C24 Dynamical Systems – Notes

December 20, 2014

1 Linear algebra

Definition 1.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then*

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{1}$$

Proposition 1.1. *If* $\mathbf{S}$ *and* $\mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *which commute, i.e.* $\mathbf{ST} = \mathbf{TS}$*, then* $e^{\mathbf{S}+\mathbf{T}} = e^{\mathbf{ST}}$*.*

Proof. Not interesting. □

Proposition 1.2. *If* $\mathbf{P}, \mathbf{T}$ *are linear transformations on* $\mathbb{R}^n$ *(i.e.* $\mathbf{P}, \mathbf{T} \in \mathbb{R}^{n\times n}$*) and* $\mathbf{S} = \mathbf{PTP}^{-1}$ *then* $e^{\mathbf{S}} = \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}$*.*

Proof.

$$\begin{aligned}
e^{\mathbf{S}} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathbf{PTP}^{-1})^k}{k!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mathbf{PT}^k\mathbf{P}^{-1}}{k!} \\
&= \mathbf{P}\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!}\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{T}}\mathbf{P}^{-1}
\end{aligned}$$

□

Proposition 1.3. *If*

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

then

$$e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Proof. By induction

$$\mathbf{A}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

for $k = 0, 1, \ldots$. Then we can write

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} a^k/k! & 0 \\ 0 & b^k/k! \end{bmatrix} \\ &= \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned}$$

□

Proposition 1.4. *If*

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

Proof. Write $\mathbf{A} = a\mathbf{I} + \mathbf{B}$ where

$$\mathbf{B} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

Then $a\mathbf{I}$ commutes with $\mathbf{B}$ and by Proposition 1.1,

$$e^{\mathbf{A}} = e^{a\mathbf{I}} e^{\mathbf{B}} = e^a e^{\mathbf{B}}$$

And from the definition

$$e^{\mathbf{B}} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2/2! + \cdots = \mathbf{I} + \mathbf{B}$$

since by direct computation $\mathbf{B}^2 = \mathbf{B}^3 = \cdots = 0$. □

Proposition 1.5. *If*

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$e^{\mathbf{A}} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix}$$

explicitly: it is true for $k = 0$ and assuming true for $k - 1$, we can write

$$\begin{aligned} \lambda^{k-1} &= a_{k-1} + ib_{k-1} \\ \lambda^k &= \lambda^{k-1}\lambda \\ &= (a_{k-1} + ib_{k-1})(a + ib) \\ &= (a_{k-1}a - b_{k-1}b) + i(b_{k-1}a + a_{k-1}b) \end{aligned}$$

and so

$$\begin{aligned} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k &= \begin{bmatrix} \mathrm{Re}(\lambda^{k-1}) & -\mathrm{Im}(\lambda^{k-1}) \\ \mathrm{Im}(\lambda^{k-1}) & \mathrm{Re}(\lambda^{k-1}) \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} a_{k-1} & -b_{k-1} \\ b_{k-1} & a_{k-1} \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} (a_{k-1}a - b_{k-1}b) & (-a_{k-1}b - b_{k-1}a) \\ (b_{k-1}a + a_{k-1}b) & (-b_{k-1}b + a_{k-1}a) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix} \end{aligned}$$

Using this

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}(\frac{\lambda_k}{k!}) & -\mathrm{Im}(\frac{\lambda_k}{k!}) \\ \mathrm{Im}(\frac{\lambda_k}{k!}) & \mathrm{Re}(\frac{\lambda_k}{k!}) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(e^\lambda) & -\mathrm{Im}(e^\lambda) \\ \mathrm{Im}(e^\lambda) & \mathrm{Re}(e^\lambda) \end{bmatrix} \\ &= e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} \end{aligned}$$

□

Theorem 1.1 (The Jordan Canonical Form)**.** *Let* $\mathbf{A} \in \mathbb{R}^{2n-k}$ *be a real matrix with*

- *real eigenvalues* $\lambda_j, j = 1, \ldots, k$ *and*
- *complex eigenvalues* $\lambda_j = a_j + ib_j$ *and* $\bar{\lambda}_j = a_j - ib_j, j = k+1, \ldots, n$.

Then there exists a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \ldots, \mathbf{v}_n, \mathbf{u}_n\}$ *for* $\mathbb{R}^{2n-k}$*, where*

- $\mathbf{v}_j, j = 1, \ldots, k$ *and*
- $\mathbf{u}_j + i\mathbf{v}_j, j = k+1, \ldots, n$

are generalised eigenvectors of $\mathbf{A}$ *such that the matrix*

$$\mathbf{W} = [\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \dots, \mathbf{v}_n, \mathbf{u}_n]$$

is invertible and

$$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_r \end{bmatrix} \tag{2}$$

where the elementary Jordan blocks $\mathbf{J} = \mathbf{J}_j, j = 1, \dots, r$ *are either of the form*

$$\mathbf{B} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & & \lambda & 1 \\ 0 & \cdots & & 0 & \lambda \end{bmatrix} \tag{3}$$

for one of the real eigenvalues of $\mathbf{A}$, λ, *or of the form*

$$\mathbf{B} = \begin{bmatrix} \mathbf{D} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & & & & \\ \mathbf{0} & \cdots & & \mathbf{D} & \mathbf{I} \\ \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{4}$$

with

$$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for one of the complex eigenvalues of $\mathbf{A}$, $\lambda = a + ib$.

Corollary 1.1. *For any* $\mathbf{A} \in \mathbb{R}^{2\times 2}$, *there exists an invertible matrix* $\mathbf{W} \in \mathbb{R}^{2\times 2}$ *(described in the proof) such that the matrix*

$$\mathbf{\Lambda} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$$

has one of the following forms

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \text{ or } \mathbf{\Lambda} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

It then follows from Propositions 1.3, 1.4 and 1.5 that

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, e^{\mathbf{\Lambda}t} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \text{ or } e^{\mathbf{\Lambda}t} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1.2, the matrix $e^{\mathbf{A}t}$ *is then given by*

$$e^{\mathbf{A}t} = \mathbf{W}e^{\mathbf{\Lambda}t}\mathbf{W}^{-1}$$

Proof. We analyse the Jordan Canonical Form of three possible cases of eigendecomposition of $\mathbf{A} \in \mathbb{R}^{2\times 2}$ (characteristic equation $|\mathbf{A} - \lambda\mathbf{I}|$ a quadratic equation in λ):

1. Eigenvalues are real and distinct: λ, μ. Eigenvectors are $\mathbf{w}_1, \mathbf{w}_2$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

   where $J_1 = \lambda, J_2 = \mu$.

2. Eigenvalues are real and equal: λ. There is only one eigenvector, $\mathbf{w}$.

   In this case $n = k = 2$. For the matrix $\mathbf{W} = [\mathbf{w}, \mathbf{w}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{J}_1]$$

   where

   $$\mathbf{J}_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

3. Eigenvalues are complex conjugates: $\lambda_1 = a + ib, \lambda_2 = a - ib$. Eigenvectors are $\mathbf{w}_1 = \mathbf{u} + i\mathbf{v}, \mathbf{w}_2 = \mathbf{u} - i\mathbf{v}$.

   In this case $k = 0, n = 1$. For the matrix $\mathbf{W} = [\mathbf{v}, \mathbf{u}]$:

   $$\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = [\mathbf{D}]$$

   where

   $$\mathbf{D} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

□

2 Lecture 1 – Introduction

We have: state $\mathbf{x}(t) \in \mathbb{R}^n$, function $\mathbf{f} : D \to \mathbb{R}^n, D \subseteq \mathbb{R}^n$. Then a dynamical system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5}$$

3 Lecture 2 – Equilibria and stability

4 Lecture 3 – Invariant Manifolds

5 Lecture 4 – Lyapunov Functions

6 Lecture 5 – Asymptotic behaviour

7 Lecture 6 – Limit Cycles and Index Theory

8 Lecture 7 – Local Bifurcations

9 Lecture 8 – Chaos