C24 Dynamical Systems
Notes based on Ron Daniel's lectures and lecture notes in MT 2014

1 Linear algebra

Definition 1.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then*

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{1}$$

Proposition 1.1. *If* $\mathbf{S}$ *and* $\mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *which commute, i.e.* $\mathbf{ST} = \mathbf{TS}$*, then* $e^{\mathbf{S}+\mathbf{T}} = e^{\mathbf{S}}e^{\mathbf{T}}$.

Proof. Not interesting. □

Proposition 1.2. *If* $\mathbf{P}, \mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *(i.e.* $\mathbf{P}, \mathbf{T} \in \mathbb{R}^{n\times n}$*) and* $\mathbf{S} = \mathbf{PTP}^{-1}$ *then* $e^{\mathbf{S}} = \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}$.

Proof.

$$\begin{aligned}
e^{\mathbf{S}} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathbf{PTP}^{-1})^k}{k!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mathbf{PT}^k\mathbf{P}^{-1}}{k!} \\
&= \mathbf{P}\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!}\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}
\end{aligned}$$

□

Proposition 1.3. *If*

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

then

$$e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Proof. By induction

$$\mathbf{A}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

for $k = 0, 1, \ldots$. Then we can write

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} a^k/k! & 0 \\ 0 & b^k/k! \end{bmatrix} \\ &= \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned}$$

□

Proposition 1.4. *If*

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

Proof. Write $\mathbf{A} = a\mathbf{I} + \mathbf{B}$ where

$$\mathbf{B} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

Then $a\mathbf{I}$ commutes with $\mathbf{B}$ and by Proposition 1.1,

$$e^{\mathbf{A}} = e^{a\mathbf{I}} e^{\mathbf{B}} = e^a e^{\mathbf{B}}$$

And from the definition

$$e^{\mathbf{B}} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2/2! + \cdots = \mathbf{I} + \mathbf{B}$$

since by direct computation $\mathbf{B}^2 = \mathbf{B}^3 = \cdots = 0$. □

Proposition 1.5. *If*

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix}$$

explicitly: it is true for $k = 0$ and assuming true for $k - 1$, we can write

$$\begin{aligned} \lambda^{k-1} &= a_{k-1} + ib_{k-1} \\ \lambda^k &= \lambda^{k-1}\lambda \\ &= (a_{k-1} + ib_{k-1})(a + ib) \\ &= (a_{k-1}a - b_{k-1}b) + i(b_{k-1}a + a_{k-1}b) \end{aligned}$$

and so

$$\begin{aligned}
\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k &= \begin{bmatrix} \mathrm{Re}(\lambda^{k-1}) & -\mathrm{Im}(\lambda^{k-1}) \\ \mathrm{Im}(\lambda^{k-1}) & \mathrm{Re}(\lambda^{k-1}) \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\
&= \begin{bmatrix} a_{k-1} & -b_{k-1} \\ b_{k-1} & a_{k-1} \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\
&= \begin{bmatrix} (a_{k-1}a - b_{k-1}b) & (-a_{k-1}b - b_{k-1}a) \\ (b_{k-1}a + a_{k-1}b) & (-b_{k-1}b + a_{k-1}a) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix}
\end{aligned}$$

Using this

$$\begin{aligned}
e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}(\frac{\lambda_k}{k!}) & -\mathrm{Im}(\frac{\lambda_k}{k!}) \\ \mathrm{Im}(\frac{\lambda_k}{k!}) & \mathrm{Re}(\frac{\lambda_k}{k!}) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Re}(e^{\lambda}) & -\mathrm{Im}(e^{\lambda}) \\ \mathrm{Im}(e^{\lambda}) & \mathrm{Re}(e^{\lambda}) \end{bmatrix} \\
&= e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}
\end{aligned}$$

□

Theorem 1.1 (The Jordan Canonical Form). *Let $\mathbf{A} \in \mathbb{R}^{(2n-k)\times(2n-k)}$ be a real matrix with*

- *real eigenvalues $\lambda_j, j = 1, \dots, k$ and*
- *complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j, j = k+1, \dots, n$.*

Then there exists a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \dots, \mathbf{v}_n, \mathbf{u}_n\}$ for $\mathbb{R}^{2n-k}$, where

- *$\mathbf{v}_j, j = 1, \dots, k$ and*
- *$\mathbf{u}_j + i\mathbf{v}_j, j = k+1, \dots, n$*

are generalised eigenvectors of $\mathbf{A}$ such that the matrix

$$\mathbf{W} = [\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \dots, \mathbf{v}_n, \mathbf{u}_n]$$

is invertible and

$$\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_r \end{bmatrix} \tag{2}$$

where the elementary Jordan blocks $\mathbf{J} = \mathbf{J}_j, j = 1, \dots, r$ are either of the form

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & & \lambda & 1 \\ 0 & \cdots & & 0 & \lambda \end{bmatrix} \tag{3}$$

for one of the real eigenvalues of $\mathbf{A}$, λ, or of the form

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & & & & \\ \mathbf{0} & \cdots & & \mathbf{D} & \mathbf{I} \\ \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{4}$$

with

$$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for one of the complex eigenvalues of $\mathbf{A}$*,* $\lambda = a + ib$*.*

Corollary 1.1. *For any* $\mathbf{A} \in \mathbb{R}^{2\times 2}$*, there exists an invertible matrix* $\mathbf{W} \in \mathbb{R}^{2\times 2}$ *(described in the proof) such that the matrix*

$$\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$$

has one of the following forms

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \textit{or } \mathbf{\Lambda} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

It then follows from Propositions 1.3, 1.4 and 1.5 that

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, e^{\mathbf{\Lambda}t} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \textit{or } e^{\mathbf{\Lambda}t} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1.2, the matrix $e^{\mathbf{A}t}$ *is then given by*

$$e^{\mathbf{A}t} = \mathbf{W}e^{\mathbf{\Lambda}t}\mathbf{W}^{-1}$$

Proof. We analyse the Jordan Canonical Form of three possible cases of eigendecomposition of $\mathbf{A} \in \mathbb{R}^{2\times 2}$ (characteristic equation $|\mathbf{A} - \lambda\mathbf{I}|$ a quadratic equation in λ):

1. Eigenvalues are real and distinct: λ, μ. Eigenvectors are $\mathbf{w}_1, \mathbf{w}_2$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

   where $J_1 = \lambda, J_2 = \mu$.

2. Eigenvalues are real and equal: λ. There is only one eigenvector, $\mathbf{w}$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}, \mathbf{w}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{J}_1]$$

   where

   $$\mathbf{J}_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

3. Eigenvalues are complex conjugates: $\lambda_1 = a + ib, \lambda_2 = a - ib$. Eigenvectors are $\mathbf{w}_1 = \mathbf{u} + i\mathbf{v}, \mathbf{w}_2 = \mathbf{u} - i\mathbf{v}$.

   In this case $k = 0, n = 1$. For the matrix $\mathbf{W} = [\mathbf{v}, \mathbf{u}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{D}]$$

   where

   $$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

□

2 Introduction

We have: state $\mathbf{x}(t) \in \mathbb{R}^n$, function $\mathbf{f} : D \to \mathbb{R}^n, D \subseteq \mathbb{R}^n$. Then a dynamical system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5}$$

In this lecture course, we assume that the solution exists and is unique locally, i.e. we will assume that the vector fields are sufficiently smooth to allow this.

We call $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ a linear autonomous system.

Theorem 2.1 (The Fundamental Theorem for Linear Systems)**.** *Let $\mathbf{A}$ be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbb{R}^n$, the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$
$$\mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{8}$$

3 Equilibria and stability

Equilibrium solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a solution $\mathbf{x}^* \in \mathbb{R}^n$ which is constant, i.e. $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. $\mathbf{x}^*$ is called a fixed point, stationary point, rest point, critical point or steady state.

For maps, $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$, the equilibrium solution is a solution $\mathbf{x}^* \in \mathbb{R}^n$ such that $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$.

Definition 3.1. *An equilibrium point $\mathbf{x}^*$ is said to be stable if, given $\epsilon > 0$, there exists a real number $\delta > 0$ such that for any other solution $\mathbf{y}(t)$ satisfying $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$, $|\mathbf{y}(t) - \mathbf{x}^*| < \epsilon$ for all $t \geq 0$. Otherwise it is called unstable.*

Definition 3.2. *An equilibrium point $\mathbf{x}^*$ is said to be asymptotically stable if it is stable and there is a $b > 0$ such that if $|\mathbf{y}(0) - \mathbf{x}^*| < b$ then $\lim_{t\to\infty} |\mathbf{y}(t) - \mathbf{x}^*| = 0$.*

Definition 3.3. *An equilibrium point $\mathbf{x}^*$ is said to be exponentially stable if it is asymptotically stable and there exist finite $\alpha, \beta, \delta > 0$ such that if $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$ then $|\mathbf{y}(t) - \mathbf{x}^*| \leq \alpha e^{-\beta t}|\mathbf{y}(0) - \mathbf{x}^*|$ for $t \geq 0$.*

3.1 Analysing stability in 2x2 systems

In a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x} \in \mathbb{R}^2$ where we can decompose $\mathbf{A} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}$ according to Corollary 1.1, we can use the coordinate transformation $\mathbf{z} = \mathbf{W}^{-1}\mathbf{x}$ and $\mathbf{z}_0 = \mathbf{W}^{-1}\mathbf{x}_0$ to analyse the system in the transformed coordinates $\mathbf{z}$:

$$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z}$$
$$\mathbf{z}(0) = \mathbf{z}_0$$

hence according to Theorem 2.1:

$$\mathbf{z}(t) = e^{\mathbf{\Lambda}t}\mathbf{z}_0$$

We now analyse the three possible cases of $\mathbf{\Lambda}$:

Real, distinct eigenvalues. We have for $\lambda, \mu, \lambda \neq \mu$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda} t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}$$

hence

$$\begin{aligned} \mathbf{z}(t) &= \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{z}_0 \\ &= \begin{bmatrix} z_{1,0} e^{\lambda t} \\ z_{2,0} e^{\mu t} \end{bmatrix} \end{aligned}$$

There can be three subcases:

- $\lambda < \mu < 0$: stable.
- $\lambda < 0 < \mu$: saddle shape (separatrices).
- $0 < \lambda < \mu$: unstable (opposite arrows to stable case).

Duplicate real eigenvalue. We have for $\lambda \neq 0$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda} t} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

hence

$$\mathbf{z}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{z}_0$$

There can be two subcases:

- $\lambda < 0$: stable.
- $\lambda > 0$: unstable.

Complex eigenvalues. We have for $\lambda_1 = a + ib, \lambda_2 = a - ib$:

$$\mathbf{\Lambda} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and

$$e^{\mathbf{\Lambda} t} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

hence

$$\mathbf{z}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{z}_0$$

We can look at the norm of $\mathbf{z}(t)$ to analyse the shape of the solution:

$$\begin{aligned}\|\mathbf{z}(t)\|^2 &= |e^{at}|^2 \left\| \begin{bmatrix} z_{1,0}\cos bt - z_{2,0}\sin bt \\ z_{1,0}\sin bt + z_{2,0}\cos bt \end{bmatrix} \right\|^2 \\ &= |e^{at}|^2 (z_{1,0}^2 \cos^2 bt - 2z_{1,0}z_{2,0}\sin bt \cos bt + z_{2,0}^2 \sin^2 bt + \\ & z_{1,0}^2 \sin^2 bt + 2z_{1,0}z_{2,0}\sin bt\cos bt + z_{2,0}^2\cos^2 bt) \\ &= |e^{at}|^2 (z_{1,0}^2(\sin^2 bt + \cos^2 bt) + z_{2,0}^2(\sin^2 bt + \cos^2 bt)) \\ &= |e^{at}|^2 (z_{1,0}^2 + z_{2,0}^2) \\ &= |e^{at}|^2 \|\mathbf{z}_0\|^2\end{aligned}$$

i.e. $\|\mathbf{z}(t)\| = |e^{at}|\|\mathbf{z}_0\|$. So

- $a = 0$: circle.
- $a < 0$: stable spiral.
- $a > 0$: unstable spiral.
- $b > 0$: anticlockwise.
- $b < 0$: clockwise.

3.2 Nonlinear systems

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an equilibrium $\mathbf{x}^*$, such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$ and assume $\mathbf{f}$ is differentiable and not that the equilibrium point doesn't change with time. Then for $j = 1, \ldots, n$:

$$f_j(\mathbf{x} + \mathbf{w}) = f_j(\mathbf{x}^*) + (\operatorname{grad}_{\mathbf{x}} f_j(\mathbf{x}^*))^T \mathbf{w} + \cdots$$

Hence we can write

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} + \cdots$$

where

$$D\{\mathbf{f}(\mathbf{x}^*)\} = \begin{bmatrix} (\operatorname{grad}_{\mathbf{x}} f_1(\mathbf{x}^*))^T \\ \vdots \\ (\operatorname{grad}_{\mathbf{x}} f_n(\mathbf{x}^*))^T \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Hence we can analyse the following small perturbation model

$$\dot{\mathbf{w}} \approx D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} = \mathbf{A}\mathbf{w}$$

Definition 3.4. *Let $\mathbf{x}^*$ be an equilibrium of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then $\mathbf{x}^*$ is called a hyperbolic fixed point if none of the eigenvalues of $D\{\mathbf{f}(\mathbf{x}^*)\}$ have zero real part.*

Theorem 3.1. *Suppose $\mathbf{x}^*$ is a hyperbolic fixed point and all the real parts of the eigenvalues are negative. Then the equilibrium solution $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable.*

4 Invariant Manifolds

4.1 Linear systems: Stable, Unstable and Centre subspaces

In an $n \times n$ linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

if $\mathbf{A} \in \mathbb{R}^n, n = 2m - k$ is eigendecomposed to

- real eigenvalues $\lambda_j = a_j$ with corresponding eigenvectors $\mathbf{w}_j = \mathbf{u}_j, \mathbf{v}_j = \mathbf{0}$ for $j = 1, \ldots, k$, and
- complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ with corresponding eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\mathbf{w}_j = \mathbf{u}_j - i\mathbf{v}_j$ for $j = k+1, \ldots, m$

then

$$\begin{aligned}
\mathbf{W} &= [\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \ldots, \mathbf{v}_n, \mathbf{u}_n] \\
\mathbf{\Lambda} &= \mathbf{W}^{-1}\mathbf{A}\mathbf{W} \\
&= \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_r \end{bmatrix}
\end{aligned}$$

by Theorem 1.1.

Definition 4.1. *E^s, E^c and E^u are the subspaces of $\mathbb{R}^n$ spanned by the real and imaginary parts of the generalised eigenvectors corresponding to eigenvalues with negative, zero and positive real parts respectively:*

$$\begin{aligned}
E^s &= \operatorname{span}\{\mathbf{u}_j, \mathbf{v}_j : a_j < 0\} \\
E^u &= \operatorname{span}\{\mathbf{u}_j, \mathbf{v}_j : a_j > 0\} \\
E^c &= \operatorname{span}\{\mathbf{u}_j, \mathbf{v}_j : a_j = 0\}
\end{aligned}$$

Definition 4.2. *If all eigenvalues of the $n \times n$ matrix $\mathbf{A}$ have nonzero real part, then the flow $e^{\mathbf{A}t} : \mathbb{R}^n \to \mathbb{R}^n$ (mapping $\mathbf{x}_0$ to $\mathbf{x}(t)$) is called a hyperbolic flow and $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is called a hyperbolic linear system.*

Definition 4.3. *A subspace $E \subseteq \mathbb{R}^n$ is said to be invariant with respect to the flow $e^{\mathbf{A}t}$ if for any $\mathbf{x} \in E$: $e^{\mathbf{A}t}\mathbf{x} \in E, \forall t$ (in other words, E is closed under the linear transformation $e^{\mathbf{A}t}$ for all t; or: what starts in E stays in E).*

Theorem 4.1. *Let $\mathbf{A}$ be a real $n \times n$ matrix. Then*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c$$

where E^s, E^u and E^c are the stable, unstable and center subspaces of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ respectively; furthermore, E^s, E^u and E^c are invariant with respect to the flow $e^{\mathbf{A}t}$ of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ respectively.

Proof. (fake (Perko p55 for the real deal)) Let

$$\begin{aligned}
\{\mathbf{w}_i^s, i = 1, \ldots, n_s\} &= \{\mathbf{u}_j, \mathbf{v}_j : a_j < 0, \mathbf{u}_j, \mathbf{v}_j \neq \mathbf{0}\} \\
\{\mathbf{w}_i^u, i = 1, \ldots, n_u\} &= \{\mathbf{u}_j, \mathbf{v}_j : a_j > 0, \mathbf{u}_j, \mathbf{v}_j \neq \mathbf{0}\} \\
\{\mathbf{w}_i^c, i = 1, \ldots, n_c\} &= \{\mathbf{u}_j, \mathbf{v}_j : a_j = 0, \mathbf{u}_j, \mathbf{v}_j \neq \mathbf{0}\}
\end{aligned}$$

then we can rearrange (questionable) $\mathbf{W}$ and $\mathbf{\Lambda}$ in the following way

$$\begin{aligned}
\mathbf{W} &= [\mathbf{w}_1^s, \ldots, \mathbf{w}_{n_s}^s, \mathbf{w}_1^u, \ldots, \mathbf{w}_{n_u}^u, \mathbf{w}_1^c, \ldots, \mathbf{w}_{n_c}^c] \\
\mathbf{\Lambda} &= \begin{bmatrix} \mathbf{J}_1^s & & & & & & & & \\ & \ddots & & & & & & & \\ & & \mathbf{J}_r^s & & & & & & \\ & & & \mathbf{J}_1^u & & & & & \\ & & & & \ddots & & & & \\ & & & & & \mathbf{J}_r^u & & & \\ & & & & & & \mathbf{J}_1^c & & \\ & & & & & & & \ddots & \\ & & & & & & & & \mathbf{J}_r^c \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{\Lambda}^s & & \\ & \mathbf{\Lambda}^u & \\ & & \mathbf{\Lambda}^c \end{bmatrix}
\end{aligned}$$

where $\mathbf{\Lambda}^s \in \mathbb{R}^{n_s \times n_s}$, $\mathbf{\Lambda}^u \in \mathbb{R}^{n_u \times n_u}$ and $\mathbf{\Lambda}^c \in \mathbb{R}^{n_c \times n_c}$ such that $\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$ still holds.

Using the coordinate transformation $\mathbf{z} = \mathbf{W}^{-1}\mathbf{x}$, we get $\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z}$ hence

$$\begin{aligned}
\dot{\mathbf{z}}^s &= \mathbf{\Lambda}^s \mathbf{z}^s \\
\dot{\mathbf{z}}^u &= \mathbf{\Lambda}^u \mathbf{z}^u \\
\dot{\mathbf{z}}^c &= \mathbf{\Lambda}^c \mathbf{z}^c
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{z}^s &= [z_1, \ldots, z_{n_s}]^T \\
&= [z_1^s, \ldots, z_{n_s}^s]^T \\
\mathbf{z}^u &= [z_{n_s+1}, \ldots, z_{n_s+n_u}]^T \\
&= [z_1^u, \ldots, z_{n_u}^u]^T \\
\mathbf{z}^c &= [z_{n_s+n_u+1}, \ldots, z_{n_s+n_u+n_c}]^T \\
&= [z_1^c, \ldots, z_{n_c}^c]^T
\end{aligned}$$

Fix $\mathbf{x}_0 \in E^s$. We can then write (questionable)

$$\begin{aligned}
\mathbf{x}_0 &= \mathbf{W}\mathbf{z}(0) \\
&= \sum_{i=1}^{n_s} \mathbf{w}_i^s z_i^s(0) + \sum_{j=1}^{n_u} \mathbf{w}_j^u z_j^u(0) + \sum_{k=1}^{n_c} \mathbf{w}_k^c z_k^c(0) \\
&= \sum_{i=1}^{n_s} \mathbf{w}_i^s z_i^s(0) + \mathbf{0} + \mathbf{0}
\end{aligned}$$

hence (questionable) $\mathbf{z}^u(0) = \mathbf{z}^c(0) = \mathbf{0}$. The solution in $\mathbf{z}(t)$ then becomes

$$\begin{aligned}
\mathbf{z}^s(t) &= e^{\mathbf{\Lambda}^s t}\mathbf{z}^s(0) \\
\mathbf{z}^u(t) &= \mathbf{0} \\
\mathbf{z}^c(t) &= \mathbf{0}
\end{aligned}$$

which corresponds to

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{W}\mathbf{z}(t) \\
&= \sum_{i=1}^{n_s} \mathbf{w}_i^s z_i^s(t) + \sum_{j=1}^{n_u} \mathbf{w}_j^u z_j^u(t) + \sum_{k=1}^{n_c} \mathbf{w}_k^c z_k^c(t) \\
&= \sum_{i=1}^{n_s} \mathbf{w}_i^s z_i^s(t) + \mathbf{0} + \mathbf{0}
\end{aligned}$$

which is $\in E^s$. Hence E^s is invariant with respect to the flow $e^{\mathbf{A}t}$. Similarly for E^u and E^c. $\square$

Theorem 4.2. *If $\mathbf{x}_0 \in E^s$, then $e^{\mathbf{A}t}\mathbf{x}_0 \in E^s$ for all $t \in \mathbb{R}$ and*

$$\lim_{t\to\infty} e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{0}.$$

And if $\mathbf{x}_0 \in E^u$, then $e^{\mathbf{A}t}\mathbf{x}_0 \in E^u$ for all $t \in \mathbb{R}$ and

$$\lim_{t\to-\infty} e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{0}.$$

4.2 Nonlinear system local theory

Consider the linearised model

$$\dot{\mathbf{w}} \approx D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} = \mathbf{A}\mathbf{w}$$

Theorem 4.3 (Hartman-Grobman)**.** *$\forall$ hyperbolic equilibrium points $\mathbf{x}^* \in A$ (A is an open set)*

$\exists$ bi-continuous (a mapping that is continous and whose inverse is also continuous) function $H : A \to B$ where

B is the open set containing the origin of the linearised model

so that trajectories are mapped exactly and the parameterisation of time is preserved.

4.3 Conservative systems

If there exists a non-constant function $V(\mathbf{x})$ such that $\mathrm{d}V/\mathrm{d}t = 0$ along a solution of the nonlinear differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the equations are called conservative.

If $\mathbf{x} = \mathbf{x}^*$ is an isolated equilibrium and there is a $V(\mathbf{x})$ that is a local minimum/maximum at $\mathbf{x}^*$ then there is a region around that point that contains a *closed orbit*.

5 Lyapunov Functions

Theorem 5.1. *Consider a nonlinear system of the form*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

and let $\mathbf{x}^*$ be an equilibrium point, i.e. $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Let $V : D \to \mathbb{R}$ (unrelated to V in conservative systems) be a continuously differentiable function, defined on a neighbourhood D of (open set containing) $\mathbf{x}^*$ such that

- *$V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}^*$.*
- *$\dot{V}(\mathbf{x}) = (\mathrm{grad}_{\mathbf{x}}\, V(\mathbf{x}))^T \mathbf{f}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{x}^*\}$.*

Then $\mathbf{x}^*$ is stable. If moreover

- *$\dot{V}(\mathbf{x}) < 0$ in $D \setminus \{\mathbf{x}^*\}$.*

then $\mathbf{x}^*$ is asymptotically stable.

5.1 Vector fields possesing an integral

5.2 Hamiltonian Systems

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $H = H(\mathbf{x}, \mathbf{y})$. A system of the form

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}$$
$$\dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}},$$

is called a *Hamiltonian* system with n degrees of freedom.

Theorem 5.2 (Conservation of Energy)**.** *The total energy $H(\mathbf{x}, \mathbf{y})$ of a Hamiltonian system remains constant along its trajectories.*

Proof. The total derivative of the Hamiltonian function $H(\mathbf{x}, \mathbf{y})$ along a solution trajectory $\mathbf{x}(t), \mathbf{y}(t)$ is

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial H}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial H}{\partial \mathbf{y}} \cdot \dot{\mathbf{y}} \\ &= \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial H}{\partial \mathbf{y}} - \frac{\partial H}{\partial \mathbf{y}} \cdot \frac{\partial H}{\partial \mathbf{x}} \\ &= 0\end{aligned}$$

□

5.3 Gradient Systems

A system of the form

$$\dot{\mathbf{x}} = -\operatorname{grad} V(\mathbf{x}) \tag{9}$$

is called a *gradient system.*

Theorem 5.3. *At regular points of the function $V(\mathbf{x})$, trajectories of a gradient system cross the level surfaces $V(\mathbf{x}) =$ constant orthogonally. And strict local minima of the function $V(\mathbf{x})$ are asymptotically stable equilibrium points of this gradient system.*

Also, $V(\mathbf{x}) - V(\mathbf{x}_0)$ is a strict Lyapunov function for this system, where $\mathbf{x}_0$ is a strict local minimum of $V(\mathbf{x})$.

5.4 A relationship between Gradient and Hamiltonian Systems

They are orthogonal.

6 Asymptotic behaviour

Let the non-linear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{10}$$

have a flow

$$\phi(t, \mathbf{x}) : \mathbf{f}(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(0, \mathbf{x})$$

within some open set D around the point $\mathbf{x}_0$. We require the solution to pass through the point $\mathbf{x}_0$ at $t = 0$. This solution defines a path $\Gamma_{\mathbf{x}_0}$ which is the curve

$$\Gamma_{\mathbf{x}_0} = \{\mathbf{x} \in D : \mathbf{x} = \phi(t, \mathbf{x}_0), t \in R\}.$$

Definition 6.1. *A point $p \in D$ is called an ω limit point of the trajectory $\phi(t, x)$ if there exists a sequence of times, $\{t_i\}, t_i \to \infty$ such that*

$$\lim_{i\to\infty} \phi(t_i, x) \to p. \tag{11}$$

We denote this point $\omega(x)$. α limit points are defined in a similar way but now the sequence $\{t_i\}$ is such that $t_i \to \infty$.

The α limit set of a trajectory is the set of all α limit points. Similarly, we can define the ω limit set.

Definition 6.2 (Positively invariant set)**.** *Define the positive path $\Gamma^+_{x_0} := \{x \in D : x = \phi(t, x_0), t \geq 0\}$. Set $S \subseteq D$ is called positively invariant if $x_0 \in S$ implies $\Gamma^+_{x_0} \subseteq S$.*

Definition 6.3 (Attracting set)**.** *An invariant set $A \subset D$ is attracting if there is some neighbourhood U of A which is positively invariant and all trajectories starting in U tend to A as $t \to \infty$.*

Definition 6.4 (Trapping region (informal)). *Trapping region is a region such that every trajectory that starts within the trapping region will move to the region's interior and remain there as the system evolves.*

Let $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and suppose $S \subset \mathbb{R}^n$ is a positively invariant set. Suppose the boundry of S is differentiable and S has non-empty interior. S is a trapping region.

Definition 6.5 (Basin of attraction). *The* domain *or* basin *of attraction of an attracting set A is the union of all trajectories forming a trapping region of A.*

Theorem 6.1 (La Salle's invariance principle). *Let D be a trapping region. Suppose now there exists $V(x)$ which satisfies $\dot{V} \leq 0$ on D and consider the following two sets:*

$$E = \{x \in D : \dot{V}(x) = 0\} \tag{12}$$

and

$$M = \{\textit{the union of all trajectories in } E \textit{ that are positively invariant}\}. \tag{13}$$

La Salle's invariance principle then states that for all $x \in D$, all trajectories starting at x tend to M as $t \to \infty$.

Theorem 6.2 (Poincaré-Bendixson). *Let M be a positively invariant region of a vector field, containing only a finite number of equilibria. Let $x \in M$ and consider $\omega(x)$. Then one of the following possibilities holds:*

1. *$\omega(x)$ is an equilibrium;*
2. *$\omega(x)$ is a closed orbit;*
3. *$\omega(x)$ consists of a finite number of equilibria $x_1^*, \ldots, x_n^*$ and orbits γ_k with $\alpha(\gamma_k) = x_i^*$ and $\omega(\gamma_k) = x_j^*$ for some i, j. i.e. connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.*

As a result, if inside M there are only stable equilibria, then there can only be one. If there are no equilibria, then there is a closed orbit inside it.

7 Limit Cycles and Index Theory

Definition 7.1 (Periodic orbit). *A solution of $\dot{x} = f(x)$ through x_0 is said to be periodic if there exists a $T > 0$ such that $\phi(t, x_0) = \phi(t + T, x_0)$ for all $t \in \mathbb{R}$. The minimum such T is called a period of the periodic orbit.*

Theorem 7.1 (Bendixson). *If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically 0 and does not change sign on a region D of the phase-plane, then*

$$\dot{x} = f(x, y)\dot{y} = g(x, y) \tag{14}$$

has no closed orbits in this region.

Theorem 7.2 (Dulac). *Let $B(x, y)$ be a continuously differentiable function, defined on a region $D \in \mathbb{R}^2$. If $\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$ is not identically 0 and does not change sign on a region D of the phase-plane, then*

$$\dot{x} = f(x, y)\dot{y} = g(x, y) \tag{15}$$

has no closed orbits in this region.

Claim 7.1. *A gradient system has no closed orbits.*

Index of a simple closed curve $\Gamma \in \mathbb{R}^2$, the index of Γ, $I(\Gamma)$ is defined by

$$I(\Gamma) = \oint_\Gamma \frac{d\varphi(x_1, x_2)}{2\pi} \tag{16}$$

Index theory properties:

1. The index is always an integer.

2. If there are no fixed points in the interior of a simple closed curve Γ then $I(\Gamma) = 0$.

3. If Γ is a closed orbit of the system then $I(\Gamma) = 1$.

4. Let Γ encircle counterclockwisely an isolated fixed point x^*. If x^* is a saddle node, then $I(\Gamma) = -1$, otherwise $I(\Gamma) = 1$. In the case of fixed points, we also identify $I(\Gamma)$ with $I(x^*$.

5. The index of a curve Γ surrounding a number of fixed points $x_1^*, \ldots, x_n^*$ is

$$I(\Gamma) = \sum_{k=1}^{n} I(x_k^*) \tag{17}$$

6. The index of a closed trajectory enclosing fixed points $x_1^*, \ldots, x_n^*$ is

$$I(\Gamma) = \sum_{k=1}^{n} I(x_k^*) = 1. \tag{18}$$

Definition 7.2 (Stability of periodic orbits)**.** *A periodic orbit Γ is said to be stable if for every $\epsilon > 0$ there is a neighbourhood U of Γ such that for all $x \in U$, the distance between $\phi(t, x)$ and Γ is less than ϵ. Moreover, Γ is called asymptotically stable, if it is stable and if for all points $x \in U$ we have that this distance tends to zero as $t \to \infty$.*

An asymptotically stable cycle is referred to as an ω-limit cycle.

8 Local Bifurcations

9 Chaos