The Rössler System

Steven Rosendahl

Introduction

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- Strange attractors are often associated with chaotic systems.

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\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
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  - When does chaotic behavior occur?

First Glance

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Solve[f[x, y, z] == 0 && g[x, y, z] == 0 && h[x, y, z] == 0, {x, y, z}]
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$$\begin{cases} x = \frac{1}{2}\left(c - \sqrt{c^2 - 4ab}\right) \\ y = \frac{1}{2}\left(\frac{\sqrt{c^2-4ab}}{a} - \frac{c}{a}\right) \\ z = \frac{c-\sqrt{c^2-4ab}}{2a} \end{cases} \qquad \begin{cases} x = \frac{1}{2}\left(c + \sqrt{c^2 - 4ab}\right) \\ y = \frac{1}{2}\left(-\frac{\sqrt{c^2-4ab}}{a} - \frac{c}{a}\right) \\ z = \frac{c+\sqrt{c^2-4ab}}{2a} \end{cases}$$

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- Much easier to analyze stability of system

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- Other stability exists, but are often named after the bifurcations that produce them

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$$|J(0,0,0)-\lambda I| = \left| \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -c-\lambda \end{bmatrix} \right| = (c-\lambda)(\lambda+1)^2 = 0.$$

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If two of the eigenvalues of the system are pure imaginary, then the system has a center at the fixed point. This is **not** a hyperbolic equilibrium.

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- $\therefore$ $(0,0,0)$ is a center of the linear system

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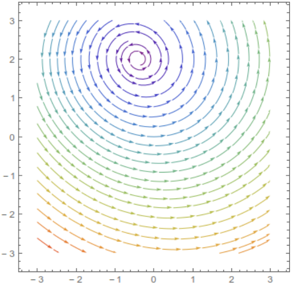
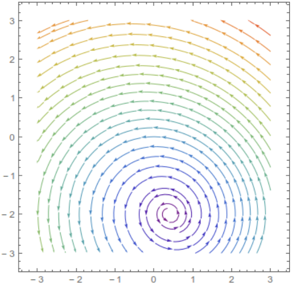


$z = -2$



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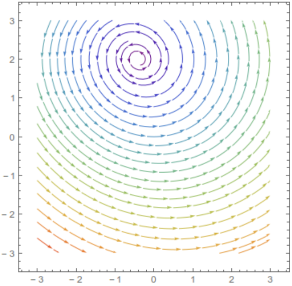

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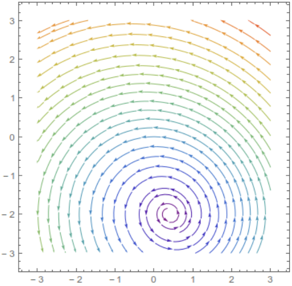

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- Actually appears to be a spiral

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- $\therefore$ fixed point is an unstable focus node

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- Mathematica demonstration

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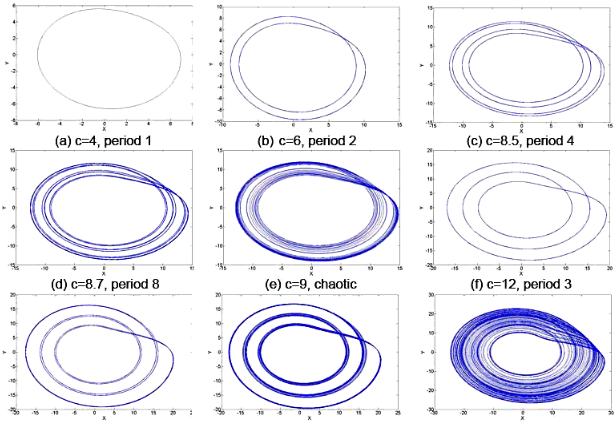

(a) c=4, period 1 (b) c=6, period 2 (c) c=8.5, period 4

(d) c=8.7, period 8 (e) c=9, chaotic (f) c=12, period 3

(g) c=12.6, period 6 (h) c=13, chaotic (i) c=18, chaotic

References

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