Tensor Analysis and Tensor Calculus

* Basics of Vectors (revision)

→ $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

→ $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\ \hat{n}$

↳ cross-product is anti symmetry

→ Gradient : $(\vec{\nabla}\phi)$

↳ directional derivative : $(\vec{\nabla}\phi)\cdot\hat{n}$

→ Divergence : $\vec{\nabla}\cdot\vec{V}$

→ Curl : $\vec{\nabla}\times\vec{V}$

→ Cartesian coordinate system: $(\hat{i},\hat{j},\hat{k})$. Calling generalized coordinal symbols for cartesian $\hat{e}_n\hat{e}_m = \delta_{nm}$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{i}\cdot\hat{k} = 0$ } (they are orthonormal)

→ Changing some notations, $x^1 = x,\ x^2 = y,\ x^3 = z$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \equiv \vec{r} = \sum_{i=1}^{3} x^i\hat{e}_i$ & $\vec{A} = \sum_{i=1}^{3} A^i\hat{e}_i$

→ A basis vectors: If they span a space and they are linearly independent

→ $\vec{A} = \sum_{n=1}^{3} A^n\hat{e}_n$; $\vec{B} = \sum_{m=1}^{3} B^m\hat{e}_m$; $\vec{A}\cdot\vec{B} = \sum_{m=1}^{3}\sum_{n=1}^{3}(A^nB^m)(\hat{e}_n\cdot\hat{e}_m) = \sum_{m=1}^{3}\sum_{n=1}^{3}A^nB^m\delta_{nm}$

$= \sum_{n=1}^{3}A^nB^n$

then $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A^x & A^y & A^z \\ B^x & B^y & B^z \end{vmatrix}$

→ ~~(x^1,x^2,x^3)~~ $(x,y,z) \longrightarrow (x',y',z')$

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$z' = z$

$\frac{\partial x'}{\partial x} = \cos\theta$ similarly others.

$\therefore x'^i = \sum_j x^j\frac{\partial x'^i}{\partial x^j} \Rightarrow x'' = x' = x\frac{\partial x'}{\partial x} + y\frac{\partial x'}{\partial y} + z\frac{\partial x'}{\partial z}$

$\therefore A'^x = A^x\cos\theta + A^y\sin\theta$

$\vdots$

$A'^i = \sum_j A^j\frac{\partial x'^i}{\partial x^j}$

→ outer product : $|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow |1\rangle\langle 1| = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$|2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $|3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = I$

→ $|A\rangle = A^1|1\rangle + A^2|2\rangle + A^3|3\rangle$; $I = \sum_\alpha|\alpha\rangle\langle\alpha|$ $\Rightarrow |A\rangle = \sum|\alpha\rangle\langle\alpha|A\rangle = \sum A^\alpha|\alpha\rangle$

→ $\hat{M} = |A\rangle\langle B|$ $\Rightarrow M^{ij} \equiv \langle i|M|j\rangle = \langle i|A\rangle\langle B|j\rangle$

→ $\langle\alpha|A\rangle = \sum_\alpha\langle\alpha|i\rangle A^i$; $A'^\alpha = \sum_j A^j\frac{\partial x'^\alpha}{\partial x^j}$ || $\frac{\partial x'^\alpha}{\partial x^i} = \langle\alpha|i\rangle = \Lambda^{\alpha i}$

$|A'\rangle = \Lambda^{\alpha i}|A\rangle$

* Two-Idex tensor example.

→ Remember Polarization? $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

Here, we assumed that $\vec{P} \parallel \vec{E}$. but this might not be the case for all materials.

We may write this as ~~an~~ matrix multiplication for more general representation.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \Rightarrow \begin{pmatrix} P^x \\ P^y \\ P^z \end{pmatrix} = \begin{pmatrix} \chi^{xx} & 0 & 0 \\ 0 & \chi^{yy} & \cancel{\chi^{yz}}0 \\ 0 & 0 & \chi^{zz} \end{pmatrix} \begin{pmatrix} E^x \\ E^y \\ E^z \end{pmatrix} \rightsquigarrow \text{for the case of } \vec{P} \parallel \vec{E}$$

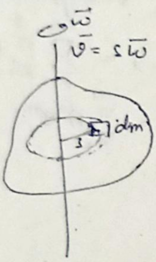
If we consider general material.

$$\vec{P} = N\langle P \rangle \quad \text{then} \quad \chi = \begin{pmatrix} \chi^{xx} & \chi^{xy} & \chi^{xz} \\ \chi^{yx} & \chi^{yy} & \chi^{yz} \\ \chi^{zx} & \chi^{zy} & \chi^{zz} \end{pmatrix}$$

$$P^y = \varepsilon_0 \left[\chi^{yx} E^x + \chi^{yy} E^y + \chi^{yz} E^z \right]$$

$$P^i = \varepsilon_0 \sum_j \chi^{ij} E^j$$

→ Inertia tensor



$$\vec{L} = \int (\vec{r} \times \vec{v})\, dm$$

$$= \vec{\omega} \int s^2 dm$$

$$I = \int s^2 dm \quad \cancel{r \sin\theta}$$

$$= \int (r \sin\theta)^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$$= \int r^4 \sin^2\theta\, dr\, d\theta\, d\phi \Rightarrow \cancel{I = \frac{4}{3}\pi R^3}$$

$$\Rightarrow I = 2\pi \frac{4}{3} \frac{R^5}{5}$$

If we set ourselves free of the restriction that we are defining the rotation is only about one axis (fixed axis)

$L^x = 0$

$L^y = 0$

$L^z = I\omega$

$$\vec{L} \cdot \vec{\omega} = I\omega^2$$

$$K = \int v^2 dm = \frac{1}{2} I\omega^2$$

$$K = \frac{1}{2} \vec{L} \cdot \vec{\omega} = \frac{1}{2} \langle L | \omega \rangle = \frac{1}{2} I \langle \omega | \omega \rangle$$

Consider that it rotates about some point.

$$\vec{L} = \int (\vec{r} \times v)\, dm = \int \vec{r} \times \vec{\omega} \times \vec{r}\, dm = \int [(\vec{r} \cdot \vec{r})\vec{\omega} - \vec{r}(\vec{r} \cdot \vec{\omega})]\, dm$$

$$\omega^i = \sum_j \omega^j \delta^{ij} \qquad L^i = \int \left[(\vec{r} \cdot \vec{r}) \sum_j \omega^j \delta^{ij} - x^i \sum_j x^j \omega^j \right] dm$$

$$\vec{r} \cdot \vec{\omega} = \sum_i x^i \omega^i \qquad = \sum_j \omega^j \int [(\vec{r} \cdot \vec{r}) \delta^{ij} - x^i x^j]\, dm$$

$$= \sum_i x^i \sum_j \omega^j \delta^{ij} \qquad I^{ij} = \int [(\vec{r} \cdot \vec{r}) \delta^{ij} - x^i x^j]\, dm$$

$$= \sum_i \sum_j x^i \omega^j \delta^{ij} \qquad L^i = \sum_j \omega^j I^{ij} = L^i = \omega^j I^{ij}$$

(Einstein Summation Rule.

$$I^{ij} = \int [\delta^{ij} r^2 - x^i x^j]\, dm$$

another representation for the inertial tensor.

$$\langle i | I | j \rangle \equiv I^{ij}$$

$\delta^{ij} = \langle i | j \rangle \quad r^2 = \langle r | r \rangle \quad x^i = \langle i | r \rangle = \langle r | i \rangle$

This definition may be used in case of any basis vectors.

$$\therefore I^{ij} = \int [\langle r | r \rangle \langle i | j \rangle - \langle i | r \rangle \langle r | j \rangle]\, dm.$$

$$\langle i | I | j \rangle = \langle i | \underbrace{\int [\langle r | r \rangle \langle i | j \rangle - \langle i | r \rangle \langle r | j \rangle] dm}_{I} | j \rangle$$

* Transformations of Two-Indexed Tensors

→ Some vector $|A\rangle = A^i |i\rangle$; say for some other coordinate system.

$\underline{1} = |\alpha\rangle\langle\alpha|$; $\langle \alpha | A \rangle = \langle \alpha | i \rangle A^i \longrightarrow \frac{\partial x'^\alpha}{\partial x^i} \equiv \Lambda^{\alpha i} A^i$

$$\langle \alpha | I | \beta \rangle = \langle \alpha | \underline{1}\, I\, \underline{1} | \beta \rangle = \langle \alpha | i \rangle \langle i | I | j \rangle \langle j | \beta \rangle$$
$$= \Lambda^{\alpha i} I^{ij} \Lambda^{j\beta}$$
$$= \Lambda^{\alpha i} I^{ij} (\Lambda^{\dagger})^{\beta j}$$

ie, we get $I' = \Lambda I \Lambda^{\dagger}$: $I'^{\alpha\beta} = \Lambda^{\alpha i} I^{ij} \Lambda^{j\beta}$

$$= \frac{\partial x'^\alpha}{\partial x^i} I^{ij} \frac{\partial x'^\beta}{\partial x^j}$$

$$\Rightarrow \boxed{I'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^i} \cdot \frac{\partial x'^\beta}{\partial x^j} I^{ij}}$$

$I^{ij} = \int [r^2 \delta^{ij} - x^i x^j]\, dm$; $I^{i'j'} = \int [r^2 \delta^{i'j'} - x^{i'} x^{j'}]\, dm$

$x' = x\cos\theta + y\sin\theta$
$y' = -x\sin\theta + y\cos\theta$
$z' = z$

$I^{xy} = -\int xy\, dm$; $I^{x'y'} = -\int x'y'\, dm$

$$\therefore I^{x'y'} = -\int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dm$$
$$= -\int [(x^2-y^2)\cos\theta\sin\theta + xy(\sin^2\theta - \cos^2\theta)]\, dm$$

fixing $z = 0$.

$$I^{yy} = \int [x^2 + y^2 - y^2]\, dm = \int x^2 dm$$
$$I^{xx} = \int y^2\, dm.$$
$$I^{x'y'} = (I^{yy} - I^{xx}) \cos\theta\sin\theta - I^{xy}(\sin^2\theta - \cos^2\theta)$$

expanding the transformation rule,

$$I'^{xy} = \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} I^{xx} + \frac{\partial x' \partial y'}{\partial x \partial y} I^{xy} + \frac{\partial x' \partial y'}{\partial y \partial x} I^{yx} + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} I^{yy}$$

$$= -\cos\theta\sin\theta\, I^{xx} + \cos^2\theta\, I^{xy} - \sin^2\theta\, I^{yx} + \cos\theta\sin\theta\, I^{yy}$$

But from defn, $I^{ij} = I^{ji}$

$$\therefore I'^{xy} = (I^{yy} - I^{xx}) \sin\theta\cos\theta - I^{xy}(\sin^2\theta - \cos^2\theta) \quad (\because \text{verified}).$$

* Diagonalizing 2nd rank Tensor

Say we have a wheel with $\vec{\omega} = \begin{pmatrix} 0 \\ \omega_z \\ \omega_z \end{pmatrix}$

$$\begin{pmatrix} L^x \\ L^y \\ L^z \end{pmatrix} = \begin{pmatrix} I^{xx} & I^{xy} & I^{xz} \\ I^{yx} & I^{yy} & I^{yz} \\ I^{zx} & I^{zy} & I^{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega^z \end{pmatrix}$$

wheels have cylindrical symmetry & use cylindrical coordinate system.

$x = r\cos\phi$
$y = r\sin\phi$
$z = z$

$$I^{ij} = \int (r^2\delta^{ij} - x^i x^j)\, dm.$$

$$I^{xz} = \iiint xz\, dm = \int_0^{2\pi}\!\!\iint r^2 \cancel{\cos\phi}\, z\, dr\, d\phi\, dz \to 0$$; similarly: $I^{yz} = 0$, $I^{zy} = I^{zx} = 0$

$$I^{xy} = \int (r^2\delta^{ij} - x^i x^j)\, dm = -\int xy\, dm = -\int_0^{\pi} r^3 \cos\phi \sin\phi\, dr\, d\phi\, dz = 0. = I^{yx}$$

$$I = \begin{pmatrix} I^{xx} & 0 & 0 \\ 0 & I^{yy} & 0 \\ 0 & 0 & I^{zz} \end{pmatrix}$$

In case of non-diagonal matrices, we cannot exploit symmetry. We choose basis vectors s.t. they are eigen vectors of the tensor.

in coordinate system: x^i, $I^{ij} = \langle i | I | j \rangle$

new system x^α, $|\alpha\rangle$, $I^{\alpha\beta} = \langle \alpha | I | \beta \rangle = \lambda \delta^{\alpha\beta}$

$$\overleftrightarrow{I}\,|\beta\rangle = \lambda|\beta\rangle \Rightarrow (\overleftrightarrow{I} - \lambda\mathbb{I})|\beta\rangle = |0\rangle$$

$$\Rightarrow |\overleftrightarrow{I} - \lambda\mathbb{I}| = 0$$

Say, $\eta = ma^2$ $\quad I = \eta\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \begin{vmatrix} 2\eta-\lambda & -\eta & 0 \\ -\eta & 2\eta-\lambda & 0 \\ 0 & 0 & 4\eta-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda = 4\eta,\ 3\eta,\ \eta$$

consider three eigen vectors, $|\rho\rangle, |\sigma\rangle, |\gamma\rangle$

$$\eta\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 4\eta\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$$

$$\left.\begin{aligned} 2\rho_1 - \rho_2 &= 4\rho_1 \Rightarrow 2\rho_1 + \rho_2 = 0 \quad -(1) \\ -\rho_1 + 2\rho_2 &= 4\rho_2 \Rightarrow \rho_1 + 2\rho_2 = 0 \quad -(2) \\ \rho_3 &= \rho_3 \quad -(3) \end{aligned}\right\} \quad |\rho\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvectors are the principle moments of Inertia in this case.

similarly, $|\sigma\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $|\gamma\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \quad U^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (T' = U^T T U)$$

$$I' = U^T I U = \eta\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Say for some tensor T,

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^n} T^{kn}$$

Just like transforming vectors,

$$\left.\begin{aligned} A'^i &= \frac{\partial x'^i}{\partial x^k} A^k \\ B'^j &= \frac{\partial x'^j}{\partial x^n} B^n \end{aligned}\right\} A'^i B'^j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^n} A^k B^n$$

$$T^{kn} \equiv A^k B^n$$

ie, we are creating a tensor by outer product of two vectors.

* Are square matrices ~~too~~ always tensors?

is $[T^{lm}] = \begin{pmatrix} (x^2)^2 & x^1x^2 \\ x^1x^2 & (x^1)^2 \end{pmatrix}$ a tensor? $\quad T'^{ij} = \frac{\partial x'^i}{\partial x^l}\frac{\partial x'^j}{\partial x^m} T^{lm}$.

objects that are represented in an ~~orthogonal~~ orthonormal basis and transform ~~under~~ as tensors under orthonormal tensors, are called Cartesian tensor.

Calling the transformed tensor, $T'^{ij} = \frac{\partial x'^i}{\partial x^l}\frac{\partial x'^j}{\partial x^m} T^{lm}$

checking it under rotation, $x'^1 = x^1\cos\theta + x^2\sin\theta$; $x'^2 = -x^1\sin\theta + x^2\cos\theta$.

Transformation coeff: $\frac{\partial x'^1}{\partial x^1} = \cos\theta$; $\frac{\partial x'^1}{\partial x^2} = \sin\theta$; $\frac{\partial x'^2}{\partial x^1} = -\sin\theta$; $\frac{\partial x'^2}{\partial x^2} = \cos\theta$

If it transforms ~~under~~ as a tensor, $T'^{11} = (x'^2)^2$ this is what we should get.

$$T'^{11} = \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}T^{11} + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^2}T^{12} + \frac{\partial x'^1}{\partial x^2}\frac{\partial x'^1}{\partial x^1}T^{21} + \frac{\partial x'^1}{\partial x^2}\frac{\partial x'^1}{\partial x^2}T^{22}$$

$$= \cos^2\theta\,(x^2)^2 + \cos\theta\sin\theta\, x^1x^2 + \cos\theta\sin\theta\, x^1x^2 + \sin^2\theta\,(x^1)^2$$

$$= \cos^2(x^2)^2 + 2\sin\theta\cos\theta\, x^1x^2 + \sin^2\theta(x^1)^2$$

$$= (\cos\theta\, x^2 + \sin\theta\, x^1)^2$$

But $(x'^2)^2 = (x^2\cos\theta - x^1\sin\theta)^2$ $\therefore$ The ~~tensor~~ it is not a tensor.

Q If $x'^1 = a_1x^1 + a_2x^2$ and $x'^2 = a_3x^1 + a_4x^2$ under what condition will the tensor T^{ij} transform as a tensor. when $[T^{lm}] = \begin{pmatrix} (x^2)^2 & x^1x^2 \\ x^1x^2 & (x^1)^2 \end{pmatrix}$

$$T'^{11} = (x'^2)^2 = (a_3x^1 + a_4x^3)^2$$

But $T'^{11} = \frac{\partial x'^1}{\partial x^1}\cdot\frac{\partial x'^1}{\partial x^1}(x^2)^2 + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^2}x^1x^2 + \frac{\partial x'^1}{\partial x^2}\frac{\partial x'^1}{\partial x^1}x^2x^1 + \frac{\partial x'^1}{\partial x^2}\frac{\partial x'^1}{\partial x^2}(x^1)^2$

$\frac{\partial x'^1}{\partial x^1} = a_1$, $\frac{\partial x'^1}{\partial x^2} = a_2$, $\frac{\partial x'^2}{\partial x^1} = a_3$, $\frac{\partial x'^2}{\partial x^2} = a_4$

$$= (a_1)^2(x^2)^2 + (a_1)(a_2)x^1x^2 + a_1a_2x^2x^1 + (a_2)^2(x^1)^2$$

$$= (a_1x^2 + a_2x^1)^2$$

But $T'^{11} = (a_3x^1 + a_4x^3)^2 = (a_1x^2 + a_2x^1)^2$

$\Rightarrow a_1 = a_3$ and $a_2 = a_4$

* The Metric Tensor

→ It is called the fundamental tensor.

→ Coordinate displacement and distance are very different things.

$(x^1, x^2, x^3) = (x, y, z) = (\rho, \phi, z) = (r, \theta, \phi)$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$
$$= d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$$
$$= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

→ Metric tensor converts coordinate displacement into

$$(ds)^2 = d\vec{r}\cdot d\vec{r}$$
$$= (dx)^2 + (dy)^2 + (dz)^2$$
$$= (d\rho)^2 + (\rho\, d\phi)^2 + (dz)^2$$
$$= (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2$$

→ in a more general way.

$(ds)^2 =$ (if we say) $dx^i dx^i = (dx)^2 + (dy)^2 + (dz)^2$ (✓)

$= (d\rho)^2 + (d\phi)^2 + (dz)^2$ (✗)

we need to find something better. But what's that? Answer: Metric tensor

$(ds)^2 = g_{ij}\, dx^i dx^j = g_{ij} dx^i dx^j\ \ g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3$

(for orthogonal systems, off-diagonal components are zero).

for cartesian, $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

for cylindrical, $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; for spherical, $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

we can write the tensor because we know what the result should be.

But how can we find the metric tensor for a new coordinate system.

Let's find for spherical system.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$= r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$

$\vec{e}_r = \dfrac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\vec{e}_\theta = \dfrac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$

$\vec{e}_\phi = \dfrac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y} + 0$

$g_{ij} = \vec{e}_i \cdot \vec{e}_j \Rightarrow g_{rr} = \vec{e}_r \cdot \vec{e}_r = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1$

$g_{\theta\theta} = \vec{e}_\theta \cdot \vec{e}_\theta = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2$

$g_{\phi\phi} = \vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta$

✱ Metric tensors in Special Theory of Relativity

→ we saw that $dx^i = (dx, dy, dz) = (dx^1, dx^2, dx^3) = d\vec{r}$

→ But in STR we have "four-vectors". $dx^\mu = (c\,dt, dx, dy, dz) = (dx^0, dx^1, dx^2, dx^3)$

$= (c\,dt, d\vec{r})$

→ The square of proper time times speed of light $= (c\,d\tau)^2 = c^2 dt^2 - d\vec{r}\cdot d\vec{r}$

$= (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$

$\Rightarrow (c\,d\tau)^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad [g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \eta_{\mu\nu}$ (minkowski matrix)

→ what is a four-velocity? $u^\mu = \dfrac{\partial x^\mu}{\partial t}$ (✗)

$x^\mu = (ct, x, y, z)$

$\dfrac{\partial x^\mu}{\partial t} = (c, \ldots\ldots) \rightarrow \dfrac{\partial x^\mu}{\partial t}$ is not a four-vector since one component is constant (i.e., c).

∴ we differentiate w.r.t proper time τ.

$$u^\mu = \frac{\partial x^\mu}{\partial \tau} = \frac{\partial x^\mu}{\partial t} \cdot \frac{\partial t}{\partial \tau}$$

$$\Rightarrow \boxed{u^\mu = v^\mu \gamma}$$

$$\Rightarrow u^\mu = \gamma(c, \vec{v})$$

$$(c\,d\tau)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

(we know this)

$$\Rightarrow (c\,d\tau)^2 = (dt)^2 (c^2 - v^2)$$

$$\Rightarrow (d\tau)^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \text{ (lorentz factor)}$$

*Upper and lower indices

$$(ds)^2 = (g_{\mu\nu}\, dx^\mu)\, dx^\nu$$

$$dx_\nu = g_{\mu\nu}\, dx^\mu \; ; \; dx_0 = g_{\mu 0}\, dx^\mu = g_{00}\, dx^0 + \cancel{g_{10}}\, dx^1 + \cancel{g_{20}\, dx^2} + \cancel{g_{30}\, dx^3}$$

$$dx_0 = dx^0$$

$$dx_1 = g_{\mu 1}\, dx^\mu = -dx^1 \text{ (same for } dx^2, dx^3)$$

→ dx^i are dual to dx_i

→ dx^i are determined by coordinates and dx_i (the duals of dx^i) are derived from the metric

→ dx^i := components of contravariant vectors ; dx_i := components of covariant vectors.

→ $(ds)^2 = \langle dx | dx \rangle$ — ket lives in a hilbert space, Bra lives in dual hilbert space.

$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= dx_\mu\, dx^\nu$$

$$\langle A|B \rangle = g_{\mu\nu} A^\mu B^\nu = A_\mu B^\nu$$

→ one natural way to look at vectors with superscripts is vectors with displacement in numerator $\left(\frac{dx}{dt} \text{ or } \frac{dv}{dt}\right)$. Similarly a natural way to look at vectors with subscript. Ex: $\nabla \phi = \frac{\partial \phi}{\partial x^i}$

Ex: $\nabla_\mu \phi = \frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi = \phi_{,\mu}$

$$\left\{ \partial_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu}} \partial_\nu \right.$$

→ say we want to transform $x^\nu \to x^\mu$: $\frac{\partial \phi}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu}$, $A_\mu = \frac{\partial x^\nu}{\partial x^\mu} A_\nu$

so we need inverse metric to go back for $A_\mu \to A^\nu$.

→ $A_\mu = g_{\mu\nu} A^\nu \to g^{-1} g = \mathbb{1}$

$g^{\mu\nu} \to \underline{\underline{g^{-1}}}$ $\qquad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$ ⟶ mixed tensor.

$$g^{\lambda\mu} A_\mu = g^{\lambda\mu} g_{\mu\nu} A^\nu = \delta^\lambda_\nu A^\nu = A^\lambda$$

$$g^{\lambda\mu} A_\mu = A^\lambda$$

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[g^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

→ mixed tensors $\underline{\underline{T^\mu_\beta}} = g^{\mu\alpha} T_{\alpha\beta}$ ← mixed indices.

$\to$ ~~$T'_{\mu\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}$~~ $\to$ $\boxed{T'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} T_{\alpha\beta}}$

$\to$ $\boxed{T'^{\mu}_{\ \nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x'^{\nu}} T^{\alpha}_{\ \beta}}$ $\to$ Transformation rules.

$\to$ Now we want to prove $g_{\mu\nu}$ is a tensor actually.

$(ds)^2 = (ds')^2 \longrightarrow$ spacetime interval is invariant quantity

$$g_{\mu\nu}\, dx^{\mu} dx^{\nu} = g'_{\alpha\beta}\, dx'^{\alpha} dx'^{\beta} = g'_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} dx^{\mu} dx^{\nu}$$

$$\Rightarrow \left\{ g_{\mu\nu} - \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta} \right\} dx^{\mu} dx^{\nu} = 0$$

$$\Rightarrow g_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}$$ $\longrightarrow \therefore g_{\mu\nu}$ is actually a tensor of rank 2.

*** Co/contravariant vectors & "ordinary" vectors** (suspending Einstein sum convention)

$\to \vec{A} = \tilde{A}_{\rho}\hat{\rho} + \tilde{A}_{\theta}\hat{\theta} + \tilde{A}_{z}\hat{z}$ (ordinary vector no covariant/contravariant idea)

$\to (ds)^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^{\mu} dx^{\nu} = (h_1)^2(dx^1)^2 + (h_2)^2(dx^2)^2 + (h_3)^2(dx^3)^2$

$\to (g_{\mu\nu}) = (h_{\mu})^2 \delta_{\mu\nu}$

$\to (ds)^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^{\mu} dx^{\nu}$, $\boxed{g_{\mu\nu} = (h_{\mu})^2 \delta_{\mu\nu}}$, $\boxed{A^{\mu} = \frac{\tilde{A}_{\mu}}{h_{\mu}}}$

$$\sum_{\mu\nu} g_{\mu\nu} A^{\mu} B^{\nu} = \sum_{\mu\nu} (h_{\mu})^2 \delta_{\mu\nu} \frac{\tilde{A}_{\mu}}{h_{\mu}} \cdot \frac{\tilde{B}_{\nu}}{h_{\nu}} = \sum_{\mu} \tilde{A}_{\mu} \tilde{B}_{\mu}$$

$\to A_{\mu} = \sum_{\nu} g_{\mu\nu} A^{\nu} = \sum_{\nu} (h_{\mu})^2 \delta_{\mu\nu} \frac{\tilde{A}_{\nu}}{h_{\nu}} \Rightarrow A_{\mu} = (h_{\mu})^2 \left(\delta_{\mu 1} \frac{\tilde{A}_1}{h_1} + \delta_{\mu 2} \frac{\tilde{A}_2}{h_2} + \cdots \right)$

$A_1 = h_1 \tilde{A}_1 \quad \therefore \boxed{A_{\mu} = h_{\mu} \tilde{A}_{\mu}}$

$\to$ increasing the length of basis decreases component and decreasing the length of basis vectors increases component of the given vector, then these are contravariant components of the vector

$\to$ Suppose if we express the vector in terms of dot product with the basis vectors, if we increase length of basis vector, the dot product increases. therefore these are called co-variant components of the vector

(watch video by: Physics Videos by Eugene Khutoryansky)

$\to \quad g_{\mu\nu} = (h_{\mu})^2 \delta_{\mu\nu}$, for spherical coordinates, $h_r = 1$, $h_{\theta} = r$, $h_{\phi} = r\sin\theta$

$\rightarrow \partial_\mu = h_\mu \nabla_\mu \Rightarrow \nabla_\mu = \frac{1}{h_\mu}\partial_\mu \Rightarrow \nabla_r = \partial_r = \frac{\partial}{\partial r}$

$\nabla_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}$

$\nabla_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

*** Tensor algebra**

→ Rank of tensor = (no. of covariant indices) + (no. of contravariant ind)

$\rightarrow a T^\alpha_\beta = S^\alpha_\beta$

$\rightarrow A^\alpha_\beta = T^\alpha_\beta + S^\alpha_\beta$ (Tensor addition)

$\rightarrow R^\alpha_\beta = T^{\alpha\gamma} S_{\gamma\beta}$ (contraction)

$\rightarrow R^{\alpha\beta\gamma\delta} = T^{\alpha\beta} S^{\gamma\delta}$ (Tensor / outer product).

Q Show that R^μ_μ is a scalar

$$R'^\mu_\mu = \frac{\partial x'^\mu}{\partial x^\rho} \cdot \frac{\partial x^\sigma}{\partial x'^\mu} R^\rho_\sigma$$

→ Since it is invariant under the coordinate transformation, it is a scalar

$$= \delta^\sigma_\rho R^\rho_\sigma = R^\rho_\rho$$

*** Jacobians**

$\int f(x,y)\,dx\,dy \longleftrightarrow \int f(r,\theta)\, r\,dr\,d\theta$

$\rightarrow dx'dy' = J\,dx\,dy$ where $J = \left|\frac{\partial X'}{\partial X}\right|$

Ex: polar coordinate.
$x = r\cos\theta$
$y = r\sin\theta$

$$\Big\} \rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$(r,\theta) \rightarrow (x,y) \Rightarrow dx\,dy = r\,dr\,d\theta$

$$\rightarrow g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \rightarrow |g'| = \left|\frac{\partial x}{\partial x'}\right|\left|\frac{\partial x}{\partial x'}\right| |g|$$

$$= J^{-2}|g| \Rightarrow J = \sqrt{\frac{|g|}{|g'|}}$$

*** Symmetric and Antisymmetric tensors.**

→ for symmetric: $(T^{\mu\nu})^T = T^{\mu\nu}$; for antisymmetric: $(T^{\mu\nu})^T = -T^{\mu\nu}$

$\Rightarrow T^{\nu\mu} = T^{\mu\nu}$; $\Rightarrow T^{\nu\mu} = -T^{\mu\nu}$

(all diagonals = 0 for anti symmetric)

$$\rightarrow T^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) + \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$$

$$T^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu}$$

Tensor Calculus

*** Derivative of tensors.**

→ How components of vector transform ? : $x'^i = \frac{\partial x'^i}{\partial x^j} x^j$

→ take derivative $\frac{dx'^i}{dt} = \frac{\partial x'^i}{\partial x^j} \cdot \frac{dx^j}{dt} + x^j \cdot \frac{d}{dt}\left(\frac{\partial x'^i}{\partial x^j}\right)$

$$= \frac{\partial x'^i}{\partial x^j} \frac{dx^j}{dt} + \underbrace{x^j \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{dx^k}{dt}}$$

WTF is this? This is the part that's actually stoping $\frac{dx'^i}{dt}$ to be a vector. But we know velocity is a vector, then what's happening? In every day physics this term goes to zero.

Ex: $x' = x\cos\theta$ system

$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = 0$

→ Imagine we jumped off a plane with a brick. w.r.t us, the brick's velocity is zero. ie, $\frac{d^2 X^\alpha}{d\tau^2} = 0$

For someone looking from ground, $X^\alpha(x^\nu) \to \frac{d}{d\tau}\left(\frac{dX^\alpha}{d\tau}\right)$

$$= \frac{d}{d\tau}\left(\frac{\partial X^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau}\right)$$

$$= \frac{\partial X^\alpha}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\nu}{d\tau} \cdot \frac{dx^\mu}{d\tau} = 0$$

$\boxed{\text{using } \frac{\partial x^\lambda}{\partial X^\alpha} \frac{\partial X^\alpha}{\partial x^\mu} = \delta^\lambda_\mu}$

$$\Rightarrow \delta^\lambda_\mu \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} = 0$$

$$\Rightarrow \boxed{\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\nu}{d\tau} \cdot \frac{dx^\mu}{d\tau} = 0}$$

→ This is called Affine connection

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu}$$

→ Generally these are interchangable $(\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu})$

→ But is the $\Gamma^\lambda_{\mu\nu}$ a tensor?

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} v^\mu v^\nu = 0$$

how X^α is connected to x^μ by Γ and we can get that by checking how X^α is connected to x'^μ by Γ'

say $\Gamma^{\lambda'}_{\mu'\nu'} = \Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x'^\mu \partial x'^\nu}$ and $X^\alpha = X^\alpha(x^\sigma)$; $x'^\mu = x'^\mu(x^\rho)$

$$= \left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial X^\alpha}\right) \frac{\partial}{\partial x'^\mu}\left(\frac{\partial X^\alpha}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\nu}\right)$$

$$= \left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial X^\alpha}\right)\left(\frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial}{\partial x^\sigma} \frac{\partial X^\alpha}{\partial x^\tau} \frac{\partial x^\tau}{\partial x'^\mu} + \frac{\partial X^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\right)$$

Term 1: $\left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial X^\alpha}\right)\left(\frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial}{\partial x^\sigma} \frac{\partial X^\alpha}{\partial x^\tau} \cdot \frac{\partial x^\tau}{\partial x'^\mu}\right) = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\rho}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x^\sigma \partial x^\tau}$

$$= \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\sigma\tau}$$

→ upto here the Affine connection transformed as a ~~vect~~ tensor of rank 3. But wait! there's another term.

Term 2: $\left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial X^\alpha}\right)\left(\frac{\partial X^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\right)$

$$= \frac{\partial x'^\lambda}{\partial x^\rho} \delta^\rho_\sigma \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}$$

$$\therefore \Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}$$ This mins the fun.

$$\Rightarrow \Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\sigma\tau} - \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu} \text{ and } \frac{\partial}{\partial x'^{\mu}} \delta^{\lambda}_{\nu} = 0$$

$$\Rightarrow \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} + \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = 0$$

* Covariant Derivative

$$\frac{\partial A'^{\lambda}}{\partial x'^{\mu}} = \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x'^{\lambda}}{\partial x^{\nu}} A^{\nu} \right)$$

$$= \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x'^{\lambda}}{\partial x^{\nu}} A^{\nu} \right)$$

$$= \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \left\{ \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\nu}} A^{\nu} + \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x^{\sigma}} \right.$$

$$D_{\mu} A'^{\mu} = \frac{\partial A'^{\lambda}}{\partial x'^{\mu}} + \Gamma^{\lambda'}_{\mu'\nu'} A'^{\nu}$$

$$= \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\nu}} A^{\nu} + \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x^{\sigma}}$$

$$+ \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\alpha} \frac{\partial x'^{\nu}}{\partial x^{\beta}}}_{\delta^{\alpha}_{\beta}} A^{\beta}$$

$$- \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \underbrace{\frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}}}_{\delta^{\rho}_{\beta}} A^{\beta}$$

$$= (\text{first two terms}) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\tau\alpha} A^{\alpha} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\alpha}} A^{\rho}$$

$$= \cancel{\frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\nu}} A^{\nu}} + \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x^{\sigma}} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\tau\alpha} A^{\alpha} - \cancel{\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\alpha}} A^{\rho}}$$

$$\Rightarrow D_{\mu} A'^{\mu} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \left(\frac{\partial A^{\nu}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\sigma\alpha} A^{\alpha} \right)$$ [By renaming the dummy indices]

what we learn is, this object $\left(\frac{\partial A^{\nu}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\sigma\alpha} A^{\alpha} \right)$ transforms as a whole as a 2nd rank mixed tensor.

$$\frac{D A'^{\lambda}}{d\tau} = \frac{\partial A'^{\lambda}}{\partial \tau} + \Gamma^{\lambda'}_{\mu'\nu'} \frac{d x'^{\mu}}{d\tau} A'^{\nu}$$

$$\boxed{D_{\alpha} V^{\lambda} = \partial_{\alpha} V^{\lambda} + \Gamma^{\lambda}_{\alpha\beta} V^{\beta}}$$

* Christoffel Symbols.

→ Taking covariant derivative of 2nd rank tensors. (ie $T_{\mu\nu}$)

$$D_{\alpha} T_{\mu\nu} = \partial_{\alpha} T_{\mu\nu} - \Gamma^{\rho}_{\alpha\mu} T_{\rho\nu} - \Gamma^{\rho}_{\alpha\nu} T_{\rho\mu}$$

Note: the sign different of Γ depends on the type of indices.

$$D_{\alpha} T^{\nu}_{\mu} = \partial_{\alpha} T^{\nu}_{\mu} + \Gamma^{\nu}_{\alpha\rho} T^{\rho}_{\mu} - \Gamma^{\rho}_{\alpha\mu} T^{\nu}_{\rho}$$

→ we take another condition called metric compatibility (for 3 permutation).

$$D_{\alpha} g_{\mu\nu} = 0$$

$(\alpha\mu\nu)$, $(\mu\nu\alpha)$, $(\nu\alpha\mu)$

$$\Rightarrow \partial_{\alpha} g_{\mu\nu} - \Gamma^{\rho}_{\alpha\mu} g_{\rho\nu} - \Gamma^{\rho}_{\alpha\nu} g_{\rho\mu} = 0 \quad \text{——— (1)}$$

$$- \partial_{\mu} g_{\nu\alpha} + \Gamma^{\rho}_{\mu\nu} g_{\rho\alpha} + \Gamma^{\rho}_{\mu\alpha} g_{\rho\nu} = 0 \quad \text{——— (2)}$$

$$- \partial_{\nu} g_{\alpha\mu} + \Gamma^{\rho}_{\nu\alpha} g_{\rho\mu} + \Gamma^{\rho}_{\nu\mu} g_{\rho\alpha} = 0 \quad \text{——— (3)}$$

Now take (1) − (2) − (3) = 0

$$\partial_{\alpha} g_{\mu\nu} - \partial_{\mu} g_{\nu\alpha} - \partial_{\nu} g_{\alpha\mu} + 2 \Gamma^{\rho}_{\mu\nu} g_{\rho\alpha} = 0 \Rightarrow \Gamma^{\rho}_{\mu\nu} g_{\rho\alpha} = \frac{1}{2} \left(\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\alpha\mu} - \partial_{\alpha} g_{\mu\nu} \right)$$

Using inverse matrix, $g^{\lambda\alpha} g_{\rho\alpha} = \delta^{\lambda}_{\rho}$

$\Rightarrow g^{\lambda\alpha} g_{\rho\alpha} \Gamma^{\rho}_{\mu\nu} = \delta^{\lambda}_{\rho} \Gamma^{\rho}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}$

$\Rightarrow \boxed{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right)}$ → Christoffel symbol of 2nd kind.

remember that $\Gamma^{\lambda}_{\mu\nu} \neq 0 \not\Rightarrow$ curved space

*** Covariant Divergence**

→ In cartesian coordinate system: $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$; $d\tilde{x}_i = (dx, dy, dz)$

$= (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2$;

$d\tilde{x}_i = (dr, r\,d\theta, r\sin\theta\, d\phi)$

i.e. $ds^2 = g_{ij} dx^i dx^j$ in sph. system $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

$dx^i = (dr, d\theta, d\phi) \equiv$ coordinate displacement.

$g_{ij} = (h_i)\, \delta_{ij}$ (no sum over i) $\quad h_i = \frac{d\tilde{x}_i}{dx_i}$ $\quad dx_i = h_i\, d\tilde{x}_i$

→ $A^\mu = \frac{\tilde{A}_\mu}{h_i}$, $A_\mu = h_\mu \tilde{A}_\mu$

→ $\partial_\mu f = h_\mu \nabla_\mu f$

→ derivative w.r.t. distance.

→ derivative w.r.t. coordinates

→ $D_\mu A^\lambda = \partial_\mu A^\lambda + \Gamma^{\lambda}_{\mu\nu} A^\nu$ $\quad \Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu k} \left[\partial_\mu g_{\lambda k} + \partial_\lambda g_{\mu k} - \partial_k g_{\mu\lambda} \right]$

$= \frac{1}{2} g^{\mu k} \partial_\lambda g_{\mu k} = \frac{1}{\sqrt{|g|}} \partial_\lambda (\sqrt{|g|})$

(prove later)

→ $D_\mu A^\mu = \partial_\mu A^\mu + \Gamma^{\mu}_{\mu\lambda} A^\lambda$

(Covariant divergence)

$= \partial_\mu A^\mu + \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g}) A^\lambda = \partial_\mu A^\mu + \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}) A^\mu$

$= \frac{1}{\sqrt{g}} \left[\sqrt{g}\, \partial_\mu A^\mu + \partial_\mu (\sqrt{g}) A^\mu \right] \Rightarrow \boxed{D_\mu A^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} A^\mu \right)}$ | if $|g| < 0$ then $\sqrt{g} = \sqrt{|\det(g)|}$

Ex: Spherical coordinates $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$: $\sqrt{g} = \sqrt{r^2 r^2 \sin^2\theta} = r^2 \sin\theta$

$$D_\mu A^\mu = \frac{1}{r^2\sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta\, A^r) + \frac{\partial}{\partial\theta} (r^2\sin\theta\, A^\theta) + \frac{\partial}{\partial\phi} (r^2\sin\theta\, A^\phi) \right]$$

$$= \frac{1}{r^2\sin\theta} \left[\frac{\partial}{\partial r} \left(r^2\sin\theta \frac{\tilde{A}^r}{1} \right) + \frac{\partial}{\partial\theta} \left(r^2\sin\theta \frac{\tilde{A}^\theta}{\sqrt{r^2}} \right) + \frac{\partial}{\partial\phi} \left(r^2\sin\theta \frac{\tilde{A}^\phi}{\sqrt{r^2\sin^2\theta}} \right) \right]$$

$$D_\mu A^\mu = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tilde{A}^r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, \tilde{A}^\theta) + \frac{1}{r\sin\theta} \frac{\partial \tilde{A}^\phi}{\partial\phi}$$

→ Laplacian: $\nabla^2\phi \equiv \vec{\nabla} \cdot (\vec{\partial}\phi)$

Covariant laplacian: $\boxed{D_\mu D^\mu \phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, \partial^\mu \phi)}$

* Covariant Curl.

-> Not talking about ~~curl~~ tensors, curl is defined : $(\nabla \times \tilde{A})_i = \epsilon_{ijk}\, \partial_j \tilde{A}_k$

-> To make this in tensor equivalent form, we may say $\partial_j \sim D$, $\tilde{A}_k \sim A_k$ but what about ϵ_{ijk}.

-> ϵ_{ijk} is not a tensor : it is a tensor density.

$$\epsilon_{ijk} = J\left(\frac{\partial x^l}{\partial x^i}\frac{\partial x^n}{\partial x^j}\frac{\partial x^m}{\partial x^k}\right)\epsilon_{lmn} \qquad w = 1.$$

But $\sqrt{g}$ and ϵ_{ijk} are not tensors individually, but

$$E_{ijk} = \sqrt{g}\,\epsilon_{ijk} \rightsquigarrow \text{a tensor}$$

$$E^{ijk} = \frac{1}{\sqrt{g}}\epsilon^{ijk}$$

$J = \sqrt{\frac{g}{g'}} \to \sqrt{g'} = \frac{1}{J}\sqrt{g}$

These kind of things are called tensor density

Suppose $T_{ij} = \int \tau^{ij}\, d^n x$ ↳ not a tensor

$\sqrt{g}\, T_{ij} \to$ tensor

$$\to C^i = E^{ijk} D_j A_k = \frac{1}{\sqrt{g}}\epsilon^{ijk} D_j A_k$$

Suppose ijk is cyclic : $C^i = \frac{1}{\sqrt{g}}\left(\underbrace{\epsilon^{ijk}}_{1} D_j A_k + \underbrace{\epsilon^{ikj}}_{-1} D_k A_j\right) = \frac{1}{\sqrt{g}}\left(D_j A_k - D_k A_j\right) = C_{jk}$

$$\Rightarrow \frac{1}{\sqrt{g}}\left(\partial_j A_k - \cancel{\Gamma^l_{jk} A_l} - \partial_k A_j + \cancel{\Gamma^l_{kj} A_l}\right) \longrightarrow \left(\Gamma^l_{jk} = \Gamma^l_{kj}\right)$$

$$\boxed{C^i = \frac{1}{\sqrt{g}}\left(\partial_j A_k - \partial_k A_j\right)} \qquad C^i = \frac{1}{h_i}\tilde{C}_i$$

say we want to know C^θ

Ex: spherical.

$$\tilde{C}_\theta = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right) \qquad A_\mu = h_\mu \tilde{A}$$

$$h_\mu = (1, r, r\sin\theta)$$

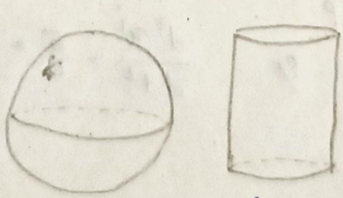
$$C^\theta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}A_\phi\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\phi}\tilde{A}_r - \frac{\partial}{\partial r} r\sin\theta\,\tilde{A}_\phi\right] \qquad \text{But } C^\theta \text{ should be } = \frac{1}{h_\theta}\tilde{C}_\theta$$

$$= \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\tilde{A}_r - \frac{\partial}{\partial r}(r\tilde{A}_\phi)\right]$$

$h_\theta = r$, $\Rightarrow h_\theta C^\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\tilde{A}_r - \frac{\partial}{\partial r}(r\tilde{A}_\phi)\right]$ (as expected).

* Riemann Curvature Tensor



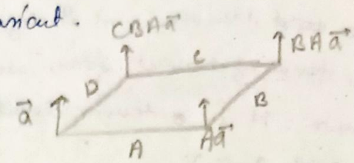
-> Are these objects curved?

The answer is the problem is ill defined

-> Normally what we understand geometrically as curvature, is extrinsic curvature - we look at the sphere from outside and say yes! it's curved.

-> Intrinsic Curvature - Can the ~~ant~~ bug on the sphere, tell if it is curved?

→ The idea is parallel transportation to the same point leaves the vector invariant.



$\vec{a} = DCBA\vec{a}$

→ another way is: $BA\vec{a} - BA\vec{a} = 0$

→ Here, we look at local scope: infinitesimal.

$$\frac{df}{dx} = \frac{f(x+dx) - f(x)}{dx} \rightsquigarrow f(x+dx) = f(x) + dx\, f'(x)$$

Derivatives are generators of infinitesimal translations

ie, covariant derivative generate infinitesimal parallel transportation

then $BA\vec{a} - B\vec{A}\vec{a} \rightsquigarrow D_\mu D_\nu A^\lambda - D_\nu D_\mu A^\lambda$ → This A is not same as the $B\vec{A}\vec{a} - AB\vec{a}$

$= [D_\mu, D_\nu] A^\lambda$

if $[D_\mu, D_\nu] A^\lambda = 0 \Rightarrow$ space is flat if not, the space is curved.

$$\boxed{\begin{aligned} \nabla_\mu A^\lambda &= \partial_\mu A^\lambda + \Gamma^\lambda_{\mu\nu} A^\nu \\ \nabla_\mu T^\lambda_\nu &= \partial_\mu T^\lambda_\nu + \Gamma^\lambda_{\alpha\mu} T^\alpha_\nu - \Gamma^\sigma_{\mu\nu} T^\lambda_\sigma \end{aligned}}$$

$D_\mu D_\nu A^\lambda - D_\nu D_\mu A^\lambda$.

$$= D_\mu (\underbrace{\partial_\nu A^\lambda + \Gamma^\lambda_{\alpha\nu} A^\alpha}_{T^\lambda_\nu}) - D_\nu (\underbrace{\partial_\mu A^\lambda + \Gamma^\lambda_{\alpha\mu} A^\alpha}_{T^\lambda_\mu})$$

$$\begin{aligned} D_\mu T^\lambda_\nu &= \partial_\mu T^\lambda_\nu + \Gamma^\lambda_{\sigma\mu} T^\sigma_\nu - \Gamma^\gamma_{\mu\nu} T^\lambda_\gamma \\ &= \partial_\mu(\partial_\nu A^\lambda + \Gamma^\lambda_{\alpha\nu} A^\alpha) + \Gamma^\lambda_{\sigma\mu}(\partial_\nu A^\sigma + \Gamma^\sigma_{\alpha\nu} A^\alpha) \\ &\quad - \Gamma^\gamma_{\mu\nu}(\partial_\gamma A^\lambda + \Gamma^\lambda_{\alpha\gamma} A^\alpha) \end{aligned}$$

$$= D_\mu T^\lambda_\nu - D_\nu T^\lambda_\mu$$

$$= A^\alpha \partial_\mu \Gamma^\lambda_{\alpha\nu} - A^\alpha \partial_\nu \Gamma^\lambda_{\alpha\nu} + \cancel{\Gamma^\lambda_{\alpha\nu} \partial_\mu A^\alpha} - \cancel{\Gamma^\lambda_{\alpha\mu} \partial_\nu A^\alpha}$$
$$+ \cancel{\Gamma^\lambda_{\sigma\mu} \partial_\nu A^\sigma} - \cancel{\Gamma^\lambda_{\sigma\nu} \partial_\mu A^\sigma} + \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\alpha\nu} A^\alpha - \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\alpha\mu} A^\alpha$$

$$= (\partial_\mu \Gamma^\lambda_{\alpha\nu} - \partial_\nu \Gamma^\lambda_{\alpha\nu} + \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\alpha\nu} - \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\alpha\mu}) A^\alpha$$

$$\Rightarrow [D_\mu, D_\nu] A^\alpha = (\partial_\mu \Gamma^\lambda_{\alpha\nu} - \partial_\nu \Gamma^\lambda_{\alpha\nu} + \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\alpha\nu} - \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\alpha\mu}) A^\alpha$$

$$\Rightarrow \boxed{[D_\mu, D_\nu] A^\alpha = R^\lambda_{\alpha\nu\mu} A^\alpha}$$ Reimann curvature tensor

→ These RCT are linear to 2nd derivative of $g_{\mu\nu}$ and quadratic in 1st derivative of $g_{\mu\nu}$.

* Metric tensor and Gravity

→ Poisson's eqn: $\nabla^2 \phi = 4\pi G \rho$

→ $\dfrac{d^2 x^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} V^\mu V^\nu = 0$ & $\dfrac{d^2 x^i}{dt^2} - \bar{g} = 0$

(will continue later)