Formal Solutions To PDE's

1 Homogeneous Solutions

1.1 The Heat Equation

Dirichlet

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

Robin

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)^2 t} \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)dx$$

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)^2 t} \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}x\right)dx$$

1.2 The Wave Equation

Dirichlet

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{a_0}{2} + \frac{b_0}{2}t + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \cos\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad b_0 = \frac{2}{n\pi c}\int_0^\ell g(x)\,dx$$

Robin

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} \left[a_n \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right) + b_n \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{\ell}\right)\right] \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{\ell}\right)dx \qquad b_0 = \frac{2}{n\pi c}\int_0^\ell g(x)\,dx$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} \left[a_n \cos\left(\left[n + \frac{1}{2} \right] \frac{\pi t}{\ell} \right) + b_n \sin\left(\left[n + \frac{1}{2} \right] \frac{\pi t}{\ell} \right) \right] \cos\left(\left[n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\left[n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right) dx \qquad a_0 = \frac{2}{\ell} \int_0^{\ell} f(x)\, dx$$

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \cos\left(\left[n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right) dx \qquad b_0 = \frac{2}{n\pi c} \int_0^{\ell} g(x)\, dx$$

1.3 Laplace Equation

Rectangular Region

Dirichlet

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\ell,y) = 0 \end{cases}$$

$$u(x,y) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\frac{n\pi y}{\ell} \right) + a_n \cosh\left(\frac{n\pi y}{\ell} \right) \right] \sin\left(\frac{n\pi x}{\ell} \right)$$

Neumann

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0,y) = u_x(\ell,y) = 0 \end{cases}$$

$$u(x,y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \left[b_n \sinh\left(\frac{n\pi y}{\ell} \right) + a_n \cosh\left(\frac{n\pi y}{\ell} \right) \right] \cos\left(\frac{n\pi x}{\ell} \right)$$

Robin

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u_x(\ell,y) = 0 \end{cases}$$

$$u(x,y) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\left[n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) + a_n \cosh\left(\left[n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) \right] \sin\left(\left[n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0,y) = u(\ell,y) = 0 \end{cases}$$

$$u(x,y) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\left[n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) + a_n \cosh\left(\left[n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) \right] \cos\left(\left[n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

Circular Region

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(h, \theta) = f(\theta) \end{cases}$$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{r^n}{a^n} \cos(n\theta) + b_n \frac{r^n}{a^n} \sin(n\theta)$$

$$a^n a_n = \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$a^n b_n = \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta$$

2 Non-Homogeneous Solutions

2.1 The Heat Equation

Dirichlet

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[b_n^{'}(t) + k\left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

Neumann

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{a_0'(t)}{2} + \sum_{n=1}^{\infty} \left[a_n'(t) + k\left(\frac{n\pi}{\ell}\right)^2 a_n\right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

2.2 The Wave Equation

Dirichlet

$$\begin{cases} u_{tt} - c^2u_{xx} = h(x,t) \\ u(0,t) = u(\ell,t) = 0 \\ u(x,t) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

Neumann

$$\begin{cases} u_{tt} - c^2u_{xx} = h(x,t) \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,t) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{b_0(t)}{2} + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{b_0''(t)}{2} + \sum_{n=1}^{\infty} \left[b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t)\right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x,t)$$

2.3 Non-Homogeneous Boundaries

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = L(t),\ u(\ell,t) = R(t) \\ u(x,0) = f(x) \end{cases}$$

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(0,t) = L(t),\ u(\ell,t) = R(t) \\ u(x,0) = f(x) \end{cases}$$

$$\delta(x,t) = L + \frac{x}{\ell}(R - L)$$

$$\gamma(x,t) = u(x,t) - \delta(x,t)$$

3 Cauchy Problems

3.1 The Heat Equation

Homogeneous

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)\, dy$$

Non-Homogeneous

$$\begin{cases} u_t - ku_{xx} = h(x,t) \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} h(y,s)\, dy\, ds$$

3.2 The Wave Equation

Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

Non-Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy\, ds$$