

1 Modeling Free Fall

1. Measure the height of the object above ground
2. Record time in seconds that it takes for object to hit the ground
3. Determine the quantitative relationship between the height and the time (a mathematical model)

Galileo makes some assumptions before experimenting:

1. $F = ma$: we already know the mathematical expression for force
2. Gravity is the only force acting on the object
3. Acceleration due to gravity is $-32\frac{m}{s^2}$.

$$\begin{aligned}mg &= my'' \\g &= y'' \\ \iint g &= \iint y'' \\ -16t^2 &= y + Ct + D \\ y &= -16t^2 + v_0t + y_0\end{aligned}$$

Galileo assumed that gravity was the only force, but what if air resistance is added? We know that air resistance acts opposite to the velocity. We now have

$$\begin{aligned}\kappa v - 32m &= mv' \\ v' &= 32 - \kappa_0 v\end{aligned}$$

2 Discrete Modeling

We can now consider a discrete case for modeling.

2.1 Credit Card Balancing

We can work with the equation

$$B_1 = B_0 - P$$

as a simple model for a credit card balance. We will eventually find that this model is very simplistic, and we will need to expand on it later. We have a recurrence relation where

$$\begin{aligned}B_2 &= B_1 - P \\ B_3 &= B_2 - P \\ &\dots\end{aligned}$$

Substituting into the recurrence relation tells us that

$$B_n = B_0 - nP.$$

We can reform the model to help use come up with a better representation of the situation. In the real world, credit cards often come with interest rates. Taking this into account gives us

$$B_{\text{new}} = (1 + r)B_{\text{old}} - P.$$

We again have a recurrence relation. We will let $s = (1 + r)$:

$$\begin{aligned} B_1 &= sB_0 - P \\ B_2 &= sB_1 - P = s^2B_0 - p(1 + s) \\ B_3 &= sB_2 - P = s^3B_0 - p(1 + s + s^2) \\ B_4 &= sB_3 - P = s^4B_0 - p(1 + s + s^2 + s^3) \\ &\dots \end{aligned}$$

We can derive a general solution of the form

$$\begin{aligned} B_n &= s^n B_0 - p(1 + s + s^2 + s^3 + \dots + s^{n-1}) \\ &= (1 + r)^n B_0 - p \left(\frac{(1 + r)^n - 1}{r} \right) \end{aligned}$$

2.1.1 Solving With Excel

We can use Excel to represent the model. We will consider the case where $r = 1.5\%$ with an initial balance of \$1000, and \$10 per month on payments. In the A1 cell, we want to have

$$=(1.015)^{\wedge}A1 * 1000 - 10*((1.015)^{\wedge}A1-1)/0.015$$

3 Modeling Ordinary Differential Equation

Consider the following system of ODE's

$$\begin{bmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \end{bmatrix}$$

We will use arbitrary values for the coefficient matrix. We want to solve the system using eigenvalues:

$$\begin{bmatrix} 5 - \lambda & 4 & 2 \\ 4 & 5 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{bmatrix} = 0$$

We want to take the determinant of the matrix, but first we can do some row reductions:

$$\begin{bmatrix} 1 - \lambda & 0 & 0 \\ 4 & 9 - \lambda & 2 \\ 2 & 4 & 2 - \lambda \end{bmatrix}$$

Taking that determinant gives us

$$\lambda = 1 \quad \text{and} \quad \lambda = 10.$$

3.1 Modeling War

We will model the Nazi party and the Soviet Union military forces. We want to model the rate at which Soviet Tanks reduce as Nazi anti-tank guns decrease.

$$\begin{aligned} \frac{dx}{dt} &= -ay \\ \frac{dy}{dt} &= -bx \end{aligned}$$

In this system, x' represents the rate at which the Soviet tanks decreased, and y' represents the rate at which German anti-tank guns decrease. We will consider the battle of Kursk, during which the number of Soviet tanks decreased by 50% in the first hour. In this model, a is referred to as the anti-tank kill rate, where b is the tank kill rate. We can use separation of variables yields

$$\begin{aligned} \frac{ay^2}{2} + \frac{ay_0^2}{2} &= \frac{bx^2}{2} + \frac{bx_0^2}{2} \\ ay^2 - bx^2 &= C \end{aligned}$$

4 Agents

A Schelling Model is used to model things like segregation. Consider the following example:

x1	x2				
x3	o1		o2		
x4	x5	o3	o4	o5	
x6	o6			x7	x8
	o7	o8	x9	x10	x11
		o9	o10	o11	

We can consider the case where, in the first iteration, all the members of the grid who do not have three directly adjacent neighbors leave the area. In addition, x's want to be with other x's, and o's want to be with other o's.

x3	x6		o1	o2	
x4	x5	o3	o4		
	o6	x2	x1	x7	x8
o11	o7	o8	x9	x10	x11
		o5	o9	o10	

We can see that as the system changes, there will be a mass exodus of both types. In the first table, there was only 1 agent with no neighbors of the opposite type. By the second iteration, there are now 6 agents with no neighbors of the opposite type.

5 The Game of Life

The game of life is a mathematical model created by John Conway. The rules are as follows:

1. Birth Rule: If at time t a cell is dead (empty) and the cell has 3 live neighbors in any direction, then at time $t + 1$ the cell becomes alive.
2. Death Rule: If at time t a live cell has 0 and 1 neighbors it dies of isolation, and if a live cell has 4 or more neighbors, it dies of overcrowding.
3. Survival Rule: If at time t a live cell has 2 or 3 live neighbors, then at time $t + 1$, the cell is still alive.

6 Tumor Growth

Tumor growth can be modeled by the differential equation

$$\frac{dV}{dt} = aV^{2/3} - bV'.$$

This equation arises from the logistic equation

$$\frac{dP}{dt} = P(a - bP),$$

which has a solution of

$$P = \frac{M}{1 + e^{-\log be^c - at}}.$$

As $t \rightarrow \infty$, P heads towards M , since the exponential decreases. The tumor equation can be solved by separation of variables:

$$\begin{aligned} \frac{dV}{aV^{2/3} - bV} &= dt \\ \int \frac{dV}{aV^{2/3} - bV} &= \int dt \\ 3 \int \frac{du}{a - bu} &= t + c. \end{aligned}$$

The previous result arises from the u -substitution where $u = V^{1/3}$. We can make another substitution, where $w = a - bu$. Then

$$\begin{aligned} 3 \int \frac{du}{a - bu} &= t + c \\ \frac{-3}{b} \ln |a - bu| &= t + c \\ \frac{-3}{b} \ln |a - bV^{1/3}| &= t + c. \end{aligned}$$

We now will specify the integration constant, c , in terms of the the volume of the tumor, V .

$$c = t - \frac{3}{b} \ln |a - bV^{1/3}|.$$

The plot of this function can be expressed as

$$t = \frac{3}{b} \ln \frac{|a - bV_0^{1/3}|}{|a - bV^{1/3}|},$$

which simplifies to

$$V = \frac{e^{-bt} \left(-a + bV_0^{1/3} + ae^{bt/3} \right)^3}{b^3}$$

This function's plot acts the same as the logistic equation.

6.1 Gompertz Curve

Gompertz Curve is an equation that arises from

$$\begin{cases} \frac{dV}{dt} = r(t)V \\ \frac{dr}{dt} = -Kr \end{cases}.$$

The model has the same curve that the logistic growth model has, but the equation differs.

$$V = V_0^{r_0/k} e^{-(r_0/k)e^{-kt}}.$$

The model can also be expressed as a single equation:

$$\begin{cases} \frac{dV}{dt} = r(t)V \\ \frac{dr}{dt} = -Kr \end{cases} \rightarrow \frac{dV}{dt} = aV - bV \ln V,$$

which has the solution

$$V(t) = V_0 e^{r_0/t} e^{-r_0/ke^{-kt}}$$

7 Voting

22%	23%	15%	29%	7%	4%
T	T	B	B	I	I
B	I	T	I	B	T
I	B	I	T	T	B

Because of the plurality, T wins the election. If the I candidate is removed, then B wins the election. Often in this model, we deal with non-transitive systems. Consider the following voter profile:

14	$A > B > C$
12	$C > A > B$
10	$B > C > A$

This profile has the following outcome:

First	A	C	B
Second	B	A	C
Third	C	B	A
	14	12	10

In this case, $A > B$, $B > C$, but $C > A$.

7.1 Pairwise Comparison

This is based on combinatorics; if we have 3 candidates, we have

$$\binom{3}{2}$$

possibilities.

7.2 Arrow's Axioms

Kenneth Arrow has several axioms that must be taken into account when modeling voting situations.

1. The model should be transitive
2. The model should not consider irrelevant alternatives
3. The voters should be able to vote for whomever they choose
4. If every voter prefers one alternative to another, so does the whole society
5. There should be no case that leads to a dictatorship

Among these axioms, (2) is the most logical of all. This axiom is stating that if we wish to know the overall preference of x vs y , then we **only** need to consider x and y . In the general case, this makes sense. The outcome should not be affected by candidate z , so we need not consider that candidate.

Example Consider the case where we have the following candidates:

Rubio: A, Trump: B, Carson: C, Cruz: D, Kaish: E.

We can consider the following setup

A	B	C	D	E	E
D	E	B	C	B	C
E	D	E	E	D	D
C	C	D	B	C	B
B	A	A	A	A	A
18	12	10	9	4	2

Plurality: Under Plurality, A has the majority vote (18), so A wins.

Single Runoff: Under the single runoff model, the two candidates with the most 1st place votes have a runoff. The winner of the runoff wins. The two top candidates here are A and B. In a single runoff, A has 18 votes and B has 37, so B wins.

Hare: Under the Hare system, C wins.

Bourda: Under the Bourda method, D wins.

Pairwise: Under pairwise comparison, there are $\binom{5}{2} = 10$ comparisons. In this case, E wins the election.