1 High Order PDEs

We can begin by talking about the heat equation in 2D space. We have the following

$$u_t = ku_{xx},$$

which, under certain conditions, yields the general solution

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right).$$

If we consider the laplacian, $\nabla^2 u$, we can express the heat equation as

$$\begin{aligned} u_t &= k\nabla^2 u \\ &= k(u_{xx} + u_{yy}) \text{ in } \mathbb{R}^3. \end{aligned}$$

We can now begin to solve a three-dimensional heat equation.

Example: Solve the following heat equation for $u(x,t)$:

$$\begin{cases} u_t = a^2(u_{xx} + u_{yy}) \\ u(x,t) = f(x,y) \\ u_{2D} = 0 \qquad \text{(there is no external source of energy in the system)} \end{cases}.$$

Solution: We begin by noticing that this equation is taking place on a cylinder. much like the process when solving Laplace's equation on a disk, we can transform our PDE from cartesian coordinates to polar coordinates. The time variable, t is not affected by this transformation, and we know

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Applying our transformation yields the new PDE

$$\begin{cases} u_t = a^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \\ u_{t=0} = f(r,\theta) \\ u_{r=r} = 0 \end{cases}.$$

We will now have a general solution of $u(r,\theta,t)$, but we cannot jump immediately to a Fourier series solution. Instead, we will use separation of variables to provide us with a solution. We have

$$\begin{aligned} u &= V(r,\theta)T(t) \\ \frac{T'}{a^2T} &= \frac{V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta}}{V} = -k^2. \end{aligned}$$

We now have two ODEs to deal with:

$$T' + a^2k^2T = 0 \qquad \text{and} \qquad V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta} + k^2V = 0$$

We can again apply separation of variables to $V(r,\theta)$, and we find that we have two more resulting ODEs.

$$\begin{cases} \theta'' + \mu\theta = 0 \\ r^2R'' + rR' + (k^2r^2 - \mu)R = 0 \end{cases}.$$

The equation involving θ is a simple ODE; the equation involving R will take a little more work to solve. We will discover a method to solve this ODE in the next section.

1.1 Power Series Solutions to ODEs

From the previous problem, we saw that we need to develop a method to solve

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$
$$p(x)y'' + q(x)y' + r(x)y = 0.$$

We will attempt to solve this using power series solutions to ODEs. There is, however, an issue that we will face. Since $p(x) = x^2$, we cannot recover y'' near zero ($p(0) = 0$). The regular power series expansion we would use would take the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

but we cannot use this method since y'' is lost near zero. We can, however modify our sequence in such a way as to preserve y''. Doing so gives us the sequence

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n,$$

where α is called the characteristic value. The strategy here will be to determine the values of α, and then determine the value of a_n.

Example: Determine the solution to the following ODE

$$x^2y'' + x^2y' + ry = 0.$$

Solution: We start by saying

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha},$$

so

$$y'(x) = \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}$$
$$y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}.$$

We now have enough to substitute back into our original ODE.

$$x^2 \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} + x^2 \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1} + r \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$