

1. (a) Let a, b and c be natural numbers with a odd. Prove that if $a|(b-c)$ and $a|(b+c)$, then $a|b$ and $a|c$.

Proof: Let $a|(b-c)$ and $a|(b+c)$. Then $b-c = ax$ and $b+c = ay$ for some $x, y \in \mathbb{N}$. We have that $b = ax + c$. Then

$$ax + c + c = ay$$

$$ax + 2c = ay$$

$$2c = ay - ax$$

$$2c = a(y-x).$$

Since a is odd, $y-x$ must be even since $2c$ is even. Then $2|(y-x)$, and we have $c = a\frac{y-x}{2}$, or $c = az$ for $z \in \mathbb{N}$. Therefore $a|c$. If we let $c = b - ax$, we have

$$b + b - ax = ay$$

$$2b - ax = ay$$

$$2b = ay + ax$$

$$2b = a(y+x)$$

Since $2b$ is even and a is odd, $y+x$ must be even, or $2|(y+x)$. Therefore $b = a\frac{y+x}{2}$, or $b = aj$ for $j \in \mathbb{N}$. Therefore $a|b$.

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- (b) Using a truth table, show that $\neg(P \wedge Q)$ and $(\neg P \vee \neg Q)$ are logically equivalent.

P	Q	$\neg(P \wedge Q)$	$(\neg P \vee \neg Q)$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

2. (a) Establish the following identity using induction.

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Proof:

Base Case: $n = 1$

$$\sum_{i=1}^1 i^3 = \left(\frac{1(1+1)}{2} \right)^2$$

$$1^3 = \left(\frac{2}{2} \right)^2$$

$$1 = 1$$

Assume:

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Prove:

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2} \right)^2$$

$$\sum_{i=1}^n i^3 + \sum_{i=n+1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2} \right)^2$$

$$\left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2} \right)^2 \quad \text{By Induction Hypothesis}$$

$$n^2(n+1)^2 + 4(n+1)^3 = (n+1)^2(n+2)^2$$

$$(n+1)^2(n^2 + 4n + 4) = (n+1)^2(n+2)^2$$

$$(n+2)^2 = (n+2)^2$$

△

- (b) Prove that if n^3 is odd, then n is odd.

Proof: Assume that if n is even, then n^3 is even. Then $n = 2k$ for $k \in \mathbb{Z}$, which mean that $n^3 = (2k)^3$. $(2k)^3 = 2(2^2k^3)$ where $(2^2k^3) \in \mathbb{Z}$. Therefore n^3 is even.

△

3. (a) Using the definition, prove that $f : \mathbb{Q} \rightarrow \mathbb{Q}$ given by $f(x) = 3x + 2$ is bijective.

Proof: Let $x, y \in \mathbb{Q}$ such that $f(x) = f(y)$. Then

$$3x + 2 = 3y + 2$$

$$3x = 3y$$

$$x = y$$

Therefore f is injective.

Let $y \in \mathbb{Q}$ such that $y = \frac{x-2}{3}$. Then

$$\begin{aligned} f(y) &= 3 \left(\frac{x-2}{3} \right) + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

Therefore f is surjective.

Therefore f is bijective.

△

- (b) Define a relation on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (c, d)$ if $a + d = b + c$. Prove that \sim is an equivalence relation.

Symmetric: Let $(a, b) \sim (c, d)$. Then

$$a + d = b + c$$

$$-b - c = -a - d$$

$$(-1)(b + c) = (-1)(a + d)$$

$$b + c = a + d$$

Therefore $(c, d) \sim (a, b)$, and \sim is symmetric.

Reflexive: Let $(a, b) \in \mathbb{N} \times \mathbb{N}$. If $(a, b) \sim (a, b)$, then $a + b = a + b$. Therefore $(a, b) \sim (a, b)$, and \sim is reflexive.

Transitive: Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Solving for c yields $c = d + e - f$, and substituting gives us $a + d = b + d + e - f$. Then $a + f = e + b$, and $(a, b) \sim (e, f)$. Therefore, \sim is transitive.

Therefore \sim is an equivalence relation.

△

4. (a) Let \mathcal{X} be a finite set with cardinality n . Prove that the power set, $\mathcal{P}(\mathcal{X})$, has cardinality 2^n .

Proof:

Base Case: A set of size 1.

Let \mathcal{X} be the set $\{x\}$. Then $\mathcal{P}(\mathcal{X})$ is $\{\emptyset, \{x\}\}$, which has a cardinality of $2^{|\mathcal{X}|}$, or 2^1 .

Assume: $|\mathcal{P}(\{x_0, x_1, x_2, \dots, x_n\})| = 2^{|\mathcal{X}|}$.

Prove: $|\mathcal{P}(\{x_0, x_1, x_2, \dots, x_n, x_{n+1}\})| = 2^{|\mathcal{X}|+1}$.

We know that \mathcal{P} is the set of all subsets of \mathcal{X} . If we count the number of subsets of $\{x_0, x_1, x_2, \dots, x_n, x_{n+1}\}$, we know that the subset will either contain x_{n+1} , or it will not contain x_{n+1} . If the subset γ does not contain x_{n+1} , then $\gamma \subseteq \{x_0, x_1, x_2, \dots, x_n\}$, and there are $2^{|\mathcal{X}|}$ γ by the induction hypothesis. If the subset λ contains x_{n+1} , then it is the result of some set $\gamma \cup \lambda$. Since $\gamma \subseteq \{x_0, x_1, x_2, \dots, x_n\}$, we only need $\gamma \cup \{x_{n+1}\}$ to account for all possible sets. Therefore $|\mathcal{P}(\gamma \cup \{x_{n+1}\})|$ is $|\mathcal{P}(\gamma)| \cdot |\mathcal{P}(\{x_{n+1}\})|$, or $2^{|\mathcal{X}|} \cdot 2^{|\{x_{n+1}\}|}$. This is equivalent to $2^{|\mathcal{X}|} \cdot 2^1$, or $2^{|\mathcal{X}|+1}$.

△

- (b) Let $n \geq 2$ be an integer. Prove that $a \equiv b \pmod{n}$ if $n|(b-a)$ is an equivalence relation on \mathbb{Z} .

Proof: Let $n \geq 2$ and $n|(b-a)$. Therefore $b-a = nk$, $k \in \mathbb{Z}$. Then $b-a \equiv 0 \pmod{n}$. It follows that $b \equiv a$, and since \pmod{n} is an equivalence relation, $a \equiv b$ by symmetry.

△

- (c) Let A and B be sets. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof:

$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$: Let $x \in \overline{A \cup B}$. Then $x \notin A$ or B . Since $x \notin A$, $x \in \overline{A}$. Since $x \notin B$, $x \in \overline{B}$. Therefore $x \in \overline{A}$ and $x \in \overline{B}$, or $x \in \overline{A} \cap \overline{B}$.

$\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$: Let $x \in \overline{A} \cap \overline{B}$. Then $x \notin A$ and $x \notin B$. Therefore $x \notin A \cup B$, or $x \in \overline{A \cup B}$.

△

- (d) Prove that $\sqrt{5}$ is irrational.

Proof: Assume $\sqrt{5}$ is rational. Then $\sqrt{5} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$, and p and q have opposite parity. It follows that $5 = \frac{p^2}{q^2}$, or $5q^2 = p^2$. If q is odd, then p must be even. However, an odd number squared and multiplied by an odd number is odd, which means that $5q^2 \in \{2k+1 | k \in \mathbb{Z}\}$. This implies that p^2 is odd, but p is even, so p^2 is even. Therefore by contradiction, $\sqrt{5}$ must be irrational.

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