Infinite Series

Let $\{u_n\}$ be a sequence.

Then the sequence $\{s_n\}$ defined by

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \quad s_3 = u_1 + u_2 + u_3, \ \dots$$

→ The term $u_1 + u_2 + u_3 + \cdots$ is said to be an infinite series (or a series) generated by the sequence $\{u_n\}$, it is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

→ u_n : the n^{th} term of the series.

→ The sequence $\{s_n\}$ is called the sequence of partial sums of the series $\sum u_n$.

→ The infinite series $\sum u_n$ is said to be convergent or divergent according as the sequence $\{s_n\}$ is convergent or divergent.

In case of convergence, if $\lim_{n\to\infty} s_n = s$ then s is said to be the sum of the series $\sum u_n$.

If, however, $\lim s_n = \infty$ (or $-\infty$) the series $\sum u_n$ is said diverge to ∞ (or $-\infty$).

Examples.

1. Let us consider the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

Let $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

$\Rightarrow S_n = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}$

$\Rightarrow \lim_{n\to\infty} S_n = 2$. [Since $\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$]

$\therefore$ The given series $1+\frac{1}{2}+\frac{1}{2^2}\cdots$ is convergent and the sum of the series is 2.

2. The series $1+a+a^2+\cdots$ is convergent when $|a|<1$

[because $S_n = 1+a+a^2+\cdots+a^{n-1}$

$$= \frac{1-a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a}$$

$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-a}$, Since $\lim_{n\to\infty} a^n = 0$]

This series is divergent if $|a|\geq 1$.

Theorem:

A necessary condition for the convergence of a series $\sum_{n=1}^{\infty} u_n$ is $\lim_{n\to\infty} u_n = 0$.

Proof: Let $\sum u_n$ is convergent.

$\Rightarrow$ The sequence of partial sum $S_n = u_1+u_2+\cdots+u_n$ is convergent.

$\Rightarrow \{S_n\}$ is a Cauchy sequence.

~~Let~~ Then for a pre-assigned positive ε there exists a natural number k such that

$$|S_{n+p} - S_n| < \varepsilon \quad \text{for all } n \geq k \text{ and for every natural number } p.$$

[this is the (another) definition of Cauchy sequence]

take $p = 1$, $|u_{n+1}| < \varepsilon$ for all $n \geq k$.

This implies $\lim_{n\to\infty} u_n = 0$

Example: Prove that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

$\Rightarrow$ Here let $u_n = \frac{n}{n+1} \Rightarrow \lim_{n\to\infty} u_n = 1$.

Since $\lim u_n$ is NOT 0, $\sum u_n$ is divergent because a ~~convergent~~ necessary condition for convergence of the series $\sum u_n$ is $\lim_{n\to\infty} u_n = 0$.

The converse of the above theorem is NOT true.

That is $\lim u_n = 0$ does not necessarily imply convergence of the series $\sum u_n$.

consider the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum \frac{1}{n}$, that is

$$u_n = \frac{1}{n}$$

Here, $\lim u_n = 0$.

But $\sum u_n$ is a divergent series.

* A series of positive real numbers $\sum u_n$ (that is for each $n \in \mathbb{N}$, $u_n > 0$) is convergent if and only if the sequence $\{S_n\}$ of partial sums is bounded above.

<u>Proof</u>: $S_n = u_1 + u_2 + \cdots + u_n$.

Then $S_{n+1} - S_n = u_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence the sequence $\{S_n\}$ is a monotone increasing sequence.

Therefore $\{S_n\}$ is convergent if and only if it is bounded above.

Consequently, the series $\sum u_n$ is convergent iff the sequence $\{S_n\}$ is bounded above.

<u>Definition</u>: $$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$ and $0! = 1$.

Since $$S_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdots n}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} < 3$$
[Because, $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2$]

$\Rightarrow$ $2 < S_n < 3$

$\Rightarrow$ The sequence of partial sum $\{S_n\}$ of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is bounded above (by 3).

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent.

* Note

So, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$

Let $S_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ [n^{th} member of the [sequence of partial sum] of $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$]

$$\Rightarrow \quad e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$= \frac{1}{(n+1)!}\left\{1 + \frac{1}{(n+2)} + \frac{1}{(n+3)(n+2)} + \cdots\right\}$$

$$< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right\}$$

$$= \frac{1}{(n+1)!} \; \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!\, n}$$

$$\Rightarrow \quad 0 < e - S_n < \frac{1}{n!\,n} \quad \text{—— (i)}$$

* Show that e is irrational.

Suppose e is rational. Then $e = \frac{p}{q}$, where p and q are positive integers.

$\therefore$ by (i), $\quad 0 < e - S_q < \frac{1}{q!\,q}$, —— (ii)

where $S_q = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}$.

Now, $e = \frac{p}{q} \Rightarrow p = e \cdot q$ is an integer

$\Rightarrow eq!$ is also an integer [As p is an integer and $eq = p$. $eq! = \underbrace{eq}_{\text{integer}} \times \underbrace{(q-1) \times \cdots \times 2 \times 1}_{\text{integer}}$]

From (ii) $\quad 0 < q!\,e - q!\,S_q < \frac{1}{q}$

Note that $q!\,S_q = q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$ is also an integer.

Since, $q \geq 1$, a contradiction arises, because there is NO integer between 0 and 1. $\left[0 < \underbrace{(q!\,e - q!\,S_q)}_{\text{integer}} < \frac{1}{q}\right]$

* Tests for convergence of a series of positive terms :

Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers. ~~and~~

1. Comparison test.

Let $u_n \leq k\, v_n$ for all $n \geq m$ and $k > 0$ be a fixed real number

Then i) $\sum u_n$ is convergent if $\sum v_n$ is convergent,

ii) $\sum v_n$ is divergent if $\sum u_n$ is divergent.

2. Limit form.

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l$$, where l is a non-zero finite number.

Then the two series $\sum u_n$ and $\sum v_n$ converge or diverge together.

3. D'Alembert's ratio test.

Let $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$. Then $\sum u_n$ is convergent if $l < 1$,

$\sum u_n$ is divergent if $l > 1$.

If $l = 1$, the test fails to give a decision.

4. Cauchy's root test.

Let $\lim_{n \to \infty} u_n^{\frac{1}{n}} = l$. Then $\sum u_n$ is convergent if $l < 1$,

$\sum u_n$ is divergent if $l > 1$.

The test fails to give decision if $l = 1$.

5. Cauchy's condensation test.

Let $\{f(n)\}$ be a monotone decreasing sequence of positive real numbers and a be a positive integer >1. Then the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ converge or diverge together.

Example: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent by the above test.

Let $a = 2 > 1$.

Here, $f(n) = \frac{1}{n(\log n)^2}$ is monotone decreasing.

ie. $\frac{1}{(n+1)(\log(n+1))^2} < \frac{1}{n(\log n)^2}$ [Since $\log x$ is an increasing function]

By the Cauchy's condensation test $\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together.

Note that $\sum 2^n f(2^n) = \sum 2^n \frac{1}{2^n (\log 2^n)^2}$

$= \sum \frac{1}{n^2 (\log 2)^2} = \frac{1}{(\log 2)^2} \sum \frac{1}{n^2}$

and this is a convergent series.

$\therefore$ $\sum f(n)$ or $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.

6. Raabe's test.

Let $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$. Then

$\sum u_n$ is convergent if $l > 1$,
$\sum u_n$ is divergent if $l < 1$.
If $l = 1$, the test is inconclusive.

Ex. Show that the series $1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \frac{1\cdot 3.5}{2\cdot 4\cdot 6}\cdot\frac{1}{7} + \dots$ is convergent by Raabe's test.