1 Mathematical Models

1. A bank charges 18% annual interest on credit card balances. What is the value of an original balance of \$1,000 after 1 year if the bank computes new balances

   (a) once a year?

   We can use the equation $B_n = (1+r)B_{n-1}$ to calculate the balance. The original balance was \$1000, so after 1 year, we have $B_1 = (1+0.18)(\$1000)$, so $B_1 = \$1180$.

   (b) twice a year?

   Similarly, we can solve the problem using the same equation as above. We know that after the first 6 months, the balance is \$1180. If we calculate again, we find that the new balance is $B_2 = \$1392.40$.

   (c) every three months?

   After the first 8 months, the balance is \$1392.40. Calculating again yields $B_3 = \$1643.03$.

   (d) every week?

   In this case, it helps to come up with a more generalized model, since there are 52 weeks in a year. We can abstract the model into $B_n = (1+r)^n B_0$, so we have that $B_{52} = (1+0.18)^{52}(\$1000)$, so $B_{52} = \$5468451.69$.

   (e) every day?

   We can again use our abstracted model, so we have $B_{365} = (1+0.18)^{365}(\$1000)$, so $B_{365} = \$1.73 \times 10^{29}$.

2 Stable and Unstable Arms Races

1. Determine the outcome of an arms race governed by the Richardson model:

$$\begin{cases} \frac{dx}{dt} = 10y - 14x - 12 \\ \frac{dy}{dt} = 8x - 4y - 24 \end{cases}$$

when the initial level is

   (a) (4,4)

   (b) (13,6)

   We will begin by finding a general solution to the provided system of ODE's. As it stands now, we need to find a substitution that will allow us to solve the system. We can do so by finding the stable point of the system.

$$\left[\begin{array}{cc|c} -14 & 10 & 12 \\ 8 & -4 & 24 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & 18 \end{array}\right].$$

   We can make the substitution of $\alpha = x - 12$ and $\gamma = y - 18$. If we substitute in terms of α and γ we get the system

$$\begin{cases} \alpha' = 10\gamma - 14\alpha \\ \gamma' = 8\alpha - 12\gamma \end{cases}.$$

   We can solve this. Our A matrix is

$$\begin{bmatrix} -14 & 10 \\ 8 & -12 \end{bmatrix} \Longrightarrow \begin{bmatrix} -14-\lambda & 10 \\ 8 & -12-\lambda \end{bmatrix}.$$

   Our eigenvalues can be found by

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda^2 + 26\lambda + 88 = 0$$
$$\lambda = -22 \qquad \lambda = -4.$$

Our first eigenvector is given by

$$\begin{bmatrix} -14+22 & 10 \\ 8 & -12+22 \end{bmatrix} \to \begin{bmatrix} 8 & 10 \\ 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{5}{4} \\ 0 & 0 \end{bmatrix} \implies \vec{\lambda}_1 = \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix}.$$

We can find the second eigenvector in a similar fashion:

$$\vec{\lambda}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We now have solutions in terms of α and γ.

$$\begin{cases} \alpha = c_1 e^{-22t} + c_2 e^{-4t} \\ \gamma = -\frac{4}{5} c_1 e^{-22t} + c_2 e^{-4t} \end{cases}.$$

We can back substitute, and we find that $x = \alpha + 12$ and $y = \gamma + 18$, so

$$\begin{cases} x = c_1 e^{-22t} + c_2 e^{-4t} + 12 \\ y = -\frac{4}{5} c_1 e^{-22t} + c_2 e^{-4t} + 18 \end{cases}.$$

Finally, we can apply our initial conditions, as specified by the problem. With the initial vector $(4, 4)$, we have that $c_1 = 10/3$ and $c_2 = -34/3$.

3 Single Species Ecological Models

1. The rate of growth of a certain population of bacteria in a culture is directly proportional to the size of the population. If an experiment begins with 1,000 bacteria and one hour later the count is 1,500 bacteria, then how many bacteria are present at the end of 24 hours?

   We can model population growth with the equation $P = P_0 e^{rt}$. To find the rate, we can solve for r, which gives us

   $$r = \frac{\log \frac{P}{P_0}}{t}.$$

   Using the information provided, we have that

   $$r = \frac{\log \frac{1500}{1000}}{1} = 0.4054.$$

   After 24 hours, we will have

   $$P = 1000 e^{(0.4054)(24)} = 16834112.19.$$

2. Suppose that 20 years ago the population of a town was 2,000, and that the population increased continuously at a rate proportional to the existing population. If the population of the town is now 6,000, what has been the rate of growth?

   If we take 20 years ago to be $t = 0$, then we have $P_0 = 2000$. We can similarly find the rate as above. Doing so yields $r = 0.0549$.

3. If the population of a country is undergoing exponential growth at a rate of r percent per year, show that the population doubles every $(\log 2)/r$ years. This number is called the "doubling time." Compute the doubling time if r = 2.

   We know that population growth can be modeled by the equation $P = P_0 e^{rt}$. We can divide and take the natural log of both sides to get $\log(P/P_0) = rt$. P is the current population, so if it double the original, $P = 2P_0$. This yields $\log(2) = rt$, or $(\log(2))/r = t$. When $r = 2$, we have $t \approx 0.34$.

Assume that the U.S. population has grown exponentially. Estimate the growth rate using each of the following years in place of the year 1830 as done in the text.

(a) 1800

From the text, we have that the population of the United States can be modeled by the equation

$$P(t) = 3.929e^{0.029643(t-1790)}.$$

For the year 1800, we have that $P(t) = 5.28$ million.

(b) 1850

$$P(t) = 23.25 \text{ million.}$$

(c) 1900

$$P(t) = 102.32 \text{ million.}$$

(d) 1970

$$P(t) = 814.44 \text{ million.}$$