Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

REAL ANALYSIS & CALCULUS (MA 111)

Week 3 / September 2023	Problem Set 5	GR, PD

Real Analysis

Continuity and Differentiability (of real-valued functions)

■ Tutorial Problems

1. Show that (Using ϵ-δ *definition* or *sequential definition*) the following functions are continuous at any $c \in \mathbb{R}$.

   i. $f(x) = \sqrt{x}, \qquad c > 0$

   ii. $f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x \neq 2 \\ 12 & \text{if } x = 2. \end{cases}$

2. Show that $f(x) = [x],\ 0 < x < 2$ is not continuous at $x = 1$.

3. Show that (*Dirichlet's function*) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is nowhere continuous in $\mathbb{R}$.

4. Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2-x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ is continuous at $x = 1$ only.

5. Show that if f is continuous then $|f|$ is continuous, where $|f|(x) = |f(x)|$. Give an example that the converse of this is NOT true.

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at one point $c \in \mathbb{R}$, prove that f is continuous at any point in $\mathbb{R}$.

7. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x = 0$, prove that f is continuous on $\mathbb{R}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

9. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and let $f(a) < g(a)$, $f(b) > g(b)$. Show that there exists a point c in (a,b) such that $f(c) = g(c)$.

   Deduce that $\cos x = x^2$ for some $x \in \left(0, \frac{\pi}{2}\right)$.

10. A real valued function f is continuous on $[0,2]$ and $f(0) = f(2)$. Prove that there exists at least a point c in $[0,1]$ such that $f(c) = f(c+1)$.

11. Find the n^{th} derivative of the function $x^3 \log(x+1)$.

12. If $x + y = 1$ find the n^{th} derivative of $x^n y^n$.

13. If $y = \left(x + \sqrt{1+x^2}\right)^m$, find the value of $y^{(n)}(0)$, where $y^{(n)}(0)$ denotes the n^{th} derivative of y (with respect to x) at 0.

    [**Hint:** Calculate y', y'' and get the equation $(1+x^2)y'' + xy' - m^2 y = 0$, then differentiate n times by Leibnitz's theorem/rule.]

■ Assignment Problems

1. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ having that same range $[0,1]$, prove that $f(c) = g(c)$ for some c in $[a,b]$.

2. A function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and assumes only rational values. If $f\left(\frac{1}{2}\right) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ for all $x \in [0,1]$.

3. Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ is continuous at $x = 0$ only.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

5. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that either $f(x) = 0$ for all $x \in \mathbb{R}$, or $f(x) = a^x$ for all $x \in \mathbb{R}$, where a is some positive real number.

6. If $y = \frac{\log x}{x}$, prove that $y^{(n)} = \frac{(-1)^n n!}{x^{n+1}} \left[\log x - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right]$.

7. If $y = \cos\left(m \sin^{-1} x\right)$, prove that

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} + (m^2 - n^2)y^{(n)} = 0.$$

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