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Homework 3

1. Suppose that $a, b \in \mathbb{Z}$ and define the set $J = ax + by | x, y \in \mathbb{Z}$. Prove that $J = Z$ if and only if $gcd(a, b) = 1$.

2. We define the *Fibonacci Sequence* to be the sequence of integers $x_0, x_1, x_2, \ldots$ satisfying the properties

$$x_0 = 0, \quad x_1 = 1, \quad \text{and} \quad x_n = x_{n-1} + x_n \text{ for all } n \geq 2.$$

   Prove that $gcd(x_n, x_{n-1}) = 1$ for all $n \geq 1$. (Hint: Try using induction on n.)

3. Let a, b, x be positive integers with $x \geq 2$ and set $d = gcd(a, b)$.
   (a) Prove that $x^d - 1$ divides $gcd(x^a - 1, x^b - 1)$.
   (b) Prove that $x^d - 1$ is a multiple of $gcd(x^a - 1, x^b - 1)$ and conclude that $x^d - 1 = gcd(x^a - 1, x^b - 1)$. (Hint: We know that there exist integers u and v such that $d = au + bv$. Now show that there exist integers α and β such that $x^d - 1 = \alpha(x^a - 1) + \beta(x^b - 1)$.)

4. Show that the equation $1495x + 50060y = 4$ has no solutions for $x, y \in \mathbb{Z}$.

$$\begin{aligned}
50060 &= 1495(33) + 725 \\
1495 &= 725(2) + 45 \\
725 &= 45(16) + 5 \\
45 &= 5(9) + 0 \\
gcd(50060, 1495) &= 5
\end{aligned}$$

   However, $5 \nmid 4$, so there is no solution.

5. Find all solutions to the equation $1485x + 1745y = 15$ for $x, y \in \mathbb{Z}$.

$$\begin{aligned}
1745 &= 1485(1) + 260 \\
1485 &= 260(5) + 185 \\
260 &= 185(1) + 75 \\
185 &= 72(2) + 35 \\
75 &= 35(2) + 5 \\
35 &= 7(5) + 0 \\
gcd(1745, 1485) &= 5
\end{aligned}
\qquad
\begin{aligned}
5 &= 75 - 35(2) \\
5 &= 75 - (185 - 75(2))(2) \\
&= 75(5) - 185(2) \\
5 &= (260 - 185(1))(5) - 185(2) \\
&= 260(5) - 187(7) \\
5 &= 260(5) - (1485 - 260(5))(7) \\
&= 260(40) + 1485(-7) \\
5 &= (1745 - 1485(1))(40) + 1485(-7) \\
&= 1745(40) + 1485(-47)
\end{aligned}$$

   $x = -47 + 349n$ and $y = 40 + 297n$ are solutions to the equation.

6. Suppose you have two small champagne glasses, one holding 8 ounces and another holding 5 ounces. Is it possible to fill one of the glasses with exactly 1 ounce of champagne? If so, how can this be done? If not, prove that it cannot be done.

   We need to find a solution to the Diophantine Equation

$$8x + 5y = 1.$$

We know that $gcd(8, 5) = 1$, and $1|1$, so there is a solution. Working backwards through the Euclidian Algorithm gives us

$$\begin{aligned} 1 &= 3 - 2(1) \\ &= 3(2) - 5 \\ &= 8(2) + 5(-3). \end{aligned}$$

Therefore, if we fill the first glass up twice and empty the second glass three times, we will end up with 1 ounce leftover.