Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

REAL ANALYSIS & CALCULUS (MA 111)

Week 4 / August 2023 Problem Set 2 GR

# Real Analysis

## Real sequences

### Tutorial Problems

1. Using the definition of limit, show that $\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0$.

2. Find $\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

3. Use Sandwich theorem to prove that

   (i) $\lim\,(2^n+3^n)^{\frac{1}{n}}=3$,

   (ii) $\lim\left(\sqrt{n+1}-\sqrt{n}\right)=0$.

4. Show that

   (i) $\lim\sqrt[n]{n+1}=1$,

   (ii) $\lim\sqrt[n+1]{n}=1$,

   (iii) $\lim\frac{(n+1)^{2n}}{(n^2+1)^n}=e^2$,

   (iv) $\lim\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\right\}^{n^2}=e^6$.

   [Hint. Use $\lim n^{\frac{1}{n}}=1$, $\lim\left(1+\frac{1}{n}\right)^n=e$. If $x_n>0$ and $\lim x_n=x>0$ for all $n\in\mathbb{N}$ and $\lim y_n=y$, then $\lim\,(x_n)^{y_n}=x^y$.]

5. A sequence $\{u_n\}$ is defined by $u_1>0$ and $u_{n+1}=\sqrt{6+u_n}$ for $n\geq 1$. Show that

   (i) the sequence $\{u_n\}$ is monotone increasing if $0<u_1<3$;

   (ii) the sequence $\{u_n\}$ is monotone decreasing if $0<u_1>3$.

   Find $\lim u_n$.

6. Prove that the sequence $\{x_n\}$ defined by $x_1=\sqrt{7}$ and $x_{n+1}=\sqrt{7+x_n}$ for all $n\geq 1$ converges to the positive root of the equation $x^2-x-7=0$.

   [Hint. Monotone increasing and bounded above implies convergent.]

7. If $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)$ for all $n \geq 1$. Prove that the sequence $\{x_n\}$ converges to 3.

   [Hint. Monotone decreasing and bounded below implies convergent.]

8. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

   (i) Prove that the sub-sequences $\{u_{2n+1}\}$ and $\{u_{2n}\}$ converges to a common limit.

   (ii) Find $\lim u_n$.

9. Establish the convergence and find the limits of the following sequences

   (i) $\left(1 + \frac{1}{3n+1}\right)^n$,

   (ii) $\left(1 + \frac{1}{n^2+2}\right)^{n^2}$.

   [Hint. Approach through sub-sequence.]

10. Let $\{u_n\}$ be a bounded sequence and $\lim v_n = 0$. Prove that $\lim u_n v_n = 0$. Utilise this to prove that

    (i) $\lim \frac{\sin n}{n} = 0$.

    (ii) $\lim \frac{(-1)^n n}{n^2+1} = 0$.

11. Prove that

    (i) $\lim n^{\frac{1}{n}} = 1$.

    (ii) $\lim \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

    [Hint. Use the **Theorem**: Let $\{u_n\} > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \ell$ (finite or infinite). Then $\lim \sqrt[n]{u_n} = \ell$]

12. Establish from definition that $\{u_n\}$ is a Cauchy sequence, where

    (i) $u_n = \frac{n}{n+1}$,

    (ii) $u_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$,

    (iii) $|u_{n+2} - u_{n+1}| \leq \frac{1}{2}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$.

    [Hint. (ii) $(n+1)! \geq 2^n$ for $n \geq 2$. (iii) $|u_{n+2} - u_{n+1}| \leq \left(\frac{1}{2}\right)^n |u_2 - u_1|$.]

## Assignment Problems

1. Prove that the sequence $\{u_n\}$ defined by

   (i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1+3u_2}{4}$,

   (ii) $0 < u_1 < u_2$ and $u_{n+2} = \frac{u_{n+1}+2u_n}{3}$ for $n \geq 1$, converges to $\frac{2u_1+3u_2}{5}$

   [**Hint.** Observe that $u_3 - u_2 = \left(-\frac{1}{3}\right)(u_2 - u_1), \ldots, u_n - u_{n-1} = \left(-\frac{1}{3}\right)^{n-2}(u_2 - u_1)$. Add all these equations and get $u_n - u_1 = \frac{3}{4}(u_2 - u_1)\left[1 - \left(-\frac{1}{3}\right)^{n-1}\right]$]

2. Prove that the sequence $\{u_n\}$ defined by

   (i) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to the limit $\sqrt[3]{u_1u_2^2}$,

   (ii) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to the limit $\frac{3}{\left(\frac{1}{u_1}+\frac{2}{u_2}\right)}$.

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Department of Mathematical Sciences

**Rajiv Gandhi Institute Of Petroleum Technology, Jais**

**Real Analysis and Calculus (MA 111)**

Week 4 / ~~November~~ August 2023 | Problem Set 2 | Solutions

# Real Analysis

## Real Sequences

1. To show $\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0$.

   Let $\varepsilon>0$ be given. Now

$$\left|\left(\sqrt{n^2+1}-n\right)-0\right| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n}.$$

   Note that $\frac{1}{2n}<\varepsilon$ if $n>\frac{1}{2\varepsilon}$. Choose $k_0=\left[\frac{1}{2\varepsilon}\right]+1$. Then

$$\left|\left(\sqrt{n^2+1}-n\right)-0\right|<\varepsilon, \quad \text{for all } n\geq k_0.$$

   Since $\varepsilon$ is arbitrary, 0 is the limit of $\left\{\sqrt{n^2+1}-n\right\}$.

2. To find $\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

   Note that

$$\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

   Let $u_n=1$ and $v_n=\sqrt{1+\frac{1}{n}}+1$ for all $n\in\mathbb{N}$. Since $\lim v_n=2$, $\lim\frac{u_n}{v_n}=\frac{1}{2}$.

3. Use Sandwich Theorem to prove that

> **Sandwich/Squeeze Theorem**
>
> Suppose $\{u_n\},\{v_n\}$ and $\{w_n\}$ be three sequences of real numbers and there exists $m\in\mathbb{N}$ such that
>
> $$u_n\leq v_n\leq w_n, \qquad \text{for all } n\geq m.$$
>
> If $\lim_{n\to\infty}u_n=\lim_{n\to\infty}w_n=l$, then the sequence $\{v_n\}$ is convergent and $\lim_{n\to\infty}v_n=l$.

(i) $\lim(2^n+3^n)^{\frac{1}{n}}=3$,

**Solution.** Let $v_n=(2^n+3^n)^{\frac{1}{n}}$, for all $n\in\mathbb{N}$.
Since

$$3^n<2^n+3^n<2\cdot 3^n, \quad \forall n\in\mathbb{N}$$

$$\Rightarrow (3^n)^{\frac{1}{n}} < (2^n+3^n)^{\frac{1}{n}} < (2\cdot 3^n)^{\frac{1}{n}}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 3 < (2^n+3^n)^{\frac{1}{n}} < (2^{\frac{1}{n}}\cdot 3), \quad \forall n \in \mathbb{N}$$

Hence, take $u_n = 3,\ v_n = (2^n+3^n)^{\frac{1}{n}}$ and $w_n = 3\cdot 2^{1/n}$ for all $n \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} u_n = 3, \quad \lim_{n\to\infty} w_n = \lim_{n\to\infty}(3\cdot 2^{\frac{1}{n}}) = 3, \quad \left[\text{since } \lim_{n\to\infty} 2^{\frac{1}{n}} = 0\right]$$

i.e.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = 3.$$

Hence, by Sandwich Theorem,

$$\lim_{n\to\infty} v_n = 3,$$

$$\Rightarrow \lim_{n\to\infty}(2^n+3^n)^{\frac{1}{n}} = 3.$$

□

(ii) $\lim(\sqrt{n+1}-\sqrt{n}) = 0.$

**Solution.** For all $n \in \mathbb{N}$,

$$\begin{aligned}\sqrt{n+1}-\sqrt{n} &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}\\ &= \frac{1}{(\sqrt{n+1}+\sqrt{n})}\end{aligned}$$

Note that

$$\frac{1}{2\sqrt{n+1}} < \frac{1}{(\sqrt{n+1}+\sqrt{n})} < \frac{1}{2\sqrt{n}}, \quad \forall\, n \geq 1$$

i.e.

$$\frac{1}{2\sqrt{n+1}} < (\sqrt{n+1}-\sqrt{n}) < \frac{1}{2\sqrt{n}}, \quad \forall\, n \geq 1$$

Take

$$u_n = \frac{1}{2\sqrt{n+1}}, \quad v_n = \sqrt{n+1}-\sqrt{n}, \quad w_n = \frac{1}{2\sqrt{n}} \quad \text{for all } n \in \mathbb{N}.$$

Then $u_n < v_n < w_n$, $\forall\, n \in \mathbb{N}$. [OR one can take $u_n = 0$, $\forall n$. In that case also $u_n < v_n < w_n$ holds and the followings are true.] Clearly,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = 0.$$

Hence, by Sandwich Theorem,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0.$$

□

4. Show that

(i) $\lim_{n\to\infty} \sqrt[n]{n+1} = 1.$

**Hint**

Use $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$

**Theorem.** If $x_n > 0$ and $\lim_{n\to\infty} x_n = x > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} (x_n)^{y_n} = x^y.$$

**Solution.** Note that

$$\sqrt[n]{n+1} = n^{\frac{1}{n}} \left(1+\frac{1}{n}\right)^{\frac{1}{n}}$$

We have

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{\frac{1}{n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Therefore,

$$\lim_{n\to\infty} \sqrt[n]{n+1} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{\frac{1}{n}} = 1.$$

□

(ii) $\lim \sqrt[n+1]{n} = 1.$

**Solution.** Write

$$(n)^{\frac{1}{n+1}} = \left(n^{\frac{1}{n}}\right)^{\frac{1}{1+\frac{1}{n}}} = (x_n)^{y_n},$$

where $x_n = n^{\frac{1}{n}}$ and $y_n = \dfrac{1}{1+\frac{1}{n}}$ for all $n \in \mathbb{N}$. Clearly, $\{x_n\}$ and $\{y_n\}$ are sequences of positive real numbers. We also have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} n^{\frac{1}{n}} = 1 > 0$, and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$ Therefore,

$$\lim_{n\to\infty} (n)^{\frac{1}{n+1}} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^{\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}} = 1.$$

□

(iii) $\lim_{n\to\infty} \frac{(n+1)^{2n}}{(n^2+1)^n} = e^2.$

**Solution.** Observe that,

$$\frac{(n+1)^{2n}}{(n^2+1)^n} = \frac{\left\{\left(1+\frac{1}{n}\right)^n\right\}^2}{\left\{\left(1+\frac{1}{n^2}\right)^{n^2}\right\}^{\frac{1}{n}}}.$$

If $x_n = \left(1+\dfrac{1}{n}\right)^n$, then $\left(1+\dfrac{1}{n^2}\right)^{n^2} = x_{n^2}$. We know that, if a sequence $\{x_n\}$ converges to $\ell$, then all its subsequences converge to $\ell$. Since

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e, \quad \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{n^2} = e,$$

$$\Longrightarrow \lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)^{n^2}\right\}^{\frac{1}{n}} = e^0 = 1.$$

Therefore,

$$\lim_{n\to\infty}\frac{(n+1)^{2n}}{(n^2+1)^n} = \frac{e^2}{e^0} = e^2.$$

□

(iv) $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\right\}^{n^2} = e^6.$

**Solution.** Observe that,

$$\begin{aligned}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\right\}^{n^2} &= \left(1+\frac{1}{n^2}\right)^{n^2}\left(1+\frac{1}{\frac{n^2}{2}}\right)^{n^2}\left(1+\frac{1}{\frac{n^2}{3}}\right)^{n^2}\\ &= \left(1+\frac{1}{n^2}\right)^{n^2}\left\{\left(1+\frac{1}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}}\right\}^2\left\{\left(1+\frac{1}{\frac{n^2}{3}}\right)^{\frac{n^2}{3}}\right\}^3\end{aligned}$$

We know that

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = e, \quad \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}} = e, \quad \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{3}}\right)^{\frac{n^2}{3}} = e.$$

Therefore,

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\right\}^{n^2} = e\cdot e^2\cdot e^3 = e^6.$$

□

5. Prove that the sequence $\{u_n\}$ defined by

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1+3u_2}{4}$.

**Solution.** Given sequence $\{u_n\}$ is defined by,

$$u_{n+2} = \frac{2u_{n+1}+u_n}{3}; \qquad 0 < u_1 < u_2, \qquad \forall n \geq 1$$

To show that $\lim_{n\to\infty} u_n = \frac{u_1+3u_2}{4}$.

$$\begin{aligned} 0 < u_1 &< u_2\\ \Longrightarrow \quad u_2 - u_1 &> 0\\ u_3 - u_2 &= \frac{2u_2+u_1}{3} - u_2 = \frac{u_1-u_2}{3} = -\frac{1}{3}(u_2-u_1)\\ u_4 - u_3 &= \frac{2u_3+u_2}{3} - u_3 = \frac{u_2-u_3}{3} = \left(-\frac{1}{3}\right)^2(u_2-u_1)\\ &\vdots\\ u_n - u_{n-1} &= \left(-\frac{1}{3}\right)^{n-2}(u_2-u_1).\end{aligned}$$

This implies (adding all the above equations)

$$
\begin{aligned}
u_n - u_1 &= (u_2-u_1)\left[1-\frac{1}{3}+\left(-\frac{1}{3}\right)^2+\cdots+\left(-\frac{1}{3}\right)^{n-2}\right]\\
&= (u_2-u_1)\left[\frac{1-\left(-\frac{1}{3}\right)^{n-1}}{1+\frac{1}{3}}\right]\\
&= \frac{3(u_2-u_1)}{4}\left[1-\left(-\frac{1}{3}\right)^{n-1}\right]\\
\text{Hence,}\quad \lim_{n\to\infty}(u_n-u_1) &= \frac{3}{4}(u_2-u_1) \qquad \left[\because \lim_{n\to\infty}\left(-\frac{1}{3}\right)^{n-1}=0\right]\\
\Rightarrow \lim_{n\to\infty} u_n &= u_1+\frac{3}{4}(u_2-u_1)\\
\Rightarrow \lim_{n\to\infty} u_n &= \frac{u_1+3u_2}{4}.
\end{aligned}
$$

□

(ii) $0<u_1<u_2$ and $u_{n+2}=\frac{u_{n+1}+2u_n}{3}$ for $n\geq 1$, converges to $\frac{2u_1+3u_2}{5}$.

**Solution.** Given sequence $\{u_n\}$ is defined by

$$u_{n+2}=\frac{u_{n+1}+2u_n}{3}; \qquad 0<u_1<u_2, \qquad \forall n\geq 1$$

To show: $\lim_{n\to\infty} u_n = \frac{2u_1+3u_2}{5}$.
Here,

$$
\begin{aligned}
0<u_1 &< u_2\\
\Rightarrow \quad u_2-u_1 &> 0\\
u_3-u_2 &= \frac{u_2+2u_1}{3}-u_2=\frac{2u_1-2u_2}{3}=-\frac{2}{3}(u_2-u_1)\\
u_4-u_3 &= \frac{u_3+2u_2}{3}-u_3=\frac{2u_2-2u_3}{3}=\left(-\frac{2}{3}\right)^2(u_2-u_1)\\
&\vdots\\
u_n-u_{n-1} &= \left(-\frac{2}{3}\right)^{n-2}(u_2-u_1).\\
\text{Therefore,}\quad u_n-u_1 &= (u_2-u_1)\left[1-\frac{2}{3}+\left(-\frac{2}{3}\right)^2+\cdots+\left(-\frac{2}{3}\right)^{n-2}\right]\\
&= (u_2-u_1)\left[\frac{1-\left(-\frac{2}{3}\right)^{n-1}}{1+\frac{2}{3}}\right]\\
&= \frac{3(u_2-u_1)}{5}\left[1-\left(-\frac{2}{3}\right)^{n-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence,}\quad \lim(u_n - u_1) &= \frac{3}{5}(u_2 - u_1) \qquad\qquad [\because \lim\left(-\frac{2}{3}\right)^{n-1} = 0]\\
\Rightarrow \lim u_n &= u_1 + \frac{3}{5}(u_2 - u_1)\\
\Rightarrow \lim u_n &= \frac{2u_1 + 3u_2}{5}.
\end{aligned}
$$

□

6. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \geq 1$. Show that
   (i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$;
   (ii) the sequence $\{u_n\}$ is monotone decreasing if $u_1 > 3$.
   Find $\lim\limits_{n\to\infty} u_n$.

**Solution.** Note that

$$
\begin{aligned}
u_{n+1}^2 - u_n^2 &= 6 + u_n - u_n^2\\
\Longrightarrow (u_{n+1} + u_n)(u_{n+1} - u_n) &= (2 + u_n)(3 - u_n), \qquad \text{for all } n = 1, 2, 3, \ldots \quad (1)
\end{aligned}
$$

From the above equation, we obtain $(u_2 + u_1)(u_2 - u_1) = (2 + u_1)(3 - u_1)$. Since $u_1 > 0$, it is clear that

$$u_2 > u_1 \text{ if } u_1 < 3 \quad \text{and} \quad u_2 < u_1 \text{ if } u_1 > 3 \qquad (2)$$

As $u_1 > 0$, it is easy to see that $u_n > 0$ for all $n$. Since $u_{n+1} = \sqrt{6 + u_n}$,

$$u_{n+1}^2 - u_n^2 = u_n + 6 - u_{n-1} - 6 \Longrightarrow (u_{n+1} + u_n)(u_{n+1} - u_n) = (u_n - u_{n-1}) \quad \forall n \geq 1.$$

Since $u_n > 0$ for all $n$, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

(i) From (2), $\mathbf{0 < u_1 < 3}$ implies $u_2 > u_1$, consequently $u_3 > u_2$, $u_4 > u_3$, $\ldots$, and therefore $\{u_n\}$ is monotonic increasing sequence in this case. Now from (1), $\{u_n\}$ monotonic increasing, that is $u_{n+1} > u_n$ implies $u_n < 3$ for all $n \in \mathbb{N}$. This shows that, when $0 < u_1 < 3$, $\{u_n\}$ is monotonic increasing and bounded above by 3. Therefore $\{u_n\}$ is convergent.

(ii) On the other hand, if $\mathbf{u_1 > 3}$, then $u_2 < u_1$. Consequently $u_3 < u_2$, $u_4 < u_3$, $\ldots$, and therefore $\{u_n\}$ is monotonic decreasing sequence in this case. Now from (1), $\{u_n\}$ monotonic decreasing, that is $u_{n+1} < u_n$ implies $u_n > 3$ for all $n \in \mathbb{N}$. This shows that, when $u_1 > 3$, $\{u_n\}$ is monotonic decreasing and bounded below by 3. Therefore $\{u_n\}$ is convergent.

In both the cases (i) and (ii), the sequence $\{u_n\}$ is convergent. Let $\lim\limits_{n\to\infty} u_n = \ell$. Since, $u_{n+1}^2 = u_n + 6$, taking limit $n \to \infty$ both side, we obtain

$$\ell^2 = 6 + \ell \Longrightarrow \ell = -2 \text{ or } 3.$$

Since $\{u_n\}$ is a sequence of positive real numbers, $\lim\limits_{n\to\infty} u_n$ can not be negative real number. Therefore $\ell \neq -2$, but $\ell = 3$. □

7. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7+x_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

**Monotone Convergence Theorem**

A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound (supremum).

A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound (infimum).

**Solution.** The sequence is $(\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots)$
$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$.
or, $(u_{n+1}+u_n)(u_{n+1}-u_n) = u_n - u_{n-1}$.
Since, $u_n > 0$ for all $n \in \mathbb{N}$, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.
But $u_2 > u_1$. Consequently, $u_3 > u_2$, $u_4 > u_3, \ldots$ and therefore $\{u_n\}$ is a monotone increasing sequence.
We have $u_n^2 < u_{n+1}^2 = 7 + u_n \; \forall \, n \in \mathbb{N}$
or, $u_n^2 - u_n - 7 < 0$
or, $(u_n - \alpha)(u_n - \beta) < 0$, where $\alpha, \beta$ are the roots of the equation $x^2 - x - 7 = 0$. One of the roots is negative and the other is positive.
Let $\alpha > 0$.
Since $u_n > 0 \; \forall \, n \in \mathbb{N}, u_n - \alpha > 0$. Consequently, $u_n < \beta \; \forall \, n \in \mathbb{N}$.
This proves that the sequence $(u_n)$ is bounded above and therefore this sequence is $\{u_n\}$ is convergent.
Let $\lim u_n = l$. By definition, $u_{n+1}^2 = 7 + u_n, \forall \, n \in \mathbb{N}$.
Taking limit as $n \to \infty$, we have $l^2 = 7 + l$.
Therefore, $(l - \alpha)(l - \beta) = 0$.
But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$. Therefore, $l = \beta$.
That is, the sequence converges to the positive root of the equation $x^2 - x - 7 = 0$. □

8. If $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)$ for all $n \geq 1$. Prove that the sequence $\{x_n\}$ converges to 3.

**Solution.** Given $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)$ for all $n \geq 1$.
Then, we have

$$x_n{}^2 - 2x_{n+1}x_n + 9 = 0.$$

This is quadratic equation in $x_n$ having real roots. Therefore

$$4x_{n+1}^2 - 36 \geq 0$$

$\implies x_{n+1} \geq 3$, for all $n \geq 1$ [since $x_{n+1} > 0$; for all $n \geq 1$.]
Now

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{9}{x_n}\right) = \frac{1}{2}\left(x_n - \frac{9}{x_n}\right) \\ &= \frac{1}{2}\left(\frac{x_n^2 - 9}{x_n}\right) \geq 0, \qquad \text{for all } n \geq 2. \end{aligned}$$

Therefore

$$x_{n+1} \leq x_n, \qquad \text{for all } n \geq 2.$$

That is, $\{x_n\}_{n=2}^{\infty}$ is monotonic decreasing sequence which is bounded below. Therefore $\{x_n\}_{n=2}^{\infty}$ is convergent.

Let $\lim\limits_{n\to\infty} x_n = l$. Note that $l$ cannot be 0, because $x_n \geq 3$ for all $n \geq 2$. Then

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{9}{\lim\limits_{n\to\infty} x_n}\right) \\
\Longrightarrow l &= \frac{1}{2}\left(l + \frac{9}{l}\right) \\
\Longrightarrow l^2 &= 2l^2 + 9 \\
\Longrightarrow l &= \pm 3 \qquad [\because x_n > 0, \text{ for all } n \geq 1] \\
\Longrightarrow l &= 3.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} x_n = 3$. □

9. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

   (i) Prove that the sub-sequences $\{u_{2n+1}\}$ and $\{u_{2n}\}$ converges to a common limit.

   (ii) Find $\lim\limits_{n\to\infty} u_n$.

**Theorem**

If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit $l$ then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$. The converse is also true.

**Solution.** (i) Consider

$$u_{n+1} - u_n = \frac{6}{1+u_n} - u_n = \frac{6 - u_n - u_n^2}{1+u_n} = \frac{(2-u_n)(3+u_n)}{1+u_n}; \ \forall n \in \mathbb{N}$$

Therefore,

$$u_n < 2 \Longrightarrow u_{n+1} > u_n$$
$$u_n > 2 \Longrightarrow u_{n+1} < u_n.$$

Thus

$$u_n < 2 \Longrightarrow u_{n+1} = \frac{6}{1+u_n} > 2$$
$$u_n > 2 \Longrightarrow u_{n+1} = \frac{6}{1+u_n} < 2.$$

It follows that,

$$u_n < 2 \Longrightarrow u_n < 2 < u_{n+1}, \quad u_n > 2 \Longrightarrow u_{n+1} < 2 < u_n \tag{3}$$

Now

$$u_{n+2} - u_n = \frac{6(1+u_n)}{7+u_n} - u_n = \frac{6 - u_n - u_n^2}{7+u_n} = \frac{(2-u_n)(3+u_n)}{7+u_n}.$$

Therefore,

$$u_n < 2 \Longrightarrow u_n < u_{n+2}; \quad u_n > 2 \Longrightarrow u_n > u_{n+2}. \tag{4}$$

*Case (I)*. When $u_1 < 2$. Then $u_2 > 2$.
From (1);

$$\begin{aligned} u_1 < 2 &\Rightarrow u_1 < 2 < u_2 \\ u_2 > 2 &\Rightarrow u_3 < 2 < u_2 \\ u_3 < 2 &\Rightarrow u_3 < 2 < u_4 \\ u_4 > 2 &\Rightarrow u_5 < 2 < u_4 \\ &\vdots \end{aligned}$$

From (2);

$$\begin{aligned} u_1 < 2 &\Rightarrow u_1 < u_3 \\ u_3 < 2 &\Rightarrow u_3 < u_5 \\ u_2 > 2 &\Rightarrow u_2 > u_4 \\ u_4 > 2 &\Rightarrow u_4 > u_6 \\ &\vdots \end{aligned}$$

Therefore,

$$u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2.$$

This shows that the subsequence $\{u_{2n+1}\}$ is monotonic increasing, bounded above and the subsequence $\{u_{2n}\}$ is monotonic decreasing, bounded below. Hence, both the subsequences $\{u_{2n+1}\}$ and $\{u_{2n}\}$ are convergent. Now check whether both the subsequence limits are same.

Let

$$\lim_{n\to\infty} u_{2n+1} = l \text{ and } \lim_{n\to\infty} u_{2n} = m.$$

Note that

$$u_{2n} = \frac{6}{1+u_{2n-1}}, \quad u_{2n+1} = \frac{6}{1+u_{2n}} \qquad \forall n \in \mathbb{N}.$$

Taking $n \to \infty$, we have

$$m = \frac{6}{1+l} \text{ and } l = \frac{6}{1+m}.$$

Therefore, $l = m$ and hence the subsequences $\{u_{2n+1}\}$ and $\{u_{2n}\}$ converge to a common limit. Therefore the sequence $\{u_n\}$ is convergent.

*Case (II)*. When $u_1 > 2$.
From (1) and (2) it follows that

$$u_2 < u_4 < u_6 < \cdots < u_5 < u_3 < u_1.$$

The subsequence $\{u_{2n}\}$ is monotonic increasing, bounded above and the subsequence $\{u_{2n+1}\}$ is monotonic decreasing, bounded below. Hence, both the subsequences are convergent.
Proceeding similarly to *Case (I)*, it can be shown that they converge to a common limit.

(ii) To find, $\lim_{n\to\infty} u_n$
Let $\lim_{n\to\infty} u_n = l$. We have

$$u_{n+1} = \frac{6}{1+u_n}, \qquad \forall n \in \mathbb{N}$$

$$\begin{aligned}
&\Longrightarrow \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty}\left(\tfrac{6}{1+u_n}\right)\\
&\Longrightarrow l = \tfrac{6}{1+l}\\
&\Longrightarrow l^2+l-6=0\\
&\Longrightarrow l=2 \text{ or } l=-3
\end{aligned}$$

As $u_n > 0$ for all $n \in \mathbb{N}$, $l \neq -3$.
Therefore, $l = 2$.
Hence, $\lim_{n\to\infty} u_n = 2$. □

10. Establish the convergence and find the limits of the following sequences
[Hint. Approach through sub-sequences.]

(i) $\left(1+\frac{1}{3n+1}\right)^n$,

**Solution.** Let the sequence $\{u_n\}$ be defined by,

$$u_n = \left(1+\frac{1}{3n+1}\right)^n, \quad \forall n \in \mathbb{N}.$$

Let

$$v_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad w_n = \left(1+\frac{1}{3n+1}\right)^{3n+1} \qquad \text{for all } n \in \mathbb{N}.$$

Then we know that

$$\lim_{n\to\infty} v_n = e.$$

Note that

$$w_n = v_{3n+1} \qquad \text{for all } n \in \mathbb{N}.$$

Since the sequence $\{v_n\}$ converges to $e$, all its subsequences are convergent and converge to $e$. Therefore $\lim_{n\to\infty} w_n = e$. Now

$$w_n = u_n^3 \cdot \left(1+\frac{1}{3n+1}\right) \qquad \text{for all } n \in \mathbb{N}.$$

This implies, $u_n = w_n^{\frac{1}{3}} \cdot \left(1+\frac{1}{3n+1}\right)^{-\frac{1}{3}}$. Since $\{w_n\}$ and $\left\{\left(1+\frac{1}{3n+1}\right)^{-\frac{1}{3}}\right\}$ both are convergent, $\{u_n\}$ is convergent.
Taking limit $n\to\infty$ both side of the above equation, we get

$$\lim_{n\to\infty} u_n = e^{\frac{1}{3}}.$$

□

(ii) $\left(1+\frac{1}{n^2+2}\right)^{n^2}$.

**Solution.** The same way as the above problem, it can be shown that the sequence $\left\{\left(1+\frac{1}{n^2+2}\right)^{n^2}\right\}$ is convergent and $\lim_{n\to\infty}\left(1+\frac{1}{n^2+2}\right)^{n^2} = e$. □

11. Let $\{u_n\}$ be a bounded sequence and $\lim_{n\to\infty} v_n = 0$. Prove that $\lim_{n\to\infty} u_n v_n = 0$. Utilise this to prove that

(i) $\lim_{n\to\infty} \frac{\sin n}{n}$.

(ii) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1}$.

**Solution.** To show, if $\{u_n\}$ be a bounded sequence and $\lim_{n\to\infty} v_n = 0$ then $\lim_{n\to\infty} u_n v_n = 0$.
Given $\{u_n\}$ is bounded $\implies$ there exists $M > 0$ such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} v_n = 0 \implies$ there exists $k \in \mathbb{N}$ such that

$$|v_n - 0| < \frac{\varepsilon}{M}, \quad \text{for all } k \geq n$$

$$\text{or, } |v_n| < \frac{\varepsilon}{M}, \quad \text{for all } k \geq n.$$

Now

$$|u_n v_n - 0| = |u_n v_n| = |u_n||v_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon, \quad \text{for all } k \geq n.$$

That is

$$|u_n v_n - 0| < \varepsilon \quad \text{for all } k \geq n.$$

Hence, $\lim_{n\to\infty} u_n v_n = 0$.

(i) Consider $\{u_n\} = \{\sin n\}$, which is a bounded sequence as $|\sin n| \leq 1, \forall n \in \mathbb{N}$ and, $\{v_n\} = \{\frac{1}{n}\}$, which is converging to 0. This implies from the above, $\lim_{n\to\infty} u_n v_n = 0$. That is $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(ii) Consider $\{u_n\} = \{(-1)^n\}$, which is a bounded sequence by 1. Also, $\{v_n\} = \{\frac{n}{n^2+1}\}$, which is convergent with $\lim_{n\to\infty} v_n = 0$. Then by above result, we get $\lim_{n\to\infty} u_n v_n = 0$ or $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$. □

12. Prove that

(i) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(ii) $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

**Theorem**

Let $\{u_n\} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ (finite or infinite). Then $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$.

**Solution.**

(i) Take $u_n = n$ for all $n \in \mathbb{N}$.
Observe that,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$

which is finite. Hence, by given theorem

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}(n)^{\frac{1}{n}} = 1.$$

□

(ii) Take $u_n = \dfrac{n!}{n^n}$ for all $n \in \mathbb{N}$. Observe that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e},$$

which is finite. Hence, by given theorem

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}.$$

□

13. Establish from definition that $\{u_n\}$ is a Cauchy sequence, where

(i) $u_n = \frac{n}{n+1}$,

(ii) $u_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$,

(iii) $|u_{n+2} - u_{n+1}| \leq \frac{1}{2}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$.

**Cauchy Sequence**

A sequence $\{u_n\}$ is said to be a *Cauchy sequence* if for any pre-assigned $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$|u_n - u_m| < \varepsilon, \qquad \forall m,n \geq k$$

**Cauchy's General Principle of Convergence.** A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for a pre-assigned $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that,

$$|u_{n+p} - u_n| < \varepsilon; \qquad \forall n \geq k \ \ \& \ \ p = 1,2,3,\ldots$$

A sequence satisfying above *Cauchy's General Principle of Convergence* is a *Cauchy sequence.*

**Solution.**

(i) To show, $\{u_n\} = \{\frac{n}{n+1}\}$ is a Cauchy sequence.
Let $\varepsilon > 0$ be arbitrary. Choose a natural number $k$ such that $\frac{2}{k} < \varepsilon$ (such $k$ exist by Archimedean Property). Then

$$\begin{aligned}
|u_n - u_m| &= \left|\frac{n}{n+1} - \frac{m}{m+1}\right| \\
&= \left|\frac{mn+n-mn-m}{(m+1)(n+1)}\right| \\
&= \left|\frac{n-m}{(m+1)(n+1)}\right|
\end{aligned}$$

Since $nm < (n+1)(m+1) \implies \frac{1}{(n+1)(m+1)} < \frac{1}{nm}$, for all $m,n \in \mathbb{N}$. Thus,

$$|u_n - u_m| < \left|\frac{n-m}{mn}\right| < \frac{1}{m} + \frac{1}{n} \text{ which is } < \varepsilon \quad \text{for all } m,n \geq k$$

Hence, $\{\frac{n}{n+1}\}$ is a Cauchy sequence. □

(ii) To show, $\{u_n\}$ is a Cauchy sequence, where $u_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$.
Let $\varepsilon > 0$ be arbitrary. Let $p$ be a natural number. Then

$$\begin{aligned}
|u_{n+p} - u_n| &= \left|\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}\right| \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}} \qquad [\text{ since } (n+1)! \geq 2^n \text{ for all } n \geq 2] \\
&= \frac{1}{2^n}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right] \\
&= \frac{1}{2^n}\left[\frac{1-(\frac{1}{2})^p}{1-\frac{1}{2}}\right] = \frac{1}{2^{n-1}}\left[1 - \left(\frac{1}{2}\right)^p\right] \\
&\leq \frac{1}{2^{n-1}} \qquad \text{for all } p \in \mathbb{N}.
\end{aligned}$$

Now $\frac{1}{2^{n-1}} < \varepsilon$ if $n > 1 - \frac{\ln \varepsilon}{\ln 2}$. Choose

$$k = \left[1 - \frac{\ln \varepsilon}{\ln 2}\right] + 1.$$

Then

$$|u_{n+p} - u_n| \leq \frac{1}{2^{n-1}} < \varepsilon \quad \text{for all } n \geq k \quad \text{and} \quad \text{for all } p \in \mathbb{N}.$$

Hence, $\left\{1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right\}$ is a Cauchy sequence. □

(iii) To show, $\{u_n\}$ is a Cauchy sequence, where

$$|u_{n+2} - u_{n+1}| \leq \frac{1}{2}|u_{n+1} - u_n| \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Note that

$$\begin{aligned}
|u_{n+2} - u_{n+1}| &\leq \frac{1}{2}|u_{n+1} - u_n| \\
&\leq \left(\frac{1}{2}\right)^2 |u_n - u_{n-1}| \\
&\leq \cdots \\
&\leq \left(\frac{1}{2}\right)^n |u_2 - u_1|
\end{aligned}$$