General Solutions To IBVP's

The Heat Equation

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases} \qquad \begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = \textstyle\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \qquad u(x,t) = \frac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u_x(L,t) = 0 \\ u(x,0) = h(x) \end{cases} \qquad \begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u(L,t) = 0 \\ u(x,0) = k(x) \end{cases}$$

$$u(x,t) = \textstyle\sum_{n=1}^{\infty} e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}\right)^2 t} \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right) \quad u(x,t) = \textstyle\sum_{n=1}^{\infty} e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{L}\right)^2 t} \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right)$$

The Wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{cases} \qquad \begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi tc}{L}\right) + b_n \sin\left(\frac{n\pi tc}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \quad u(x,t) = \frac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi tc}{L}\right) + b_n \sin\left(\frac{n\pi tc}{L}\right)\right] \cos\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{L}\right) + b_n \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right)\right] \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right)$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = a_0 + b_0 t + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi t}{L}\right) + b_n \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right)\right] \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi x}{L}\right)$$

The Laplace Equation

Cartesian Coordinates

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(L,y) = 0 \end{cases} \qquad \begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0,y) = u_x(L,y) = 0 \end{cases}$$

$$u(x,y) = \textstyle\sum_{n=1}^{\infty} \left[a_n \sinh\left(\frac{n\pi y}{L}\right) + b_n \cosh\left(\frac{n\pi y}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \quad u(x,y) = \textstyle\sum_{n=1}^{\infty} \left[a_n \sinh\left(\frac{n\pi y}{L}\right) + b_n \cosh\left(\frac{n\pi y}{L}\right)\right] \cos\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = u(L, y) = 0 \end{cases}$$

$$u(x, t) = \textstyle\sum_{n=1}^{\infty} \left[a_n \sinh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right) + b_n \cosh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right)\right] \cos\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right)$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = u_x(L, y) = 0 \end{cases}$$

$$u(x, t) = \textstyle\sum_{n=1}^{\infty} \left[a_n \sinh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right) + b_n \cosh\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right)\right] \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi y}{L}\right)$$

Polar Coordinates

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(h, \theta) = f(\theta) \quad \text{on } x < \theta < L \end{cases}$$

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos\left(\frac{n\pi\theta}{L}\right) + b_n r^n \sin\left(\frac{n\pi\theta}{L}\right)$$