Assignment - 3

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Define an implicit function given a set of Points? By using Concept of normal- Poisson surface reconstruction

- Intution :- Surface reconstruction is to transform a finite sample into a surface model. Althogh there are several obstacles while designing such algorithm to do this task. The sample could very large, noisy or to sparse too capture all the feature of the solid. Also, accessibility Constrants during scanning may leave some regions. so the algorithm try to fit noisy data accurately, fill holes accordin-gly by figuring out the topology of the unknown surface.

several approaches are based on Combinatorial structures such as Delauny triangulations, alpha shapes or voronoi diagrams. These Schemes typically create a triange mesh that interpolates all or most of the Points. In the presence of noisy data, the resulting surface is often jagged, and is therefore smoothed or refit to the Points in subsequent Processing.

While there are some schemes directly recons-truct an approximating surface, typically represented in implicit form. We can classify these schemes in either global or local approaches.

- Global fitting Methods - Define implicit function as the sum of radial basis functions centered at the Points.
- Local fitting methods - Try to estimate tangent planes. This is done by taking a subset of k nearby Points at a time. After this try to define the implicit function as the signed distance to the tangent plane of the closest Point.

An implicit model is a function $f: R^3 \to R$ such that the zero set of f i.e. $f^{-1}(0)$, is a surface that interpolates or approximate the sample.

We will define an implicit function approach to solve the problem of surface reconstruction. We can express surface reconstruction as the solution to a Poisson eq^n. Poisson reconstruction uses both global fitting method and local fitting method.

3D Indicator function (χ) - Defined as ~~the~~ 1 for Points inside the surface model and 0 for Points outside the surface model.

we will use 3D indicator function to obtain the reconstructed surface by extracting the appropriate iso surface

Mathematical background -
Poisson reconstruction Approach -
The input data S is a set of samples $s \in S$, each consisting of a point s.p. and an inward-facing normal $s.\vec{N}$. These points are supposed to lie on or near the surface M of an unknown model M. Task is to reconstruct a tight, triangulated approximation to the surface by approximating the indicator function of the model and extracting the iso surface.
Implicit function Approach
Now, we define a function with value less than zero outside the model and greater than zero inside the model. Then we extract the zero-set.
we construct the surface of the model by solving for the indicator function of the shape.

$$\chi_m(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases}$$

In practice, we define the indicator function to be $-1/2$ outside the shape and $1/2$ inside, so that the surface is at the zero level set. we do this to smooth the function a little.

The challenge is to Calculate the indicator function. we have to derive a relation between the gradient of the indicator function and an integral of the surface normal field. we then approximate this surface integral by a summation over the given oriented point samples. finally, we reconstruct the indicator function from this gradient field ~~to~~ as a Poisson Problem.
since, the indicator function is constant (piecewise constant) function, explicit computation of its gradient field ~~would~~ would result in a vector field with unbounded values at the surface boundary. To avoid this, we compute the indicator function with a smoothing filter and consider the gradient field of the smoothed function.

The following lemma formalizes the ~~indicator~~ relation ship between the gradient of the smoothed indicator function and the surface normal field.
Lemma :- Given a solid M with boundary ∂M, let χ_M ~~be~~ denote the indicator function of M, $\vec{N}_{\partial M}(p)$ be the inward surface normal at $p \in \partial M$, $\vec{F}(q)$ be a smoothing filter, and $\tilde{F}_p(q) = \tilde{F}(q-p)$ its translation to the point p. The gradient of the smoothed indicator function is equal to the vector field obtained by smoothing the surface normal

field.

$$\nabla(\chi_M * \tilde{F})(q_0) = \int_{\partial M} \tilde{F}_p(q_0)\, \vec{N}_{\partial M}(p)\, dp \quad \text{— (1)}$$

This can be proved by showing equality of each field. Computing the partial derivative of the smoothed function with respect to x

$$\frac{\partial}{\partial x}\bigg|_{q_0} (\chi_M * \tilde{F}) = \frac{\partial}{\partial x}\bigg|_{q=q_0} \int_M \tilde{F}(q-p)\, dp$$

$$= \int_{\partial M} \langle \tilde{F}_p(q_0), 0, 0),\ \vec{N}_{\partial M}(p) \rangle\, dp$$

Here the first equality follows from the fact that χ_M is equal to zero outside of M and one inside. The second follows from the fact that $(\partial/\partial q)\tilde{F}(q-p) = -(\partial/\partial p)\tilde{F}(q-p)$.
A similar argument can be shown for y and z.

- Approximating the gradient field.

we can get surface integral using the point set S to partition δM into distinct patches $P_s \subset \delta M$. we can approximate the integral over a patch P_s by the value at point sample s.p.

$$\nabla(\chi_m * \tilde{F})(q) = \sum_{s \in S} \int_{P_s} \vec{F}_p(q)\, \vec{N}_{\delta M}(p)\, dp \quad (2)$$

$$\approx \sum_{s \in S} |P_s|\, \tilde{F}_{s.p.}(q)\, s.\vec{N} \equiv \vec{V}(q) \quad \text{— (3)}$$

Solving the Poisson Problem – Having formed a vector field $\vec{V}$, we want to solve for the function $\tilde{\chi}$ such that $\nabla\tilde{\chi} = \vec{V}$. However, $\vec{V}$ is generally not integrable, so an exact solution does not ~~guarantee~~ generally exist. To find the best least-square approximate solution, we apply the divergence operator to form the Poisson equation

$$\Delta\tilde{\chi} = \nabla\cdot\vec{V}$$

Now, the problem of computing the indicator function χ whose gradient best approximates a vector field $\vec{V}$ defined by the samples.

i.e.

$\min_{\chi} \|\nabla\chi - \vec{V}\|$.

If we apply the divergence operator, the variational problem transforms into a standard Poisson eqn: compute the scalar function χ whose Laplacian equals the divergence of the vector field $\vec{V}$.

$$\Delta\chi \equiv \nabla\cdot\nabla\chi = \nabla\cdot\vec{V}$$

for any Poisson eqn of the form $\nabla^2 f(x_i) = \nabla\cdot g_i$ subject to some boundary conditions, we can represent f as following

$$f(x) = \sum_{j=1}^{K} a_j B_j(\|x - c_j\|)$$

we solve linear system for the coefficients a_j and compute 2nd derivatives of B_j to solve the eqn

Implementation- first Assume that the Point Samples are uniformly distributed over the model Surface. Then define a Space of functions with high resolution near the surface of the model and Coarser resolution away from it, express the $\vec{\ }$ as a linear sum of functions in this Space, set up and solve the Poisson eq^n, and extract an isosurface of the resulting indicator fu^n.