Steven Rosendahl
Homework 9

1. Directly calculate $\sum_{d|12} \phi(d)$ and verify that you obtain 12 as your answer.

$$\begin{aligned}\sum_{d|12} \phi(d) &= \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) \\ &= 1 + 1 + 2 + 2 + 2 + 4 \\ &= 12\end{aligned}$$

2. Suppose that $p_1, p_2, \ldots, p_N$ are distinct primes. Prove that

$$\frac{\phi(p_1 p_2 \ldots p_N)}{p_1 p_2 \ldots p_N} = \prod_{n=1}^{N} \left(1 - \frac{1}{p_n}\right)$$

$$\begin{aligned}\frac{\phi(p_1 p_2 \ldots p_N)}{p_1 p_2 \ldots p_N} &= \frac{\phi(p_1)\phi(p_2)\ldots\phi(p_N)}{p_1 p_2 \ldots p_N} \\ &= \prod_{n=1}^{N} \frac{\phi(p_n)}{p_n} \\ &= \prod_{n=1}^{N} \frac{p_n - 1}{p_n} \\ &= \prod_{n=1}^{N} \left(1 - \frac{1}{p_n}\right)\end{aligned}$$

3. Find a value of n such that $\phi(n)/n < 1/4$. What do you think is a good strategy for choosing n so that $\phi(n)/n$ is close to zero?

   One value for which this holds true is $n = 210$. $\phi(210)$ is 48, and $48/210 = 8/35 < 1/4$. One strategy for finding these numbers would be to

4. Suppose that p is prime and m and n are non-negative integers.

   (a) Prove that $\phi(p^{m+n}) \geq \phi(p^m)\phi(p^n)$.

   We can consider $\phi(p^{m+n})$. If we let $m+n = j$, then we have $\phi(p^j)$, which can be expressed as $p^j - p^{j-1}$. If we consider $\phi(p^m)\phi(p^n)$, we have

$$\begin{aligned}\phi(p^m)\phi(p^n) &= (p^m - p^{m-1})(p^n - p^{n-1}) \\ &= p^{m+n} - 2p^{m+n-1} + p^{m+n-2} \\ &= p^j - 2p^{j-1} + p^{j-2}.\end{aligned}$$

   If we compare the two values, we get

$$\begin{aligned}p^j - p^{j-1} &\stackrel{?}{\geq} p^j - 2p^{j-1} + p^{j-2} \\ p^{j-1} &\geq p^{j-2}.\end{aligned}$$

   We know this is true since $m, n > 0$, so $j > 0$.