Department of Mathematical Sciences

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MA111 / Real Analysis and Calculus

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Q1. Check the convergence of the sequence $\left\{ \frac{\cos n}{1+n^2} \right\}$ and find its limit if exists. [3]

Solution: Note that $-1 \leq \cos n \leq 1$. It follows that

$$\frac{-1}{1+n^2} \leq \frac{\cos n}{1+n^2} \leq \frac{1}{1+n^2}.$$

Since $\lim_{n\to\infty} \frac{-1}{1+n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{1+n^2} = 0$, by *Sandwich theorem* the sequence $\left\{ \frac{\cos n}{1+n^2} \right\}$ is convergent and

$$\lim_{n\to\infty} \frac{\cos n}{1+n^2} = 0$$

Q2. Check the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{\cos n}{1+n^2}$. [3]

Solution: Let $u_n = \frac{\cos n}{1+n^2}$. Then $|u_n| \leq \frac{1}{1+n^2} \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} |u_n|$ is a series of positive terms and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by comparison test $\sum_{n=1}^{\infty} |u_n|$ is convergent. That means $\sum_{n=1}^{\infty} u_n$ is *absolutely convergent*.

We know that an **absolutely convergent series is convergent**. Therefore $\sum_{n=1}^{\infty} u_n$ is convergent.

******Note that one can apply comparison test to check the convergence of series of positive terms (real numbers). As $\sum_{n=1}^{\infty} \frac{\cos n}{1+n^2}$ is a series of positive and negative terms, comparison test can not be applied to check the convergence of this series.**

Q3. A function $f : [0,1] \to [0,1]$ is continuous. Prove that there is a point c in $[0,1]$ such that $f(c) = c^{2023}$. [4]

Solution: If $f(0) = 0$ or $f(1) = 1$, the existence of c is proved.
Let $f(0) \neq 0$ and $f(1) \neq 1$.
Consider a function $\varphi : [0,1] \to \mathbb{R}$ by

$$\varphi(x) = f(x) - x^{2023}.$$

Then φ is continuous on $[0,1]$. Also $\varphi(0) = f(0) > 0$ (since $f(0) \in [0,1]$ and $f(0) \neq 0$) and $\varphi(1) = f(1) - 1 < 0$ (since $f(1) \in [0,1]$ and $f(1) \neq 1$).
By *Bolzano's theorem* (or *Intermediate Value Property*) there is a point c in $(0,1)$ such that $\varphi(c) = 0$. It follows that

$$f(c) = c^{2023}.$$