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Homework 3

1. Suppose F is a field with a non-Archimedean absolute value $|\cdot|$ and \mathcal{O}_F is the ring of integers of F . Prove that a point $x \in \mathcal{O}_F$ is a unit if and only if $|x| = 1$.

Suppose $x \in \mathcal{O}_F$ is a unit and consider $x^{-1} \in \mathcal{O}_F$ as the inverse of x . Since $x \in \mathcal{O}_F$, we know

$$\begin{aligned} |x| &\leq 1 \\ |x^{-1}||x| &\leq |x^{-1}| \leq 1 \quad (\text{since } x^{-1} \in \mathcal{O}_F) \\ |x^{-1}x| &\leq |x^{-1}| \leq 1 \\ |1| &\leq |x^{-1}| \leq 1 \\ 1 &\leq |x^{-1}| \leq 1. \end{aligned}$$

The only choice we have for x^{-1} is 1, so

$$\begin{aligned} x^{-1} &= 1 \\ xx^{-1} &= x \\ x &= 1 \\ |x| &= 1. \end{aligned}$$

Suppose $|x| = 1$.

2. If n is a non-negative integer a p is prime, define $\mathcal{I}_n = \{x \in \mathcal{O}_p : |x|_p \leq p^{-n}\}$.

- (a) If $x, y \in \mathcal{I}_n$ prove that $x + y \in \mathcal{I}_n$.

Suppose $x, y \in \mathcal{I}_n$, and consider $|x + y|_p$. Then

$$\begin{aligned} |x + y|_p &\leq \max\{|x|_p, |y|_p\} \\ &\leq \max\{p^{-n}, p^{-n}\} \\ &= p^{-n} \end{aligned}$$

Therefore $|x + y|_p \leq p^{-n}$, so $x + y \in \mathcal{I}_n$.

- (b) If $x \in \mathcal{I}_n$ and $r \in \mathcal{O}_p$ prove that $rx \in \mathcal{I}_n$.

Consider $|rx| = |r| \cdot |x|$. Since $|x| \leq p^{-n}$ and $|r| \leq 1$, we have $|r| \cdot |x| \leq 1 \cdot p^{-n} = p^{-n}$, so $|rx| \leq p^{-n}$ which implies that $rx \in \mathcal{I}_n$.

- (c) What is another way to describe the set $\mathbb{Z} \cap \mathcal{I}_n$?