(iv) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx$ converges to $\pi \log 2$.

8. Assuming convergence of the integral $\int_0^{\frac{\pi}{2}} \cos 2nx \log \sin x \, dx$ to $-\frac{\pi}{4n}$, when n is a positive integer, prove that

(i) $\int_0^{\frac{\pi}{2}} \cos 2nx \log \cos x \, dx$ converges to $(-1)^{n+1} \frac{\pi}{4n}$, when n is a positive integer;

(ii) $\int_0^{\pi} \cos nx \log 2(1 + \cos x) \, dx$ converges to $(-1)^{n+1} \frac{\pi}{n}$, when n is a positive integer.

12.11. Beta function and Gamma function.

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent if $m > 0, n > 0$. The integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$ is called the *Beta function* and it is denoted by $B(m,n)$.

Thus $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$.

The improper integral $\int_0^\infty e^{-x} x^{n-1} dx$ is convergent if $n > 0$. The integral $\int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$ is called the *Gamma function* and it is denoted by $\Gamma(n)$.

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$.

Properties.

1. $B(1,1) = 1$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$.

Therefore $B(1,1) = \int_0^1 dx = 1$.

2. $B(m,n) = B(n,m)$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$.

$$= \lim_{\epsilon \to 0 \delta \to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1} dx.$$

Let $x = 1 - y$. Then $dx = -dy$.

$$\lim_{\epsilon \to 0 \delta \to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1} dx = \lim_{\epsilon \to 0 \delta \to 0} \int_\delta^{1-\epsilon} (1-y)^{m-1} y^{n-1} dy.$$

$$= \lim_{\delta \to 0 \epsilon \to 0} \int_\delta^{1-\epsilon} y^{n-1}(1-y)^{m-1} dy = \int_0^1 y^{n-1}(1-y)^{m-1} dy = B(n,m).$$

Therefore $B(m,n) = B(n,m)$.

3. $B(m+1,n) = \frac{m}{m+n} B(m,n)$, $m > 0, n > 0$.

Proof. $B(m+1,n) = \int_0^1 x^m(1-x)^{n-1}dx$

$$= \left[\frac{x^m(1-x)^n}{-n}\right]_0^1 + \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n dx$$
$$= \frac{m}{n}\int_0^1 (1-x)x^{m-1}(1-x)^{n-1}dx$$
$$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}dx - \frac{m}{n}\int_0^1 x^m(1-x)^{n-1}dx$$
$$= \frac{m}{n}B(m,n) - \frac{m}{n}B(m+1,n).$$

Therefore $(1+\frac{m}{n})B(m+1,n) = \frac{m}{n}B(m,n)$

or, $B(m+1,n) = \frac{m}{m+n}B(m,n)$.

4. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m>0, n>0$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0, n>0$.

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$.

As $x \to 0+, \theta \to 0+$; as $x \to 1-, \theta \to \frac{\pi}{2}-$.

Therefore $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m>0, n>0.$$

Deductions.

(i) $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2}), m>-1, n>-1$.

(ii) $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{1}{2}B(\frac{n+1}{2}, \frac{1}{2}), n > -1$.

(iii) $B(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$.

5. $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx,\ m>0, n>0$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0, n>0$.

Let $x = \frac{t}{1+t}$. Then $dx = \frac{1}{(1+t)^2}$.

As $x \to 0+, t \to 0+$; as $x \to 1-, t \to \infty$.

Therefore $B(m,n) = \int_0^{\infty}(\frac{t}{1+t})^{m-1}(\frac{1}{1+t})^{n-1}\frac{1}{(1+t)^2}dt$

$$= \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}dt$$
$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$

6. $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$.

Proof. We have $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$$

Let $x = \frac{1}{t}$ in the second integral. Then $dx = -\frac{1}{t^2}$.

As $x \to 1+, t \to 1-$; as $x \to \infty, t \to 0+$.

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{1}{t^{m-1}} \frac{t^{m+n}}{(1+t)^{m+n}} \frac{1}{t^2} dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

Therefore $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx.$$

7. $\Gamma(1) = 1$.

Proof. $\Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{X\to\infty} \int_0^X e^{-x} dx = \lim_{X\to\infty} [1 - e^{-X}]_0^X = 1.$

8. $\Gamma(n+1) = n\Gamma(n),\ n > 0$.

Proof. $\int_\epsilon^X x^n e^{-x} dx = [\frac{x^n e^{-x}}{-1}]_\epsilon^X + n \int_\epsilon^X x^{n-1} e^{-x} dx$

$$= -X^n e^{-X} + \epsilon^n e^\epsilon + n \int_\epsilon^X x^{n-1} e^{-x} dx.$$

Proceeding to limit as $X \to \infty$ and $\epsilon \to 0$, we have

$\int_0^\infty e^{-x} x^n dx = n \int_0^\infty e^{-x} x^{n-1} dx$

or, $\Gamma(n+1) = n\Gamma(n),\ n > 0$.

Corollary. If n be a positive integer then $\Gamma(n+1) = n!$.

$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)...2.1\Gamma(1) = n!$.

9. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ m > 0, n > 0$.

The proof of the property is beyond the scope of this book.

Deductions.

(i). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$B(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})$.

Therefore $(\Gamma(\frac{1}{2}))^2 = B(\frac{1}{2}, \frac{1}{2}) = \pi$ and this gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(ii). If m, n be positive integers, $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

$B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)},\ m > -1, n > -1.$

If m, n are postive integers, $\Gamma(m+1) = m!$, $\Gamma(n+1) = n!$ and therefore $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

10. Legender's Duplication formula.

$\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$, $n > 0$.

Proof. $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$

$= 2\int_0^{\pi} 2\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$, $m > 0, n > 0$ ... (i)

Taking $m = n$, we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta$

$= \frac{1}{2^{2n-2}}\int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta\, d\theta$

$= \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n-1}\phi\, d\phi$ $[2\theta = \phi]$

$= \frac{1}{2^{2n-1}}.2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\phi\, d\phi$... (ii)

Taking $m = \frac{1}{2}$ in (i), we have $\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\, d\theta$ ... (iii)

From (ii) and (iii) we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}$, $n > 0$

or, $\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

11. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$, $0 < m < 1$.

Proof. We have $B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\Gamma(1-m)$.

Since $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx$, $m > 0, n > 0$, $B(m, 1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx$, $0 < m < 1$.

Therefore $\Gamma(m)\Gamma(1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin m\pi}$, $0 < m < 1$. [worked Ex.9, page 524]

12. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt = \frac{\Gamma(n)}{k^n}$, $k > 0, n > 0$;

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}} = \frac{\Gamma(n)}{k^n}$, $k > 0, n > 0$.

Proof. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt$

$= \int_0^{\infty} e^{-y}(\frac{y}{k})^{n-1}\frac{1}{k}dy$ [Let $y = kt$. As $t \to \infty, t \to \infty$ since $k > 0$.]

$= \frac{1}{k^n}\int_0^{\infty} e^{-y}y^{n-1}dy$, $n > 0$

$= \frac{\Gamma(n)}{k^n}$.

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}}dy$

$= \int_0^\infty t^{n-1} e^{-kt} dt$ [Let $\log y = t$. Then $y = e^t$. $y = 1 \Rightarrow t = 0$]

$= \frac{\Gamma(n)}{k^n}$, since $k > 0, n > 0$. [using (i)]

Worked Examples.

1. Prove that (i) $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$; (ii) $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$;

(i) Let $x^2 = t$. Then $dx = \frac{1}{2\sqrt{t}} dt$. As $x \to \infty, t \to \infty$.

$\int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$.

(ii) Let $f(x) = e^{-x^2}, x \in \mathbb{R}$. Then f is an even function on $\mathbb{R}$.

Therefore $\int_{-\infty}^\infty e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx$, assuming convergence of the integral on the right

$= \sqrt{\pi}$.

2. Prove that $\int_0^{\frac{\pi}{2}} \sin^p x \, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x \, dx = \frac{\pi}{2(p+1)}, p > -1$.

We have $\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} B(\frac{m+1}{2}, \frac{n+1}{2}), m > -1, n > -1$.

Therefore $\int_0^{\frac{\pi}{2}} \sin^p x \, dx = \frac{1}{2} B(\frac{p+1}{2}, \frac{1}{2}) = \frac{1}{2} \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}, p > -1$

and $\int_0^{\frac{\pi}{2}} \sin^{p+1} x \, dx = \frac{1}{2} B(\frac{p+2}{2}, \frac{1}{2}) = \frac{1}{2} \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})}, p > -2$.

$\int_0^{\frac{\pi}{2}} \sin^p x \, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x \, dx = \frac{1}{4} \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})} \cdot \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})}, p > -1$

$= \frac{1}{4} \frac{\pi\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+3}{2})}$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$= \frac{1}{4} \frac{2\pi}{p+1}$, since $\Gamma(\frac{p+3}{2}) = \frac{p+1}{2}\Gamma(\frac{p+1}{2})$

$= \frac{\pi}{2(p+1)}$.

3. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = (b-a)^{m+n-1} B(m,n)$, $m > 0, n > 0$.

We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$.

$(x-a) + (b-x) = b - a \Rightarrow \frac{x-a}{b-a} + \frac{b-x}{b-a} = 1$.

Let $\frac{x-a}{b-a} = y$. Then $\frac{b-x}{b-a} = 1 - y$, $dx = (b-a)dy$.
As $x \to a, y \to 0$; as $x \to b, y \to 1$.

$\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = \int_0^1 (b-a)^{m+n-1} y^{m-1}(1-y)^{n-1} dy$

$= (b-a)^{m+n-1} B(m,n)$.

4. Prove that $\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}} dx = \frac{\pi}{n} \text{cosec } \frac{\pi}{n}$, $n > 1$.

Let $x^n = t$. Then $dx = \frac{1}{nt^{\frac{n-1}{n}}} dt$.

$$\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}} dx = \int_0^1 (1-t)^{-\frac{1}{n}} . \frac{1}{nt^{\frac{n-1}{n}}} dt$$
$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1}(1-t)^{(1-\frac{1}{n})-1} dt$$
$$= \frac{1}{n} B(\tfrac{1}{n}, 1 - \tfrac{1}{n}), \text{ since } 0 < \tfrac{1}{n} < 1$$
$$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})}{\Gamma(1)}$$
$$= \frac{1}{n} \frac{\pi}{\sin \frac{\pi}{n}} = \frac{\pi}{n} \text{cosec } \frac{\pi}{n}.$$

5. If n be a positive integer, prove that $\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n}) \ldots \Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

Let $P = \Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n}) \ldots \Gamma(\frac{n-1}{n})$.

Then $P = \Gamma(1 - \frac{1}{n})\Gamma(1 - \frac{2}{n}) \ldots \Gamma(\frac{2}{n})\Gamma(\frac{1}{n})$. [taking the factors in the reverse order]

$$P^2 = [\Gamma(\tfrac{1}{n})\Gamma(1 - \tfrac{1}{n})][\Gamma(\tfrac{2}{n})\Gamma(1 - \tfrac{2}{n}] \ldots [\Gamma(\tfrac{n-1}{n})\Gamma(\tfrac{1}{n})]$$
$$= \frac{\pi}{\sin \frac{\pi}{n}} . \frac{\pi}{\sin \frac{2\pi}{n}} \cdots \frac{\pi}{\sin \frac{(n-1)\pi}{n}} = \frac{\pi^{n-1}}{\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n}} \ldots \text{(i)}$$

We prove the following lemma.

Lemma. $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

Proof. $x^{2n} - 2x^n \cos 2n\theta + 1 = 0$ gives $x^n = \cos 2n\theta + i \sin 2n\theta$, i.e., $x = \cos(2\theta + \frac{2k\pi}{n}) + i \sin(2\theta + \frac{2k\pi}{n})$, where $k = 0, 1, \ldots, n-1$.

Therefore $x^{2n} - 2x^n \cos 2n\theta + 1 = \prod\limits_{k=0}^{n-1} [x^2 - 2x \cos(2\theta + \dfrac{2k\pi}{n}) + 1]$.

Taking $x = 1$, we have $4 \sin^2 n\theta = \prod\limits_{k=0}^{n-1} 4 \sin^2(\theta + \dfrac{k\pi}{n})$.

$\sin^2 n\theta = 4^{n-1} \sin^2 \theta \sin^2(\theta + \frac{\pi}{n}) \ldots \sin^2(\theta + \frac{(n-1)\pi}{n})$

or, $\sin n\theta = 2^{n-1} \sin \theta \sin (\theta + \frac{\pi}{n}) \sin (\theta + \frac{2\pi}{n}) \ldots \sin(\theta + \frac{(n-1)\pi}{n})$

or, $\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin (\theta + \frac{\pi}{n}) \sin (\theta + \frac{2\pi}{n}) \ldots \sin(\theta + \frac{(n-1)\pi}{n})$.

Proceeding to limit as $\theta \to 0$, we have

$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n}$

or, $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

This proves the lemma.