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Homework 6

1. Find the radius of convergence of the series

$$f(x) = \sum_{i=0}^{\infty} \left(\frac{x}{7}\right)^n$$

in $\mathbb{Q}_p$.

We will consider the case in $\mathbb{Q}_7$ first. We need to find

$$\begin{aligned} r &= \left(\limsup_{n\to\infty}\left(\left|\frac{1}{7^n}\right|_7\right)^{1/n}\right)^{-1} \\ &= \left(\limsup_{n\to\infty}(7^n)^{1/n}\right)^{-1} \\ &= \left(\limsup_{n\to\infty} 7\right)^{-1} \\ &= \frac{1}{7}. \end{aligned}$$

By the p-adic root test, this means that the series converges for all $x \in \mathbb{Q}_p$ such that $|x|_p < 1/7$. To check convergence in $\mathbb{Q}_p$ when $p \neq 7$, we can consider

$$\begin{aligned} \left|\frac{1}{7^n}\right| &= p^{-v_p(1/7^n)} \\ &= p^{-v_p(1)+v_p(7^n)} \\ &= p^{0+0} \quad \text{since there will be no factors of } p \text{ in } 7^n \text{ when } p \neq 7 \\ &= 1, \end{aligned}$$

so

$$r = \frac{1}{\limsup_{n\to\infty}\left|\frac{1}{7^n}\right|_p^{1/n}} = \frac{1}{\limsup_{n\to\infty} 1} = 1,$$

and the radius of convergence is 1. Therefore, $f(x)$ converges on $\{x : |x|_p < 1\}$, also known as $\mathbb{Z}_p$.

2. Suppose that $\{a_n\}$ is the sequence of points in $\{1, 2, 3, 4, 5\}$ defined so that $a_n \equiv n + 1 \mod 5$. That is, $\{a_n\}$ is given by

$$1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, \ldots$$

Find the radius of convergence r of the series

$$g(x) = \sum_{n=0}^{\infty}\left(\frac{x}{a_n}\right)^n$$

in $\mathbb{Q}_5$ and in $\mathbb{Q}_3$. In each case, determine whether the series converges when $|x|_p = r$.

Consider the case in $\mathbb{Q}_5$. We have that

$$\begin{aligned}
r &= \frac{1}{\limsup_{n\to\infty} \left|\frac{1}{a_n^n}\right|_5^{1/n}} \\
&= \frac{1}{\limsup_{n\to\infty} \left(|a_n|_5^{-n}\right)^{1/n}} \\
&= \frac{1}{\limsup_{n\to\infty} |a_n|_5^{-1}} \\
&= \frac{1}{\limsup_{n\to\infty} \left|\frac{1}{a_n}\right|_5} \\
&= \frac{1}{5}.
\end{aligned}$$

We know that $1/5$ is the right choice, since the 5-adic absolute value of $1/5$ is the largest value that appears in the sequence 5-adically. Therefore, the radius of convergence of the sequence is $1/5$. We need a value that has $|x|_5 = r$, so we will let $x = 5$. Evaluating $g(5)$ gives us

$$\sum_{n=0}^{\infty} \left(\frac{5}{a_n}\right)^n,$$

which we can analyze for convergence by looking at

$$\begin{aligned}
\lim_{n\to\infty} \left|\left(\frac{5}{a_n}\right)^n\right|_5 &= \lim_{n\to\infty} 5^{-v_5(5^n)+v_5(a_n^n)} \\
&= \lim_{n\to\infty} \frac{5^{nv_5(a_n)}}{5^n}
\end{aligned}$$

If we consider the subsequence of a_n consisting of $\{5, 5, 5, 5, \dots\}$, then the limit tends to 1 since

$$\lim_{n\to\infty} \frac{5^{nv_5(a_n)}}{5^n} = \lim_{n\to\infty} \frac{5^n}{5^n} = 1$$

Similarly, for the subsequence defined by $\{1, 2, 3, 4, 1, 2, 3, 4, \dots\}$, we have

$$\lim_{n\to\infty} \frac{5^{nv_5(a_n)}}{5^n} = \lim_{n\to\infty} \frac{5^0}{5^n} = 0,$$

So the limit does not exist. Therefore, the sequence does not converge on the boundary.

Consider the case in $\mathbb{Q}_3$. Then we can say that

$$\frac{1}{\limsup_{n\to\infty} \left(|a_n|_3^{-n}\right)^{1/n}} = \frac{1}{\limsup_{n\to\infty} \left|\frac{1}{a_n}\right|_3} = \frac{1}{3}.$$

We need to pick a point on the boundary (i.e. $|x|_3 = 1/3$), so let $x = 3$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{3^n}{a_n^n}\right|_3 &= \lim_{n\to\infty} 3^{-v_3(3^n)+v_3(a_n^n)} \\
&= \lim_{n\to\infty} \frac{3^{nv_3(a_n)}}{3^n}.
\end{aligned}$$

If we consider the subsequence of a_n given by $\{3, 3, 3, 3, \dots\}$, we see that

$$\lim_{n\to\infty} \frac{3^{nv_3(a_n)}}{3^n} = \lim_{n\to\infty} \frac{3^n}{3^n} = 1,$$

and if we consider the subsequence given by $\{1, 2, 4, 5, 1, 2, 4, 5, \dots\}$, then we find that

$$\lim_{n\to\infty} \frac{3^{nv_3(a_n)}}{3^n} = \lim_{n\to\infty} \frac{3^0}{3^n} = 0.$$

Therefore, the limit does not exist, so the series diverges on the boundary of the region of convergence.

3. Find the radius of convergence of the power series

$$h(x) = \sum_{n=0}^{\infty} n! x^n$$

in $\mathbb{Q}_p$.

To begin, we want a formula for $v_p(n!)$. Luckily, there' a nice theorem that can help us here.

Legendre's Theorem:

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

In order to determine the radius of convergence, we need to evaluate

$$\frac{1}{\limsup_{n\to\infty}(|n!|_p)^{1/n}}.$$

Consider the quantity inside the lim sup expression. We need to calculate $(p^{-v_p(n!)})^{(1/n)}$. Note that

$$\begin{aligned}
\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor &\le \sum_{k=1}^{\infty} \frac{n}{p^k} \\
-\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor &\ge -\sum_{k=1}^{\infty} \frac{n}{p^k} \\
p^{-\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor} &\ge p^{-\sum_{k=1}^{\infty} \frac{n}{p^k}} \\
\frac{1}{p^{-\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor}} &\le \frac{1}{p^{-\sum_{k=1}^{\infty} \frac{n}{p^k}}},
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{\limsup_{n\to\infty} |n!|_p^{1/n}} &= \frac{1}{\limsup_{n\to\infty} p^{\left(-\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor\right)^{1/n}}} \\
&\le \frac{1}{\limsup_{n\to\infty} p^{\left(-\sum_{k=1}^{\infty} \frac{n}{p^k}\right)^{1/n}}} \\
&= \frac{1}{\limsup_{n\to\infty} p^{-\sum_{k=1}^{\infty} \frac{1}{p^k}}}.
\end{aligned}$$

By the geometric series test, we find that

$$\sum_{k=1}^{\infty} \frac{1}{p^k} = \frac{1}{1-p},$$

so

$$\frac{1}{\limsup_{n\to\infty} |n!|_p^{1/n}} \le p^{\frac{1}{p-1}}$$

To determine a lower bound for the expression, consider

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \geq \sum_{k=1}^{\infty} \frac{n}{p^k} - 1.$$

Using a similar argument as above, we find that

$$\frac{1}{p^{-\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor}} \geq \frac{1}{p^{\left(-\sum_{k=1}^{\infty} \frac{n}{p^k} - 1\right)}}$$

so that

$$\begin{aligned}
\frac{1}{\limsup_{n\to\infty} \left(p^{-\sum_k \left\lfloor \frac{n}{p^k} \right\rfloor}\right)^{1/n}} &\geq \frac{1}{\limsup_{n\to\infty} \left(p^{-\sum_k \left(\frac{n}{p^k} - 1\right)}\right)^{1/n}} \\
&= \frac{1}{\limsup_{n\to\infty} p^{-\sum_k \frac{1}{p^k} - \frac{1}{n}}} \\
&= \frac{1}{p^{-\sum_{k=1}^{\infty} \frac{1}{p^k}}} \\
&= \frac{1}{p^{-\frac{1}{p-1}}} \\
&= p^{\frac{1}{p-1}}
\end{aligned}$$

Hence

$$p^{\frac{1}{p-1}} \leq \frac{1}{\limsup_{n\to\infty} (|n!|_p)^{1/n}} \leq p^{\frac{1}{p-1}} \implies \frac{1}{\limsup_{n\to\infty} (|n!|_p)^{1/n}} = p^{\frac{1}{p-1}}$$