

# Exam Review

1. Find the general solutions to the following ordinary differential equations.

(a)  $y'(t) + 4y(t) = 1 + 2t$

Homogeneous Solution:

$$\begin{aligned}y' + 4y &= 0 \\r + 4 &= 0 \\r &= -4 \\y_g(t) &= ce^{-4t}\end{aligned}$$

Particular Solution:

$$\begin{aligned}y' + 4y &= 1 + 2t \\y &= At + B \quad y' = A \\A + 4At + 4B &= 1 + 2t \\4A &= 2 \quad A + 4B = 1 \\A &= \frac{1}{2} \quad B = \frac{1}{8} \\y_p(t) &= \frac{1}{2}t + \frac{1}{8}\end{aligned}$$

$$y(t) = ce^{-4t} + \frac{1}{2}t + \frac{1}{8}$$

(b)  $y''(t) + y(t) = \cos \omega t$

Homogeneous Solution:

$$\begin{aligned}y''(t) + y(t) &= 0 \\r^2 + 1 &= 0 \\r &= \pm i \\y_g(t) &= c_1 \cos t + c_2 \sin t\end{aligned}$$

Particular Solution:

2. Here is another Question

3. Solve the initial boundary value problem.

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = 0 \\ u(0, t) = u(1, t) = 0 \end{cases} \quad f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x) \\u(x, 0) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x) \\u_t(x, 0) &= \sum_{n=1}^{\infty} n\pi b_n \sin(n\pi x) = 0\end{aligned}$$

$$\begin{aligned}
a_n &= 2 \left( \int_0^1 f(x) \sin(n\pi x) dx \right) \\
&= 2 \left( \int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 (1-x) \sin(n\pi x) dx \right) \\
&= 2 \left( \int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 \sin(n\pi x) dx - \int_{1/2}^1 x \sin(n\pi x) dx \right) \\
&= 2 \left( \left[ \frac{\sin(n\pi x)}{(n\pi)^2} - \frac{x \cos(n\pi x)}{n\pi} \right] \Big|_0^{1/2} + \left[ \frac{-\cos(n\pi x)}{n\pi} \right] \Big|_{1/2}^1 + \left[ \frac{x \cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{(n\pi)^2} \right] \Big|_{1/2}^1 \right) \\
&= 2 \left( \frac{\sin \frac{n\pi}{2}}{(n\pi)^2} - \frac{\cos \frac{n\pi}{2}}{2n\pi} + \frac{\cos \frac{n\pi}{2}}{n\pi} - \frac{\cos n\pi}{n\pi} - \frac{\cos \frac{n\pi}{2}}{2n\pi} + \frac{\sin \frac{n\pi}{2}}{(n\pi)^2} + \frac{\cos n\pi}{n\pi} - \frac{\sin n\pi}{(n\pi)^2} \right) \\
&= \frac{4 \sin \frac{n\pi}{2}}{(n\pi)^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{\frac{1}{\pi} \int_0^1 0 dx}{n\pi} \\
&= 0
\end{aligned}$$

$$u(x, t) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos(2k\pi t + \pi t) \sin(2k\pi t + \pi t)$$

4. Solve the following partial differential equation.

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = 0 \\ u(0, t) = A \cos t, \quad u(1, t) = 0 \end{cases}$$

$$\delta(x, t) = A \cos t - xA \cos t$$

$$\gamma(x, t) = u(x, t) - \delta(x, t)$$

$$= u - A \cos t + xA \cos t$$

$$\gamma_t(x, t) = u_t + A \sin t - xA \sin t$$

$$\gamma_{xx}(x, t) = u_{xx}$$