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Homework 2

1. Let $\vec{a_1} = (1, 1)$ and $\vec{a_2} = (1, -1)$.

   (a) Write the vector $\vec{b_1} = (3, 1)$ as $c_1 a_1 + c_2 a_2$, where c_1 and c_2 are appropriate scalars.

$$(3, 1) = (c_1 + c_2, c_1 - c_2)$$
$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$
$$\vec{b} = 2(1, 1) + 1(1, -1)$$

   (b) Do the same for $(3, -5)$.

$$(3, 1) = (c_1 + c_2, c_1 - c_2)$$
$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right]$$
$$\vec{b} = -1(1, 1) + 4(1, -1)$$

   (c) Show that any vector $\vec{b} = (b1, b2)$ in $\mathbb{R}^2$ may be written in the form $c_1 a_1 + c_2 a_2$ for appropriate choices of the scalars c_1, c_2.

   Assume that we can express any vector (x_1, x_2) as $(c_1 + c_2) + (c_1 - c_2)$, where $x_1, x_2 \in \mathbb{R}$. Then the solution for the corresponding matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{x_1+x_2}{2} \\ 0 & 1 & \frac{x_1-x_2}{2} \end{array}\right].$$

   Since $\frac{x_1+x_2}{2}$ and $\frac{x_1-x_2}{2}$ are both in $\mathbb{R}$, any vector of the form (x_1, x_2) can be written as a linear combination of $\vec{a_1}$ and $\vec{a_2}$.

2. Write the following as a set of parametric equations: The line in $\mathbb{R}^3$ through the point $(2, 1, 5)$ that is parallel to the vector $i + 3j - 6k$.

$$\begin{cases} x = t + 2 \\ y = 3t - 1 \\ z = 5 - 6t \end{cases}$$

3. Write the following as a set of parametric equations: The line in $\mathbb{R}^3$ through the points $(1, 4, 5)$ and $(2, 4, 1)$.

$$\begin{cases} x = t + 1 \\ y = 4 \\ z = 5 - 6t \end{cases}$$

4. Write the following as a set of parametric equations: Write a set of parametric equations for the line in $\mathbb{R}^4$ through the point $(1, 2, 0, 4)$ and parallel to the vector $(2, 5, 3, 7)$.

$$\begin{cases} x_1 = 1 - 2t \\ x_2 = 5t + 2 \\ x_3 = 3t \\ x_4 = 7t + 4 \end{cases}$$

5. Give a symmetric form for the line having parametric equations $x = t + 7$, $y = 3t - 9$, $z = 6 - 8t$.

$$\begin{cases} x = t + 7 \\ y = 3t - 9 \\ z = 6 - 8t \end{cases} \to \begin{cases} t = x - 7 \\ t = \frac{y}{3} + 3 \\ t = \frac{6-z}{8} \end{cases}$$

6. Give a set of parametric equations for the line with symmetric form

$$\frac{x+5}{3} = \frac{y-1}{7} = \frac{z+10}{-2}.$$

$$\begin{cases} t = \frac{x+5}{3} \\ t = \frac{y-1}{7} \\ t = \frac{z+10}{-2} \end{cases} \rightarrow \begin{cases} x = 3t - 5 \\ y = 7t + 1 \\ z = -2t - 10 \end{cases}$$

7. Show that the two sets of equations

$$\frac{x-2}{3} = \frac{y-1}{7} = \frac{z}{5} \text{ and } \frac{x+1}{-6} = \frac{y+6}{-14} = \frac{z+5}{10}$$

represent the same line in $\mathbb{R}^3$.

If we move to parametric coordinates, we have

$$\begin{cases} x = 3t + 2 \\ y = 7t + 1 \\ z = 5t \end{cases} \text{ and } \begin{cases} x = -6t - 1 \\ y = -14t - 6 \\ z = -10t - 5 \end{cases}.$$

If we take the second vector and scale it by $-\frac{1}{2}$, we get

$$\begin{cases} x = 3t + \frac{1}{2} \\ y = 7t + 3 \\ z = 5t = \frac{5}{2} \end{cases}.$$

We can now replace t with an arbitrary linear combination of t that we will call s. We can now rewrite the vector in terms of s as such:

$$\begin{cases} x = 3s + 2 \\ y = 7s + 1 \\ z = 5s \end{cases}.$$

This vector lies along the same line described by the original equation.

8. Do the parametric equations $x = 5t^2 - 1, y = 2t^2 + 3, z = 1t^2$ determine a line? Explain?

No, the given parametric equations do not form a line. If we represent the equations in symmetric form, they will have nonlinear terms.

9. Find the points of intersection of the line $x = 2t3, y = 3t + 2, z = 5 - t$ with each of the coordinate planes $x = 0, y = 0$, and $z = 0$.

The line intersects the plane at $(\frac{3}{2}, -\frac{2}{3}, 5)$.

10. Does the line $x = 5 - t, y = 2t - 3, z = 7t + 1$ intersect the plane $x - 3y + z = 1$? Why?

Substituting in our values for x, y, z into the equation of the plane yields

$$(5 - t) - 3(2t - 3) + (7t + 1) = 2.$$

Solving for t gives us

$$-3 = 2,$$

which is impossible. Therefore, the line does not intersect the plane.

11. Find the point of intersection of the two lines $l_1 : x = 2t+3, y = 3t+3, z = 2t+1$ and $l_2 : x = 15-7t, y = t - 2, z = 3t - 7$.

To find the point of intersection, we can set the values in each equation equal to each other.

$$\begin{cases} 2s + 3 = 15 - 7t \\ 3s + 3 = t - 2 \\ 2s + 1 = 3t - 7 \end{cases}.$$

We only have two unknowns, so we only need two equations:

$$\begin{cases} 2s + 7t & = 12 \\ 2s - 3t & = 8 \end{cases}.$$

Solving the system yields $t = 2$. Therefore the two lines intersect at $t = 2$.