Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

Real Analysis and Calculus (MA 111)

Week 1 / September 2023	Problem Set 3	Solutions

Real Analysis

(Infinite) Series

1. If $\sum u_n$ be a convergent series of positive real numbers prove that $\sum \frac{u_n}{n}$ is convergent.

Comparison Test

Suppose $\sum u_n$ and $\sum v_n$ be two series of positive real numbers such that there exists $m \in \mathbb{N}$ for which

$$u_n \le kv_n, \quad \text{for all } n \ge m,$$

for some positive k. Then,
(i) $\sum u_n$ is convergent if $\sum v_n$ is convergent.
(ii) $\sum v_n$ is divergent if $\sum u_n$ is divergent.

p-Test. The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges for $p > 1$ and diverges for $p \le 1$.

Solution. Given $\sum u_n$ be convergent series.

Note that;

$$\begin{aligned} A.M. &\ge G.M. \\ \frac{(u_n + \frac{1}{n})}{2} &\ge \left(u_n \cdot \frac{1}{n}\right)^{\frac{1}{2}}; \qquad \forall n \in \mathbb{N} \\ \frac{(u_n + \frac{1}{n})^2}{4} &\ge u_n \cdot \frac{1}{n}; \qquad \forall n \in \mathbb{N} \\ u_n \cdot \frac{1}{n} &< \frac{u_n^2 + \frac{1}{n^2}}{2}; \qquad \forall n \in \mathbb{N} \end{aligned}$$

In R.H.S., $\sum u_n^2$ is convergent (**Verify!**) and $\sum \frac{1}{n^2}$ is convergent by p-test. Thus, by comparison test $\sum \frac{u_n}{n}$ is convergent. □

2. If $\sum u_n$ be a convergent series of positive real numbers prove that $\sum \frac{u_n}{1+u_n}$ is convergent.

Limit Comparison Test

Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series of positive real numbers and $\lim_{n\to\infty} \frac{u_n}{v_n} = l\ (\neq 0, \infty)$. Then the two series $\sum u_n$ and $\sum v_n$ converge or diverge together.

If $l = 0$, $\sum v_n$ is convergent $\implies$ $\sum u_n$ is convergent.

If $l = \infty$, $\sum v_n$ is divergent $\implies$ $\sum u_n$ is divergent.

Note that, if a series $\sum u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$.

Solution. Given series $\sum u_n$ is convergent.
Take $v_n = \frac{u_n}{1+u_n}$.
Then $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (Since $\sum_{n\geq 1} u_n$ is convergent, $\lim_{n\to\infty} u_n = 0$)
$\Rightarrow \sum v_n$ is convergent (by limit comparison test). $\square$

3. If $\sum u_n$ be a series of positive real numbers and $v_n = \frac{u_1+u_2+u_3+\ldots u_n}{n}$, prove that $\sum v_n$ is divergent.

Theorem

Suppose $\sum u_n$ be series of positive real numbers. Then the series converges if and only if the sequence of its partial sum $\{s_n\}$ is bounded above.

Solution. Given $\sum u_n$ be a series of positive real numbers. By definition,

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= \frac{u_1+u_2}{2} > \frac{u_1}{2} \\
&\vdots \\
v_n &= \frac{u_1+u_2+\cdots+u_n}{n} > \frac{u_1}{n}, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore by comparison test, since $u_1 \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum v_n$ is divergent. $\square$

4. Test the convergence of the following series.

(i) $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \ldots,$

Solution. Observe that $\dfrac{1}{1+2^n} < \dfrac{1}{2^n}$ for all $n = 1, 2, 3, \ldots$

Let $u_n = \dfrac{1}{1+2^n}$ and $v_n = \dfrac{1}{1+2^n}$ for all $n = 1, 2, 3, \ldots$

Then $u_n < v_n$ for all $n = 1, 2, 3, \ldots$ Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent series (being geometric series with common ratio $\frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ is convergent by comparison test. $\square$

(ii) $\sin\frac{\pi}{2}+\sin\frac{\pi}{4}+\sin\frac{\pi}{6}\ldots,$

Solution. Note that
$$1 \le \frac{x}{\sin x} \le \frac{\pi}{2}, \qquad 0 < x \le \frac{\pi}{2}$$
Choose $x=\frac{\pi}{2n}$. Then
$$\frac{1}{n} \le \sin\frac{\pi}{2n} \quad \text{for all } n=1,2,3,\ldots.$$
Let $u_n=\dfrac{1}{n}$ and $v_n=\sin\dfrac{\pi}{2n}$ for all $n=1,2,3,\ldots$

Then $u_n \le v_n$ for all $n=1,2,3,\ldots$ Since the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, by comparison test the series $\displaystyle\sum_{n=1}^{\infty}\sin\frac{\pi}{2n}$ is divergent. □

(iii) $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots,$

Solution. We can write above series as,
$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$
Let $u_n=\frac{1}{n(n+1)(n+2)}$ and $v_n=\frac{1}{n^3}$ then $\lim\frac{u_n}{v_n}=1$.
Since $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent (by p-test).
Hence, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ is convergent (by limit comparison test). □

(iv) $\frac{1}{1+2^{-1}}+\frac{1}{1+2^{-2}}+\frac{1}{1+2^{-3}}\cdots$

Solution. Given series is $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{1+2^n}=\frac{1}{1+2^{-1}}+\frac{1}{1+2^{-2}}+\frac{1}{1+2^{-3}}+\ldots$

Let $u_n=\dfrac{2^n}{1+2^n}$. Note That $\lim\limits_{n\to\infty}u_n=1\neq 0$. Since $\lim\limits_{n\to\infty}u_n\neq 0$, $\sum u_n$ is divergent. □

(v) $\sum_{n=1}^{\infty}\sqrt[3]{n^3+1}-n,$

Solution. Let
$$\begin{aligned} u_n &= (n^3+1)^{\frac{1}{3}}-n \qquad [x^3-y^3=(x-y)(x^2+xy+y^2)] \\ &= \frac{1}{(n^3+1)^{\frac{2}{3}}+n(n^3+1)^{\frac{1}{3}}+n^2} > 0 \quad \text{for all } n=1,2,3,\ldots \end{aligned}$$

Let $v_n=\dfrac{1}{n^2}$ for all $n=1,2,3,\ldots$. Then $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\dfrac{1}{3}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, by comparison test (limit form), the series $\sum u_n$ is convergent. □

(vi) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n},$

Solution. Taking $u_n=\frac{\sqrt{n+1}-\sqrt{n-1}}{n}=\frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$ and $v_n=\frac{1}{n^{\frac{3}{2}}}$, $\forall n\in\mathbb{N}$.
Then
$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2n^{\frac{1}{2}}}{\sqrt{n+1}+\sqrt{n-1}}=1$$
Since $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is convergent (by p-test with $p>1$), the series $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ is convergent by limit comparison test. □

(vii) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$,

Solution. Since $\sin x \leq x$ for all $x \geq 0$, $0 < \frac{1}{n} \sin \frac{1}{n} < \frac{1}{n^2}$, for all $n \in \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test), the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$ is convergent by comparison test. □

(viii) $\frac{1}{\log 2} + \frac{1}{\log 3} + \ldots$, [**Hint.** $\log(1+x) < x$ for $x > 0$.]

Solution. Note that

$$\begin{aligned} \log(1+x) &< x, \text{ for all } x > 0 \\ \implies \frac{1}{n} &< \frac{1}{\log(1+n)}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{\log(1+n)}$ is divergent by comparison test. □

5. Use **Cauchy's Condensation Test** to discuss the convergence of the following series:

> **Cauchy's Condensation Test**
>
> Let $\{f(n)\}$ be a **monotone decreasing** sequence of positive real numbers and a be a positive integer > 1. Then the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ converge or diverge together.

(i) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$; $p > 0$.

Solution. Let $f(n) = \frac{1}{n(\log n)^p} > 0$; $\quad p > 0$; $n \geq 2$
As $\{\log n\}$ is an increasing sequence and $p > 0$ then $\forall n \in \mathbb{N}$;

$$\begin{aligned} \{\log(n+1)\}^p &> \{\log(n)\}^p \\ (n+1)\{\log(n+1)\}^p &> n\{\log(n)\}^p \\ \frac{1}{n\{\log(n)\}^p} &< \frac{1}{(n+1)\{\log(n+1)\}^p} \end{aligned}$$

Therefore $\{f(n)\}_{n=2}^{\infty}$ is monotonically decreasing sequence of positive real numbers.
By Cauchy's condensation test, the two series $\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together.
Here, $\sum_{n=2}^{\infty} 2^n f(2^n) = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}$ and this series converges when $p > 1$ and diverges when $p \leq 1$.
Therefore, the series $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent when $p > 1$ and divergent when $p \leq 1$. □

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Solution. Let $f(n) = \frac{1}{n \log n} > 0$; $\quad n \geq 2$
As $\{\log n\}$ is an increasing sequence then $\forall n \in \mathbb{N}$;

$$\begin{aligned} \log(n+1) &> \log(n) \\ (n+1)\log(n+1) &> n\log(n) \\ \frac{1}{n\log(n)} &< \frac{1}{(n+1)\log(n+1)} \end{aligned}$$

Therefore $\{f(n)\}_{n=2}^{\infty}$ is monotonically decreasing sequence of positive real numbers.
By Cauchy's condensation test, the two series $\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together.
Here $\sum_{n=2}^{\infty} 2^n f(2^n) = \frac{1}{\log 2}\sum_{n=2}^{\infty}\frac{1}{n}$ which is divergent.
Therefore, the series $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty}\frac{1}{n\log n}$ is divergent. □

(iii) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^2}$

Solution. Let $f(n) = \frac{1}{(\log n)^2} > 0; \quad n \geq 2$
As $\{\log n\}$ is an increasing sequence then $\forall n \in \mathbb{N}$;

$$\begin{aligned} \log(n+1) &> \log(n) \\ (\log(n+1))^2 &> (\log(n))^2 \\ \frac{1}{(\log(n))^2} &< \frac{1}{(\log(n+1))^2} \end{aligned}$$

Therefore $\{f(n)\}_{n=2}^{\infty}$ is monotonically decreasing sequence of positive real numbers.
By Cauchy's condensation test, the two series $\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together.
Here, $\sum_{n=2}^{\infty} 2^n f(2^n) = \frac{1}{(\log 2)^2}\sum_{n=2}^{\infty}\frac{2^n}{n^2}$ is divergent (Verify by *D'Alembert ratio test*!).
Therefore the series $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty}\frac{1}{(\log n)^2}$ is divergent. □

6. Use general form of root test to show that the series

$$a + b + a^2 + b^2 + a^3 + b^3 + \ldots,$$

where $0 < a < b < 1$, is convergent.

General form of Root Test

Let $\sum u_n$ be a series of positive real numbers and let $\overline{\lim}\,(u_n)^{\frac{1}{n}} = l$. Then $\sum u_n$ is convergent if $l < 1, \sum u_n$ is divergent if $l > 1$.

Solution. Let $\sum u_n = a + b + a^2 + b^2 + a^3 + b^3 + \ldots$
Here

$$\frac{u_{2n}}{u_{2n-1}} = \left(\frac{b}{a}\right)^n, \qquad \frac{u_{2n+1}}{u_{2n}} = a\left(\frac{a}{b}\right)^n$$

$\Rightarrow$

$$\lim\frac{u_{2n}}{u_{2n-1}} = \infty, \qquad \lim\frac{u_{2n+1}}{u_{2n}} = 0.$$

It follows that

$$\overline{\lim}\frac{u_{n+1}}{u_n} = \infty, \qquad \underline{\lim}\frac{u_{n+1}}{u_n} = 0$$

That means, Ratio test gives no decision.
Now,

$$\begin{aligned} \lim(u_{2n})^{\frac{1}{2n}} &= \lim(b^n)^{\frac{1}{2n}} = \sqrt{b}, \\ \lim(u_{2n+1})^{\frac{1}{2n+1}} &= \lim(a^{n+1})^{\frac{1}{2n+1}} = \sqrt{a}. \end{aligned}$$

Therefore, $\limsup(u_n)^{\frac{1}{n}} = \sqrt{b} < 1$.
Hence, by general root test, $\sum u_n$ is convergent. □

7. Show that the following series are convergent.

Absolute Convergence

Let $\sum u_n$ be a series of positive and negative real terms. Let $u_n' = |u_n|$ then $\sum u_n'$ is a series of positive real numbers. If $\sum u_n'$ is convergent then the series $\sum u_n$ is said to be *absolutely convergent*.

Theorem. An absolutely convergent series is convergent.

(i) $1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \ldots$

Solution. Given series is

$$\sum u_n = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \ldots$$

which is series of arbitrary terms.
Here, $\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$
Clearly, $\sum |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.
$\Rightarrow \sum u_n$ is absolutely convergent.
$\Rightarrow \sum u_n$ is convergent. □

(ii) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} - \ldots$

Solution. Given series is

$$\sum u_n = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} - \ldots,$$

which is series of arbitrary terms. Here $\sum |u_n| = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$.
Clearly $\sum |u_n| = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$ is a convergent series by D'Alembert ratio test. [Actually $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{1}{2}(e - e^{-1})$].
It follows that $\sum u_n$ is absolutely convergent.
$\Rightarrow \sum u_n$ is convergent. □

(iii) $\frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \ldots$

Solution. Given series is

$$\sum u_n = \frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \ldots$$

which is series of arbitrary terms.
Here, $\sum |u_n| = \frac{1}{2^2 \log 2} + \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$
Clearly, $\sum |u_n| = \sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ is convergent by Cauchy's condensation test.
$\Rightarrow \sum u_n$ is absolutely convergent.
$\Rightarrow \sum u_n$ is convergent. □

8. Show that the following series are conditionally convergent.

Conditional Convergence

A series $\sum u_n$ is said to be *conditionally convergent* if $\sum u_n$ is convergent but $\sum |u_n|$ is NOT convergent.

(i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

Solution. Let $u_n = \frac{1}{\sqrt{n}}$ for all $n = 1, 2, 3, \dots$. Note that the sequence of positive real numbers $\{u_n\}$ is monotonic decreasing and $\lim_{n\to\infty} u_n = 0$.
Therefore by *Leibnitz's Test*, the given (alternating) series

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

is convergent.
Note that $|v_n| = u_n$. But the series $\sum_{n=1}^{\infty} |v_n| = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (by p-test with $p = \frac{1}{2}$). Hence, the series $\sum v_n$ is conditionally convergent. □

(ii) $\sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \sin\frac{\pi}{6} - \dots$

Solution. Let $u_n = (-1)^{n+1} \sin\frac{\pi}{2n}$ for all $n = 1, 2, 3, \dots$. The given alternating series $\sum u_n = \sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \sin\frac{\pi}{6} - \dots$ is convergent (by *Leibnitz's Test* as the previous problem).
But the series $\sum |u_n| = \sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \sin\frac{\pi}{6} + \dots$ is divergent (See Problem 4(ii) for poof).
Hence, the series $\sum u_n$ is conditionally convergent. □

(iii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$

Solution. Given series $\sum u_n = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$ is convergent (by *Leibnitz's Test*).
But the series $\sum |u_n| = \frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \dots$ is divergent by comparison test (See Problem 4($viii$) for proof).
Hence, the series $\sum u_n$ is conditionally convergent. □

9. Show that the following series are convergent.

Leibnitz's Test

If $\{u_n\}$ is a sequence of positive real numbers which is a monotone decreasing and converging to 0, then the alternating series $u_1 - u_2 + u_3 - u_4 + \dots$ is convergent.

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Solution. Given series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

where $u_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $u_{n+1} = \frac{1}{n+1}$. Since $\frac{1}{n+1} < \frac{1}{n}$, $\forall n \in \mathbb{N}$, that is $u_{n+1} < u_n, \quad \forall n \in \mathbb{N}$

$\Longrightarrow \{u_n\}$ is monotonic decreasing sequence.
Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$.
Therefore, by *Leibnitz's Test*, the given alternating series is convergent. $\square$

(ii) $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \dots$

Solution. Given series is $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = \dfrac{1}{1+a^2} - \dfrac{1}{2+a^2} + \dfrac{1}{3+a^2} - \dots$, where $u_n = \frac{1}{n+a^2}$ for all $n \in \mathbb{N}$. Then $u_{n+1} = \frac{1}{n+1+a^2}$. Since $(n+1)+a^2 > n+a^2$ for all $n \in \mathbb{N}$, $u_{n+1} < u_n, \quad \forall n \in \mathbb{N}$
$\Rightarrow \{u_n\}$ is monotonic decreasing sequence.
Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+a^2} = 0$.
Therefore, by *Leibnitz's Test*, the given series is convergent. $\square$