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Homework 5

1. Let $f(x,y) = x^2 + y^2$. Observe that $(x,y) = (0,0)$ defines a solution to the equation $f(x,y) = 0$. For the remainder of this problem, we shall call $(0,0)$ the trivial solution to $f(x,y) = 0$.

   (a) Prove that $f(x,y) = 0$ has no non-trivial solutions with $x, y \in \mathbb{Q}$.

   Suppose that $f(x,y)$ has a non-trivial zero in $\mathbb{Q}$. Then $x^2 + y^2 = 0$ for some pair $(x,y) \neq (0,0)$. For any choice of $x \neq 0$ or $y \neq 0$, we know that x^2 and y^2 are both positive. The sum of two positive numbers is still positive, so $x^2 + y^2 > 0$ for $x \neq 0$ and $y \neq 0$. Therefore there cannot be any other zeros besides $(0,0)$.

   (b) Prove that there exists a prime p such that $f(x,y) = 0$ has a nontrivial solution with $x, y \in \mathbb{Q}_p$.

   Let $p = 2$ and $y = 1$. We want to show by Hensel's Lemma that $f(x,1) = g(x) = x^2 + 1$ has a solution near $x = 1$. We can see that the first property of Hensel's Lemma is satisfied since

   $$g(1) = 1^2 + 1 = 2 \equiv 0 \mod 2.$$

   Similarly, the second property is satisfied, since

   $$g'(1) = 1 \not\equiv 0 \mod 2.$$

   Hence, there is a non-trivial zero of f at $(x,y) = (1,1)$.

   (c) Verify that f satisfies the Hasse Principle without referring to the Hasse-Minkowski Theorem.

   We have shown that f has has no non-trivial solutions in $\mathbb{Q}$ and that f has a non-trivial solution in $\mathbb{Q}_2$, so it satisfies the negation of the Hasse Principle.

2. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of points in $\mathbb{Z}_p$. Prove that the series

   $$\sum_{n=0}^{\infty} a_n p^n$$

   converges in $\mathbb{Q}_p$.

   Proof: We know that a series will converge if and only if its sequence of partial sums converges. We can test for convergence by calculating

   $$\lim_{n\to\infty} \left|a_{n+1}p^{n+1} - a_n p^n\right|_p.$$

   We can say

   $$\begin{aligned}\left|a_{n+1}p^{n+1} - a_n p^n\right|_p &\leq \max\left\{\left|a_{n+1}p^{n+1}\right|_p, \left|a_n p^n\right|_p\right\}\\ &= \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\}.\\ \lim_{n\to\infty} \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\} &= \max\{0,0\}\\ &= 0.\end{aligned}$$

   Since the p-adic absolute value is greater than or equal to 0, we know

   $$\lim_{n\to\infty} \left|a_{n+1}p^{n+1} - a_n p^n\right|_p = 0.$$

   Therefore, the sequence of partial sums is Cauchy, and thus convergent, so the series converges.

3. Suppose that $k \in \mathbb{N}$.

(a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: If we can show that the sequence of partial sums does not converge, then the series will not converge. We want to evaluate (and eventually take the limit of)

$$\left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p .$$

We know by definition that

$$\begin{aligned}\left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p &= \left| \frac{n^k - (n+1)^k}{n^k (n+1)^k} \right|_p \\ &= p^{-\left[v_p(n^k-(n+1)^k) - v_p(n^k) - v_p((n+1)^k)\right]}.\end{aligned}$$

Through some tedious calculation, we can solve for the valuation of each term in the exponent.

$$\begin{aligned} v_p\left(n^k - (n+1)^k\right) &= \frac{\log\left(n^k - (n+1)^k\right) - \log(m_0)}{\log(p)} \\ v_p(n^k) &= \frac{k\log(n) - \log(m_1)}{\log(p)} \\ v_p((n+1)^k) &= \frac{k\log(n+1) - \log(m_2)}{\log(p)}, \end{aligned}$$

where m_0, m_1, and m_2 do not divide p. Combining these valuations as required yields

$$-\left[v_p(n^k - (n+1)^k) - v_p(n^k) - v_p((n+1)^k)\right] = \frac{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)}{\log p} + c$$

where c is some constant number independant of n. We can now evaluate the real valued limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)}{\log p} + c = c + \frac{1}{\log p} \lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right).$$

The function on the inside of the limit is continuous for all $k \in \mathbb{N}$, so we can determine the limit as follows:

$$\begin{aligned} \lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right) &= L \\ e^{\lim_{n\to\infty} \log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)} &= e^L \\ \lim_{n\to\infty} e^{\log\left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right)} &= e^L \\ \lim_{n\to\infty} \left(\frac{n^k(n+1)^k}{n^k-(n+1)^k}\right) &= e^L. \end{aligned}$$

For the fraction inside the limit, notice that after multiplication, the highest order term in the numerator will be n^{2k}, while the highest order term in the denominator will be n^k, so the limit behaves like

$$\lim_{n\to\infty} \frac{n^{2k}}{n^k} = \lim_{n\to\infty} n^k = \infty = e^L.$$

Taking the logarithm of both sides changes nothing, nor does dividing by $\log p$ where $p \neq \infty$, or adding c. Hence, we have that

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)^k} - \frac{1}{n^k} \right|_p = \infty,$$

so the sequence of partial sums does **not** converge, and therefore the series does not converge.

(b) Prove that the series

$$\sum_{n=1}^{\infty} n^k$$

does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

Proof: We want to show that the individual terms in the series do not tend towards 0 p-adically. Consider all the elements of $\mathbb{Z}/p\mathbb{Z}$. We know that all multiples of p will live in $\overline{0}$ and as such will have a p-adic absolute value less than 1. All other elements will have a p-adic absolute value of 1, since they contain a factor of p^0. Hence, the sum tends towards infinity, so the series does not converge.