***Steven Rosendahl***
***Homework 5***

1. Let $f(x, y) = x^2 + y^2$. Observe that $(x, y) = (0, 0)$ defines a solution to the equation $f(x, y) = 0$. Do the remainder of this problem, we shall call $(0, 0)$ the trivial solution to $f(x, y) = 0$.

   (a) Prove that $f(x, y) = 0$ has no non-trivial solutions with $x, y \in \mathbb{Q}$.

   (b) Prove that there exists a prime $p$ such that $f(x, y) = 0$ has a nontrivial solution with $x, y \in \mathbb{Q}_p$.

   (c) Verify that $f$ satisfies the Hasse Principle without referring to the Hasse-Minkowski Theorem.

2. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of points in $\mathbb{Z}_p$. Prove that the series

$$\sum_{n=0}^{\infty} a_n p^n$$

   converges in $\mathbb{Q}_p$.

   ***Proof:*** We know that a series will converge if and only if its sequence of partial sums converges. We can test for convergence by calculating

$$\lim_{n\to\infty} \left|a_{n+1}p^{n+1} - a_n p^n\right|_p .$$

   We can say

$$\begin{aligned}
\left|a_{n+1}p^{n+1} - a_n p^n\right|_p &= \max\left\{\left|a_{n+1}p^{n+1}\right|_p, \left|a_n p^n\right|_p\right\} \\
&= \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\}. \\
\lim_{n\to\infty} \max\left\{\frac{1}{p^{v_p(a_{n+1})+n+1}}, \frac{1}{p^{n+v_p(a_n)}}\right\} &= \max\{0, 0\} \\
&= 0.
\end{aligned}$$

   Therefore, the sequence of partial sums is Cauchy, and thus convergent, so the series converges.

3. Suppose that $k \in \mathbb{N}$.

   (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

   does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.

   (b) Prove that the series

$$\sum_{n=1}^{\infty} n^k$$

   does not converge in $\mathbb{Q}_p$ for any prime $p \neq \infty$.