# Problems Involving Legendre Polynomials

1. Show $\int_{-1}^{1} x^2 p_l(x)\, dx = 0$ for $l \geq 3$.

We know that the general form for a Legendre polynomial $p_l(x)$ is given by Rodrigues' Formula to be

$$p_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left[(x^2-1)^l\right].$$

We can use integration by parts to perform the integration:

$$\begin{array}{ccc} (+) & x^2 & \frac{d^{l-1}}{dx^{l-1}} \left[(x^2-1)^l\right] \\ (-) & 2x & \frac{d^{l-2}}{dx^{l-2}} \left[(x^2-1)^l\right] \\ (+) & 2 & \frac{d^{l-3}}{dx^{l-3}} \left[(x^2-1)^l\right]. \end{array}$$

We now have that

$$\int_{-1}^{1} x^2 p_l(x)\, dx = x^2 \frac{d^{l-1}}{dx^{l-1}} \left[(x^2-1)^l\right] - 2x \frac{d^{l-2}}{dx^{l-2}} \left[(x^2-1)^l\right] + 2 \frac{d^{l-3}}{dx^{l-3}} \left[(x^2-1)^l\right].$$

It is clear that the $l \geq 3$ requirement is necessary, as any $l < 3$ would lead to a negative order derivative. We now need to show that this value is equal to 0; to do so, we will start by substituting $Q(x) = (x^2-1)^l$. We now have

$$\begin{aligned} & x^2 \frac{d^{l-1}}{dx^{l-1}} \left[(x^2-1)^l\right] - 2x \frac{d^{l-2}}{dx^{l-2}} \left[(x^2-1)^l\right] + 2 \frac{d^{l-3}}{dx^{l-3}} \left[(x^2-1)^l\right] \\ &= x^2 \frac{d^{l-1}}{dx^{l-1}} Q(x) - 2x \frac{d^{l-2}}{dx^{l-2}} Q(x) + 2 \frac{d^{l-3}}{dx^{l-3}} Q(x) \\ &= x^2 Q^{(l-1)}(x) - 2x Q^{(l-2)}(x) + 2 Q^{(l-3)}(x). \end{aligned}$$

Let's consider the case where $l = 3$ to show that this integral is indeed 0. Substituting 3 for $l$ yields

$$x^2 Q^{''}(x) - 2x Q^{'}(x) + 2Q(x).$$

Through some careful integration, we find that

$$\begin{aligned} Q &= (x^2-1) \\ Q^{'} &= 6x(x^2-1)^2 \\ Q^{''} &= 24x^2(x^2-1) + 6(x^2-1)^2. \end{aligned}$$

Notice that in every term of each polynomial there is an $(x^2-1)^n$ term, where $n \in \mathbb{N}$. If we factor this term, we find that we have $(x+1)^n(x-1)^n$. The original integral was over the interval $(-1, 1)$, which yields a 0 in each term when we evaluate the anti-derivative at those bounds. Hence,

$$\begin{aligned} x^2 Q^{''}(x) - 2x Q^{'}(x) + 2Q(x) \Big|_{-1}^{1} &= x^2(0) - 2x(0) + 2(0) \\ &= 0. \end{aligned}$$

We have shown that for $l = 3$, the value of the integral is 0. However, we need to show that $\forall l \geq 3$, the value of the integral is 0. We can use the Leibniz Rule to express the $l^{\text{th}}$ derivative in terms of a summation.

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

We are trying to prove that $x^2 Q^{(l-1)}(x) - 2xQ^{(l-2)}(x) + 2Q^{(l-3)}(x) = 0$ for all $l \geq 3$. We know $Q = (x^2-1)^l$, so we can take one derivative of $Q$, which gives us $2xl(x^2-1)^{l-1}$. Let $f(x) = 2xl$ and $g(x) = (x^2-1)^{l-1}$. Our goal is to find the $l^{\text{th}}$ derivative of $Q$. By the Leibniz Rule we have

$$(fg)^{(l)} = \sum_{k=0}^{l} \binom{l}{k} f^{(l-k)} g^{(k)}.$$

Consider the case where we take another derivative. We know that we will have a $(x^2-1)^{(l-2)}$ term. Again, we can take another derivative, which would provide us with $(x^2-1)^{(l-3)}$. We know that this term will continue to appear until $k > l$. However, the Leibniz Rule will stop when $k = l$. Therefore, every product $f^{(l-k)}g^{(k)}$ will have a $(x^2-1)^n$ term where $n$ is the $l-k$ power. We can also rewrite $(x^2-1)^n$ as $(x-1)^n(x+1)^n$, so we now know that every term in the summation will have an $(x-1)^n(x+1)^n$ term. Recall that we are evaluating this term from $-1$ to 1, so when we ultimately substitute into the equation, we will either have $(x-1)^n = 0$ or $(x+1)^n = 0$. Therefore, we can conclude that every term in the summation will be 0. This implies that the $l^{\text{th}}$ derivative of $Q$ is 0 for any $l \geq 3$. Therefore $x^2Q^{(l-1)}(x) - 2xQ^{(l-2)}(x) + 2Q^{(l-3)}(x) = 0$.

2. We want to find $a_l$ in the series

$$f(x) = \sum_{l=0}^{\infty} a_l p_l(x)$$

where

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & -1 < x \leq 1 \end{cases} .$$

We know that we can express $a_l$ as

$$a_l = \frac{\int_{-1}^{1} f(x) p_l(x)\, dx}{\int_{-1}^{1} p_l^2(x)\, dx}.$$

Since $f$ is piecewise defined to be 0 on $-1 < x \leq 1$, we are only dealing with

$$a_l = \frac{\int_{0}^{1} x p_l(x)\, dx}{\int_{-1}^{1} p_l^2(x)\, dx}.$$

We can use integration by parts to solve the numerator. If we let $u = x$ and $dv = p_l(x)\,dx$, we have that

$$\frac{\frac{x}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right]\Big|_0^1 - \int_0^1 \frac{1}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right] dx}{\int_{-1}^{1} p_l^2(x)\, dx}.$$

Let's consider the first term in the numerator: $\frac{x}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right]\Big|_0^1$. In order to evaluate the differentiated term, we need to express the derivative in a closed form. If we make the substitution $y = x^2$, we have

$$(y-1)^l = (y-1)(y^{l-1} + y^{l-2} + \cdots + 1).$$

By back substitution, we have

$$\begin{aligned}(x^2-1)^l &= (x^2-1)(x^{2l-2}+x^{2l-4}+\cdots+1) \\ &= (x^2-1)(\sum_{j=0}^{l} x^{(2l-2j)}) \\ &= x^{2l+2}-1\end{aligned}$$