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Homework 8

1. Determine whether each of the following polynomials has a zero in the given $\mathbb{Z}_p$. Either find a zero or prove that no zero exists.

   (a) $f(x) = x^3 + x^2 + x + 1$ in $\mathbb{Z}_2$.

   We can factor $x^3 + x^2 + x + 1$ into $(x^2 + 1)(x + 1)$. The only two options for a root in $\mathbb{Z}_2$ are 1 and 0. If we plug in zero we get $(0 + 1)(0 + 1) \equiv 1 \mod 2$, so 1 is not a zero of the polynomial. If we try 1, we get $(1 + 1)(1 + 1) \equiv (2)(2) \equiv (0)(0) \equiv 0 \mod 2$, so 1 is a zero of the polynomial.

   (b) $f(x) = x^3 + x^2 + x + 1$ in $\mathbb{Z}_3$.

   $f(x)$ factors into $(x^2 + 1)(x + 1) \equiv (x^2 - 2)(x - 2) \mod 3$. We have one zero at $x = 2$. If we test 0, we get $(0 + 1)(0 + 1) \equiv 1 \mod 3$, so 0 is not a zero of the polynomial, and if we test 1 we get $(-1)(-1) \equiv 1 \mod 3$, so 1 is not a zero either.

   (c) $f(x) = x^2 + 2x + 3$ in $\mathbb{Z}_5$.

   The polynomial $f(x) = x^2 + 2x + 3$ cannot be factored in $\mathbb{Z}_5$. Therefore, it has no zeros since we cannot express it in the form $g(x)(x - \alpha)$.

2. Let $f(x) = x^2 + 1$. Prove that f has a zero in $\mathbb{Z}_5$, but not in $\mathbb{Z}_7$.

   If we consider the polynomial in $\mathbb{Z}_5$, we have that

   $$x^2 + 1 \equiv x^2 - 4 \equiv (x + 2)(x - 2) \equiv (x - 3)(x - 2) \mod 5.$$

   Since f was reducible in $\mathbb{Z}_5$, we know it has at least one zero, and in this case it has two zeros, $x = 3$ and $x = 5$. In $\mathbb{Z}_7$, we have

   $$x^2 + 1 \equiv x^2 - 6 \mod 7.$$

   The polynomial $x^2 - 6$ is irreducible in $\mathbb{Z}_7$, so there are no roots.