CS461: Computer Graphics

Assignment 3

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Quaternions

In mathematics, the **quaternions** are a number system that extends the complex numbers. They were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space. He used quaternions as a method to multiply and divide vectors, rotating and stretching them.

Hamilton defined a quaternion as the quotient of two directed lines in a three-dimensional space or equivalently as the quotient of two vectors.

Quaternions can be represented in the form:

$$s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

where s, x, y, and z are real numbers, and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the fundamental quaternion units.

Quaternions are used in pure mathematics, and also have practical uses in applied mathematics—in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics, computer vision, and crystallographic texture analysis. In practical applications, they can be used alongside other methods, such as Euler angles and rotation matrices, or as an alternative to them, depending on the application.

History and Motivation

Complex numbers can be represented as points in a plane (Basically in 2D). Hamilton was looking for a way to do the same for 3D Space. We can represent complex number in 2d, by taking horizontal axis as real part and vertical axis as imaginary part. We can also perform rotation in the

complex plane by multiplying complex number and the rotor of the form:

$$q = cos\theta + \boldsymbol{i}\sin\theta$$

Visit for more on Complex numbers.

Now we have this knowledge of Complex plane, we can extend this to 3D Space by adding two imaginary numbers to number system in addition to $\boldsymbol{i}$(iota).

Quaternions are composed of a scalar and a vector. They can be represented in vector form as:

$$q = s + \mathbf{a}$$

$$q = [s, \mathbf{a}]$$

Where s is a scalar number and $\mathbf{a}$ is a vector representing an axis.

or in another form as:

$$q = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad s, x, y, z \in \mathbb{R}$$

Where, according to Hamilton's famous expression:

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

Basic Operations

Additions and Subtractions

Quaternions can be added and subtracted similar to Complex numbers-

Let q_a and q_b are two quaternions represented as:

$$\begin{array}{l} q_a = s_a + x_a\boldsymbol{i} + y_a\boldsymbol{j} + z_a\boldsymbol{k} \\ q_b = s_b + x_b\boldsymbol{i} + y_b\boldsymbol{j} + z_b\boldsymbol{k} \end{array}$$

Then

$$q_a \pm q_b = (s_a \pm s_b) + (x_a \pm x_b)\boldsymbol{i} + (y_a \pm y_b)\boldsymbol{j} + (z_b \pm z_b)\boldsymbol{k}$$

Or in Vector form as

$$q_a = [s_a \, , \mathbf{a}]$$
$$q_b = [s_b \, , \mathbf{b}]$$

$$q_a + q_b = [s_a + s_b \, , \mathbf{a} + \mathbf{b}]$$
$$q_a - q_b = [s_a - s_b \, , \mathbf{a} - \mathbf{b}]$$

Scalar Multiplication

Multiplication with scalar is also like Complex Numbers. Let q be a Quaternion and k be a Scalar then Scalar Multiplication will be-

$$q = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \text{or} \quad q = [s, \mathbf{a}]$$

Then,

$$kq = ks + kx\boldsymbol{i} + ky\boldsymbol{j} + kz\boldsymbol{k} \quad \text{or} \quad kq = [ks, k\mathbf{a}]$$

Product

Let q_a and q_b are two quaternions represented as:

$$q_a = s_a + x_a\boldsymbol{i} + y_a\boldsymbol{j} + z_a\boldsymbol{k} \ \text{ or } q_a = [s_a \, , \mathbf{a}]$$
$$q_b = s_b + x_b\boldsymbol{i} + y_b\boldsymbol{j} + z_b\boldsymbol{k} \ \text{ or } \ q_b = [s_b \, , \mathbf{b}]$$

Then

$$q_a q_b = [s_a s_b - \mathbf{a} \, . \mathbf{b}, s_a\mathbf{b} + s_b\mathbf{a} + \mathbf{a} \times \mathbf{b} \,]$$

Also, can be expressed as -

$$\begin{aligned}
q_a q_b &= [s_a \, , \mathbf{a}][s_b, \mathbf{b}] \\
&= (s_a + x_a i + y_a j + z_a k)(s_b + x_b i + y_b j + z_b k) \\
&= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\
&\quad +(s_a x_b + s_b x_a + y_a z_b - y_b z_a)i \\
&\quad +(s_a y_b + s_b y_a + z_a x_b - z_b x_a)j \\
&\quad +(s_a z_b + s_b z_a + x_a y_b - x_b y_a)k
\end{aligned}$$

Norm

Norm of $q = [s, \mathbf{a}]$ can be defined as:

$$q = \sqrt{s^2 + a^2}$$

Where a is the norm of vector $\mathbf{a}$.

Unit Norm

Unit Norm (Normalized quaternion) of $q = [s, \mathbf{a}]$ can be defined as:

$$q = \frac{q}{\sqrt{s^2+a^2}}$$

Where a is the norm of vector $\mathbf{a}$.

Conjugate and Inverse

Conjugate of $q = [s, \mathbf{a}]$ can be defined as:

$$q^* = [s, -\mathbf{a}]$$

And Inverse is defined as:

$$q^{-1} = \frac{q^*}{q}$$

Modulus

The absolute value of a Quaternion is the scalar quantity that determines the length of the quaternion from the origin.

Modulus of $q = [s, \mathbf{a}]$ can be defined as:

$$|q| = \sqrt{q.q} = \sqrt{q^*q} = s^2 + x^2 + y^2 + z^2$$

Pure Quaternion

Hamilton defined the Pure Quaternion as a quaternion that has a zero Scalar term.

$$q = [0, \mathbf{a}]$$

Rotations

In Order to correctly rotate a vector q by an angle θ about an arbitrary axis $\mathbf{u} = [x, y, z]$, we should consider the half-angle. Unit quaternions which represent rotation are also called unit quaternions.

These quaternions can be defined as –

$$q = [c, sx, sy, sz] \quad \text{or} \quad q = [c, s\mathbf{u}]$$

where $c = cos\frac{\theta}{2}$ and $s = sin\frac{\theta}{2}$

Proof

Let $\vec{u}$ be the rotation axis and α be the rotation angle. Then we want to prove that quaternion $q = [cos(\frac{\alpha}{2}), sin(\frac{\alpha}{2})\vec{u}]$ rotates the vector $\vec{v}$ by angle α radians by using the following equations

$$\vec{v'} = q\vec{v}q^{-1} = \left(cos(\frac{\alpha}{2}) + sin(\frac{\alpha}{2})\right)\vec{v}\left(cos(\frac{\alpha}{2}) - sin(\frac{\alpha}{2})\right)$$

$$= \vec{v}cos^2(\frac{\alpha}{2}) + (\vec{u}\vec{v} - \vec{v}\vec{u})cos(\frac{\alpha}{2})sin(\frac{\alpha}{2}) - \vec{u}\vec{v}\vec{u}sin^2\frac{\alpha}{2}$$

$$= \vec{v}cos^2(\frac{\alpha}{2}) + 2(\vec{u} \times \vec{v})cos(\frac{\alpha}{2})sin(\frac{\alpha}{2}) - (\vec{v}(\vec{u}.\vec{u}) - 2\vec{u}(\vec{u}.\vec{v}))sin^2\frac{\alpha}{2}$$

$$= \vec{v}(cos^2(\frac{\alpha}{2}) - sin^2(\frac{\alpha}{2})) + (\vec{u} \times \vec{v})(2cos(\frac{\alpha}{2})sin(\frac{\alpha}{2})) + \vec{u}(\vec{u}.\vec{v})2sin^2\frac{\alpha}{2}$$

$$= \vec{v}cos(\alpha) + (\vec{u} \times \vec{v})sin(\alpha) + (\vec{u}.\vec{v})(1 - cos(\alpha))$$

$$= (\vec{v} - \vec{u}(\vec{u}.\vec{v}))cos(\alpha) + (\vec{u} \times \vec{v})sin(\alpha) + \vec{u}(\vec{u}.\vec{v})$$

$$= \vec{v_\perp}cos(\alpha) + (\vec{u} \times \vec{v})sin(\alpha) + \vec{v_\|}$$

where $\vec{v}_\perp$ and $\vec{v}_\|$ are the components of $\vec{v}$ perpendicular and parallel to $\vec{u}$ respectively. Hence it can be seen that we get the same result as we would have gotten by matrix multiplication. This proves correctness of rotation. Also we can see that we didn't have to do the rotation axis by axis.

Use in Computer graphics

Quaternions are used in Computer Graphics to represent rotations and orientations of objects in 3D space. They are smaller than other representations (matrices etc.) and operations on them like composition can be computes more efficiently. They are also used in other areas such as control theory, signal processing, attitude control, physics and orbital mechanics, where they are mainly used for representing rotations/orientations in 3D. benefit of using quaternions is that combining many quaternion transformations is more numerically stable than combining many matrix transformations, avoiding such phenomena as Gimbal lock, which can occur when Euler angles are used.

Using quaternions also reduces overhead from that when rotation matrices are used, because one varies only four components, not nine, and the multiplication algorithms to combine successive rotations are faster.

There are several methods which can be defined to represent a rotational interpolation in 3D space using Quaternions. For example- SLERP, used to smoothly interpolate a point between two orientations and SQUAD, extension of SLERP which is used to interpolate through a sequence of orientations that define a path.