C24 Dynamical Systems – notes

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0.1 Linear algebra

Definition 0.1.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then for* $t \in \mathbb{R}$,

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \tag{1}$$

Proposition 0.1.1. *If* $\mathbf{P}, \mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *(i.e.* $\mathbf{P}, \mathbf{T} \in \mathbb{R}^{n \times n}$*) and* $\mathbf{S} = \mathbf{P}\mathbf{T}\mathbf{P}^{-1}$ *then* $e^{\mathbf{S}} = \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}$.

Proof.

$$\begin{aligned}
e^{\mathbf{S}} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathbf{P}\mathbf{T}\mathbf{P}^{-1})^k}{k!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mathbf{P}\mathbf{T}^k\mathbf{P}^{-1}}{k!} \\
&= \mathbf{P}\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!}\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}
\end{aligned}$$

□

Proposition 0.1.2. *For any* $\mathbf{A} \in \mathbb{R}^{2\times 2}$*, there exists an invertible matrix* $\mathbf{W} \in \mathbb{R}^{2\times 2}$ *(whose columns consist of generalised eigenvectors of* $\mathbf{A}$*) such that the matrix*

$$\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$$

has one of the following forms

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \text{ or } \Lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

It then follows from above that

$$e^{(\Lambda t)} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, e^{(\Lambda t)} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \text{ or } e^{(\Lambda t)} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 0.1.1, the matrix $e^{\mathbf{A}t}$ *is then given by*

$$e^{\mathbf{A}t} = \mathbf{W}e^{\mathbf{\Lambda}t}\mathbf{W}^{-1}$$

We have: state $\mathbf{x}(t) \in \mathbb{R}^n$, function $\mathbf{f} : D \to \mathbb{R}^n, D \subseteq \mathbb{R}^n$. Then a dynamical system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2}$$