Example: If $x_n = (a^n + b^n)^n$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim\limits_{n \to \infty} x_n = b$.

Solution: $x_n = b\left[\left(\frac{a}{b}\right)^n + 1\right]^{\frac{1}{n}} > b$ for all $n \in \mathbb{N}$,

$\left[\text{Since } \left(\frac{a}{b}\right)^n + 1 > 1 \ \forall n \in \mathbb{N}\right]$

Again, $0 < a < b \Rightarrow a^n < b^n$ for all $n \in \mathbb{N}$.

$\therefore \ a^n + b^n < 2b^n$

or, $x_n < 2^{\frac{1}{n}} b$ for all $n \in \mathbb{N}$.

Let $u_n = b$ for all $n \in \mathbb{N}$.

$v_n = 2^{\frac{1}{n}} b$ for all $n \in \mathbb{N}$.

Then $\lim\limits_{n \to \infty} u_n = b$ and $\lim\limits_{n \to \infty} v_n = b$ $\left[\text{as } \lim\limits_{n \to \infty} 2^{\frac{1}{n}} = 1\right]$.

Now $u_n < x_n < \cancel{b_n}\, v_n$ for all $n \in \mathbb{N}$.

$\therefore$ By Sandwich theorem $\lim\limits_{n \to \infty} x_n = b$.

Cauchy Sequence.

A sequence $\{u_n\}$ is said to be a Cauchy sequence if for a pre-assigned positive ε the exists a natural number k such that

$$|u_n - u_m| < \varepsilon \quad \forall\, n,m \geq k.$$

Theorem 1.

A convergent sequence is a Cauchy sequence.

[Proof: ~~Not~~ Left as a reading exercise.]

Theorem 2.

A Cauchy sequence of real numbers is convergent.

[Proof: left as a reading exercise]

Example: Prove that $\{\frac{1}{n}\}$ is a Cauchy sequence.

Let $u_n = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. There is a natural number k such that

$\frac{2}{k} < \varepsilon$.

Then $|u_m - u_n| = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon$ if $m,n \geq k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

* Prove that the sequence $\{(-1)^n\}$ is NOT a Cauchy sequence.

Soln: ~~Exercise.~~ Let $u_n = (-1)^n$.

Then $|u_m - u_n| = |(-1)^m - (-1)^n|$.

$|u_m - u_n| = 0$ if m and n are both odd or both even.

$|u_m - u_n| = 2$ if one of m,n is odd and the other is even.

Let us choose $\varepsilon = \frac{1}{2}$. Then it is NOT possible to find a natural number k such that

$$|u_m - u_n| < \varepsilon \quad \text{for all } m, n \geq k.$$

Hence $\{u_n\}$ is NOT a Cauchy sequence.

* Prove that the sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is NOT convergent.

$\Rightarrow$ ~~let p be a natural number~~

Note that $|u_{2m} - u_m|$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m}$$

$$= \frac{1}{2}$$

$\therefore \; |u_{2m} - u_m| > \frac{1}{2}$ for any $m \in \mathbb{N}$.

If we choose $\varepsilon = \frac{1}{2}$ then no natural number k can be found such that $|u_m - u_n| < \varepsilon$ will hold for all $m, n \geq k$.

$\therefore$ This shows that $\{u_n\}$ is NOT a Cauchy sequence, therefore $\{u_n\}$ is NOT convergent.

* Prove that the sequence $\{u_n\}$ where $u_1 = 0$, $u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \geqslant 1$ is a Cauchy sequence.

Solution :

$$u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1} = -\frac{1}{2}(u_{n+1} - u_n)$$

or, $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$.

$$\therefore |u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n| = \frac{1}{2^2}|u_n - u_{n-1}| = \cdots$$

$$= \frac{1}{2^n}|u_2 - u_1| = \frac{1}{2^n}$$

Let $m > n$. Then $|u_m - u_n|$

$$\leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \cdots + |u_{n+1} - u_n|$$

$$= \left(\frac{1}{2}\right)^{m-2} + \left(\frac{1}{2}\right)^{m-3} + \cdots + \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{4}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] < \frac{4}{2^n}$$

Let $\varepsilon > 0$. Then there exists a natural number k such that $\frac{4}{2^n} < \varepsilon$ for all $n \geqslant k$.

Hence $|u_m - u_n| < \varepsilon$ for all $m, n \geqslant k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

[what if $u_{n+2} = c(u_{n+1} + u_n)$, $u_1 = 0$, $u_2 = 1$, where $0 < c < 1$.]