Riemann Integral.

- Let $[a,b]$ be a closed and bounded interval.

A partition of $[a,b]$ is a finite set $P = \{t_0, t_1, \cdots, t_n\}$ of points of $[a,b]$ such that

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

For example, $P = \{0, \frac{1}{4}, 1\}$ is a partition of $[0,1]$.

$\wp = \{0, \frac{1}{100}, \frac{3}{4}, 1\}$ is a partition of $[0,1]$.

* Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded on $[a,b]$.

Let $P = \{\underset{\overset{\shortparallel}{a}}{t_0} < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$, f is bounded on $[t_{r-1}, t_r]$ for $r = 1, 2, \ldots, n$.

Let $M = \sup\limits_{t \in [a,b]} f(t)$, $M_r = \sup\limits_{t \in [t_{r-1}, t_r]} f(t)$, $r = 1, 2, \ldots, n$.

$m = \inf\limits_{t \in [a,b]} f(t)$, $m_r = \inf\limits_{t \in [t_{r-1}, t_r]} f(t)$,

Then we have $m \leq m_r \leq M_r \leq M$.

- The sum $M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1})$ is said to be the upper sum of f corresponding to the partition P, denoted by $U(P, f)$.

- The sum $m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1})$ is said to be the lower sum of f corresponding to the partition P, denoted by $L(P, f)$.

We have, $m \le m_r \le M_r \le M$, for $r = 1, 2, \ldots, n$.

$$\Rightarrow m(t_r - t_{r-1}) \le m_r (t_r - t_{r-1}) \le M_r (t_r - t_{r-1}) \le M (t_r - t_{r-1})$$

$$\Rightarrow \sum_{r=1}^{n} m(t_r - t_{r-1}) \le \sum_{r=1}^{n} m_r (t_r - t_{r-1}) \le \sum_{r=1}^{n} M_r (t_r - t_{r-1}) \le \sum_{r=1}^{n} M (t_r - t_{r-1})$$

$$\Rightarrow m \sum_{r=1}^{n} (t_r - t_{r-1}) \le L(P,f) \le U(P,f) \le M \sum_{r=1}^{n} (t_r - t_{r-1})$$

$$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a) \qquad \text{———— } (*)$$

* The above relation shows that

if f is bounded on $[a,b]$, then the sets

$\{ L(P,f) : P \text{ is a partition of } [a,b] \}$

and $\{ U(P,f) : P \text{ is a partition of } [a,b] \}$

are bounded.

Let $\mathcal{P}[a,b]$ be the collection of all partition of $[a,b]$.

$\therefore \quad m(b-a) \le L(P,f) \le M(b-a)$ for all $P \in \mathcal{P}[a,b]$

and $\quad m(b-a) \le U(P,f) \le M(b-a)$ for all $P \in \mathcal{P}[a,b]$.

✓ $\therefore$ The supremum of the set $\{ L(P,f) : P \in \mathcal{P}[a,b] \}$ exists (as bounded above by $M(b-a)$) and is called as <u>Lower integral</u>, denoted by $\underline{\int_a^b} f$ or $\underline{\int_a^b} f(t)\,dt$

✓ Also the infimum (or greatest lower bound) of the set $\{ U(P,f) : P \in \mathcal{P}[a,b] \}$ exists (as bounded below by $m(b-a)$) and is called <u>upper integral</u>, denoted by $\overline{\int_a^b} f$ or $\overline{\int_a^b} f(t)\,dt$.

$\rightarrow$ f is said to be <u>Riemann Integrable</u> on $[a,b]$

if $\underline{\int_a^b} f = \overline{\int_a^b} f$.

We also have the following relation

$$\boxed{\cancel{m(b-a) \leq \underline{\int_a^b} f}}$$

$\rightarrow$ $$m(b-a) \leq L(P,f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(P,f) \leq M(b-a)$$

for any partition of $[a,b]$.

<u>Example</u>: $f:[0,1] \rightarrow \mathbb{R}$ defined by

$$f(x) = 1, \text{ if } x \text{ is rational in } [0,1]$$
$$= -1, \text{ '' } x \text{ '' irrational '' }.$$

Show that f is NOT integrable.

$\Rightarrow$ Since. $-1 \leq f(x) \leq 1$ for all $x \in [0,1]$,

f is bounded on $[0,1]$.

Consider the partition $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$.

Let $M = \sup\limits_{t \in [0,1]} f(t)$, $m = \inf\limits_{t \in [0,1]} f(t)$,

$M_r = \sup\limits_{t \in [t_{r-1}, t_r]} f(t)$, $m_r = \inf\limits_{t \in [t_{r-1}, t_r]} f(t)$ for $r = 1, 2, \ldots, n$.

Then $M = 1$, $m = -1$, $M_r = 1$, $m_r = -1$ for all $r = 1, 2, \ldots, n$.

$$\therefore L(P,f) = m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1})$$
$$= -(t_1 - t_0) + (-)(t_2 - t_1) + \cdots + (-1)(t_n - t_{n-1}) \quad [\text{As } m_r = -1]$$
$$= -(t_n - t_0)$$
$$= -1$$

$$U(P,f) = M_1(t_1-t_0) + M_2(t_2-t_1) + \cdots + M_n(t_n-t_{n-1})$$
$$= t_1-t_0+t_2-t_1+\cdots+t_n-t_{n-1} \qquad [\text{As } M_r = 1]$$
$$= t_n - t_0$$
$$= 1.$$

$\therefore \quad L(P,f) = -1$ for any partition P of $[0,1]$.

$$\Rightarrow \sup\{L(P,f) : P \in \mathcal{P}[0,1]\} = -1.$$

that is, $\underline{\int_0^1} f = -1$

Also, $U(P,f) = 1$ for any partition P of $[0,1]$

$$\Rightarrow \inf\{U(P,f) : P \in \mathcal{P}[a,b]\} = 1.$$

$$\Rightarrow \overline{\int_0^1} f = 1.$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, f is NOT integrable on $[0,1]$.

(*) ~~If~~ $P_n = \{a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b\}$

denote norm P_n as $\|P_n\| = \max\{t_k - t_{k-1} : k = 1,2,\ldots,n\}$

~~Theorem~~ $\boxed{\text{Theorem}}$: If $\|P_n\| \to 0$ as $n \to \infty$, then

$$U(P_n, f) \longrightarrow \overline{\int_a^b} f \text{ as } n \to \infty \text{ and}$$

$$\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$$

* Riemann integrable functions.

1. Any function $f: [a,b] \to \mathbb{R}$ is monotone, f is integrable on $[a,b]$ (increasing / decreasing)

2. A continuous function $f: [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$.

3. If a bounded function $f: [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ except ~~for~~ at a finite number of points in $[a,b]$, then f is integrable on $[a,b]$.

4. Let $f: [a,b] \to \mathbb{R}$ be bounded and let f be continuous on $[a,b]$ except on a infinite subset $S \subset [a,b]$ such that the number of limit points of S is finite. Then f is integrable on $[a,b]$.

[for example: Let $f: [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2^{n-1}}, \quad \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}, \text{ for } n = 1,2,3,\ldots$$
$$= 0, \quad x = 0$$

Rewriting the function,
$$f(x) = 1, \quad \frac{1}{2} < x \leq 1$$
$$= \frac{1}{2}, \quad \frac{1}{4} < x \leq \frac{1}{2}$$
$$= \frac{1}{4}, \quad \frac{1}{8} < x \leq \frac{1}{4}$$
$$\vdots$$
$$= 0 \quad x = 0$$

Here ~~for~~ $S = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$, where the function f is NOT continuous. $S = \left\{\frac{1}{2^n} : n = 1,2,\ldots\right\}$.
S has only one limit point 0. Therefore f is integrable on $[0,1]$.

*** Theorem:** Let $f : [a,b] \to R$ be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$.

- Converse of this theorem is NOT true. For example,

$f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \text{ is rational in } [0,1] \\ -1 & x \text{ is irrational in } [0,1]. \end{cases}$$

Then $|f|(x) = 1$ for all $x \in [0,1]$.

Since $|f|$ is a constant function (so continuous) on $[0,1]$, f is integrable on $[0,1]$.

But f is NOT integrable on $[0,1]$.

(already proved)

$\boxtimes$

*1. A function f is defined on $[a,b]$ by $f(x) = e^x$. Find $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. Deduce that f is integrable on $[a,b]$.

$\Rightarrow$ Note that f is bounded on $[a,b]$.

Let $P_n = (a, a+h, a+2h, \dots, a+nh)$ where $nh = b-a$.

Then P_n is a partition with $\|P_n\| = \frac{b-a}{n}$.

Let $M_r = \sup\{f(x) : x \in [a+\overline{r-1}h, a+rh]$

and $m_r = \inf\{f(x) : x \in [a+\overline{r-1}h, a+rh]\}$,

$r = 1, 2, \dots, n$.

Then $M_r = f(a+rh)$ [Since $f(x) = e^x$ is increasing function]

$= e^{a+rh}$,

$m_r = e^{a+\overline{r-1}h}$, $r = 1, 2, \dots, n$.

$$\therefore U(P_n, f) = hM_1 + hM_2 + \dots + hM_n$$

$$= h[e^{a+h} + e^{a+2h} + \dots + e^{a+nh}]$$

$$= h e^{a+h} \cdot \frac{e^{nh}-1}{e^h-1} = h \cdot e^{a+h} \cdot \frac{e^{b-a}-1}{e^h-1}$$

$$= \frac{h e^h}{e^h-1} \cdot (e^b - e^a)$$

$$L(P_n, f) = h[e^a + e^{a+h} + \dots + e^{a+\overline{n-1}h}]$$

$$= he^a\left[\frac{e^{nh}-1}{e^h-1}\right] = \frac{h}{e^h-1} \cdot (e^b - e^a)$$

Consider the sequence of partitions $\{P_n\}$ of $[a,b]$.

Here $\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0$

Then $\overline{\int_a^b} f = \lim_{n\to\infty} U(P_n, f)$ and $\underline{\int_a^b} f = \lim_{n\to\infty} L(P_n, f)$.

So, $\overline{\int_a^b} f = \lim_{n\to\infty} \frac{he^h}{e^h-1}(e^b-e^a) = e^b-e^a$

and $\underline{\int_a^b} f = \lim_{n\to\infty} \frac{e^h}{e^h-1}(e^b-e^a) = e^b-e^a$.

As $\overline{\int_a^b} f = \underline{\int_a^b} f$, f is integrable on $[a,b]$ and

$$\int_a^b f = e^b - e^a.$$

Theorem: Let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$ and let f be continuous on $[a,b]$ except on a infinite set $S\subset[a,b]$ such that the number of limit points of S is finite. Then f is integrable on $[a,b]$

Example: 2.

$$f(x) = \frac{1}{n}, \quad \frac{1}{n+1} < x \le \frac{1}{n} \quad \text{for } n=1,2,3,\dots$$
$$= 0, \quad x = 0$$

$f:[0,1]\to\mathbb{R}$ be a function such that

Here, ~~Note that~~

$$
\begin{aligned}
f(x) &= 1, & \tfrac{1}{2} < x \le 1 \\
&= \tfrac{1}{2}, & \tfrac{1}{3} < x \le \tfrac{1}{2} \\
&= \tfrac{1}{3}, & \tfrac{1}{4} < x \le \tfrac{1}{3} \\
&\ \vdots \\
&= 0 & x = 0
\end{aligned}
$$

Note that f is bounded on $[0,1]$ and is continuous on $[0,1]$ except at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$.

The set of points of discontinuity of f on $[0,1]$ is an infinite set having only one limit point. Therefore f is integrable on $[0,1]$.

* Let $f : [a,b] \to \mathbb{R}$, $\varphi : [a,b] \to \mathbb{R}$ be both bounded on $[a,b]$ such that $f(x) = \varphi(x)$ except for a finite number of points in $[a,b]$.
If f be integrable on $[a,b]$ then φ is also integrable on $[a,b]$ and $\int_a^b \varphi = \int_a^b f$.

Ex: 3. Let $f(x) = [x]$, $x \in [0,3]$. To evaluate $\int_0^3 f$.

Note that
$$f(x) = 0 \; , \; 0 \le x < 1$$
$$= 1 \; , \; 1 \le x < 2$$
$$= 2 \; , \; 2 \le x < 3$$
$$= 3 \quad x = 3$$

f is clearly bounded on $[0,3]$. f is continuous on $[0,3]$ except for the points $1, 2, 3$.

So f is integrable on $[0,3]$.

Define functions $\varphi_1, \varphi_2, \varphi_3$ on $[0,1], [1,2], [2,3]$ respectively by
$$\varphi_1(x) = 0 \; , \; x \in [0,1];$$
$$\varphi_2(x) = 1 \; , \; x \in [1,2];$$
$$\varphi_3(x) = 2 \; , \; x \in [2,3]$$

Then $\int_0^3 f = \int_0^1 f + \int_1^2 f + \int_2^3 f$ ——— (i)

Since $\left(\begin{array}{ll} f(x) = 0 & [0,1) \\ \quad\;\; = 1 & x = 1 \end{array} \right)$ $f(x) = \varphi_1(x)$ on $[0,1]$ except at $x = 1$,
$\Rightarrow \int_0^1 f = \int_0^1 \varphi$

Similarly, $f(x) = \varphi_2(x)$ on $[1,2]$ except at $x = 2$, $\Rightarrow \int_1^2 f = \int_1^2 \varphi$
and $\int_2^3 f = \int_2^3 \varphi$

$\therefore$ From (i) $\displaystyle\int_0^3 f = \int_0^1 \varphi_1 + \int_1^2 \varphi_2 + \int_2^3 \varphi_3$

$$= \int_0^1 0 + \int_1^2 1 + \int_2^3 2$$

$$= 0 + 1 + 2 = 3$$

Ex:

Consider example 2, that is

$f(x) = \frac{1}{n}, \quad \frac{1}{n+1} < x \leq \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

We have shown that $f : [0,1] \to \mathbb{R}$ is integrable.

$$\begin{aligned} f(x) &= 1, & \tfrac{1}{2} < x \leq 1 \\ &= \tfrac{1}{2}, & \tfrac{1}{3} < x \leq \tfrac{1}{2} \\ &= \tfrac{1}{3}, & \tfrac{1}{4} < x \leq \tfrac{1}{3} \\ &\vdots \\ &= 0 & x = 0 \end{aligned}$$

define $\varphi_1, \varphi_2, \ldots$ on $[\frac{1}{2}, 1], [\frac{1}{3}, \frac{1}{2}], \ldots$ respectively by

$\varphi_1(x) = 1$ on $[\frac{1}{2}, 1]$, $\varphi_2(x) = \frac{1}{2}$ on $[\frac{1}{3}, \frac{1}{2}]$, ...

$$\therefore \int_{\frac{1}{2}}^1 \varphi_1 = 1\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}, \quad \int_{\frac{1}{3}}^{\frac{1}{2}} \varphi_2 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right), \ldots,$$

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi_n = \tfrac{1}{n}\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right), \ldots$$

Now, $\displaystyle\int_0^1 f = \lim_{n\to\infty} \left[1\left(1-\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \tfrac{1}{n}\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)\right]$

$$= \lim_{n\to\infty} \left[\left(\tfrac{1}{1^2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{n^2}\right) - \left(\tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \cdots + \tfrac{1}{n\cdot(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\tfrac{1}{1^2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{n^2}\right] - \lim_{n\to\infty}\left(1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \tfrac{1}{n+1}\right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} - \lim_{n\to\infty}\left(1 - \tfrac{1}{n+1}\right)$$

$$= \frac{\pi^2}{6} - 1 \qquad \text{where } \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

$\blacksquare$

Fundamental Theorem.

Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$.
Then for each $x \in [a,b]$, f is integrable on $[a,x]$.

$\therefore$ $\int_a^x f(t)\,dt$ exists for each $x \in [a,b]$, and depends on x.

* Define F on $[a,b]$ by

$$F(x) = \int_a^x f(t)\,dt \quad \text{for all } x \in [a,b].$$

Theorem:

1. F is continuous on $[a,b]$.
2. If f is continuous on $[a,b]$, then F is differentiable on $[a,b]$, and in this case $F'(x) = f(x) \quad \forall x \in [a,b]$.

[In particular, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.].

Example: Let

$$f(x) = 0, \quad -1 \leq x \leq 0$$
$$= 1, \quad 0 < x \leq 1.$$

Then $f : [-1,1] \to \mathbb{R}$ be integrable on $[-1,1]$.
(As f has only one point of discontinuity which is 0)

Then for $-1 \leq x \leq 0$,

$$F(x) = \int_{-1}^{x} f(t)\, dt$$
$$= \int_{-1}^{x} 0\, dt = 0,$$

for $0 < x \leq 1$,

$$F(x) = \int_{-1}^{x} f(t)\, dt$$
$$= \int_{-1}^{0} f(t)\, dt + \int_{0}^{x} f(t)\, dt$$
$$= \int_{-1}^{0} 0\, dt + \int_{0}^{x} 1\, dt = 0 + x$$
$$= x$$

$$\therefore F(x) = 0, \quad -1 \leq x \leq 0$$
$$= x, \quad 0 < x \leq 1.$$

* **Definition:** A function φ is called antiderivative or a primitive of a function f on an interval $[a,b]$, if $\varphi'(x) = f(x)$ for all $x \in [a,b]$.

Example 1. $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = 2x \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right), \quad x \neq 0$$
$$= 0, \quad x = 0$$

Let $\varphi : [-1,1] \to \mathbb{R}$ by, $\varphi(x) = x^2 \sin\left(\frac{1}{x}\right)$, $x \neq 0$
$= 0$, $x = 0$.

Then $\varphi'(x) = f(x)$ $\forall x \in [-1,1]$.

Thus φ is an antiderivative of f on $[-1,1]$.

Example 2.

Let $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right), \quad x \neq 0$$
$$= 0, \quad x = 0$$

Here f is unbounded on every neighbourhood of 0.

Therefore f is NOT integrable on $[-1,1]$.

Define $\varphi : [-1,1] \to \mathbb{R}$ by

$$\varphi(x) = x^2 \sin\left(\frac{1}{x^2}\right), \quad x \neq 0$$
$$= 0, \quad x = 0.$$

Then $\varphi'(x) = f(x)$ on $[-1,1]$.

So, φ is an antiderivative of f on $[-1,1]$.

Note :

i) Example 1 shows that the continuity of f is NOT necessary to have an antiderivative of f in some interval $[a,b]$.

ii) Example 2 shows that integrablity of f on $[a,b]$ is NOT necessary to have an antiderivative of f on $[a,b]$.

* ~~But~~ If $f : [a,b] \to \mathbb{R}$ is continuous, then f always has an antiderivative F on $[a,b]$, which is

$$F(x) = \int_a^x f(t)\, dt \quad \forall\, x \in [a,b].$$

[Fundamental Theorem of Integral Calculus]

If i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$,

and ii) f possesses an antiderivative φ on $[a,b]$,

then
$$\int_a^b f(t)\,dt = \varphi(b) - \varphi(a).$$

* Differentiation under sign integral.

If f is continuous on $[a,b]$ and u, v are differentiable on I with $u(I) \subset [a,b]$, $v(I) \subset [a,b]$.

then
$$g(x) = \int_{u(x)}^{v(x)} f(t)\,dt$$

implies,
$$g'(x) = f(v(x))\,v'(x) - f(u(x))\,u'(x), \quad \forall x \in I.$$