

# Formal Solutions To PDE's

## 1 Homogeneous Solutions

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### 1.1 The Heat Equation

**Dirichlet**

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0, t) = u(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right)$$
$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

**Neumann**

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0, t) = u_x(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\left(\frac{n\pi x}{\ell}\right)$$
$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx$$
$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx$$

**Robin**

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0, t) = u_x(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}\right)^2 t} \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell} x\right)$$
$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell} x\right) dx$$

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0, t) = u(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell}\right)^2 t} \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell} x\right)$$
$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\left[n+\frac{1}{2}\right]\frac{\pi}{\ell} x\right) dx$$

## 1.2 The Wave Equation

### Dirichlet

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0, t) = u(\ell, t) = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right) \right] \sin\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

### Neumann

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0, t) = u_x(\ell, t) = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = \frac{a_0}{2} + \frac{b_0}{2}t + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right) \right] \cos\left(\frac{n\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \quad a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx$$

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \quad b_0 = \frac{2}{n\pi c} \int_0^{\ell} g(x) dx$$

### Robin

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0, t) = u_x(\ell, t) = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = a_0 + b_0 t + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\left[n + \frac{1}{2}\right] \frac{\pi t}{\ell}\right) + b_n \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi t}{\ell}\right) \right] \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi x}{\ell}\right)$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi x}{\ell}\right) dx \quad a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx$$

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \sin\left(\left[n + \frac{1}{2}\right] \frac{\pi x}{\ell}\right) dx \quad b_0 = \frac{2}{n\pi c} \int_0^{\ell} g(x) dx$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0, t) = u(\ell, t) = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = a_0 + b_0 t + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \left[ n + \frac{1}{2} \right] \frac{\pi t}{\ell} \right) + b_n \sin \left( \left[ n + \frac{1}{2} \right] \frac{\pi t}{\ell} \right) \right] \cos \left( \left[ n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \left( \left[ n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right) dx \quad a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx$$

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \cos \left( \left[ n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right) dx \quad b_0 = \frac{2}{n\pi c} \int_0^{\ell} g(x) dx$$

### 1.3 Laplace Equation

#### Rectangular Region

##### Dirichlet

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = u(\ell, y) = 0 \end{cases}$$

$$u(x, y) = \sum_{n=1}^{\infty} \left[ b_n \sinh \left( \frac{n\pi y}{\ell} \right) + a_n \cosh \left( \frac{n\pi y}{\ell} \right) \right] \sin \left( \frac{n\pi x}{\ell} \right)$$

##### Neumann

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = u_x(\ell, y) = 0 \end{cases}$$

$$u(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \left[ b_n \sinh \left( \frac{n\pi y}{\ell} \right) + a_n \cosh \left( \frac{n\pi y}{\ell} \right) \right] \cos \left( \frac{n\pi x}{\ell} \right)$$

##### Robin

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = u_x(\ell, y) = 0 \end{cases}$$

$$u(x, y) = \sum_{n=1}^{\infty} \left[ b_n \sinh \left( \left[ n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) + a_n \cosh \left( \left[ n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) \right] \sin \left( \left[ n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = u(\ell, y) = 0 \end{cases}$$

$$u(x, y) = \sum_{n=1}^{\infty} \left[ b_n \sinh \left( \left[ n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) + a_n \cosh \left( \left[ n + \frac{1}{2} \right] \frac{\pi y}{\ell} \right) \right] \cos \left( \left[ n + \frac{1}{2} \right] \frac{\pi x}{\ell} \right)$$

#### Circular Region

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(h, \theta) = f(\theta) \end{cases}$$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{r^n}{a^n} \cos(n\theta) + b_n \frac{r^n}{a^n} \sin(n\theta)$$

## 2 Non-Homogeneous Solutions

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### 2.1 The Heat Equation

Dirichlet

$$\begin{cases} u_t - ku_{xx} = h(x, t) \\ u(0, t) = u(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[ b_n'(t) + k \left(\frac{n\pi}{\ell}\right)^2 b_n(t) \right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x, t)$$

Neumann

$$\begin{cases} u_t - ku_{xx} = h(x, t) \\ u_x(0, t) = u_x(\ell, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$

$$u(x, t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{a_0'(t)}{2} + \sum_{n=1}^{\infty} \left[ a_n'(t) + k \left(\frac{n\pi}{\ell}\right)^2 a_n(t) \right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x, t)$$

### 2.2 The Wave Equation

Dirichlet

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x, t) \\ u(0, t) = u(\ell, t) = 0 \\ u(x, t) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \left[ b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t) \right] \sin\left(\frac{n\pi x}{\ell}\right) = h(x, t)$$

Neumann

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x, t) \\ u_x(0, t) = u_x(\ell, t) = 0 \\ u(x, t) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

$$u(x, t) = \frac{b_0(t)}{2} + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{b_0''(t)}{2} + \sum_{n=1}^{\infty} \left[ b_n''(t) + \left(\frac{n\pi}{\ell}\right)^2 b_n(t) \right] \cos\left(\frac{n\pi x}{\ell}\right) = h(x, t)$$

### 2.3 Non-Homogeneous Boundaries

$$\begin{cases} u_t - ku_{xx} = h(x, t) \\ u(0, t) = L(t), \quad u(\ell, t) = R(t) \\ u(x, 0) = f(x) \end{cases} \quad \begin{cases} u_t - ku_{xx} = h(x, t) \\ u(0, t) = L(t), \quad u(\ell, t) = R(t) \\ u(x, 0) = f(x) \end{cases}$$

$$\delta(x, t) = L + \frac{x}{\ell}(R - L)$$

$$\gamma(x, t) = u(x, t) - \delta(x, t)$$

### 3 Cauchy Problems

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#### 3.1 The Heat Equation

Homogeneous

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y) dy$$

Non-Homogeneous

$$\begin{cases} u_t - ku_{xx} = h(x, t) \\ u(x, 0) = f(x) \end{cases}$$
$$u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y) dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} h(y, s) dy ds$$

#### 3.2 The Wave Equation

Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x, 0) = f(x), u_t(x, 0) = g(x) \end{cases}$$
$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

Non-Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x, t) \\ u(x, 0) = f(x), u_t(x, 0) = g(x) \end{cases}$$
$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y, s) dy ds$$