Continuity of Functions

Let $D \subseteq \mathbb{R}$. consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that f is continuous at c if $\{x_n\}$ any sequence in D such that

$$x_n \to c \Rightarrow f(x_n) \to f(c).$$

If f is NOT continuous at c, we say that f is discontinuous at c.

Ex. 1 $f(x) = |x|$, $x \in \mathbb{R}$.

Let $x_n \to c$ as $n \to \infty$

$\Rightarrow |x_n| \to |c|$

This shows that $f(x_n) \to f(c)$.

$\therefore$ f is continuous at any point $c \in \mathbb{R}$.

Ex. 2 $f : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let $c \in \mathbb{R}$.

Let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to c$,

and $\{y_n\}$ be a sequence of irrational numbers such that $y_n \to c$.

Now $f(x_n) = 1 \to 1$
$f(y_n) = 0 \to 0$

This shows that f is discontinuous at any point $c \in \mathbb{R}$.

* Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f, g : D \to \mathbb{R}$ be function that are continuous at c. Then

i) $f+g$ is continuous at c
ii) rf " " " c for every $r \in \mathbb{R}$,
iii) fg " " " c
iv) $|f|$ " " " c

- What about the converse of the above four statement?
[Left as exercise]

[ε-δ definition]

Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f : D \to \mathbb{R}$ be a function. Then f is continuous at c if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon .$$

[Note that this can be derived from the sequential definition of continuity]

<u>Note</u>: Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f : D \to \mathbb{R}$ be a function that is continuous at c.

If $f(c) > 0$ then there is a neighbourhood $N(c, \delta_1)$ of c such that $f(x) > 0$ for all $x \in N(c, \delta_1)$

If $f(c) < 0$ then there is a neighbourhood $N(c, \delta_2)$ of c such that $f(x) < 0$ for all $x \in N(c, \delta_2)$.

* Properties of Continuous Functions.

- Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then

i) f is bounded on $[a,b]$

ii) f attains its supremum and infimum (bounds),

that is

$$f(c) = \sup\{f(x) : x \in [a,b]\} \text{ and}$$

$$f(d) = \inf\{f(x) : x \in [a,b]\}$$

for some c and $d \in [a,b]$.

Ex. for example, $f(x) = x^2$, $x \in [-1,2]$

then f attains its greatest lower bound at $x = 0$ and least upper bound at $x = 2$.

Ex. 2. $f(x) = \dfrac{1}{x-1}$, $x \in (1,2]$

Then f is continuous on $(1,2]$ but it is NOT bounded on $(1,2]$.

Similarly, $f(x) = x^2$, $x \in (1,2]$.

Then f is continuous and bounded on $(1,2]$. But $\inf\{f(x) : x \in (1,2]\} = 1$ which is NOT attained by f in $(1,2]$.

$\longrightarrow$ This is because $(1,2]$ is NOT closed and bounded interval.

* [Intermediate Value Theorem]

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ in ~~the of~~ (a,b).

Proof: ~~Let~~ Without loss of generality, we assume that $f(a) < f(b)$.

Let k be a real number such that

$$f(a) < k < f(b)$$

~~Let~~ consider $\varphi(x) = f(x) - k \quad \forall x \in [a,b]$.

Then $\varphi : [a,b] \to \mathbb{R}$ continuous.

Now
$$\varphi(a) = f(a) - k < 0,$$
$$\varphi(b) = f(b) - k > 0$$

$$\therefore \varphi(a) \cdot \varphi(b) < 0$$

$\therefore$ by Bolzano theorem, there is a $c \in (a,b)$ such that $\varphi(c) = 0$.

$$\Rightarrow f(c) = k$$

$\boxtimes$

* Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I. Then show that $f(I)$ is an interval. [Left as Exercise]

[A subset S of $\mathbb{R}$ is an interval if for any two points $x_1, x_2 \in S$ with $x_1 < x_2$, the closed interval $[x_1, x_2] \subset S$]

* A function $f : [a,b] \to [a,b]$ is continuous on $[a,b]$. Prove that there exists a point c in $[a,b]$ such that $f(c) = c$.

Solution :- If $f(a) = a$, then $c = a$.

If $f(b) = b$, then $c = b$.

If $f(a) \neq a$ and $f(b) \neq b$, then consider

$\varphi(x) = f(x) - x$, $x \in [a,b]$.

Note that $\varphi(a) = f(a) - a > 0$ as $f(a) \in [a,b]$ and $f(a) \neq a$

$\varphi(b) = f(b) - b < 0$ as $f(b) \in [a,b)$.

So, $\varphi(a) \cdot \varphi(b) < 0$.

therefore by Bolzano's theorem, there is a $c \in (a,b)$ such that $\varphi(c) = 0$

$\Rightarrow f(c) = c$. [Proved].

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