Real Analysis and Calculus.

MA111

- $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

$\mathbb{Q} = \{ \frac{p}{q} : q \neq 0, p, q \in \mathbb{Z}\}$

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$

- Density property of $\mathbb{Q}$.

If x and y be any two rational numbers and $x < y$, there exists a rational number r such that $x < r < y$.

~~Ans~~ $x < \frac{1}{2}(x+y) < y$

Actually between any two rational numbers x and y we can interpolate infinitely many rational numbers.

Def^n^: Let S be a subset of $\mathbb{R}$. A real number u is said to be an upper bound if $x \in S \Rightarrow x \leq u$.

A real number l is said to be an lower bound of S if $x \in S \Rightarrow x \geq l$.

Let S be a subset of $\mathbb{R}$. S is said to be bounded above if S has an upper bound.

S is said to be bounded below if S has a lower bound.

S is said to be a bounded set if S be bounded above as well as bounded below.

- Example: $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is bounded above, 1 being an upper bound. S is bounded below, 0 being a lower bound.

Definition: Let S be a subset of $\mathbb{R}$. If S be bounded above, then an upper bound of S is said to be the <u>supremum</u> of S (or the <u>least upper bound</u> of S) if it is less than every upper bound of S.

If S be bounded below then a lower bound of S is said to be the <u>infimum</u> of S (or <u>the greatest lower bound</u> of S) if it is greater than every other lower bound of S.

Example: Let $S = \phi$. Every real number x is an upper bound of the set S and every real number x is a lower bound of S. S is a bounded set.

- <u>The supremum property of $\mathbb{R}$.</u>

  Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Theorem: Let S be a non-empty subset of $\mathbb{R}$, bounded below. Then S has an infimum.

Proof: Let l_0 be a lower bound of S. Let
$$T = \{l \in \mathbb{R} : l \text{ is a lower bound of } S\}.$$
Then T is a non-empty subset of $\mathbb{R}$ because $l_0 \in T$.
Moreover, $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above.
Thus T is a non-empty subset of $\mathbb{R}$, bounded ~~below~~ above. By the supremum property of $\mathbb{R}$, T has a ~~sp~~ supremum. Let
$$\sup T = L.$$
Then (i) $t \leq L$ for every $t \in T$, since L is an upper bound of T.
and (ii) since every $s \in S$ is an upper bound of T and $L = \sup T$,
$L \leq s$ for every $s \in S$.
(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. Consequently, $L = \inf S$.

Properties of supremum and infimum

Let S be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup S$ exists. Let $M = \sup S$.

Then $M \in \mathbb{R}$ and M satisfies the following conditions:

(i) $x \in S \Rightarrow x \leq M$, and

(ii) for each $\varepsilon > 0$, there exists an element $y(\varepsilon)$ in S such that $\cancel{M - \varepsilon \leq y(\varepsilon)}$ $M - \varepsilon < y \leq M$.

Let S be a non-empty subset of $\mathbb{R}$, bounded below. Then $\inf S$ exists. Let $m = \inf S$. Then $m \in \mathbb{R}$ and m satisfies the following conditions:

(i) $x \in S \Rightarrow x \geq m$, and

(ii) for each $\varepsilon > 0$, there exists an element $y(\varepsilon)$ in S such that $m \leq y < m + \varepsilon$.

* Archimedean property of $\mathbb{R}$.

If $x, y \in \mathbb{R}$ and $x > 0, y > 0$ then there exists a natural number n such that $ny > x$.

or if $x \in \mathbb{R}$ then there exists a natural number n such that $n > x$.

or if $x \in \mathbb{R}$ and $x > 0$ then there exists a natural number n such that $0 < \frac{1}{n} < x$.

- If $x \in \mathbb{R}$ and $x > 0$ there exists a natural number m such that $m - 1 \leq x < m$.

* Density property of $\mathbb{R}$.

i) If x, y are real numbers with $x < y$ then there exists a rational number r such that

$$x < r < y.$$

ii) If x, y are real numbers with $x < y$ then there exists an irrational number s such that

$$x < s < y.$$

[Proof: Application of Archimedean property of $\mathbb{R}$]

* Trichotomy Law:

For any two real number x, y ~~are~~

either $x < y$ or $x = y$ or $y < x$.

* Real Sequence.

Defn: A mapping $f: \mathbb{N} \to \mathbb{R}$ is said to be a sequence in $\mathbb{R}$, or a real sequence.

The f-images $f(1), f(2), f(3), \cdots$ are real numbers.

A sequence f is generally denoted by the symbol $\{f(n)\}$.

Also the symbol $\{f(1), f(2), f(3), \cdots\}$ is used to denote the sequence f.

The symbols like $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, etc. shall also be used to denote a sequence.

Examples: $\{n\}$, $\{n^2\}$, $\left\{\frac{n}{n+1}\right\}$, $\{(-1)^n\}$, $\left\{\sin\frac{n\pi}{2}\right\}$ — Oscillatory Sequence

$\{2\}$. — constant sequence.

✓ The sequence $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = \sin\frac{n\pi}{2}, n \in \mathbb{N}$.

✓ The same sequence is $\{1, 0, -1, 0, 1, 0, \ldots\}$. The range of this sequence is $\{-1, 0, 1\}$

✓ Sometimes it is convenient to specify $f(1)$ and describe $f(n+1)$ in terms of $f(n)$ for all $n \geq$. Ex: $f(1) = \sqrt{2}$, $f(n+1) = \sqrt{2f(n)}$ $\forall n \geq 1$

$\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$

* Bounded Sequence.

A sequence $\{f(n)\}$ is said to be bounded above if there exists a real number G such that $f(n) \leq G$ for all $n \in \mathbb{N}$. G is said to be a upper bound of the sequence.

A real sequence $\{f(n)\}$ is said to be bounded below if there exists a real number g such that $f(n) \geq g$ for all $n \in \mathbb{N}$. g is said to be the lower bound of the sequence.

A real sequence $\{f(n)\}$ is said to be a bounded sequence if there exist real numbers G and g such that $g \le f(n) \le G \quad \forall n \in \mathbb{N}$.

Examples: ✓ $\{\frac{1}{n}\} = \{x_n\}$

0 is the greatest lower bound, and
1 " " least upper bound of this sequence $\{x_n\}$

✓ Let $\{y_n\} = \{n^2\}$
then $\sup\limits_{n\ge1} y_n = \infty \quad \inf\limits_{n\ge1} y_n = 1$.

* The least upper bound of a real sequence $\{f(n)\}$ is a real number M satisfying the following conditions:

(i) $f(n) \le M \quad \forall n \in \mathbb{N}$,

(ii) for each (pre-assigned) given $\varepsilon > 0$, there exists a natural number k such that $f(k) > M - \varepsilon$.

• For a real sequence $\{x_n\}$ bounded below, there exists a greatest lower bound and it is denoted by ~~$\inf\{f(n)\}$~~ $\inf x_n$ or $\text{glb } x_n$.

The greatest lower bound of a real sequence x_n is a real number m satisfying the following conditions:

(i) $x_n \ge m \quad \forall n \in \mathbb{N}$,

(ii) for any given $\varepsilon > 0$, there exists a natural number k such that $x_k < m + \varepsilon$.

Ex: $\{x_n\} = \{(-1)^n\}$, $\{y_n\} = \{(-1)^n n\}$