1 Logarithmic Equations

It can often be helpful to express a logarithm in terms of base e. The following relationship can help us get to the natural log, ln.

$$\log_n x = \frac{\ln x}{\ln n}$$

We can also use the properties of logarithmic functions to simplify exponential functions.

$$n^x = e^{\ln x}$$

Applying calculus to logarithmic and exponential functions can yield interesting results.

$$\frac{d}{dx}(\log_n(x)) = \frac{1}{x \ln(n)}$$

$$\int n^x dx = \frac{n^x}{\ln(n)}$$

2 Inverse Trigonometry Functions

The derivatives of inverse trig functions are useful for integration. We have the following derivatives:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\text{arcsec}(x)) = \frac{1}{|x|\sqrt{1+x^2}}$$

3 Integration Techniques

The main focus of Calculus II is integration. In Calculus I, we learned how to integrate by using a u substitution, but that technique is often not powerful enough to handle more complex integrals. Luckily, we have several different methods that allow us to integrate.

3.1 Integration By Parts

$$\int u dv = uv = \int v du$$

This technique is useful for integrating two functions, $f(x)$ and $g(x)$. Suppose we want to find the integral $\int xe^{x^2} dx$, or $\int x \sin \pi x dx$, or even $\int (3t+5) \ln(t/5) dt$. All of these examples are set up to be solved by integration by parts. We want to split the integrand into two separate functions. We then differentiate one (u), and integrate the other (dv). Applying our formula will often simplify the original integral into something that we know how to easily integrate. Typically, when we choose our u and dv, we want to choose the function whose integral we know for dv, and the other one for u (remember, it is often easier to take a derivative than integrate).

3.2 Trig Combination

Ultimately, this technique will end up using integration by parts in most cases. There are a few trig identities that we need to know first:

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

There are some common patterns that often appear in integrals that we can solve using theis method.

1. If we see a $\cos x$ to an odd power, then we should let $u = \sin x$.
2. If we see a $\sin x$ to an odd power, then we should let $u = \cos x$.
3. If we see a $\sec x$ to an odd power, then we should let $u = \tan x$.
4. If we have a $\tan x$ to an odd power, and we have $\sec x$ present, then we should let $u = \sec x$.
5. If we have $\sec x$ to an odd power and $\tan x$ to an even power, then convert the $\tan x$ to $\sec x$.
6. If we **only** have $\sec x$ to an odd power, then we need to use integration by parts.
7. If we have a $\tan x$ and do **not** have a $\sec x$ present, then we need to convert the $\tan x$ into $\sec x$.
8. If we see a trig function to an even power, then we need to use the half angle formulas first.

3.3 Trig Substitution

This technique for integration is applicable mostly to integrals that contain functions that look like the derivatives of inverse trig functions. We have a few special cases that tell us what substitution to make.

$$\sqrt{a^2 - u^2} \to 1 - \sin^2\theta = \cos^2\theta \to u = a\sin\theta$$
$$\sqrt{a^2 + u^2} \to 1 + \tan^2\theta = \sec^2\theta \to u = a\tan\theta$$
$$\sqrt{u^2 + a^2} \to \sec^2\theta - 1 = \tan^2\theta \to u = a\sec\theta$$

3.4 Partial Fractions

We want to use this technique to solve fraction of functions:

$$\int \frac{\mathcal{P}(x)}{\mathcal{Q}(x)} dx \text{ Where the degree of } \mathcal{P} \text{ is less than the degree of } \mathcal{Q}$$

To use this method, we need to

1. Factor the denomiator
2. Determine the decomposition
3. Determine the unknown values of the numerator

4 Series and Sequences

4.1 Telescoping Series

A telescoping series takes the form of

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

The series will expand in such a way that all the terms will cancel except for the first and last terms. This allows us to say that the series will converge at

$$\lim_{n\to\infty} 1 - \frac{1}{n+1}$$

4.2 Geometric Series

A geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1}$$

where $a \neq 0$ and r is constant. We know that

$$|r| \geq 1 \implies \text{Divergence} \qquad\qquad |r| < 1 \implies \text{Convergence}$$

If the sequence converges, it converges to

$$\frac{s_0}{1-r}$$

where s_0 is the first term in the series.

4.3 P Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p \leq 1$, then the series diverges. If $p > 1$, then the series converges.

4.4 Tests of Convergence

4.4.1 The Divergence Test

If

$$\lim_{n\to\infty} a_n \neq 0$$

then the series $\sum a_n$ diverges. Note that this test **only** determines when the series diverges. It says nothing about the series converging.

4.4.2 Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive termed series.

- If b_n converges, $a_n < b_n \implies a_n$ converges.
- If b_n diverges, $a_n > b_n \implies a_n$ diverges.

4.4.3 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive termed series. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \mathcal{L}$$

then

1. If $\mathcal{L} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\mathcal{L} \to \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.
3. If $\mathcal{L} > 0$, then both series either both converge or both diverge.

4.4.4 Ratio and Root Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \mathcal{L} \qquad\qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \mathcal{L}$$

1. $\mathcal{L} < 1 \implies a_n$ converges
2. $\mathcal{L} > 1 \implies a_n$ diverges
3. $\mathcal{L} = 1$ is inconclusive

4.4.5 Alternating Series Test

The alternating series

$$\sum (-1)^{n+1} a_n$$

converges if

1. $a_{n+1} \leq a_n$
2. $\lim_{n\to\infty} a_n = 0$

4.4.6 Integral Test

Given the series

$$\sum_{n=j}^{\infty} a_n$$

let $f(n) = a_n$. The series will converge if and only if

$$\int_j^{\infty} f(n)dn$$

exists.

4.5 Types of Convergence

4.5.1 Absolute Convergence

A series converges absolutely if both $\sum a_n$ and $\sum |a_n|$ converge.

4.5.2 Conditional Convergence

A series converges conditionally if $\sum a_n$ converges and $\sum |a_n|$ diverges.

4.6 Special Series

4.6.1 Power Series

A power series is a special type of p-series that takes the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

All power series converge for $x = x_0$. With power series, we often want to know the radius of convergence of the series. The radius of convergence, $\mathcal{R}$, can be determined by finding the distance that x can move from x_0 while the series still converges. The interval $\mathcal{I} = [x_0 - \mathcal{R}, x_0 + \mathcal{R}]$ is called the interval of convergence.

4.6.2 Taylor Series

The Taylor series representation of a function can be given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

where $f^{(n)}$ is the n^{th} derivative of f. Taylor series provide an appromation of a function.

4.6.3 Binomial Series

A binomial series is given by

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$$