C24 Dynamical Systems — Notes based on Ron Daniel's lectures and lecture notes in MT 2014

1 Linear algebra

Definition 1.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then*

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{1}$$

Proposition 1.1. *If* $\mathbf{S}$ *and* $\mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *which commute, i.e.* $\mathbf{ST} = \mathbf{TS}$*, then* $e^{\mathbf{S}+\mathbf{T}} = e^{\mathbf{ST}}$.

Proof. Not interesting. □

Proposition 1.2. *If* $\mathbf{P}, \mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *(i.e.* $\mathbf{P}, \mathbf{T} \in \mathbb{R}^{n \times n}$*) and* $\mathbf{S} = \mathbf{PTP}^{-1}$ *then* $e^{\mathbf{S}} = \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}$.

Proof.

$$\begin{aligned}
e^{\mathbf{S}} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathbf{PTP}^{-1})^k}{k!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mathbf{PT}^k\mathbf{P}^{-1}}{k!} \\
&= \mathbf{P}\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!}\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}
\end{aligned}$$

□

Proposition 1.3. *If*

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

then

$$e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Proof. By induction

$$\mathbf{A}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

for $k = 0, 1, \ldots$. Then we can write

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} a^k/k! & 0 \\ 0 & b^k/k! \end{bmatrix} \\ &= \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned}$$

$\square$

Proposition 1.4. *If*

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

Proof. Write $\mathbf{A} = a\mathbf{I} + \mathbf{B}$ where

$$\mathbf{B} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

Then $a\mathbf{I}$ commutes with $\mathbf{B}$ and by Proposition 1.1,

$$e^{\mathbf{A}} = e^{a\mathbf{I}} e^{\mathbf{B}} = e^a e^{\mathbf{B}}$$

And from the definition

$$e^{\mathbf{B}} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2/2! + \cdots = \mathbf{I} + \mathbf{B}$$

since by direct computation $\mathbf{B}^2 = \mathbf{B}^3 = \cdots = 0$. $\square$

Proposition 1.5. *If*

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix}$$

explicitly: it is true for $k = 0$ and assuming true for $k - 1$, we can write

$$\begin{aligned} \lambda^{k-1} &= a_{k-1} + ib_{k-1} \\ \lambda^k &= \lambda^{k-1}\lambda \\ &= (a_{k-1} + ib_{k-1})(a + ib) \\ &= (a_{k-1}a - b_{k-1}b) + i(b_{k-1}a + a_{k-1}b) \end{aligned}$$

and so

$$\begin{aligned} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k &= \begin{bmatrix} \mathrm{Re}(\lambda^{k-1}) & -\mathrm{Im}(\lambda^{k-1}) \\ \mathrm{Im}(\lambda^{k-1}) & \mathrm{Re}(\lambda^{k-1}) \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} a_{k-1} & -b_{k-1} \\ b_{k-1} & a_{k-1} \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} (a_{k-1}a - b_{k-1}b) & (-a_{k-1}b - b_{k-1}a) \\ (b_{k-1}a + a_{k-1}b) & (-b_{k-1}b + a_{k-1}a) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix} \end{aligned}$$

Using this

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}(\frac{\lambda_k}{k!}) & -\mathrm{Im}(\frac{\lambda_k}{k!}) \\ \mathrm{Im}(\frac{\lambda_k}{k!}) & \mathrm{Re}(\frac{\lambda_k}{k!}) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(e^\lambda) & -\mathrm{Im}(e^\lambda) \\ \mathrm{Im}(e^\lambda) & \mathrm{Re}(e^\lambda) \end{bmatrix} \\ &= e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} \end{aligned}$$

□

Theorem 1.1 (The Jordan Canonical Form). *Let* $\mathbf{A} \in \mathbb{R}^{2n-k}$ *be a real matrix with*

- *real eigenvalues* $\lambda_j, j = 1, \ldots, k$ *and*
- *complex eigenvalues* $\lambda_j = a_j + ib_j$ *and* $\bar{\lambda}_j = a_j - ib_j, j = k+1, \ldots, n.$

Then there exists a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \ldots, \mathbf{v}_n, \mathbf{u}_n\}$ *for* $\mathbb{R}^{2n-k}$*, where*

- $\mathbf{v}_j, j = 1, \ldots, k$ *and*

- $\mathbf{u}_j + i\mathbf{v}_j, j = k+1, \ldots, n$

are generalised eigenvectors of $\mathbf{A}$ *such that the matrix*

$$\mathbf{W} = [\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \ldots, \mathbf{v}_n, \mathbf{u}_n]$$

is invertible and

$$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_r \end{bmatrix} \tag{2}$$

where the elementary Jordan blocks $\mathbf{J} = \mathbf{J}_j, j = 1, \ldots, r$ *are either of the form*

$$\mathbf{B} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & & \lambda & 1 \\ 0 & \cdots & & 0 & \lambda \end{bmatrix} \tag{3}$$

for one of the real eigenvalues of $\mathbf{A}$, λ, *or of the form*

$$\mathbf{B} = \begin{bmatrix} \mathbf{D} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & & & & \\ \mathbf{0} & \cdots & & \mathbf{D} & \mathbf{I} \\ \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{4}$$

with

$$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for one of the complex eigenvalues of $\mathbf{A}$, $\lambda = a + ib$.

Corollary 1.1. *For any* $\mathbf{A} \in \mathbb{R}^{2\times 2}$, *there exists an invertible matrix* $\mathbf{W} \in \mathbb{R}^{2\times 2}$ *(described in the proof) such that the matrix*

$$\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$$

has one of the following forms

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \textit{or } \mathbf{\Lambda} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

It then follows from Propositions 1.3, 1.4 and 1.5 that

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, e^{\mathbf{\Lambda}t} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \textit{or } e^{\mathbf{\Lambda}t} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1.2, the matrix $e^{\mathbf{A}t}$ *is then given by*

$$e^{\mathbf{A}t} = \mathbf{W}e^{\mathbf{\Lambda}t}\mathbf{W}^{-1}$$

Proof. We analyse the Jordan Canonical Form of three possible cases of eigendecomposition of $\mathbf{A} \in \mathbb{R}^{2\times 2}$ (characteristic equation $|\mathbf{A} - \lambda \mathbf{I}|$ a quadratic equation in λ):

1. Eigenvalues are real and distinct: λ, μ. Eigenvectors are $\mathbf{w}_1, \mathbf{w}_2$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

   where $J_1 = \lambda, J_2 = \mu$.

2. Eigenvalues are real and equal: λ. There is only one eigenvector, $\mathbf{w}$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}, \mathbf{w}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{J}_1]$$

   where

   $$\mathbf{J}_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

3. Eigenvalues are complex conjugates: $\lambda_1 = a+ib, \lambda_2 = a-ib$. Eigenvectors are $\mathbf{w}_1 = \mathbf{u} + i\mathbf{v}, \mathbf{w}_2 = \mathbf{u} - i\mathbf{v}$.

   In this case $k = 0, n = 1$. For the matrix $\mathbf{W} = [\mathbf{v}, \mathbf{u}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{D}]$$

   where

   $$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

□

2 Introduction

We have: state $\mathbf{x}(t) \in \mathbb{R}^n$, function $\mathbf{f} : D \to \mathbb{R}^n, D \subseteq \mathbb{R}^n$. Then a dynamical system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5}$$

In this lecture course, we assume that the solution exists and is unique locally, i.e. we will assume that the vector fields are sufficiently smooth to allow this.

We call $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ a linear autonomous system.

Theorem 2.1 (The Fundamental Theorem for Linear Systems)**.** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then for a given* $\mathbf{x}_0 \in \mathbb{R}^n$*, the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{8}$$

3 Equilibria and stability

Equilibrium solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a solution $\mathbf{x}^* \in \mathbb{R}^n$ which is constant, i.e. $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. $\mathbf{x}^*$ is called a fixed point, stationary point, rest point, critical point or steady state.

For maps, $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$, the equilibrium solution is a solution $\mathbf{x}^* \in \mathbb{R}^n$ such that $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$.

Definition 3.1. *An equilibrium point* $\mathbf{x}^*$ *is said to be stable if, given* $\epsilon > 0$*, there exists a real number* $\delta > 0$ *such that for any other solution* $\mathbf{y}(t)$ *satisfying* $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$*,* $|\mathbf{y}(t) - \mathbf{x}^*| < \epsilon$ *for all* $t \geq 0$*. Otherwise it is called unstable.*

Definition 3.2. *An equilibrium point* $\mathbf{x}^*$ *is said to be asymptotically stable if it is stable and there is a* $b > 0$ *such that if* $|\mathbf{y}(0) - \mathbf{x}^*| < b$ *then* $\lim_{t\to\infty} |\mathbf{y}(t) - \mathbf{x}^*| = 0$*.*

Definition 3.3. *An equilibrium point* $\mathbf{x}^*$ *is said to be exponentially stable if it is asymptotically stable and there exist finite* $\alpha, \beta, \delta > 0$ *such that if* $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$ *then* $|\mathbf{y}(t) - \mathbf{x}^*| \leq \alpha e^{-\beta t}|\mathbf{y}(0) - \mathbf{x}^*|$ *for* $t \geq 0$*.*

3.1 Analysing stability in 2x2 systems

In a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x} \in \mathbb{R}^2$ where we can decompose $\mathbf{A} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}$ according to Corollary 1.1, we can use the coordinate transformation $\mathbf{z} = \mathbf{W}^{-1}\mathbf{x}$ and $\mathbf{z}_0 = \mathbf{W}^{-1}\mathbf{x}_0$ to analyse the system in the transformed coordinates $\mathbf{z}$:

$$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z}$$
$$\mathbf{z}(0) = \mathbf{z}_0$$

hence according to Theorem 2.1:

$$\mathbf{z}(t) = e^{\mathbf{\Lambda}t}\mathbf{z}_0$$

We now analyse the three possible cases of $\mathbf{\Lambda}$:

Real, distinct eigenvalues. We have for $\lambda, \mu, \lambda \neq \mu$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}$$

hence

$$\begin{aligned} \mathbf{z}(t) &= \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{z}_0 \\ &= z_{1,0} e^{\lambda t} + z_{2,0} e^{\mu t} \end{aligned}$$

There can be three subcases:

- $\lambda < \mu < 0$: stable.
- $\lambda < 0 < \mu$: saddle shape (separatrices).
- $0 < \lambda < \mu$: unstable (opposite arrows to stable case).

Duplicate real eigenvalue. We have for $\lambda \neq 0$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda}t} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

hence

$$\mathbf{z}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{z}_0$$

There can be two subcases:

- $\lambda < 0$: stable.
- $\lambda > 0$: unstable.

Complex eigenvalues. We have for $\lambda_1 = a + ib, \lambda_2 = a - ib$:

$$\mathbf{\Lambda} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda}t} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

hence

$$\mathbf{z}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{z}_0$$

We can look at the norm of $\mathbf{z}(t)$ to analyse the shape of the solution:

$$\begin{aligned}
\|\mathbf{z}(t)\|^2 = |e^{at}|^2 & \left\| \begin{bmatrix} z_{1,0}\cos bt - z_{2,0}\sin bt \\ z_{1,0}\sin bt + z_{2,0}\cos bt \end{bmatrix} \right\|^2 \\
&= |e^{at}|^2 (z_{1,0}^2 \cos^2 bt - 2z_{1,0}z_{2,0}\sin bt \cos bt + z_{2,0}^2 \sin^2 bt + \\
& z_{1,0}^2 \sin^2 bt + 2z_{1,0}z_{2,0}\sin bt\cos bt + z_{2,0}^2\cos^2 bt) \\
&= |e^{at}|^2 (z_{1,0}^2(\sin^2 bt + \cos^2 bt) + z_{2,0}^2(\sin^2 bt + \cos^2 bt)) \\
&= |e^{at}|^2 (z_{1,0}^2 + z_{2,0}^2) \\
&= |e^{at}|^2 \|\mathbf{z}_0\|^2
\end{aligned}$$

i.e. $\|\mathbf{z}(t)\| = |e^{at}|\|\mathbf{z}_0\|$. So

- $a = 0$: circle.
- $a < 0$: stable spiral.
- $a > 0$: unstable spiral.
- $b > 0$: anticlockwise.
- $b < 0$: clockwise.

3.2 Nonlinear systems

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an equilibrium $\mathbf{x}^*$, such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$ and assume $\mathbf{f}$ is differentiable and not that the equilibrium point doesn't change with time. Then for $j = 1, \ldots, n$:

$$f_j(\mathbf{x} + \mathbf{w}) = f_j(\mathbf{x}^*) + (\text{grad}_{\mathbf{x}}\, f_j(\mathbf{x}^*))^T \mathbf{w} + \cdots$$

Hence we can write

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\{\mathbf{x}^*\}\mathbf{w} + \cdots$$

where

$$D\{\mathbf{f}(\mathbf{x}^*)\} = \begin{bmatrix} (\text{grad}_{\mathbf{x}}\, f_1(\mathbf{x}^*))^T \\ \vdots \\ (\text{grad}_{\mathbf{x}}\, f_n(\mathbf{x}^*))^T \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Hence we can analyse the following small perturbation model

$$\dot{\mathbf{w}} \approx D\{\mathbf{x}^*\}\mathbf{w} = \mathbf{A}\mathbf{w}$$

Definition 3.4. *Let* $\mathbf{x}^*$ *be an equilibrium of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. *Then* $\mathbf{x}^*$ *is called a hyperbolic fixed point if none of the eigenvalues of* $D\mathbf{f}(\mathbf{x}^*)$ *have zero real part.*

Theorem 3.1. *Suppose* $\mathbf{x}^*$ *is a hyperbolic fixed point and all the real parts of the eigenvalues are negative. Then the equilibrium solution* $\mathbf{x} = \mathbf{x}^*$ *is asymptotically stable.*

4 Invariant Manifolds

4.1 Linear systems: Stable, Unstable and Centre subspaces

4.2 Nonlinear system local theory

5 Lyapunov Functions

6 Asymptotic behaviour

7 Limit Cycles and Index Theory

8 Local Bifurcations

9 Chaos