* Real Sequence.

Def^n: A mapping $f: \mathbb{N} \to \mathbb{R}$ is said to be a sequence in $\mathbb{R}$, or a real sequence.

The f-images $f(1), f(2), f(3), \ldots$ are real numbers.

A sequence f is generally denoted by the symbol $\{f(n)\}$.

Also the symbol $\{f(1), f(2), f(3), \cdots\}$ is used to denote the sequence f.

The symbols like $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, etc. shall also be used to denote a sequence.

Examples.

$\{n\}$, $\{n^2\}$, $\{\frac{n}{n+1}\}$, $\{(-1)^n\}$, $\{\sin\frac{n\pi}{2}\}$

$\{2\}$.

constant sequence. Oscillatory Sequence

✓ The sequence $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = \sin\frac{n\pi}{2}$, $n \in \mathbb{N}$.

✓ The same sequence is $\{1, 0, -1, 0, 1, 0, \ldots\}$. The range of this sequence is $\{-1, 0, 1\}$

✓ Sometimes it is convenient to specify $f(1)$ and describe $f(n+1)$ in terms of $f(n)$ for all $n \geq$. Ex: $f(1) = \sqrt{2}$, $f(n+1) = \sqrt{2f(n)}$ $\forall n \geq 1$

$\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$

* Bounded Sequence.

A sequence $\{f(n)\}$ is said to be bounded above if there exists a real number G such that $f(n) \leq G$ for all $n \in \mathbb{N}$. G is said to be a upper bound of the sequence.

A real sequence $\{f(n)\}$ is said to be bounded below if there exists a real number g such that $f(n) \geq g$ for all $n \in \mathbb{N}$. g is said to be the lower bound of the sequence.

A real sequence $\{f(n)\}$ is said to be a bounded sequence if there exist real numbers G and g such that $g \leq f(n) \leq G \quad \forall n \in \mathbb{N}$.

Examples: ✓ $\{\frac{1}{n}\} = \{x_n\}$

0 is the greatest lower bound, and

1 " " least upper bound of this sequence $\{x_n\}$

✓ Let $\{y_n\} = \{n^2\}$

then $\sup_{n \geq 1} y_n = \infty \quad \inf_{n \geq 1} y_n = 1$.

* The least upper bound of a real sequence $\{f(n)\}$ is a real number M satisfying the following conditions:

(i) $f(n) \leq M \quad \forall n \in \mathbb{N}$,

(ii) for each (pre-assigned) given $\varepsilon > 0$, there exists a natural number k such that $f(k) > M - \varepsilon$.

- For a real sequence $\{x_n\}$ bounded below, there exists a greatest lower bound and it is denoted by ~~inf{f(n)}~~ $\inf x_n$ or $\text{glb}\, x_n$.

The greatest lower bound of a real sequence x_n is a real number m satisfying the following conditions:

(i) $x_n \geq m \quad \forall n \in \mathbb{N}$,

(ii) for any given $\varepsilon > 0$, there exists a natural number k such that $x_k < m + \varepsilon$.

Ex: $\{x_n\} = \{(-1)^n\}$, $\{y_n\} = \{(-1)^n n\}$

* Limit of a sequence:

Let $\{f(n)\}$ be a real sequence. A real number l is said to be a limit of the sequence $\{f(n)\}$ if for corresponding to a given $\varepsilon > 0$ there exists a natural number k (depending on ε) such that

$$|f(n) - l| < \varepsilon \quad \text{for all } n \geq k$$

i.e. $$l - \varepsilon < f(n) < l + \varepsilon \quad \text{for all } n \geq k.$$

$(l-\varepsilon, l+\varepsilon)$ is the ε-neighbourhood of l.

$$\therefore \quad f(n) \in (l-\varepsilon, l+\varepsilon) \quad \forall n \geq k$$

ie, $$\{f(k), f(k+1), f(k+2), \dots\} \subset (l-\varepsilon, l+\varepsilon).$$

Theorem: A sequence can have at most one limit.

Proof: If possible, let a sequence $\{x_n\}$ have two distinct limits l_1 and l_2 where $l_1 < l_2$.

Let $\varepsilon = \frac{1}{2}(l_2 - l_1)$. Then $\varepsilon > 0$ and $l_1 + \varepsilon = l_2 - \varepsilon$.

Therefore the neighbourhoods $(l_1-\varepsilon, l_1+\varepsilon)$ and $(l_2-\varepsilon, l_2+\varepsilon)$ are disjoint.

Since l_1 is a limit of the sequence, for the chosen ε, $\exists$s $k_1 \in \mathbb{N}$ such that $x_n \in (l_1-\varepsilon, l_1+\varepsilon)$ $\forall n \geq k_1$

ie, $$\{x_{k_1}, x_{k_1+1}, x_{k_1+2}, \dots\} \subset (l_1-\varepsilon, l_1+\varepsilon).$$

Since l_2 is a limit of the sequence, for the same chosen ε there exists a natural number k_2 such that

$$l_2 - \varepsilon < x_n < l_2 + \varepsilon, \quad \forall n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$

Then $\quad l_1 - \varepsilon < x_n < l_1 + \varepsilon \quad$ for all $n \geq k$

and $\quad l_2 - \varepsilon < x_n < l_2 + \varepsilon \quad$ for all $n \geq k$

i.e., $\quad \{x_k, x_{k+1}, x_{k+2}, \ldots\} \subset (l_1 - \varepsilon, l_1 + \varepsilon)$

and $\quad \{x_k, x_{k+1}, x_{k+2}, \ldots\} \subset (l_2 - \varepsilon, l_2 + \varepsilon)$

This can not happen. since the neighbourhoods $(l_1 - \varepsilon, l_1 + \varepsilon)$ and $(l_2 - \varepsilon, l_2 + \varepsilon)$ are disjoint.

Therefore our assumption that $l_1 \neq l_2$ is wrong.

Hence $l_1 = l_2$ and this proves the theorem.

- Convergent sequence.

A sequence $\{x_n\}$ is said to be a convergent sequence if it has a limit $l \in \mathbb{R}$.
In this case the sequence is said to converge to l.

We write $\quad \lim x_n = l \quad$ or $\quad \lim\limits_{n \to \infty} x_n = l$.

A sequence is said to be a divergent sequence if it is NOT convergent.

Example: i) $\{x_n\} = \{\frac{1}{n}\}$ $\quad \lim \frac{1}{n} = 0$

ii) $\{y_n\} = \{\frac{n^2+1}{n^2}\}$ $\quad \lim y_n = 1$

$\Big\}$ Prove this.

iii) $\{x_n\} = \{2\}$ i.e. $x_n = 2$ for all $n = 1, 2, 3, \ldots$

$\lim x_n = 2$.

Theorem: A convergent sequence is bounded.

Proof: Let $\{x_n\}$ be a convergent sequence and let

$$\lim x_n = l.$$

Let us choose $\varepsilon = 1$. For this chosen ε there exists a natural number k such that

$$l-1 < x_n < l+1, \quad \text{for all } n \geq k.$$

Let

$$M = \max\{x_1, x_2, \dots, x_k, l+1\}$$

$$m = \min\{x_1, x_2, \dots, x_k, l-1\}$$

Then

$$m \leq x_n \leq M \quad \text{for all } n \in \mathbb{N}.$$

This proves that the sequence $\{x_n\}$ is bounded.

Example: ~~The~~ $\left\{\frac{n^2+1}{n^2}\right\}$ is convergent because $\lim \frac{n^2+1}{n^2} = 1$.

It is bounded because,

$$1 \leq \frac{n^2+1}{n^2} \leq 2.$$

Corollary: An unbounded sequence can not be convergent.

$\boxed{\text{Note :}}$ A bounded sequence may not be a convergent sequence. For example,

$$\{x_n\} = \{(-1)^n\}$$

Note that, $\{x_n\}$ is a bounded sequence but NOT convergent.

The sequence $\left\{\sin \frac{n\pi}{2}\right\}$ is bounded but NOT convergent sequence.

* Show (by definition) that the sequence $\{\frac{1}{n}\}$ converges to 0.

Pf: Let $\varepsilon > 0$ be given.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that

$$0 < \frac{1}{k} < \varepsilon .$$

This implies $0 < \cdots < \frac{1}{k+2} < \frac{1}{k+1} < \frac{1}{k} < \varepsilon$

i.e. $0 < \frac{1}{n} < \varepsilon$ for all $n \geqslant k$.

It follows that $\left|\frac{1}{n} - 0\right| < \varepsilon$ for all $n \geqslant k$

This proves $\lim \frac{1}{n} = 0$

* Show (by definition) that the sequence $\left\{\frac{n^2+1}{n^2}\right\}$ converges to 1.

Pf: Let $\varepsilon > 0$ be given.

Now $\left|\frac{n^2+1}{n^2} - 1\right| < \varepsilon$ will hold

if $\left|1 + \frac{1}{n^2} - 1\right| < \varepsilon$, i.e., if $\left|\frac{1}{n^2}\right| < \varepsilon$ or $\frac{1}{n^2} < \varepsilon$

i.e. if $n > \frac{1}{\sqrt{\varepsilon}}$

Let $k = \left(\left[\frac{1}{\sqrt{\varepsilon}}\right] + 1\right)$, [where $[x]$ is the greatest integer $\leq x$]

[For example if $\varepsilon = 0.01$ then $k = 11$;
if $\varepsilon = 0.001$ then $k = 32$.]

$\therefore$ $k \in \mathbb{N}$ and $k > \frac{1}{\sqrt{\varepsilon}}$

$\therefore$ $\ldots > k+2 > k+1 > k > \frac{1}{\sqrt{\varepsilon}}$

i.e. $n > \frac{1}{\sqrt{\varepsilon}}$ $\forall$ $n \geqslant k$

This implies $\left|\frac{n^2+1}{n^2} - 1\right| < \varepsilon$ $\forall$ $n \geqslant k$.

This proves $\lim \frac{n^2+1}{n^2} = 1$.

* Limit theorems.

Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to u and v respectively.

Then (i) $\lim (u_n + v_n) = u + v$;

(ii) if $c \in \mathbb{R}$, $\lim (c u_n) = cu$;

(iii) $\lim (u_n v_n) = uv$;

(iv) $\lim \left(\frac{u_n}{v_n}\right) = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of non-zero real numbers and $v \neq 0$.

Proof: Left as exercise.

[Hint: (i) Let $\varepsilon > 0$ be given. $|(u_n + v_n) - (u+v)| = |(u_n - u) + (v_n - v)|$

$\leq |u_n - u| + |v_n - v|$

[triangular inequality]

Since $\lim u_n = u$, for $\frac{\varepsilon}{2} > 0$, there exists $k_1 \in \mathbb{N}$ such that

$|u_n - u| < \frac{\varepsilon}{2} \quad \forall n \geq k_1$.

Since $\lim v_n = v$, for $\frac{\varepsilon}{2} > 0$, there exists $k_2 \in \mathbb{N}$ such that

$|v_n - v| < \frac{\varepsilon}{2} \quad \forall n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then for all $n \geq k$,

$|u_n - u| < \frac{\varepsilon}{2}$ and $|v_n - v| < \frac{\varepsilon}{2}$.

$\therefore |(u_n - u) + (v_n - v)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq k$.

$\Rightarrow \lim(u_n + v_n) = (u+v)$

Similar way (ii), (iii) and (iv) can be proved.]

Theorem Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{|u_n|\}$ converges to $|u|$

Proof: [Hint: Use $\big||u_n| - |u|\big| \leq |u_n - u|$.].

Note: The converse of the above theorem is NOT true. That is, if $\{|u_n|\}$ is convergent sequence it does NOT necessarily imply that $\{u_n\}$ is a convergent sequence.

For example: Let $u_n = (-1)^n$.

Note that $|u_n| = |(-1)^n| = 1, \forall n \in \mathbb{N}$.

$\therefore$ $\{|u_n|\}$ is convergent and $\lim |u_n| = 1$,

but $\{u_n\}$ is NOT convergent.

Theorem: Let $\{u_n\}$ be a convergent sequence of ~~real~~ real no.s and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$

Proof: [Left as exercise.] [Hint! Let $\lim u_n = u$, and if possible $u < 0$; choose $\varepsilon > 0$ such that $u + \varepsilon < 0$.

Since $\lim u_n = u$, for the above $\varepsilon > 0$

$\exists k_1 \in \mathbb{N}$ such that $u - \varepsilon < u_n < u + \varepsilon \quad \forall n \geq k_1$.

Let $k = \max\{m, k_1\}$.]

Corollary: Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences and there exists a natural number m such that $u_n > v_n \;\forall n \geq m$.

Then $\lim u_n \geq \lim v_n$. [Hint: Let $w_n = u_n - v_n$].

* Sandwich Theorem.

Let $\{u_n\}, \{v_n\}, \{w_n\}$ be three sequences of real numbers and there is a natural number m such that

$$u_n < v_n < w_n \quad \text{for all } n \geq m.$$

If $\lim u_n = \lim w_n = l$ then $\{v_n\}$ is convergent and

$$\lim v_n = l.$$

example: [$\lim u_n = u$, $\lim v_n = v \Rightarrow \lim\left(\frac{u_n}{v_n}\right) = \frac{u}{v}$ provided $v \neq 0$ and $v_n \neq 0$ $\forall n$]

* show that $\lim\limits_{n\to\infty} \dfrac{3n^2+2n+1}{n^2+1} = 3$

Pf: Let $\lim\limits_{n\to\infty} \dfrac{3n^2+2n+1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{u_n}{v_n}$ where

$$u_n = 3 + \frac{2}{n} + \frac{1}{n^2} \quad \text{and}$$

$$v_n = 1 + \frac{1}{n^2}.$$

But $\lim u_n = 3$ and $\lim v_n = 1$.

$$\therefore \lim \frac{3n^2+2n+1}{n^2+1} = \lim\left(\frac{u_n}{v_n}\right) = \frac{3}{1} = 3.$$

→ Application of Sandwich Theorem.

* Prove that $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}\right) = 1$

Let $u_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}$

we have $\dfrac{1}{\sqrt{n^2+2}} < \dfrac{1}{\sqrt{n^2+1}}$, $\dfrac{1}{\sqrt{n^2+3}} < \dfrac{1}{\sqrt{n^2+1}}$, ..., $\dfrac{1}{\sqrt{n^2+n}} < \dfrac{1}{\sqrt{n^2+1}}$.

$$\therefore u_n < \frac{n}{\sqrt{n^2+1}} \quad \forall n \geq 2$$

Again $\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} > \dfrac{2}{\sqrt{n^2+2}}$, $\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \dfrac{1}{\sqrt{n^2+3}} > \dfrac{3}{\sqrt{n^2+3}}$

.... Therefore, $u_n > \dfrac{n}{\sqrt{n^2+n}} \quad \forall n \geq 2$.

Thus $\dfrac{n}{\sqrt{n^2+n}} < u_n < \dfrac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

* ## Some important limits

1> $\lim r^n = 0$ if $|r| < 1$.

Pf: [Hint : Case I : $r = 0$ then $r^n = 0 \ \forall n$, so $\lim r^n = 0$

Case II : $r \neq 0$ and $|r| < 1$

Then $\frac{1}{|r|} > 1$.

Let $\frac{1}{|r|} = a + 1$ where $a > 0$

$|r|^n = |r^n|$

Then $|r^n - 0| = \frac{1}{(1+a)^n} \quad \forall n = 1, 2, 3, \dots$

Note that $(1+a)^n > na$

$\therefore |r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$.]

2> $\lim a^{\frac{1}{n}} = 1$ if $a > 0$.

3> If $\lim x_n = 0$ and $a > 0$, then $\lim a^{x_n} = 1$.

Actually, If $\lim x_n = l$ and $a > 0$, then $\lim a^{x_n} = a^l$.

In particular, if $\lim x_n = l$, then $e^{x_n} \longrightarrow e^l$ as $n \to \infty$.

4> If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \log(1+x_n) = 0$.

5> If $u_n > 0$ and $\lim u_n = u > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v$, then $\lim (u_n)^{v_n} = u^v$.

6> $\lim n^{\frac{1}{n}} = 1$.

[Proofs of 1> to 6> are left as exercise]