# The Rössler System

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In 1976, Otto Rössler proposed a system of nonlinear ordinary differential equations that illustrated the simplest possible strange attractor. An attractor is a set of values towards which a system moves when initial conditions are *near* the attractor; to call an attractor *strange* means that the attractor exhibits fractal behavior. Often, strange attractors are associated with chaotic systems. Rössler's strange attractor is a chaotic attractor that solves his proposed system

$$f(x,y,z) = \dot{x} = -y - z \tag{1}$$

$$g(x,y,z) = \dot{y} = x + ay \tag{2}$$

$$h(x,y,z) = \dot{z} = b + z(x - c) \tag{3}$$

where $a, b$, and $c$ are arbitrary values. Rössler studied the effects that small (i.e. less than 1) $a$ and $b$ paired with a relatively large $c$ had on the system's chaotic behavior. One such plot is shown in figure 1.

We first want to analyze the stability of the critical points of the system. The Jacobian for this system can be found by constructing a matrix of partial derivatives for $f, g$, and $h$.

$$J(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}.$$

We now need to find the critical points of the system. We can accomplish this by solving $f = 0, g = 0$, and $h = 0$. Through the use of Mathematica's `Solve` command, we get

```
Solve[f[x, y, z] == 0 && g[x, y, z] == 0 && h[x, y, z] == 0, {x, y, z}]
```

$$\left[x = \frac{1}{2}\left(c - \sqrt{c^2 - 4ab}\right), y = \frac{1}{2}\left(\frac{\sqrt{c^2 - 4ab}}{a} - \frac{c}{a}\right), z = \frac{c - \sqrt{c^2 - 4ab}}{2a}\right] \tag{4}$$

$$\left[x = \frac{1}{2}\left(c + \sqrt{c^2 - 4ab}\right), y = \frac{1}{2}\left(-\frac{\sqrt{c^2 - 4ab}}{a} - \frac{c}{a}\right), z = \frac{c + \sqrt{c^2 - 4ab}}{2a}\right]. \tag{5}$$

Rössler's analysis focused on the behavior of the system for small $a$ and $b$. For simplicity, we can consider the solutions to (4) and (5) for very small $a$ and $b$. We now have

$$[x = 0, y = 0, z = 0] \tag{6}$$

$$\left[x = c, y = -\frac{c}{a}, z = \frac{c}{a}\right]. \tag{7}$$

To find the behavior around the critical points given by (6), we evaluate the Jacobian at the appropriate points and find the eigenvalues of the system.

$$|J(0,0,0) - \lambda I| = \left|\begin{bmatrix} -\lambda & -1 & -1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -c-\lambda \end{bmatrix}\right| = (c-\lambda)(\lambda-1)(\lambda+1) = 0.$$

Hence, the system has three eigenvalues: $\lambda = 1, \lambda = -1$, and $\lambda = c$ (we will assume here that $c > 1$). This implies that the system has an unstable saddle at the origin [2]. Figure 4 shows various cuts of the system for various $z$ values; the system spirals in the $x - y$ plane, while moving towards and then away from the origin as $z$ moves from $-\infty$ to $\infty$. This represents only one of the equilibrium points of the system, however. We now need to analyze the linearized system near (7).

$$\left|J\left(c, -\frac{c}{a}, \frac{c}{a}\right) - \lambda I\right| = \left|\begin{bmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ \frac{c}{a} & 0 & -\lambda \end{bmatrix}\right| = \frac{ac - a\lambda - c\lambda + a^2\lambda^2 - a\lambda^3}{a}.$$

Setting the determinant to 0 and solving in Mathematica yields three distinct (and incredibly verbose) eigenvalues with $\lambda_1 \in \mathbb{R}$ the other two complex conjugates of each other. Such behavior lends itself towards what is known as a *saddle focus*, which is always unstable [2]. Figure 2 displays the behavior of both equilibrium points. The spiral behavior can be seen in the $x - y$ plane, and the saddle behavior can be seen predominantly in the $z - x$ plane.

We now know that both equilibrium points are unstable when we ignore $a$ and $b$ in comparison to $c$. In fact, it turns out that this stability is preserved even when we reintroduce $a$ and $b$ back into the system. Figure 3 shows us what one of the raw eigenvalues looks like, according to Mathematica. The three eigenvalues resulting from the unsimplified system still exhibit the saddle focus behavior that we saw when we allowed $a$ and $b$ to drift towards zero. We now want to turn our attention towards the actions of the system when we change the values for $a$, $b$, and $c$.

Attractors are often associated with chaotic systems; as such, we will now examine various bifurcations on the Rössler system. We have three variables that contribute to the chaos of the system: $a$, $b$, and $c$. Rössler analyzed the system for relatively small $a$ and $b$, and we analyzed the stability of critical points when we neglected $a$ and $b$. We will first analyze the effect that small changes to $a$ have on the system.

The $a$ parameter affects our $g(x, y, z)$. For simplicity, we notice that $g$ is only dependent on $x$ and $y$, so we will let $g(x, y, z) = \gamma(x, y)$.

# Plots and Graphics

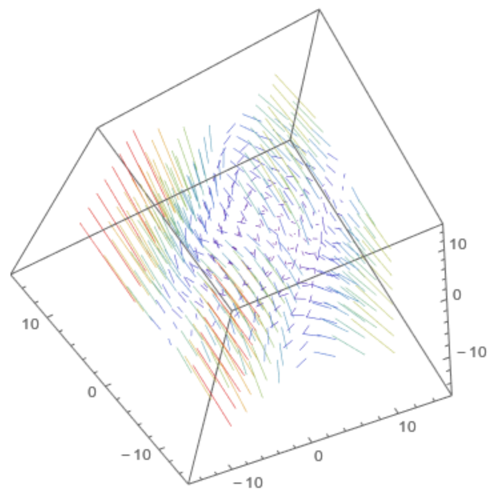


Figure 1: The Rössler system with $a = 0.2$, $b = 0.1$, and $c = 2.3$

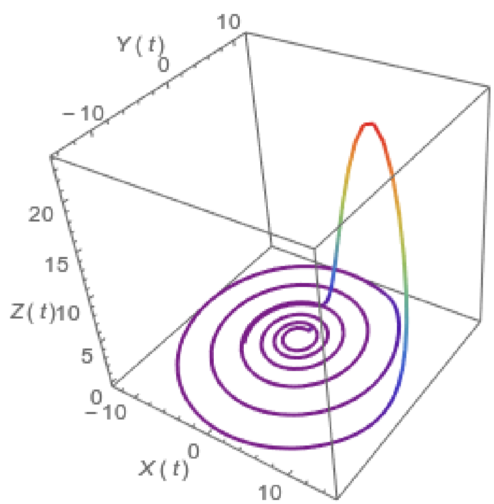


Figure 2: The Rössler attractor with $a = 0.14$, $b = 0.2$, and $c = 8.8$, courtesy of [4]

$$
\begin{gathered}
\{\lambda \to -(1/(6\times 2 \wedge (1/3)\ a)) \\
((-16\,a^6 - 12\,a^5\sqrt{(c^2 - 4\,a\,b)} - 24\,a^5\,b - 12\,a^5\,c - 8\,a^4\,b\sqrt{(c^2 - 4\,a\,b)} + 12\,a^4\,c\sqrt{(c^2 - 4\,a\,b)} - \\
24\,a^4\,b\,c + 12\,a^4\,c^2 + 72\,a^4 + 8\,a^3\,c^2\sqrt{(c^2 - 4\,a\,b)} + 144\,a^3\sqrt{(c^2 - 4\,a\,b)} - 72\,a^3\,b + 8\,a^3\,c^3 + \\
\sqrt{(16\,a^6(4\,a^3 + 3\,a^2(\sqrt{(c^2 - 4\,a\,b)} + 2\,b + c) + a(2\,b(\sqrt{(c^2 - 4\,a\,b)} + 3\,c) - 3(c\sqrt{(c^2 - 4\,a\,b)} +} \\
c^2 + 6)) - 2(c^2\sqrt{(c^2 - 4\,a\,b)} + 18\sqrt{(c^2 - 4\,a\,b)} - 9\,b + c^3))^2 - 4\,a^3 \\
(6(a^2(\sqrt{(c^2 - 4\,a\,b)} + c) + \sqrt{(c^2 - 4\,a\,b)} - 2\,a - c) + a(\sqrt{(c^2 - 4\,a\,b)} - 2\,a + c)^2)^3)) \wedge (1/3)) + \\
(-6(a^2(\sqrt{(c^2 - 4\,a\,b)} + c) + \sqrt{(c^2 - 4\,a\,b)} - 2\,a - c) - a(\sqrt{(c^2 - 4\,a\,b)} - 2\,a + c)^2)/ \\
(3\times 2^{2/3}(-16\,a^6 - 12\,a^5\sqrt{(c^2 - 4\,a\,b)} - 24\,a^5\,b - 12\,a^5\,c - 8\,a^4\,b\sqrt{(c^2 - 4\,a\,b)} + 12\,a^4\,c\sqrt{(c^2 - 4\,a\,b)} - \\
24\,a^4\,b\,c + 12\,a^4\,c^2 + 72\,a^4 + 8\,a^3\,c^2\sqrt{(c^2 - 4\,a\,b)} + 144\,a^3\sqrt{(c^2 - 4\,a\,b)} - 72\,a^3\,b + 8\,a^3\,c^3 + \\
\sqrt{(16\,a^6(4\,a^3 + 3\,a^2(\sqrt{(c^2 - 4\,a\,b)} + 2\,b + c) + a(2\,b(\sqrt{(c^2 - 4\,a\,b)} + 3\,c) - 3(c\sqrt{(c^2 - 4\,a\,b)} +} \\
c^2 + 6)) - 2(c^2\sqrt{(c^2 - 4\,a\,b)} + 18\sqrt{(c^2 - 4\,a\,b)} - 9\,b + c^3))^2 - \\
4\,a^3(6(a^2(\sqrt{(c^2 - 4\,a\,b)} + c) + \sqrt{(c^2 - 4\,a\,b)} - 2\,a - c) + a(\sqrt{(c^2 - 4\,a\,b)} - 2\,a + c)^2)^3)) \wedge \\
(1/3)) + (1/6)(-\sqrt{(c^2 - 4\,a\,b)} + 2\,a - c)\}
\end{gathered}
$$

$$
\begin{gathered}
\{l \to -4.98425\} \\
\{l \to 0.0961307\ - 0.995341i\} \\
\{l \to 0.0961307\ + 0.995341i\}
\end{gathered}
$$

Figure 3: Example of one raw eigenvalue and its corresponding evaluation at $a = 0.2$, $b = 0.2$, and $c = 0.2$.

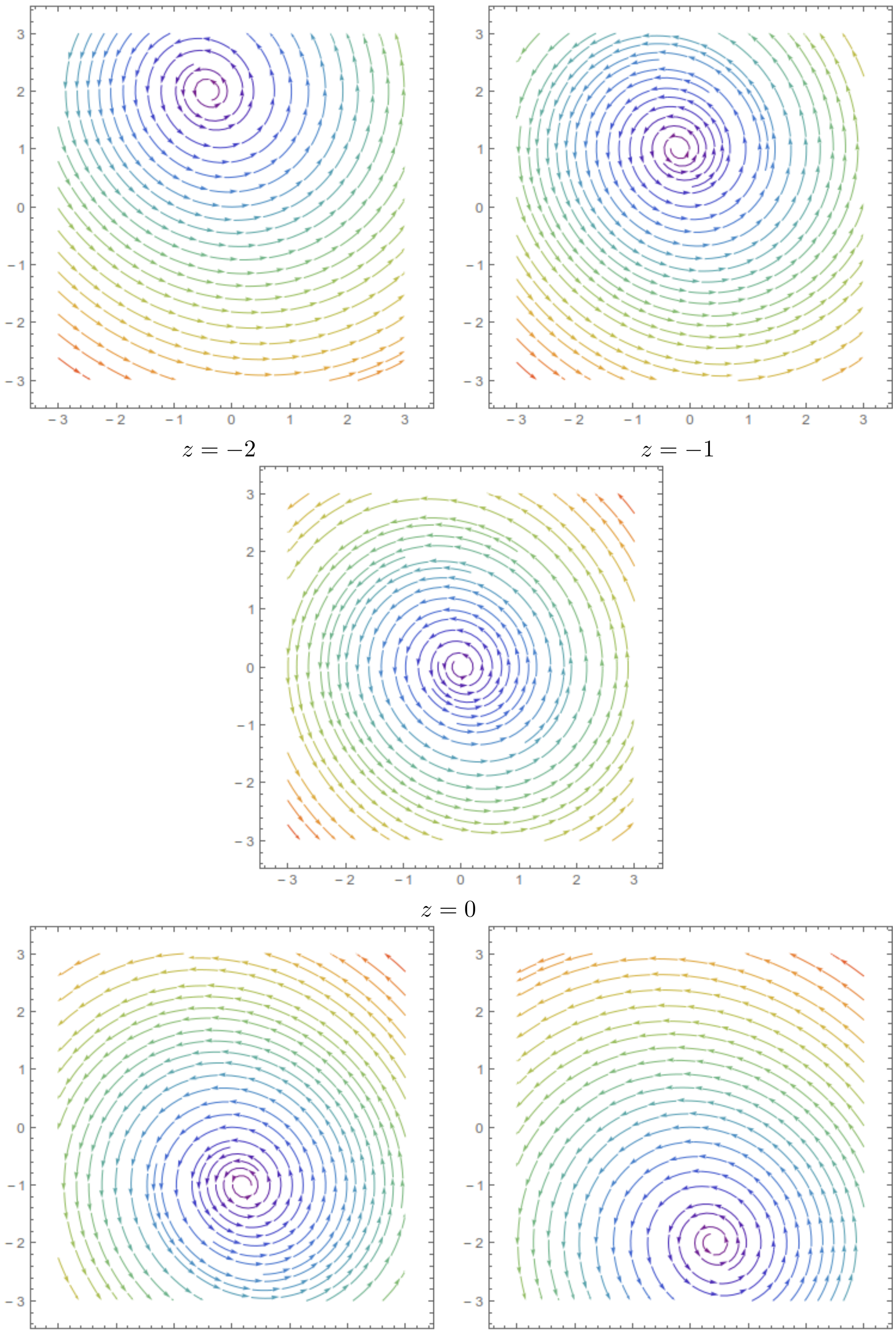


Figure 4: Several cuts of the stream plot given by Mathematica.

## References


[1] Housam Binous. *Bifurcation Diagram for the Rssler Attractor*. URL: http://demonstrations.wolfram.com/BifurcationDiagramForTheRoesslerAttractor/.

[2] Eugene M. Izhikevich. *Equilibrium*. 2007. URL: http://www.scholarpedia.org/article/Equilibrium#Three-Dimensional_Space.

[3] Christophe Letellier and Otto E. Rossler. *Rossler Attractor*. 2007. URL: http://www.scholarpedia.org/article/Rossler_attractor#Bifurcation_diagram.

[4] Daniel de Souza Carvalho and Eric W. Weisstein. *The Rossler Attractor*. URL: http://demonstrations.wolfram.com/TheRosslerAttractor/.