1. Solve the first order Cauchy problem

$$\begin{cases} u_t + u_x - 3u = t \\ u(x,0) = x^2 \end{cases}$$

From the coefficients on the u_t and u_x, we know that

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \int dx &= \int dt \\ x &= t + c \\ x - t &= c. \end{aligned}$$

Now we need to find our substitutions s and w. If we let $s = x - t$, then we can arbitrarily let $w = t$. Now we need to find u_t and u_x.

$$\begin{aligned} u_t &= u_s \frac{\partial s}{\partial t} + u_w \frac{\partial w}{\partial t} \\ &= -u_s + u_w \\ &= u_w - u_s \end{aligned} \qquad\qquad \begin{aligned} u_x &= u_s \frac{\partial s}{\partial x} + u_w \frac{\partial w}{\partial x} \\ &= u_s \end{aligned}$$

Now we substitute back into our originial equation.

$$\begin{aligned} -u_s + u_w + u_s - 3u &= w \\ u_w - 3u &= w \end{aligned}$$

We now have a first order ODE that we can solve by

$$y' - 3y = x$$

$$y_g = c_1 e^{3x} \qquad\qquad y_p = \frac{1}{3}x - \frac{1}{9}$$

$$\begin{aligned} y &= c_1 e^{3x} + \frac{1}{3}x - \frac{1}{9} \\ u(s,w) &= f(s)e^{3w} + \frac{1}{3}w - \frac{1}{9} \\ u(x,t) &= f(x-t)e^{3t} + \frac{1}{3}t - \frac{1}{9} \end{aligned}$$

Next we solve for $f(x-t)$.

$$\begin{aligned} u(x,0) = f(x) - \frac{1}{9} &= x^2 \\ f(x) &= x^2 + \frac{1}{9} \end{aligned}$$

Our final solution becomes

$$u(x,t) = \left[(x-t)^2 + \frac{1}{9}\right] e^{3t} + \frac{1}{3}t - \frac{1}{9}$$

2. Find the eigenvalues and eigenfunctions for the following Cauchy-Euler equation.

$$\begin{cases} x^2y'' + xy' + \lambda^2 y = 0 \\ y(1) = y(2) = 0 \end{cases}$$

Since this is a Cauchy-Euler equation, we have a solution of the general form

$$y = x^m$$

By substitution, we get

$$\begin{aligned} &x^2\left[m(m-1)x^m\right] + x(m)x^{m-1} + \lambda^2 x^m = 0 \\ &x^m\left[m^2 - m + m + \lambda^2\right] = 0 \\ &m^2 + \lambda^2 = 0 \\ &m = \pm\, \lambda i \end{aligned}$$

This yields the basis

$$\left\{x^a \cos\left(b \ln x\right), x^a \sin\left(b \ln x\right)\right\},$$

which leads to the general solution

$$y = c_1 \cos \lambda \ln x + c_2 \sin \lambda \ln x.$$

Applying the boundary conditions gives us

$$\begin{aligned} y(1) = 0 &= c_1 \cos\left(\lambda \ln 1\right) + c_2 \sin\left(\lambda \ln 1\right) \\ 0 &= c_1 \cos 0 + c_2 \sin 0 \\ 0 &= c_1 \\ y(2) = 0 &= c_2 \sin\left(\lambda \ln 2\right) \\ 0 &= \sin\left(\lambda \ln 2\right) \\ \lambda_n \ln 2 &= n\pi \\ \lambda_n &= \frac{n\pi}{\ln 2} \end{aligned}$$

The corresponding eigenfunction is

$$\phi_n = c_2 \sin\left[\frac{n\pi}{\ln 2} \ln x\right]$$