MA 111 - Problem Set 4

1. (i) To show $\lim_{x \to c} \sin x = \sin c$

Let $\varepsilon > 0$ be given.

Now $$|\sin x - \sin c| = \left|2 \cos \frac{x+c}{2} \cdot \sin \frac{x-c}{2}\right|$$
$$\leq 2 \frac{|x-c|}{2} = |x-c|$$

Since $\left|\cos \frac{x+c}{2}\right| \leq 1$ and

$\left|\sin \frac{x-c}{2}\right| \leq \left|\frac{x-c}{2}\right|$. $(|\sin x| \leq |x|)$

$\therefore$ $|\sin x - \sin c| < \varepsilon$

if $|x-c| < \varepsilon$.

Choose $\delta = \varepsilon$.

$\therefore$ for all $|x-c| < \delta$ implies $|\sin x - \sin c| < \varepsilon$.

This shows that $\lim_{x \to c} \sin x = \sin c$

(ii) To show $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ $\quad c > 0$

Let $\varepsilon > 0$ be given.

Now $|\sqrt{x} - \sqrt{c}| < \varepsilon$ if $\frac{|x-c|}{|\sqrt{x} + \sqrt{c}|} < \varepsilon$

that is if $|x-c| < (|\sqrt{x} + \sqrt{c}|)\varepsilon$

if $|x-c| < \sqrt{c}\,\varepsilon$

choose, $\delta = \sqrt{c}\,\varepsilon$. Then

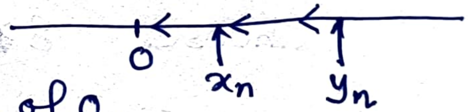
$|\sqrt{x} - \sqrt{c}| < \varepsilon$ for all $|x-c| < \delta$.

Therefore $\lim_{x \to c} \sqrt{x} = \sqrt{c}$

2. (iii) To show $\lim_{x \to \infty} x \sin x$ does NOT exist,

Show that $\lim_{y \to 0+} \frac{1}{y} \sin \frac{1}{y}$ does NOT exist.

Consider two sequences $\{x_n\}$ and $\{y_n\}$ from right side of 0,



such that $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ $\forall n \in \mathbb{N}$.

Let $f(y) = \frac{1}{y} \sin \frac{1}{y}$.

$\therefore$ ~~Now~~, $x_n \longrightarrow 0$ and $y_n \longrightarrow 0$ as $n \to \infty$,

but $f(x_n) = \frac{1}{x_n} \sin \frac{1}{x_n} = n\pi \sin n\pi = 0 \xrightarrow[\text{as } n \to \infty]{} 0$

and $f(y_n) = \frac{1}{y_n} \sin \frac{1}{y_n} = \left(2n\pi + \frac{\pi}{2}\right) \longrightarrow \infty$ as $n \to \infty$.

Therefore $\lim_{y \to 0+} \frac{1}{y} \sin \frac{1}{y}$ does NOT exist,

$\Rightarrow \lim_{x \to \infty} x \sin x$ does NOT exist.

2. (ii) To show $\lim_{x\to 0} e^{\frac{1}{x}}$ does NOT exist.

For $\lim_{x\to 0+} e^{\frac{1}{x}}$,

consider a sequence $\{x_n\}$ such that $x_n = \frac{1}{n}$ (from right side of 0).

$x_n \longrightarrow 0$ as $n\to\infty$

but $e^{\frac{1}{x_n}} = e^{n} \longrightarrow \infty$ as $n\to\infty$

$\therefore$ f is unbounded on any right side neighbourhood of 0. or nbhd $\cap (0,\infty)$

Therefore $\lim_{x\to 0+} e^{\frac{1}{x}}$ does NOT exist in $\mathbb{R}$.

So, $\lim_{x\to 0} e^{\frac{1}{x}}$ does NOT exist in $\mathbb{R}$.

Note: [for the existence of $\lim_{x\to 0} f(x)$ in $\mathbb{R}$, we should have both the $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$ exist in $\mathbb{R}$]

* **Assignment Problem :**

- Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

[Hint. Calculate $\lim_{x\to 0+} \frac{\sin x}{x} = 1$ using $0 < \sin x < x$, on $(0, \frac{\pi}{2})$

and $\lim_{x\to 0-} \frac{\sin x}{x} = 1$ using $x < \sin x < 0$ on $(-\frac{\pi}{2}, 0)$.]

3. $f : (0,1) \longrightarrow \mathbb{R}$, where

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let $a \in [0,1]$.

consider a sequence of rational numbers $\{x_n\}$ such that $x_n \longrightarrow a$, and

a sequence of irrational numbers $\{y_n\}$ such that $y_n \longrightarrow a$ as $n \to \infty$.

Then, ~~Now,~~ $f(x_n) = 1 \longrightarrow 1$ as $n \to \infty$ and

$f(y_n) = 0 \longrightarrow 0$ as $n \to \infty$.

Since, $\{x_n\}, \{y_n\}$ converge to a but $f(x_n)$ and $f(y_n)$ have different limit, $\lim_{x \to a} f(x)$ does NOT exist.

4. $$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 2-x & \text{" } x \text{ is irrational} \end{cases}$$

(i) To show $\lim_{x \to 1} f(x) = 1$

Let $\varepsilon > 0$ be given.

Now $|x-1| < \varepsilon$ or $|(2-x)-1| = |1-x| = |x-1| < \varepsilon$

if $|x-1| < \delta$ where $\delta = \varepsilon$.

$\therefore$ choose $\delta = \varepsilon$. Then $|f(x)-1| < \varepsilon$ for all $|x-1| < \delta$.

(ii) Similar as problem 3.

5. For $\lim\limits_{x \to 0+} \left[\dfrac{\sin x}{x}\right]$

For all $0 < x < \frac{\pi}{2}$, $\quad 0 < \sin x < x$

ie, $0 < \dfrac{\sin x}{x} < 1$

$$\Rightarrow \left[\frac{\sin x}{x}\right] = 0.$$

Therefore $\lim\limits_{x \to 0+} \left[\dfrac{\sin x}{x}\right] = 0$

similarly, for all $-\frac{\pi}{2} < x < 0$, $\quad x < \sin x < 0$

this implies $0 < \dfrac{\sin x}{x} < 1$ (as $x < 0$)

$$\Rightarrow \left[\frac{\sin x}{x}\right] = 0$$

$$\Rightarrow \lim_{x \to 0-} \left[\frac{\sin x}{x}\right] = 0$$

Assignment Problem.

* Find $\lim\limits_{x \to 0+} \left\{\dfrac{\sin x}{x}\right\}$ and $\lim\limits_{x \to 0-} \left\{\dfrac{\sin x}{x}\right\}$, where

$\{x\} = x - [x]$, that is fractional part of x, $\forall x \in \mathbb{R}$.

Note: For computation of $\lim\limits_{x \to a} f(x)$,

it is sufficient to know the function $f(x)$ in a neighbourhood of a,

that is in $(a-r, a+r)$ for some $r > 0$.

For $\lim\limits_{x \to a+} f(x)$, need to know $f(x)$ in $(a, a+r)$. Similarly ~~same~~ for RHL in $(a-r, r)$.