* Problem Set 1. (Tutorial Problems)

1. Recall Archimedean Property -

If $x \in \mathbb{R}$ then there exists a natural number n such that $x < n$.

[~~Now~~ In this problem, given that ~~we have~~ $a \in \mathbb{R}$ and $\cancel{a > 0}$

$0 \leq a < \frac{1}{n}$ for all $n \in \mathbb{N}$. (i)

If $a > 0$ then $\frac{1}{a} \in \mathbb{R}$.

$\therefore$ by Archimedean Property there exists a natural number n_0 such that

$$\frac{1}{a} < n_0 \Rightarrow \frac{1}{n_0} < a.$$ which contradicts (i).

Hence, $a = 0$.]

2. Given that $y \in \mathbb{R}$, $y > 0$.

$\therefore \Rightarrow \frac{1}{y} \in \mathbb{R}$ and $\frac{1}{y} > 0$

so, $\log_2\left(\frac{1}{y}\right) \in \mathbb{R}$.

By Archimedean Property there exists a natural number m such that

$$\log_2\left(\frac{1}{y}\right) < m \Rightarrow \frac{1}{y} < 2^m \Rightarrow \frac{1}{2^m} < y$$

3. The solution set is the union of two sets S_1 and S_2 where

$$S_1 = \left\{x : 2x-6 > 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\right\}$$

$$S_2 = \left\{x : 2x-6 < 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\right\}.$$

If $2x-6>0$, then

$$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow -2x+6 \leq x+3 \leq 2x-6 \quad \text{(i)}$$

If $2x-6<0$ then

$$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow 2x-6 \leq x+3 \leq -2x+6 \quad \text{(i)}$$

From (i) $x>3$ and $x \geq 1$ and $x \geq 9$ simultaneously.

From (ii) $x<3$ and $x \leq 9$ and $x \leq 1$ simultaneously.

Therefore $S_1 = \{x \in \mathbb{R} : x \geq 9\}$ and $S_2 = \{x \in \mathbb{R} : x \leq 1\}$

So, the solution set is $\{x \in \mathbb{R} : x \geq 9\} \cup \{x \in \mathbb{R} : x \leq 1\}$.

$\boxtimes$

5. Given that

i) l is a lower bound of S.

ii) for each $n \in \mathbb{N}$, there exists an element $s_n \in S$ such that $s_n < l + \frac{1}{n}$.

To show, $l = \inf S$, we need ~~the~~ to show the followings.

A) l is a lower bound of S,

B) for any positive ε; $l + \varepsilon$ is NOT a lower bound of S.

Note that A) is given.

~~Now,~~ Let $\varepsilon > 0$ be given. ~~Then,~~ Since $\frac{1}{\varepsilon} \in \mathbb{R}$, by Archimedean Property there exists a natural number k such that

$$\frac{1}{\varepsilon} < k. \Rightarrow \frac{1}{k} < \varepsilon$$

$$\Rightarrow l + \frac{1}{k} < l + \varepsilon \quad \text{——— (iii)}$$

$\therefore$ by (ii), for this k, there exists an element $s_k \in S$ such that $s_k < l + \frac{1}{k}$

$$\Rightarrow s_k < l + \varepsilon \quad (\text{using (iii)})$$

This shows that a member of S is less that $l + \varepsilon$.

$\therefore$ $l + \varepsilon$ is NOT a lower bound of S.

As l is a lower bound and $l + \varepsilon$ (for any $\varepsilon > 0$) is NOT a lower bound of S implies that

$$l = \inf S \text{ or glb } S$$