equation from the classical trinity. In this case we have an Elliptic equation, so we want to make it look like Laplace's equation. We need to find values for u_{xx} and u_{yy}, and then substitute those into the equation. This is similar to how we solved the first order PDE's; let's look at our characteristic equation:

$$\begin{aligned}
am^2 + bm + c &= 0 \\
ym^2 + 1 &= 0 \\
m &= \pm \frac{i}{\sqrt{y}}
\end{aligned}$$

So, now we have two solutions, and they are both imaginary as expected. Next we need to find values for S and T that we can substitute into the equation. We will do both of them at the same time here. Notice that we will drop the i:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{y}} & \qquad \frac{dy}{dx} &= -\frac{1}{\sqrt{y}} \\
\int \frac{dy}{dx} &= \int \frac{1}{\sqrt{y}} & \qquad \int \frac{dy}{dx} &= -\int \frac{1}{\sqrt{y}} \\
S &= \frac{2}{3} y^{\left(\frac{3}{2}\right)} - x & \qquad T &= \frac{2}{3} y^{\left(\frac{3}{2}\right)} + x
\end{aligned}$$

Now we can substitute into our equation. Unfortunately, we will need to take the partial derivative twice in order to get the substitutions we need. Let's start with u_{xx}.

$$\begin{aligned}
u_x &= u_S \frac{\partial S}{\partial x} + u_T \frac{\partial T}{\partial x} \\
&= -u_S + u_T \\
u_{xx} &= \frac{\partial}{\partial x}(-u_S + u_T) \\
&= u_{SS} + u_{TT}
\end{aligned}$$

We will skip the math for finding u_{yy}, which would give us

$$u_{yy} = yu_{SS} + yu_{TT}$$

Now we plug everything in. Remember that we were under the assumption that $y > 0$:

$$\begin{gathered}
y(u_{SS} + u_{TT}) + yu_{SS} + yu_{TT} = 0 \\
2yu_{SS} + 2yu_{TT} = 0 \\
u_{SS} + u_{TT} = 0
\end{gathered}$$

And we are finished.

3 Fourier Series and Partial Differential Equations

3.1 Finding Fourier Series

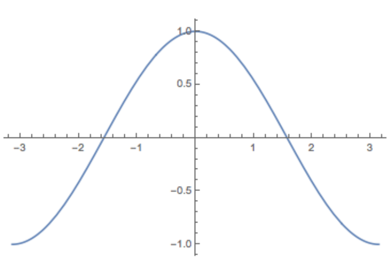
It's finally time to talk about Fourier Series. Before we start though, it may be helpful to look at some common patterns amongst the Fourier Series.

	Cosine	Sine	Full Series
Interval	$[0, L]$	$[0, L]$	$[-L, L]$
Basis	$\{1, \cos\left(\frac{n\pi x}{L}\right)\}$	$\{\sin\left(\frac{n\pi x}{L}\right)\}$	$\{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\}$
Series	$\frac{b_0}{2} + \sum_{n=1}^{\infty}[b_n \cos\left(\frac{n\pi x}{L}\right)]$	$\sum_{n=1}^{\infty}[a_n \sin\left(\frac{n\pi x}{L}\right)]$	$\frac{b_0}{2} + \sum_{n=1}^{\infty}[a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right)]$

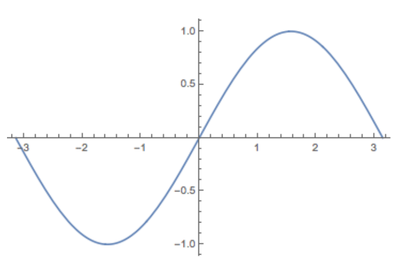
To solve for the coefficients, we have:

$$
\begin{aligned}
\textit{Sine}: \qquad & a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
\textit{Cosine}: \qquad & b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
& b_0 = \frac{1}{L}\int_0^L f(x)dx \\
\textit{Full}: \qquad & a_n = \frac{2}{L}\int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
& b_n = \frac{2}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
& b_0 = \frac{1}{L}\int_{-L}^L f(x)dx
\end{aligned}
$$

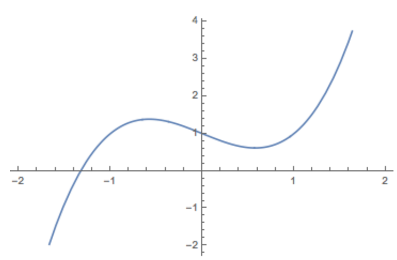
It will also help us to quickly review the concept of odd and even functions, as they will help us to simplify the process of computing Fourier series later. Specifically, let's look at the graphs for sin and cos on $[-\pi, \pi]$, as well as another function that will turn out to be neither even nor odd.



$f(x) = \cos x$



$f(x) = \sin x$



$f(x) = x^3 - x + 1$

We should recall that a function is *even* when $f(x) = f(-x)$ for all x in the domain of f. Graphically speaking, an *even* function is reflected across only the y-axis. By looking at the above graphs, we can see that $f(x) = \cos x$ is an even function.

A function is *odd* when $-f(x) = f(-x)$ for all x in the domain of f. Graphically speaking, the function is reflected across the x-axis and the y-axis. From the above graphs, we can say that $f(x) = \sin x$ is an odd function. Note that we can have a function that is neither odd nor even, but we won't see much of this in the following problems. The reason that we want to discuss odd and even functions here is because the integration that we will be seeing later will be affected by the symmetry.

The first type of series we will look at is the Full Fourier Series. Note that this series is just a combination of a Fourier Sine Series and a Fourier Cosine Series. A typical problem would look something like this:

Ex: Find the corresponding Fourier Series for

$$f(x) = \begin{cases} L & -L \leq x \leq 0 \\ 2x & 0 \leq x \leq L \end{cases}$$

Solution: So we have a piecewise function here, but that shouldn't throw us off too much; all we need to do is combine the integrals for each part of the coefficient solution. Let's start by finding the terms for the *cos* series.

$$\begin{aligned} B_0 &= \frac{1}{2L}\left[\int_{-L}^{0} L dx + \int_{0}^{L} 2x dx\right] \\ &= \frac{1}{2L}[2L^2] \\ &= L \end{aligned}$$

Now We need to find B_n, and then we will have our cosine part of the series. The integration can get pretty messy, so we will skip some steps.

$$\begin{aligned} B_n &= \frac{1}{L}\left[\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[0 + \left(\frac{2L^2}{n^2\pi^2}\right)((-1)^n - 1)\right] \\ &= \frac{2L}{(n\pi)^2}((-1)^n - 1) \end{aligned}$$

Note that the sequence will be 0 for all $n = 2k$. This allows us to substitute $2k + 1$ for n, since we don't care about the 0 terms. For our cosine part of the series, we now have:

$$\begin{aligned} B_n &= \frac{2L}{(2k+1)^2\pi^2}((-1)^{2k+1} - 1) \\ &= \frac{2L}{(2k+1)^2\pi^2}(-2) \\ &= \frac{-4L}{(2k+1)^2\pi^2} \end{aligned}$$

Now we need to find the sine part of the series. This is easier, since we only need to find A_n.

$$\begin{aligned} A_n &= \frac{1}{L}\left[\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{L}{n\pi}\left[-1 - (-1)^n\right] \end{aligned}$$

Note that we can do a similar substitution here as we did with the cosine series. This time, however, we will have 0 for $n = 2k + 1$ We now have:

$$\begin{aligned} A_n &= \frac{L}{2k\pi}\left[-1 - (-1)^{2k}\right] \\ &= \frac{L}{2k\pi}[-1 - 1] \\ &= \frac{L}{2k\pi}[-2] \\ &= \frac{-2L}{2k\pi} \end{aligned}$$

Now we have all the parts that we need. Our final answer will be the sum of all the parts:

$$\begin{aligned} f(x) &\sim \frac{L}{2} + \sum_{k=0}^{\infty}\left[\frac{-4L}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{L}\right)\right] + \sum_{k=0}^{\infty}\left[\frac{-2L}{2k\pi}\sin\left(\frac{2k\pi x}{L}\right)\right] \\ &\sim \frac{L}{2} + \sum_{k=0}^{\infty}\left[\frac{-4L}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{L}\right) + \frac{-2L}{2k\pi}\sin\left(\frac{2k\pi x}{L}\right)\right] \end{aligned}$$

And we are done.

Now lets take a look at a slightly more theoretical Fourier Series.
Ex: Find the corresponding Fourier Series for

$$f(x) = |\cos x| \qquad -\pi \leq x \leq \pi$$

Solution: Before we dive into calculations here, let's analyze the function at hand. We have the absolute value of $\cos x$. Usually, cos would be an odd function, which would imply that our cos series component to the whole Fourier Series would be 0. However, the absolute value makes our function even, and as such the sin component of the series will be 0. That's good news for us, because now we don't have to find a_n; we actually only have a Fourier Cosine Series here. Recall that we have a formula for b_n and b_0, and that our general solution will have the form of

$$\frac{b_0}{2} + \sum_{n=1}^{\infty}[b_n \cos\left(\frac{n\pi x}{L}\right)]$$

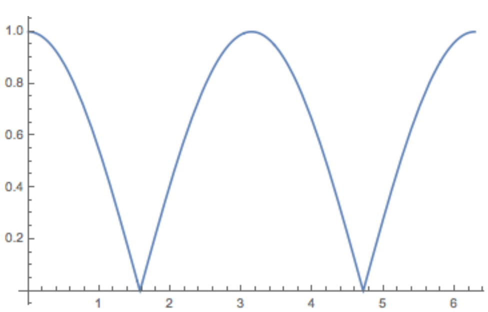
Our function is bounded by π, so we really have $L = \pi$. This gives us:

$$\frac{b_0}{2} + \sum_{n=1}^{\infty}[b_n \cos(nx)]$$

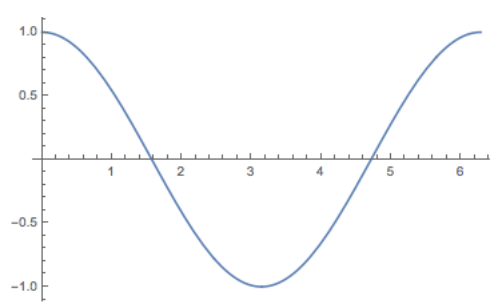
Let's go ahead and find our b_0:

$$\begin{aligned}
b_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\, dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\, dx \\
&= \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} |\cos x|\, dx \\
&= \frac{4}{\pi}\left[\sin\frac{\pi}{2} - \sin 0\right] \\
&= \frac{4}{\pi}
\end{aligned}$$

We know that since the function is even, we can say that the the integral of the entire range is really twice the integral of half the range. Since we are dealing with the cosine function, we actually have four times a quarter of the range. If we take a look at a graph, this may make more sense:



$f(x) = |\cos x|$ on $[0, 2\pi]$



$f(x) = \cos x$ on $[0, 2\pi]$

You can see that the section of the graph from $[\frac{\pi}{2}, \frac{3\pi}{2}]$ is positive for the $|\cos x|$, but negative for $\cos x$. If we simply integrated the $|\cos x|$ over $[0, \pi]$, the result would be 0, but graphically this result would not make any sense. Now let's take a look at the b_n term. Before we jump into the integration, let's think about this function:

$$|\cos x| = \begin{cases} \cos(x) & 0 \leq x \leq \frac{\pi}{2} \\ -\cos(x) & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Now we can set up the formula for the b_n term. We have as follows:

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \cos nx dx\right] \\
&= \frac{2}{\pi}\left[\frac{-2\cos\left(\frac{n\pi}{2}\right) + \sin(n\pi)}{n^2-1}\right]
\end{aligned}
$$

Note that (as we have seen before) the sine term disappears; however, the more interesting part is what happens with our cosine term. The value of $\cos\left(\frac{n\pi}{2}\right)$ will be 0 at all odd n, or for all $n = 2k+1$. So, we will let our $n = 2k$. This gives us:

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\left[\frac{-2\cos(k\pi)}{(2k)^2-1}\right] \\
&= -\frac{4}{\pi}\left[\frac{(-1)^k}{(2k)^2-1}\right]
\end{aligned}
$$

And that leaves us with the final result

$$
f(x) \sim \frac{2}{\pi} - \sum_{k=0}^{\infty}\left[\frac{4}{\pi}\left[\frac{(-1)^k}{(2k)^2-1}\right]\cos\left(\frac{2k\pi}{2}\right)\right]
$$

And we are done.

3.2 Convergence of Fourier Series

Up to this point, we have found a Fourier series and left our solutions at that. We will now look into the convergence of a Fourier series. This topic includes two questions:

1. Does the series converge?
2. If the series converges, to what value does it converge?

Before we jump into convergence, we need to review some ideas.

1. ***Jump Discontinuity:*** To say that $f(x)$ has a jump discontinuity at $x = a$ is to say that

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

   Where both the limit from the left and the limit from the right exist.

2. ***Piecewise Smooth:*** A function $f(x)$ is said to be piecewise smooth if the function can be broken into two distinct intervals, and on each interval both $f(x)$ and $f^{'}(x)$ are continuous.
3. ***Periodic Extension:*** The periodic extension of $f(x)$ on $[-L, L]$ is the repetition of the function on intervals on periods to the left and right of $[-L, L]$.

With these definitions, we can lay out a general rule for convergence of a Fourier series. We won't give a proof of these rules for the time being. We just need them to lay out some groundwork for the rest of the information we will present on Fourier series. We have the following:

> Suppose $f(x)$ is a function, and that $f(x)$ is piecewise smooth on the interval $[-L, L]$. The corresponding Fourier series for $f(x)$ converges to either
>
> 1. the periodic extension of $f(x)$, if $f(x)$ is continuous, or
> 2. the average of the two one-sided limits, if the periodic extension of $f(x)$ has a jump discontinuity at $x = a$.

We should note that the second condition can be expressed by the formula

$$\frac{1}{2}\left[\lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x)\right]$$

which gives us a nice way to calculate the value to which the series converges if we happen to meet the criteria in (2). Let's take a look at some examples.

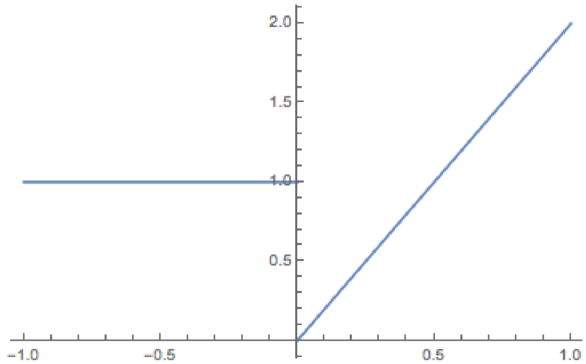
Ex: Find the Fourier series for the following equation, and determine if the series converges. If it does, tell where the series will converge.

$$f(x) = \begin{cases} L & -L \leq x \leq 0 \\ 2x & 0 \leq x \leq L \end{cases}$$

Solution: We will go ahead and skip the details of finding the Fourier series for this function. If you do want to find it, you will have to find a full piecewise Fourier series. For this particular $f(x)$, we have:

$$f(x) \sim L + \sum_{n=1}^{\infty}\left[\frac{2L}{n^2\pi^2}([-1]^n - 1)\cos\left(\frac{n\pi x}{L}\right)\right] - \sum_{n=1}^{\infty}\left[\frac{L}{n\pi}(1 + [-1]^n)\sin\left(\frac{n\pi x}{L}\right)\right]$$

Notice that we didn't simplify this solution to a nicer form. If we evaluate this series, we will run across terms where the series is 0, but that is okay for now. This function is also defined as a piecewise function, and on both intervals ($[-L, 0]$ and $[0, L]$), $f(x)$ is continuous. In other words, $f(x)$ is piecewise smooth, so we know that this series converges. Now let's take a look at the graph of $f(x)$ on an arbitrary interval of $[-1, 1]$.



$$f(x) = \begin{cases} 1 & -1 \leq x \leq 0 \\ 2x & 0 \leq x \leq 1 \end{cases}$$

We can see that there is a jump in the graph at $x = 0$. In other words, there is a jump discontinuity at $x = 0$, and that means that we need to use (2) in order to find the value of convergence. We will let $\sum$ stand in for our entire series. We have:

$$\begin{aligned} \sum &\to \frac{1}{2}\left[\lim_{x\to 0} L + \lim_{x\to 0} 2x\right] \\ &\to \frac{1}{2}[L + 0] \\ &\to \frac{L}{2} \end{aligned}$$

So, at $x = 0$, the series converges to $\frac{L}{2}$. We are not done yet, however. Now, we need to consider the periodic extension of the series. The function will repeat every $[-L, L]$, so we need to look at what the convergence of the series looks like at $-L$ and L. Let's take a look:

$$\begin{aligned} \sum &\to \frac{1}{2}\left[\lim_{x\to -L^-} f(x) + \lim_{x\to -L^+} f(x)\right] \\ &\to \frac{1}{2}[2L + L] \\ &\to \frac{3L}{2} \end{aligned}$$

$$\begin{aligned} \sum &\to \frac{1}{2}\left[\lim_{x\to L^-} f(x) + \lim_{x\to L^+} f(x)\right] \\ &\to \frac{1}{2}[2L + L] \\ &\to \frac{3L}{2} \end{aligned}$$

And we are done.

4 Second Order Partial Differential Equations

4.1 The Heat and Wave Equations

Let's take a look at the second order PDE. We will start with a homogeneous equation, which we will tackle with the separation of variables method. The general strategy is that we want to take our PDE and convert it to an ODE, which we know how to solve. The problems presented here also involve boundary conditions, which will require us to solve for eigenvalues/functions. We will also be dealing with series (specifically Fourier Series) to help us solve these problems. Let's look at an example:
Ex: Solve the following boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,t) = 0 \qquad u(1,t) = 0 \\ u(x,0) = f(x) \qquad u_t(x,0) = g(x) \end{cases}$$

$$f(x) = 2\sin(\pi x) + 3\sin(2\pi x) \qquad g(x) = 4\sin(3\pi x) - 7\sin(5\pi x)$$

given the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} [(a_n \cos[n\pi t] + b_n \sin[n\pi t]) \sin[n\pi x]]$$

Solution: There's a lot going on here, so let's break it down. First of all, notice that we have two initial conditions, one of which is a derivative. This should make sense since the equation have two second order partial derivatives included in it. Next notice that we have a Fourier Sine Series, despite both the $\sin[n\pi t]$ and $\cos[n\pi t]$ terms. If we had worked out the entire problem, we would see that our eigenvalue/function for this equation takes the form of

$$\{\lambda_n = n\pi, \sin(n\pi x)\}$$

which tells us that the corresponding series is a Fourier Sine Series. We also see the initial conditions on our equation are already representative of a series of sine functions, which is not just a conincidence. To apply the initial conditions, we will need both the general solution and the derivative with respect to t of the general solution. Let's go ahead and apply the first initial condition, $u(x,0) = f(x)$. Letting $t = 0$ gives us:

$$u(x,0) = 2\sin(\pi x) + 3\sin(2\pi x) = \sum_{n=1}^{\infty} [a_n \sin(n\pi x)]$$

We see that the term with b_n has dropped out, since $\sin(0) = 0$. Now we just have to match up the coefficients on a_n. We can see that when we have $n = 1$, $a_n = 2$, and when $n = 2$, $a_n = 3$. For all other n, we have $a_n = 0$, so we ignore those terms. Next, we can apply the initial condition for the derivative of the general solution with respect to t. We expect that all the terms related to a_n will disappear, and in fact they will. Let's take a look:

$$u_t(x,t) = \sum_{n=1}^{\infty} [(-a_n n\pi \sin[n\pi t] + b_n n\pi \cos[n\pi t]) \sin n\pi x]$$

$$u_t(x,0) = 4\sin(3\pi x) - 7\sin(5\pi x) = \sum_{n=1}^{\infty} [n\pi b_n \sin(n\pi x)]$$

We are going to do the same thing here that we did to find our a_n terms, but notice that we don't just have b_n; we actually have $n\pi b_n$. We have the following:

$$3\pi b_3 = 4 \qquad 5\pi b_5 = -7$$

$$b_3 = \frac{4}{3\pi} \qquad b_5 = -\frac{7}{5\pi}$$

Much like the solutions for a_n, all the other b_n terms are 0. Now we can construct a particular solution. In this case, the answer will not be in the form of a series, since we have definite terms. Our solution is as follows:

$$u(x,t) = 2\cos(\pi t)\sin(\pi x) + 3\cos(2\pi t)\sin(2\pi x) + \frac{4}{3\pi}\sin(3\pi t)\sin(3\pi x) - \frac{7}{5\pi}\sin(5\pi t)\sin(5\pi x)$$

And we are done.

Ex: Solve the following PDE

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ u(x,0) = x \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Solution: We will use the separation of variable technique to solve this problem. We know to use separation of variable because we are dealing with a homogeneous PDE with homogenous boundary conditions. Separation of variables tells us that our solution will be in the form of $u(x,t) = X(x)T(t)$. If we take this fact at face value, we can use the separation of variables method to solve this. First, we substitute in our solution into the PDE:

$$\begin{aligned} \frac{\partial}{\partial t}(X(x)T(t)) &= k\frac{\partial^2}{\partial x^2}(X(x)T(t)) \\ X(x)\frac{dT}{dt} &= k\frac{d^2X}{dx^2}T(t) \\ \frac{dT}{dt}\frac{1}{T(t)} &= \frac{d^2X}{dx^2}\frac{k}{X(x)} \end{aligned}$$

The only way that our function of x and our function of t can be equal is if they are equal to a constant; thus we now have:

$$\frac{dT}{dt}\frac{1}{T(t)} = \frac{d^2X}{dx^2}\frac{k}{X(x)} = -\lambda$$

And now we have a first order ODE for the time and a second order ODE for the position:

$$\frac{dT}{dt} = -k\lambda T(t) \qquad \text{and} \qquad \frac{d^2X}{dx^2} = -\lambda X(x)$$

Our next step is to find the Eigenvalues and Eigenfunctions of the second order ODE; we will come back to the time ODE when we have values for λ.

To solve for these values, we have to find non-trivial eigenvalues that correspond to the boundary values of the ODE. In this case, we know that $X(0) = 0$ and $X(L) = 0$. These boundary conditions come from the BC on the PDE which states $u(0,t) = u(L,t) = 0$. To find the eigenvalues/functions, we need to find the characteristic equation corresponding to the ODE in question. Let's go ahead and find it:

$$\begin{aligned} \frac{d^2X}{dx^2} &= -\lambda\frac{X(x)}{k} \\ \frac{d^2X}{dx^2} + \lambda\frac{X(x)}{k} &= 0 \\ r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Now, we can substitute values of λ to find out solutions. Let's start with $\lambda > 0$. Since we have a $-\lambda$ under the radical, we will have $r = \pm\sqrt{\lambda}\mathrm{i}$. This will yield the solution of

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Applying our boundary conditions gives us:

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0$$

Now we know that $c_1 = 0$. This result came from the fact that $\sin 0 = 0$, so the c_2 term disappears. Now we are left with $\cos 0 = 1$, which tells us that $c_1 = 0$. We can apply this to our second boundary condition like so:

$$y(L) = 0 = c_2 \sin\sqrt{\lambda}L$$

It looks like there will only be trivial solutions, but we know this isn't true due to the Jacobian matrix. Recall that if the determinant of the Jacobian $\neq 0$, then we only have trivial solutions, and if the determinant of the Jacobian $= 0$, then we do not have the trivial solution. Let's take a look at the Jacobian for this case:

$$\begin{aligned} \begin{vmatrix} 1 & 0 \\ \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \end{vmatrix} &= 0 \\ \sin\sqrt{\lambda}L &= 0 \\ \sqrt{\lambda}L &= \arcsin 0 \\ \sqrt{\lambda} &= \frac{\arcsin 0}{L} \\ \lambda &= \left(\frac{\arcsin 0}{L}\right)^2 \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \end{aligned}$$

Notice that we replaced the $\arcsin 0$ term with $n\pi$. Recall from trig that the sin function is 0 at every $n\pi$, starting with $n = 0$. Next, we just need to find our corresponding eigenfunction. Our c_1 was 0, so we can ignore the entire cos term. Thus, we have an eigenfunction from plugging in our λ that looks like:

$$X(x) = \sin\frac{n\pi x}{L}$$

Unfortunately, we aren't done yet. Now we have to find the corresponding eigenvalues/functions for $\lambda = 0$. This isn't too hard, and can be done in a few lines if we recall the characteristic equation. Plugging in 0 for λ gives us that $r = 0$. Using this piece of information, we now know that $y(x) = c_1 + xc_2$. Applying our boundary conditions gives us $c_1 = 0, c_2 = 0$. We can see that we have only trivial solutions for $\lambda = 0$.

Our final case is for $\lambda < 0$. I'll go ahead and tell you that this will yield another trivial solution, but we can take a look at the work behind it. Since $\lambda > 0$, we know that our r from the characteristic equation will be $r = \pm\sqrt{\lambda}$. From here we are going to do something a little unconventional that will help us. Usually, we would have a solution of the form $y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 x e^{-\sqrt{\lambda}x}$. In our case, it will be more convenient to use hyperbolic trig functions. It's not too far off from our solution from $\lambda > 0$:

$$\begin{aligned} y(x) &= c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \\ &= c_1 \cosh\sqrt{\lambda}x + c_2 \sinh\sqrt{\lambda}x \end{aligned}$$

If we apply the boundary conditions (remember that $\sinh 0 = 0$ and $\cosh 0 = 1$), we will find that we have only trivial solutions.

So, we are done with the second order ODE, and now we turn our eye to the first order time equation. We know our λ, so it isn't too hard to solve:

$$\begin{aligned} \frac{dT}{dt} &= -\lambda T(t) \\ T(t) &= ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

Finally we can put it all together. Our penultimate solution will be

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

We can analyze this solution and tell where each piece comes from. The B_n is a coefficient that results from our $c's$ in the general solutions of the differential equations. The *sin* part results from out eigen vector. Remember, we get a different value for each n, so there may or may not be 0 terms in the series. Finally, we have the exponential term. This term comes from the solution to our time equation. When we put it all together, we can see that this is, of course a series; for every n, we get a different solution. From here we can find the Fourier Series (which will be a Fourier Sine Series). We won't worry about that yet, but we will come back to it.

It will benefit us to take a moment and discuss how boundary conditions effect our Fourier series. We have three different combinations of boundary conditions, where each one will produce a different result in our series solution.

1. *Dirichlet Boundary Conditions*
   Dirichlet boundary conditions involve the function for which we are looking. In other words, the boundary conditions are presented in terms of the functions we are trying to find, rather than the derivative of that function. Typically, this type of boundary condition will take the form of (in the homogeneous case)
   $$u(0,t) = u(L,t) = 0.$$
   In regards to our Fourier series, we expect to see $\sin\left(\frac{n\pi x}{L}\right)$. Our basis will look like
   $$\left\{\lambda_n = \left(\frac{n\pi}{2}\right)^2, \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)\right\}$$
2. *Neumann Boundary Conditions*
   Neumann boundary conditions involve the derivatives of the function for which we are solving. Neumann boundary conditions will look like
   $$u_x(0,t) = u_x(L,t) = 0.$$
   This will cause our Fourier series to contain $\cos\left(\frac{n\pi x}{L}\right)$. Our basis will look like
   $$\left\{\lambda_n = \left(\frac{n\pi}{2}\right)^2, \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)\right\}$$
3. *Robin Boundary Conditions*
   Robin boundary conditions will contain a mixture of the function and it's derivative. Typically, we will see
   $$u_x(0,t) = u(L,t) = 0 \qquad \text{or} \qquad u(0,t) = u_x(L,0) = 0.$$
   These boundary conditions will give us two different results, depending on whether we have the derivative first or second. We will have
   $$u_x(0,t) = u(L,t) = 0 \implies \left\{\lambda_n = \left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}\right)^2, \phi_n(x) = \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}x\right)\right\}$$
   $$u(0,t) = u_x(L,t) = 0 \implies \left\{\lambda_n = \left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}\right)^2, \phi_n(x) = \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}x\right)\right\}$$

Now we can take a look at a slightly harder PDE. We still have the heat equation, but we will not have homogeneous boundary conditions. Our goal here is to find a way to transform our boundary conditions into a homogenous form and take it from there.

Ex: Solve the following PDE
$$\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = 0, \ u(1,t) = t \\ u(x,0) = x^2 \end{cases}$$

Solution: We would like to just jump into this PDE and solve it, but it's not quite that simple. We have to find a way to express our boundary conditions in a homogeneous manner. We can do this through a clever substitution; we will create a new PDE, denoted Υ. To find Υ, we use the following formula:
$$\Upsilon = B_1 + \frac{x}{l}\Delta B,$$
where B is a boundary condition. In our case, we have as follows:
$$\begin{aligned} \Upsilon &= B_1 + \frac{x}{l}\Delta B \\ &= 0 + \frac{x}{1}(t-0) \\ &= xt \end{aligned}$$

Now, we find a new PDE, denoted by υ. We have another formula that tells us

$$\upsilon(x,t) = u(x,t) - \Upsilon(x,t)$$

Applying this formula gives us

$$\begin{aligned} \upsilon(x,t) &= u(x,t) - \Upsilon \\ &= u(x,t) - xt \end{aligned}$$

Let's take a moment to analyze the result. We wanted to find a new PDE in terms of our old one that would allow us to have homogeneous boundary conditions. We found a solution, called υ, in terms of our original PDE, called u. Our original PDE was $u_t - u_{xx} = 0$, so to properly perform our substitution, we need to find υ_t and υ_{xx}. Through some careful differentiation, we have

$$\upsilon_t = u_t - x \qquad \text{and} \qquad \upsilon_{xx} = u_{xx}$$

To find our new PDE, we simply evaluate.

$$\upsilon_t - \upsilon_{xx} = u_t - x - u_{xx} = u_t - u_{xx} - x = 0 - x = -x$$

So we finally have a new PDE that looks like

$$\begin{cases} \upsilon_t - \upsilon_{xx} = -x \\ \upsilon(0,t) = \upsilon(1,t) = 0 \\ \upsilon(x,0) = x^2 \end{cases}$$

You may wonder how this has made anything simpler, since the PDE itself is no longer homogeneous. By forcing the boundary conditions to be homogeneous, we can now use a Fourier series to find our solution. Recall that earlier we discussed the boundary condition and their resulting eigenvalue/eigenfunction. Here, we have Dirichlet boundary conditions, which tells us that we have a Fourier sine series. We have that

$$\begin{aligned} \upsilon(x,t) &= \sum_{n=1}^{\infty} \left[a_n(t)\sin(n\pi x)\right] \\ \upsilon_t(x,t) &= \sum_{n=1}^{\infty} \left[a_n'(t)\sin(n\pi x)\right] \\ \upsilon_{xx}(x,t) &= -\sum_{n=1}^{\infty} \left[a_n(t)(n\pi)^2\sin(n\pi x)\right] \end{aligned}$$

Which tells us that

$$\upsilon_t - \upsilon_{xx} = \sum_{n=1}^{\infty} \left[(a_n'(t) + (n\pi)^2 a_n(t))\sin(n\pi x)\right] = -x$$

This may look rather complicated, but all we have is a function's Fourier sine series. Essentially, we have

$$-x = \sum_{n=1}^{\infty} \left[b_n \sin(n\pi x)\right]$$

To find b_n, we use the formula that was discussed in the earlier section about Fourier series. We have that

$$\begin{aligned} a_n'(t) + (n\pi)^2 a_n(t) &= \int_0^1 -x\sin(n\pi x)dx \\ &= \frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 \\ &= \frac{2}{n\pi} \end{aligned}$$

We now have an ODE which we can solve for a_n. Note that from out PDE, we do have an initial condition for the ODE, but it is not as simple as just saying $a_n(0) = x^2$. At $t = 0$, we have the following series:

$$\sum_{n=1}^{\infty} a_n(0) \sin(n\pi x)$$

We got this by taking our original $\upsilon(x,t)$ and plugging in 0 for t. We know how to get $a_n(0)$, since this is just a simple Fourier sine series. It is given by

$$\begin{aligned} a_n(0) &= 2\int_0^1 x^2 \sin(n\pi x)dx \\ &= \frac{(-1)^{(n+1)}}{n\pi} + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right] \end{aligned}$$

We now have all we need to solve the ODE. We will omit the work for the solution, but the final answer for a_n is

$$a_n(t) = \frac{4\left[(-1)^n - 1\right]}{(n\pi)^3} e^{-(n\pi)^2 t} + \frac{2(-1)^n}{(n\pi)^3}$$

Now all that's left to do is go back to u from υ. This may seem hard, but we saw earlier that $\upsilon = u - \Upsilon$. Solving for u gives us $u = \upsilon + \Upsilon$. Recall that Υ was xt, and υ was the series for which we just found a_n. Putting it all together gives us

$$u(x,t) = xt + \sum_{n=1}^{\infty} \left[\frac{4\left[(-1)^n - 1\right]}{(n\pi)^3} e^{-(n\pi)^2 t} + \frac{2(-1)^n}{(n\pi)^3}\right] \sin(n\pi x)$$

And we are done.

Ex: Solve the following PDE

$$\begin{cases} u_{tt} - u_{xx} = 1 + t\cos(2\pi x) \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = 2\cos(2\pi x), \quad u_t(x,0) = 1 + \cos(\pi x) - \cos(2\pi x) \end{cases}$$

Solution: This problem is an example of the wave equation with a non-homogeneous solution. We are lucky this time; the boundary conditions are already homogeneous, so there is no need to change our variable. We know that since both of the boundary conditions are derivatives, we are dealing with a Fourier cosine series. This yields the eigenvalue/function of

$$\{\lambda_n = (n\pi)^2, \cos(n\pi x)\}$$

Since we are dealing with a cosine series, we also have a leading term, a_0. Our full series looks like

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

We need u_{tt} and u_{xx}, so we have:

$$u_{tt}(x,t) = a_0''(t) + \sum_{n=1}^{\infty} a_n''(t)\cos(n\pi x) \qquad \text{and} \qquad u_{xx}(x,t) = -\sum_{n=1}^{\infty}(n\pi)^2 a_n(t)\cos(n\pi x)$$

All we have to do now is simply plug into our original equation. This gives us

$$a_0''(t) + \sum_{n=1}^{\infty} a_n''(t)\cos(n\pi x) + \sum_{n=1}^{\infty}(n\pi)^2 a_n(t)\cos(n\pi x) = 1 + t\cos(2\pi x)$$
$$a_0''(t) + \sum_{n=1}^{\infty}(a_n''(t) + (n\pi)^2 a_n(t))\cos(n\pi x) = 1 + t\cos(2\pi x)$$

Now we can play the matching game. We have a *constant+series term* on the right side, and *constant+series* on the left, so we can match up the constants. This gives us

$$a_0''(t) = 1$$

Now we match up the series term. On the right hand side, we have $\cos(2\pi x)$, so we know that for $n = 2$, we have

$$a_2''(t) + 4\pi^2 a_2(t) = t$$

a_0 and a_2 are the only terms that appear, so we can assume for now that $\forall n \neq 2$ and $3, a_n = 0$. We are left with two ODE's, so we now need to find the corresponding initial conditions. Let's apply the first initial condition to our PDE. This gives us

$$a_0(0) + \sum_{n=1}^{\infty} a_n(0)\cos(n\pi x) = 2\cos(2\pi t)$$

Which tells us

$$a_0 = 0 \qquad\qquad a_2 = 2$$

Now we can apply the second initial condition. After taking the partial derivative of u with respect to t, we have

$$a_0'(0) + \sum_{n=1}^{\infty} a_n'(0)\cos(n\pi x) = 1 + \cos(\pi x) - \cos(2\pi x)$$

Some more clever pattern matching give us

$$a_0'(0) = 1 \qquad a_1'(0) = 1 \qquad a_2'(0) = -1$$

Notice that we have a term a_1'. We cannot simply ignore this value; rather we have to take it into account when we find our solution. We know that we have an ODE for $a_1(t)$, but it may not be evident how to express it. Recall that earlier we said $\forall n \neq 2$ and $3, a_n = 0$. This means that $a_1 = 0$, and since a_1 is a coefficient in the Fourier series, we know that $a_1''(t) + \pi^2 a_1(t) = 0$. We now have three different ODE's that we must solve:

$$\begin{cases} a_0''(t) = 1 \\ a_0'(0) = 1, \quad a_0(0) = 1 \end{cases} \quad \begin{cases} a_1''(t) + \pi^2 a_1(t) = 0 \\ a_1'(0) = 1, \quad a_1(0) = 0 \end{cases} \quad \begin{cases} a_2''(t) + 4\pi^2 a_2(t) = t \\ a_2'(0) = 1, \quad a_2(0) = 2 \end{cases}$$

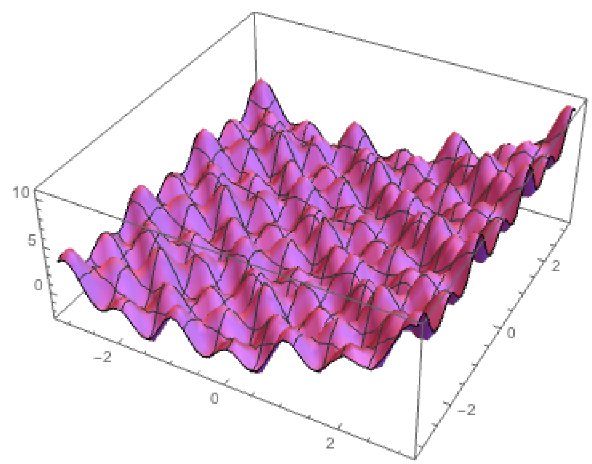
Luckily, Mathematica is here to save the day. We will skip the work and let the computer do the thinking for us. We get that

$$\begin{aligned} a_0(t) &= \frac{1}{2}t^2 + t \\ a_1(t) &= \frac{\pi\cos(t\pi) + \sin(t\pi)}{\pi} \\ a_2(t) &= \frac{2\pi t + 16\pi^3\cos(2\pi t) - \sin(2\pi t) + 4\pi^2\sin(2\pi t)}{8\pi^3} \end{aligned}$$

Now all we need to do is plug back in to our u. Our final answer becomes

$$u(x,t) = \frac{1}{2}t^2 + t + \frac{\pi\cos(t\pi) + \sin(t\pi)}{\pi}\cos(\pi x) + \frac{2\pi t + 16\pi^3\cos(2\pi t) - \sin(2\pi t) + 4\pi^2\sin(2\pi t)}{8\pi^3}\cos(2\pi x)$$

Just for interest, the plot of this solution looks like



On $-\pi \leq \theta \leq \pi$ and $-\pi \leq x \leq \pi$; and we are done.

4.2 The Laplace Equation

Up to this point, we have only looked at the wave equation and the heat equation. There is, however, one other equation in the classical trinity: the Laplace Equation. We haven't dealt with this equation yet because we have to be very careful in our approach to solving it. When we looked at the homogeneous wave and heat equations, we used the general approach called *Separation of Variables.* In the case of the homogeneous Laplace equation, we may not be able to tackle the problem using separation of variables. The problem arises from the various types of boundaries that effect the domain. We can either have a 'rectangular' bound, which will be separable, or a 'circular' bound which is **not** separable. Imagine that we have the following two problems:

$$A = \begin{cases} u_{xx} + u_{yy} = 0 \quad \text{on } [0,1] \times [0,2] \\ u(x,0) = u(x,2) = 0 \\ u(0,y) = y, u(1,y) = 0 \end{cases} \qquad B = \begin{cases} u_{xx} + u_{yy} = 0 \quad \text{on } x^2 + y^2 < a^2 \\ u(x,y) \big|_{x^2+y^2=a^2} = f(x,y) \end{cases}$$

A is an example of a separable Laplace equation. We can see graphically that the domain $[0,1] \times [0,2]$ is rectangular. B's domain is not separable, since it is circular. Let's take a look at A.

Ex: Solve the following PDE

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{on } [0,1] \times [0,2] \\ u(x,0) = u(x,2) = 0 \\ u(0,y) = y, u(1,y) = 0 \end{cases}$$

Solution: Earlier, we discussed the different boundary conditions we can encounter and their corresponding eigenvalues/functions. In this problem, we have Dirichlet boundary conditions. This tells us that

$$\left\{ \lambda_n = \left(\frac{n\pi}{2}\right)^2, \phi_n(y) = \sin\left(\frac{n\pi y}{2}\right) \right\}$$

Notice that we used y here instead of x. We did this because the boundary conditions are defined in terms of y, not x. Now we can continue to solve our problem. Recall that our separation of variables gives us two second order ODE's,

$$X'' - \lambda X = 0 \qquad \text{and} \qquad \lambda Y - Y'' = 0$$

We already solved for $Y(y)$ by finding the corresponding eigenvalues/vectors. Now we plug in our λ_n into our X ODE. Solving the resulting ODE will give us a basis of

$$\{e^{\frac{n\pi}{2}x}, e^{-\frac{n\pi}{2}x}\}$$

Now we could continue at this point, but if we think ahead, we can see that when we eventually apply our initial conditions, nothing will cancel out. At this point, we want to transform our basis. We have as follows

$$\{e^{\frac{n\pi}{2}x}, e^{-\frac{n\pi}{2}x}\} \implies \left\{\cosh\left(\frac{n\pi x}{2}\right), \sinh\left(\frac{n\pi x}{2}\right)\right\}$$

All we did here is apply an identity. Now we have everything we need to set up a solution. Our $u(x,y)$ will be

$$u(x,y) = \sum_{n=1}^{\infty} \left[a_n \cosh\left(\frac{n\pi x}{2}\right) + b_n \sinh\left(\frac{n\pi x}{2}\right)\right] \sin\left(\frac{n\pi y}{2}\right)$$

We can apply our initial conditions, which gives us b_n and a_n:

$$a_n = \frac{4(-1)^{n+1}}{n\pi}$$
$$b_n = -\coth\left(\frac{n\pi}{2}\right)\frac{4(-1)^{n+1}}{n\pi}$$

This yields a final solution of

$$u(x,y) = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n\pi} \cosh\left(\frac{n\pi x}{2}\right) - \coth\left(\frac{n\pi}{2}\right)\frac{4(-1)^{n+1}}{n\pi} \sinh\left(\frac{n\pi x}{2}\right)\right] \sin\left(\frac{n\pi y}{2}\right)$$

And we are done

Now we turn our attention to the Laplace Equation with a circular domain. As we discussed earlier, we can't simply say that our solution will be in the form of $X(x)Y(y)$. However, we can force the domain to be rectangular if we use polar coordinates. Let's consider a variation of B from above.

Ex: Solve the following PDE

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on } x^2 + y^2 < a^2 \\ u(a,\theta) = U_1 & \text{on } 0 \le \theta \le \pi \\ u(a,\theta) = U_2 & \text{on } \pi \le \theta \le 2\pi \end{cases}$$

Solution: We want to express the partial derivatives here in terms of r and θ. Recall that we have

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

First, we will find the partial derivatives of x and y with respect to r and θ. Since $x = r\cos\theta$,

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

We can compute the derivatives for y in a similar fashion:

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Now we need to find u_{rr} and $u_{\theta\theta}$. Let's focus on u_{rr} first. We will need to start by finding u_r. We have as follows

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \end{aligned}$$

We have u_r, so now we need to compute u_{rr}. We have as follows:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\ &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} \end{aligned}$$

We won't delve into finding $u_{\theta\theta}$, as the partials get a little out of hand. However, we end up with

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \end{aligned}$$

Let's go ahead and formalize it:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Now we can treat this as a 2D problem; in other words, we can now use separation of variables to solve for u. As before, we know our solution will look like

$$u(r,\theta) = R(r)T(\theta)$$

Plugging everything in and realizing by the usual trick that everything s equal to a constant gives us

$$-\frac{T''}{T} = \frac{R'' + \frac{R'}{r}}{\frac{R}{r^2}} = \lambda$$

Since we are dealing in polar coordinates, we will go ahead and walk through the eigenvalues/functions. We will be solving for $T(\theta)$, since our boundary conditions are in terms of θ. We also should note that the solutions will repeat themselves every 2π, since we are in the polar system. In addition, our u should be finite at $r = 0$. Let's take a look at the eigenvalues for the Laplace equation.

$\underline{\lambda = 0}$
When $\lambda = 0$, we have

$$T(\theta) = a\theta + b.$$

This also tells us that

$$R(r) = c\ln(r) + d.$$

Putting everything together gives us

$$u(r, \theta) = (a\theta + b)(c\ln r + d).$$

Since the solution is periodic on 2π, we know that $a = 0$. Since the solution has to remain finite at $r = 0$, we know that $c = 0$. This gives us that $u = bd$, or that u is equal to a constant.

$\underline{\lambda > 0}$
With $\lambda > 0$, we have our characteristic equation $(m = \pm\sqrt{\lambda})$ yielding a real valued solution. This tells us that

$$T(\theta) = c_1 \cosh(\sqrt{\lambda}\theta) + c_2 \sinh(\sqrt{\lambda}\theta).$$

The solution is periodic on 2π, so we can say that $c_1 = c_2 = 0$. Therefore, we have a trivial solution.

$\underline{\lambda < 0}$
$\lambda < 0$ gives us

$$T(\theta) = c_1 \cos(\sqrt{\lambda}\theta) + c_2 \sin(\sqrt{\lambda}\theta)$$

This solution is periodic with 2π, and it yields a trivial eigenvector.

So we know that the only λ that yields eigenvalues is $\lambda = 0$, and we know that this particular eigenvalue yields a constant solution. If we solve the differential equation for R, we will find that the only solution we care about is r^n. We can finally say that our solution will look like

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n[a_n \cos(n\theta) + b_n \sin(n\theta)]$$

This solution is worth memorizing, as it will generally be the same for all variations of the Laplace equation similar to this one. Now we can apply the conditions for θ. We have two conditions, but it is really just a piecewise function which we will call $f(\theta)$. We also know that our bound on the domain is a, or the length of the radius. This is enough information to find all the various variables in our $u(r, \theta)$. We have as follows:

$$u(a, \theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^n[a_n \cos(n\theta) + b_n \sin(n\theta)]$$

Note that the a which is the length of the radius is *not* the same a as a_n. Now we can solve for a_0 and a_n.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta \\ &= U_1 + U_2 \end{aligned} \qquad \begin{aligned} a_n &= \frac{1}{\pi}\left[\int_0^{\pi} U_1 \cos(n\theta)d\theta + \int_{\pi}^{2\pi} U_2 \cos(n\theta)d\theta\right] \\ &= 0 \end{aligned}$$

For b_n, we have

$$\begin{aligned} a^n b_n &= \frac{1}{\pi}\left[\int_0^{\pi} U_1 \sin(n\theta)d\theta + \int_{\pi}^{2\pi} U_2 \sin(n\theta)d\theta\right] \\ &= \frac{1-(-1)^n}{n\pi}(U_1 - U_2) \\ b_n &= \frac{[1-(-1)^n](U_1 - U_2)}{n\pi a^n} \\ &= \frac{2(U_1 - U_2)}{(2k+1)\pi a^{2k+1}} \end{aligned}$$

Putting it all together gives us a final solution of

$$u(r,\theta) = \frac{U_1 + U_2}{2} + \sum_{k=0}^{\infty} r^k \frac{2(U_1 - U_2)}{(2k+1)\pi a^{2k+1}}$$

And we are done.

Now let's look at another Laplace equation in polar coordinates. Again, we will be dealing with Dirichlet boundary conditions, so we will be able to jump straight into the solution. This one won't take us as long to solve, as we have already done a lot of the work. We are asked the following:

Ex: Solve the following PDE

$$\begin{cases} \nabla^2 u = 0 \\ u(3,\theta) = \begin{cases} 1 & 0 \le \theta \le \pi \\ \sin^2\theta & \pi \le \theta \le 2\pi \end{cases} \end{cases}$$

Solution: We already know what our separable equation will look like:

$$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = -\lambda$$

This yields an eigenvalue of $\lambda = n^2$. We know this because our function T must be periodic on 2π, and as such we get the following piecewise function:

$$T(\theta) = \begin{cases} a_0 & \text{for } n = 0 \\ a_n\cos(n\theta) + b_n\sin(n\theta) & \text{for } n > 0 \end{cases}$$

The solution to our equation for R yields

$$R(r) = \begin{cases} c_1 + c_2\ln r & \text{for } n = 0 \\ c_1r^n + c_2r^{-n} & \text{for } n > 0 \end{cases}$$

We can disregard c_2 for this equation, since it would cause our solution to go to ∞ as $r \to 0$, at the center of the disk. Putting everything together gives us

$$\begin{aligned} u(r,\theta) &= R(r)T(\theta) \\ &= \begin{cases} a_0 & \text{for } n = 0 \\ a_nr^n\cos(n\theta) + b_nr^n\sin(n\theta) & \text{for } n > 0 \end{cases} \end{aligned}$$

We should probably construct a series out of this, since that is what we usually do. The corresponding series is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_nr^n\cos(n\theta) + b_nr^n\sin(n\theta)$$

And applying the boundary conditions gives us

$$u(3,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n3^n\cos(n\theta) + b_n3^n\sin(n\theta)$$

For a_0, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\left[\int_0^{\pi} 1d\theta + \int_{\pi}^{2\pi} sin^2\theta d\theta\right] \\ &= \frac{3}{2} \end{aligned}$$

Solving for a_n gives us

$$\begin{aligned} a_n &= \frac{1}{3^n\pi}\left[\int_0^{\pi} \cos(n\theta)d\theta + \int_{\pi}^{2\pi} \cos(n\theta)\sin^2\theta d\theta\right] \\ &= \frac{[n^2 - 2 - 4\cos(n\pi)]\sin(n\pi)}{3^nn(n^2-4)\pi} \\ &= 0 \end{aligned}$$

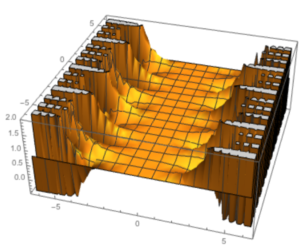
Our b_n term is

$$\begin{aligned}
b_n &= \frac{1}{3^n \pi}\left[\int_0^{\pi} \sin(n\theta)d\theta + \int_{\pi}^{2\pi} \sin(n\theta)\sin^2\theta d\theta\right] \\
&= \frac{2\left[n^2 - 6 - 4\cos(n\pi)\right]\sin^2\left(\frac{n\pi}{2}\right)}{3^n n\pi(n^2-4)} \\
&= \frac{2\left[n^2 - 6 - 4(-1)^n\right]\sin^2\left(\frac{n\pi}{2}\right)}{3^n n\pi(n^2-4)} \\
&= \begin{cases} 0 & n \in \{2k \mid k \in \mathbb{Z}\} \\ \frac{1}{3^n\pi}\left[\frac{2}{n} + \frac{4}{n(n^2-4)}\right] & n \in \{2k+1 \mid k \in \mathbb{Z}\} \end{cases}
\end{aligned}$$

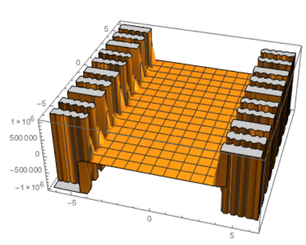
We essentially have our solution. All we have to do is combine everything:

$$u(r,\theta) = \frac{3}{4} + \sum_{k=0}^{\infty} \frac{r^{2k+1}}{3^{2k+1}\pi}\frac{2(4k^2+4k-1)}{(2k+3)(4k^2-1)}\sin\left[(2k+1)\theta\right]$$

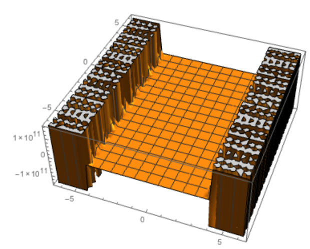
Let's take a look at some of the plots for this equation.



The first 3 terms



The first 30 terms



The first 50 terms

Plugging this infinite sum into mathematica tells us that it does not converge; the values are heading towards ∞ as the sum grows. We will leave our answer as a series, and we are done.

5 The Cauchy Problem

Up until now, we have always been given our PDE's with boundary conditions and initial values; but imagine that we did not have a boundary? The traditional method that we have been using would fail at this point. Since we would not be able to find any eigenvalues, we would not be able to form a series solution by using the *Separation of Variables* technique. Let's consider the familiar heat equation:

$$\begin{cases} \xi_t - k\xi_{xx} = 0 \\ \xi(x,0) = f(x) \end{cases}$$

Through much careful derivation and theory, we can say that the general solution for this problem will be

$$\xi(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)dy.$$

The question now should be "How can we solve this?" The first step we will take is letting

$$u = \frac{y-x}{\sqrt{4kt}}.$$

We are going to perform a *u substitution* to try and put this integral into a nicer form. This yields

$$\begin{aligned} y &= x + u\sqrt{4kt} \\ dy &= \sqrt{4kt}\ du. \end{aligned}$$

This changes our original problem into

$$\xi(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} f(x + u\sqrt{4kt})du.$$

Let's look at an example

Ex: Solve the following PDE

$$\begin{cases} \mu_t - k\mu_{xx} = 0 \\ \mu(x,0) = \cos x \end{cases}$$

Solution: From above, we know what our general solution will look like. All we need to do is substitute in our initial condition. We have the following:

$$\begin{aligned} \mu(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cos{(x + u\sqrt{4kt})}du \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \left[\cos{(x)}\cos{(u\sqrt{4kt})} - \sin{(x)}\sin{(u\sqrt{4kt})}\right] du \\ &= \frac{2}{\sqrt{\pi}} \cos{(x)} \int_{0}^{\infty} e^{-u^2} \cos{(u\sqrt{4kt})}du. \end{aligned}$$

You may be wondering what happened to the sin terms. Recall that sin is an odd function, so it will ultimately be 0 on the interval $[-\infty, \infty]$; because of this fact, we can remove it now. We also use the face that the cos is even, so it is okay to halve the interval and multiply by two. Now we need to evaluate this integral. We will substitute $\alpha = \sqrt{4kt}$, and assign a function, $\Upsilon(\alpha)$, to the integral. This gives us the following system:

$$\begin{cases} \Upsilon(\alpha) = \int_0^\infty e^{-u^2} \cos{(\alpha u)}du \\ \Upsilon(0) = \int_0^\infty e^{-u^2} du \end{cases}$$

Believe it or not, we actually know the answer to $\Upsilon(0)$. To find it, we can assign a function to the integral in a similar fashion as we have already done. Calling it Ψ, we have

$$\begin{aligned}
\Psi &= \int_0^\infty e^{-u^2} du \\
\Psi^2 &= \int_0^\infty e^{-u^2} du \int_0^\infty e^{-v^2} dv \\
\Psi^2 &= \int_0^\infty \int_0^\infty e^{-u^2-v^2} dudv \\
\Psi^2 &= \int_0^{\pi/2} \int_0^\infty re^{-r^2} drd\theta \\
\Psi^2 &= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]\Bigg|_{r=0}^{r=\infty} \\
\Psi^2 &= \frac{\pi}{4} \\
\Psi &= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

So we know that $\Upsilon(0) = \frac{\sqrt{\pi}}{2}$. Plugging this into our system gives us

$$\begin{cases} \Upsilon(\alpha) = \int_0^\infty e^{-u^2} \cos{(\alpha u)} du \\ \Upsilon(0) = \frac{\sqrt{\pi}}{2} \end{cases}$$

Now, we will take the derivative of Υ. This gives us

$$\Upsilon'(\alpha) = -\int_0^\infty ue^{-u^2} \sin{(\alpha u)} du.$$

Integration by parts gives us

$$\frac{1}{2}e^{-u^2} \sin{(\alpha u)}\Big|_0^\infty - \alpha \int_0^\infty \frac{1}{2}e^{-u^2} \cos{(\alpha u)} du.$$

The value from 0 to ∞ is zero, since the sin function odd. This leaves us with following relationship:

$$\Upsilon'(\alpha) = -\frac{\alpha}{2}\Upsilon(\alpha)\alpha.$$

We can solve this as an ODE. It's separable, so we get

$$\begin{aligned}
\frac{d\Upsilon}{\Upsilon} &= -\frac{\alpha}{2} d\alpha \\
\ln{(\Upsilon)} &= -\frac{\alpha^2}{4} + c_1 \\
\Upsilon &= c_2 e^{-\frac{\alpha^2}{4}}.
\end{aligned}$$

Applying the initial condition gives us

$$\Upsilon = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}.$$

Substituting back in gives us a final answer of

$$\begin{aligned}
\mu(x,t) &= \frac{2}{\sqrt{\pi}} \cos{x} \Upsilon(\sqrt{4kt}) \\
&= \cos{x} e^{-kt},
\end{aligned}$$

and we are done.

Now we will move on to the Cauchy wave equation. We will be presented with a similar problem: a wave equation on an unbounded domain. Consider the following problem:

Ex: Solve the following PDE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \sin x \\ u_t(x,0) = \cos x \end{cases}$$

Solution: Just as with the heat equation, we have a formal solution for the homogeneous wave equation. We will do the derivation, but I will tell you that the formal solution is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\,.$$

Now let's derive the formal solution. We want a solution in the form of

$$u(x,t) = F(x+ct) + G(x-ct).$$

Now we can say that $u_t = cF'(x+ct) - cG'(x-ct)$, and $u_t(x,0) = cF'(x) - cG'(x)$. This means

$$\begin{aligned} F(x) + G(x) &= f(x) \\ F(x) - G(x) &= \frac{1}{c}\int_0^x g(s)\,ds + c_1. \end{aligned}$$

Solving the system yields

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{c_1}{2} + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds - \frac{c_1}{2} \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, \end{aligned}$$

and that is the derivation. This problem is simply plugging into the formal solution. We have the following:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(s)\,ds \\ &= \frac{\sin(x+ct) + \sin(x-ct)}{2} + \frac{\cos(x)\sin(ct)}{c} \\ &= \frac{\cos(x)\sin(ct)}{c} + \cos(ct)\sin(x), \end{aligned}$$

and we are done.

So far, we have only dealt with homogeneous Cauchy problems. We will now look at the case where we have a non-homogeneous problem. The first variation of this problem is the *half-line* problem. In this problem, we will have a bound on our equation, and we will generalize the solution so that we are dealing with an unbounded Cauchy problem. Let's look at an example.

Ex: Solve the following PDE

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \end{cases}$$

Solution: We have both an initial condition and a boundary condition here. This problem is saying that at $x = 0$, there is an impassable boundary. This forces our domain to be $[0, \infty)$. To solve a Cauchy problem, however, we want a domain of $(-\infty, \infty)$. We can force this domain by extending our initial condition, $\phi(x)$, over the boundary. We have the following rule:

$$\phi_0(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0. \end{cases}$$

Now we are dealing with the system

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi_0(x), \end{cases}$$

which we can solve using the formal solution to the heat equation. Recall that

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_0(y)\, dy\,.$$

Substituting into the equation gives us

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4k\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} (-\phi(-y))\, dy \right] \\ &= \frac{1}{\sqrt{4k\pi t}} \end{aligned}$$

6 Transforms

Much like when we solve ODE's, we can use transforms to solve PDE's. When we use a Laplace transform to solve an ODE, we are transforming the ODE into an algebraic equation that is much easier to solve. When we transform a PDE, we get an ODE that we can solve using techniques that we already know. In this section, we will look at two transforms: the Laplace Transform and the Fourier Transform.

6.1 The Laplace Transform

Let's begin by looking at an exaple use of a Laplace Transform to solve an ODE.

Ex: Solve the following ODE using a Laplace Transform.

$$\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

Solution: We should begin by recalling the definition of the Laplace operator:

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

We also know that the Laplace transforms of a derivative and a second derivative:

$$\mathscr{L}[f'(t)] = s\mathscr{L}[f(t)] - f(0) \qquad\qquad \mathscr{L}[f''(t)] = s^2\mathscr{L}[f(t)] - sf(0) - f'(0)$$

This is really all the information we need. Applying the transform to both sides of the equation yields

$$\begin{gathered} \mathscr{L}[y'' - 5y' + 6y] = \mathscr{L}[0] \\ s^2\gamma(s) - sy(0) - y'(0) - s\gamma(s) - y(0) = 0 \\ s^2\gamma(s) - 2s - 2 - s\gamma(s) - 2 = 0 \\ s^2\gamma(s) - s\gamma(s) - 2s - 4 = 0 \\ \gamma(s) = \frac{2s-8}{s^2 - 5s + 6} \\ \gamma(s) = \frac{4}{s-2} - \frac{2}{s-3} \end{gathered}$$

We now need to transform our equation back to its ODE form; we do in fact have an inverse Laplace operator, but it involves complex variables. If we consult a table of transforms, we will see that $\gamma(s)$ becomes

$$y(t) = 4e^{2t} + -2e^{3t}$$

And we are done.

Now we want to apply the Laplace transform to a PDE. The idea is the same, but our PDE involves two variables: usually x, representative of position and t, representative of time. The Laplace transform *only* transforms with respect to our t variable. Let's take a look at an example:

Ex: Solve the following PDE using a Laplace Transform.

$$\begin{cases} u_x + u_t = x \\ u(0,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Solution: In this problem we are dealing with a first order PDE. Much like an ODE, we simply need to determine what the Laplace transforms of u_x and u_t are. We will let $\mathscr{L}[u(x,t)] = \Upsilon(x,s)$. Let's begin with $u_t(x,t)$.

$$\begin{aligned} \mathscr{L}[u_t] &= \int_0^\infty e^{-st} u_t(x,t)dt \\ &= e^{-st}u(x,t)\Big|_0^\infty + s\int_0^\infty e^{-st}u(x,t)dt \\ &= s\Upsilon(x,s) - u(x,0) \end{aligned}$$

In a similar fashion, we could find $\mathscr{L}[u_{tt}(x,t)]$. We won't derive it here, but it ends up looking very similar to our second derivative transform for our ODE:

$$\mathscr{L}[u_{tt}(x,t)] = s^2\Upsilon(x,s) - su(x,0) - u_t(x,0)$$

We also need to determine the transform for $u_x(x,t)$. Recall that the Laplace transform only deals with the time variable, so our transforms for both $u_x(x,t)$ and $u_{xx}(x,t)$ become $\Upsilon_x(x,s)$ and $\Upsilon_{xx}(x,s)$, respectivly. We now have all the information we need to continue. Applying our transform yields

$$\begin{aligned} \mathscr{L}[u_x + u_t] &= \mathscr{L}[x] \\ \Upsilon'(x,s) + s\Upsilon(x,s) - u(x,0) &= \frac{x}{s} \\ \Upsilon'(x,s) + s\Upsilon(x,s) &= \frac{x}{s} \end{aligned}$$

This is just a first order ODE. The only thing we are missing now is the initial condition, which we can determine by transforming the initial condition on the PDE. In this case, $u(0,t) = 0$, so the initial condition is still 0 after applying the transform. Now we have

$$\begin{cases} \Upsilon'(x,s) + s\Upsilon(x,s) = \frac{x}{s} \\ \Upsilon(0,s) = 0 \end{cases}$$

We can solve this ODE by using the integrating factor method. The integrating factor in this case is

$$\begin{aligned} \rho &= e^{\int s\,dx} \\ &= e^{sx} \end{aligned}$$

Our ODE now becomes

$$\frac{d}{dx}(e^{sx}\Upsilon(x,t)) = \frac{e^{xs}x}{s}$$

Integration, simplification, and applying our initial condition yields

$$\Upsilon(x,t) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{e^{-sx}}{s^3}$$

Again, we won't dive into the complexity of the inverse Laplace transform here, but if we do take the inverse, we get

$$u(x,t) = xt - \frac{t^2}{2} - H(t-x)\frac{(t-x)^2}{2}$$

The function $H(t-x)$ is the Heavyside Step Function, and is defined as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

And we are done.

6.2 The Fourier Transform

The Fourier transform also allows us to take a PDE and transform it into an ODE. In this case, however, we will be transforming with respect to the position variable. There are several variations to the Fourier transform; we will look at the regular Fourier transform, the Fourier Sine transform, and the Fourier Cosine transform.

6.2.1 The Full Fourier Transfrom

The Fourier transform of a function $f(x)$ is given by

$$\mathscr{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx$$

Just like the Laplace transform, we will need to determine the Fourier transform for the derivatives of functions as well. Recall that with the Laplace transfrom, $\mathscr{L}[u_x]$ became Υ_x. Using the same argument, we know that

$$\mathscr{F}[u_t(x,t)] = \Lambda_t(\omega, t)$$
$$\mathscr{F}[u_{tt}(x,t)] = \Lambda_{tt}(\omega, t)$$

When we apply the Fourier transfrom to our x derivatives, we get

$$\mathscr{F}[u_x(x,t)] = i\omega\Lambda(\omega, t)$$
$$\mathscr{F}[u_{xx}(x,t)] = -\omega\Lambda(\omega, t)$$

Let's consider the following example:

Ex: Solve the following PDE using a Fourier Transform.

$$\begin{cases} 2u_x + 3u_t = 0 \\ u(x,0) = 3 \end{cases}$$

Solution: We are again dealing with a first order PDE here. We will begin by taking the Fourier transform of both sides of the PDE. The Fourier transform of 0 is still 0, so we get

$$\mathscr{F}[2u_x + 3u_t] = \mathscr{F}[0]$$
$$2i\omega\Lambda(\omega, t) + 3\Lambda_t(\omega, t) = 0$$

This is just a first order ODE with respect to t. We also need to transform our inital condition. Doing so yields

$$\mathscr{F}[3] = 3\delta(\omega)$$

The delta function is defined as the derivative of the Heavyside Step Function. We now have an ODE represented by

$$\begin{cases} 2i\omega\Lambda(\omega, t) + 3\Lambda_t(\omega, t) = 0 \\ \Lambda(\omega, 0) = 3\delta(\omega) \end{cases}$$

Solving this ODE and applying the initial condition yields

$$\Lambda(\omega, t) = 3\delta(\omega)e^{-\frac{2}{3}i\omega t}$$

From this step, we would transform the equation back to the x, t coordinate system. We won't do that here, so we are done.

6.2.2 The Fourier Sine and Cosine Transform

Much like the Fourier series, the Fourier transform can also be split into a sine and cosine transform. The rules for the transformation still hold; we are still only concerned with the position variable x. The transform for the positional derivatives do change however. We have the following:

$$\mathscr{F}_s[f(x)] = \frac{2}{\pi}\int_0^\infty \sin(\omega x) f(x)\,dx \qquad \mathscr{F}_c[f(x)] = \frac{2}{\pi}\int_0^\infty \cos(\omega x) f(x)\,dx$$

Applying these definitions to our positional derivatives yields

$$\mathscr{F}_s[u_x(x,t)] = -\omega\mathscr{F}_c[u(x,t)] \qquad \mathscr{F}_c[u_x(x,t)] = -\frac{2}{\pi}u(0,t) + \omega\mathscr{F}_s[u(x,t)]$$

$$\mathscr{F}_s[u_{xx}(x,t)] = \frac{2\omega u(0,t)}{\pi} + \omega^2\mathscr{F}_s[u(x,t)] \qquad \mathscr{F}_c[u_{xx}(x,t)] = -\frac{2}{\pi}u_x(0,t) - \omega^2\mathscr{F}_c[u(x,t)]$$

Notice that with the transform of the first derivative, we end up with the opposite transform of the original function. Let's look at an example.
Ex: Solve the following PDE using a Fourier Transform.

$$2u_x + 3u_t = 0$$

Solution: We will start with the sine transform. Applying the sine transform to both sides of the equation yields

$$\begin{gathered} 2\mathscr{F}_s[u_x] + 3\mathscr{F}_s[u_t] = \mathscr{F}_s[0] \\ -2\omega\Lambda(\omega,t) + 3\Lambda_t(\omega,t) = 0 \\ -2\omega\Lambda + 3\Lambda' = 0 \end{gathered}$$

This is a first order ODE that we can easily solve. We get a final solution of

$$\Lambda(\omega,t) = f(t)e^{-\frac{2}{3}\omega}$$

And we are done.

7 Wrapping Up

The following section outlines the formal solutions to most arrangements of second order PDEs, as well as the Cauchy problems. It serves as a reference sheet for later use if need be.

Homogeneous Solutions

7.1 The Heat Equation

Dirichlet

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

Robin

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)^2 t} \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)dx$$

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)^2 t} \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)dx$$

7.2 The Wave Equation

Dirichlet

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi ct}{\ell}\right) + b_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$b_n = \frac{2}{n\pi c}\int_0^{\ell} g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{a_0}{2} + \frac{b_0}{2}t + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\cos\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad b_0 = \frac{2}{n\pi c}\int_0^\ell g(x)\,dx$$

Robin

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty}\left[a_n \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right) + b_n \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right)\right]\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad b_0 = \frac{2}{n\pi c}\int_0^\ell g(x)\,dx$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty}\left[a_n \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right) + b_n \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right)\right]\cos\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\cos\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad b_0 = \frac{2}{n\pi c}\int_0^\ell g(x)\,dx$$

7.3 Laplace Equation

Rectangular Region

Dirichlet

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\ell,y) = 0 \end{cases}$$

$$u(x,y) = \sum_{n=1}^{\infty}\left[b_n \sinh\left(\frac{n\pi y}{\ell}\right) + a_n \cosh\left(\frac{n\pi y}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right)$$

Neumann

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = u_x(\ell, y) = 0 \end{cases}$$

$$u(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \left[b_n \sinh\left(\frac{n\pi y}{\ell}\right) + a_n \cosh\left(\frac{n\pi y}{\ell}\right) \right] \cos\left(\frac{n\pi x}{\ell}\right)$$

Robin

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = u_x(\ell, y) = 0 \end{cases}$$

$$u(x, y) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{\ell}\right) + a_n \cosh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{\ell}\right) \right] \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi x}{\ell}\right)$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = u(\ell, y) = 0 \end{cases}$$

$$u(x, y) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{\ell}\right) + a_n \cosh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{\ell}\right) \right] \cos\left(\left[n + \frac{1}{2}\right] \frac{\pi x}{\ell}\right)$$

Circular Region

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(h, \theta) = f(\theta) \end{cases}$$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{r^n}{a^n} \cos(n\theta) + b_n \frac{r^n}{a^n} \sin(n\theta)$$

$$a^n a_n = \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$a^n b_n = \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta$$

Non-Homogeneous Solutions

7.4 The Heat Equation

Dirichlet

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[b_n^{'}(t) + k\left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

Neumann

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{a_0'(t)}{2} + \sum_{n=1}^{\infty} \left[a_n'(t) + k\left(\frac{n\pi}{\ell}\right)^2 a_n\right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

7.5 The Wave Equation

Dirichlet

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(0,t) = u(\ell,t) = 0 \\ u(x,t) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

Neumann

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,t) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{b_0(t)}{2} + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{b_0''(t)}{2} + \sum_{n=1}^{\infty} \left[b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

7.6 Non-Homogeneous Boundaries

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = L(t),\ u(\ell,t) = R(t) \\ u(x,0) = f(x) \end{cases}$$

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = L(t),\ u(\ell,t) = R(t) \\ u(x,0) = f(x) \end{cases}$$

$$\delta(x,t) = L + \frac{x}{\ell}(R - L)$$

$$\gamma(x,t) = u(x,t) - \delta(x,t)$$

Cauchy Problems

7.7 The Heat Equation

Homogeneous

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)\, dy$$

Non-Homogeneous

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} h(y,s)\, dy\, ds$$

7.8 The Wave Equation

Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

Non-Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy\, ds$$

8 Mathematica and Partial Differential Equations

We will briefly talk about solving PDE's in Mathematica in this section. We can solve PDE's with the tools available to solve ODE's. Let's consider an example equation:

Ex: Solve the following PDE

$$u_t + 2u_x = 0$$

Solution: We can define both u_x and u_t in Mathematica by using the D command.

```
 In[1]:= ClearAll["Global"];
         pde = D[u[x,t], t] + 2 D[u[x,t], x] == 0;
```

Here, we have assigned our PDE to the Mathematica variable pde. Note that we never gave a definition to the function $u(x, t)$. It's not necessary; Mathematica understands what we mean. To solve the equation, we will use the DSolve command, just as we would with an ODE.

```
 In[2]:= solution = DSolve[pde, u[x,t], {x,t}]
Out[2]:= {{u[x,t] -> C[1][(1/2)(2t-x)]}}
```

Mathematica represents the arbitrary function that we get when solving with C[1]. If we want to assign the solution for later use, we can create a new function and assign it to the solution that we just found.

```
 In[3]:= f[x_,t_] := u[x,t] /. solution[[1]];
```

The line above assigns u[x,t] to the function f[x,t]. We can also apply an initial condition by providing DSolve with a list of equations rather than just the PDE.

```
 In[4]:= solution = DSolve[{pde, u[0,t] == Sin[t]},u[x,t],{x,t}]
Out[4]:= {{u[x,t] -> Sin[(1/2)(2t-x)]}}
```

Here we have provided an initial condition of $\sin t$. Again, we could assign the result of Out[4] to a function that we could use later. It may benefit us, for example, to plot the result.

```
 In[5]:= Plot3D[u[x,t]/.solution,{x,-4,4},{t,-4,4}];
```

We've supressed the output here, but the result would be the graph of the function in the 3D plane.

9 High Order Partial Differential Equations

This section is concerned with high order PDE's. We will be focusing on PDE's that are still linear, but contain third, fourth, and higher derivatives. Before we continue, we will need to discuss Green's theorem.

9.1 Green's Theorem

Green's theorem stems from the harmonic function that we know as Laplace's equation. We have seen the operator ∇, which by definition means

$$\nabla f = < f_x, f_y, f_z >$$

When we dealt with Laplace's equation, we saw the operator used as

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

We ignored the u_{zz} term because we were only concerned with the 2D plane. Green's function provides a universal solution to harmonic equations in higher orders than the Laplace equation which we are used to dealing with. Before we continue, we need to discuss Green's Identities.

9.1.1 Green's First Identity

Green's first identity is represented by the equation

$$\int_D \nabla\phi \cdot \nabla\psi \, dV + \int_D \phi\nabla^2\psi \, dV = \int_C \phi\nabla\psi \cdot \mathbf{n} \, dS$$

This result follows from the *Divergence Theorem* and the product rule for partial derivatives. We have D, which is the closed surface over which we are integrating, and two function ϕ and ψ. Green's first identity provides us with a way to relate n integrals to $n-1$ integrals of the same function (it can be extended to show that the identity works for n and $n-1$ integrals).

9.1.2 Green's Second Identity

We can derive Green's second identity from the first identity realtively simply. We simply need to interchange ψ and ϕ, then subtract across. This yields

$$\int_D \psi\nabla^2\phi \, dV - \int_D \phi\nabla^2\psi \, dV = \int_C \psi\nabla\phi \cdot \mathbf{n} \, dS - \int_C \phi\nabla\psi \cdot \mathbf{n} \, dS$$

Again, this identity relates an n dimensional surface integral to an $n-1$ dimesional surface integral of the same functions.

9.1.3 The Delta Function

In addition to Green's identities, we also need to discuss the delta function, $\delta(x)$. We have seen this function before (recall $\mathscr{F}[1]$). Looking at the function in depth, we have the following definition:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty & x = 0 \end{cases}$$

We also have the following properties of the delta function:

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a) \, dx = f(a)$$

The second property, called the sifting property, is of particular interest, as we will see soon. The delta function can also be extended to the 2D case:

$$\delta(x,y) = \begin{cases} 0, & (x,y) \neq 0 \\ \infty, & (x,y) = 0 \end{cases}$$

The sifting property from the 1D case becomes

$$\iint f(x,y)\delta(x-a, y-b)\, dA = f(a,b)$$

where A is a surface over which we are integrating.

9.1.4 Green's Functions

Finally, we need to discuss the notion of a Green's Function. A Green Function is formally defined as

$$G(x,s) = \mathcal{L}^{-1}\delta(x-s)$$

For some linear differential operator $\mathcal{L}$. If we take a function $f(x)$ and multiply and integrate, we get the following identity:

$$u(x) = \int G(x,s)f(s)\, ds$$

Green's Functions can be used to solve many different types of equations; For any general ODE, we require two constraints to be fulfilled:

1. A solution to the ODE exists, and
2. the solutions are linearly independant.

Recall that we can check for linear independance of solution via the Wronskian, $\mathcal{W}$. We can derive a Green's Function for a differential equation by

$$G(x,s) = \begin{cases} \frac{y_1(s)y_2(x)}{\mathcal{W}(y_1,y_2)(s)}, & a \leq s \leq x \leq b \\ \frac{y_1(x)y_2(s)}{\mathcal{W}(y_1,y_2)(s)}, & a \leq x \leq s \leq b \end{cases}$$

which yields the solution

$$y(x) = \int_a^b G(x,s)f(s)\, ds$$

Let's consider an example.

Ex: Solve the following Boundary Value Problem:

$$\begin{cases} y''(x) = x^2 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

Solution: To solve this problem, we need to first find the Green Function associated with the problem. Let's consider the homogeneous ODE $y'' = 0$. This has the solution

$$y(x) = c_1 + c_2 x$$

We can take any arbitrary c as long as our choice does not yield a trivial solution and it satisfies the boundary conditions. Let's consider

$$y_1(x) = x \qquad \text{and} \qquad y_2(x) = 1 - x$$

We can see that this choice satisfies the boundary conditions, and that

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

which implies that the choice we made is linearly independant. We can now derive a Green's function for our ODE as follows:

$$\begin{aligned} G(x,s) &= \begin{cases} \frac{s(1-x)}{-1} \\ \frac{x(1-s)}{-1} \end{cases} \\ &= \begin{cases} x-s \\ s-x \end{cases} \end{aligned}$$

We are bounded by $[0,1]$ in this case, so we really have

$$G(x,s) = \begin{cases} x-s, & 0 \le s \le x \\ s-x, & x \le s \le 1 \end{cases}$$

We said earlier that the solution $y(x)$ can be found by

$$\begin{aligned} y(x) &= \int_a^b G(x,s) f(s)\, ds \\ &= \int_0^x (x-s)s^2\, ds + \int_x^1 (s-x)(s^2)\, ds \\ &= \frac{1}{12}(x^x - x) \end{aligned}$$

And we are done.

Green's Functions are useful for more than ODE's. We want to extend the idea to PDE's, which can be doe simply enough. Before we continue, we will mention Poisson's Equation, which is a generalized form of Laplace's Equation:

$$\nabla^2\psi = -4\pi\rho$$

If we take $\rho = 0$, we get back Laplace's Equation. Let's consider the following example:

Ex: Solve the Poisson equation

$$\begin{cases} \nabla^2 u = -f(x,y) & \text{on } \Omega = \{(x,y) | 0 < x < \pi, 0 < y < \pi\} \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Solution: We know the formal solution to this PDE:

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y) \sin{(nx)}$$