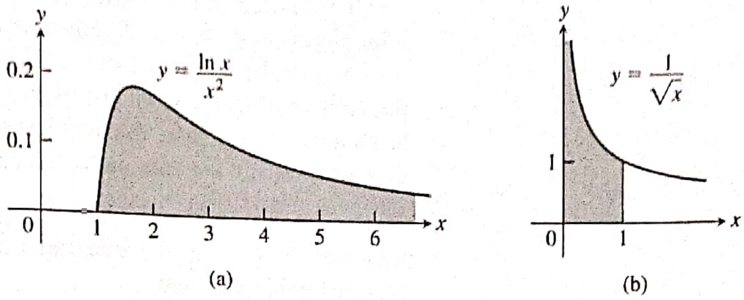

FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.13a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.13b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$

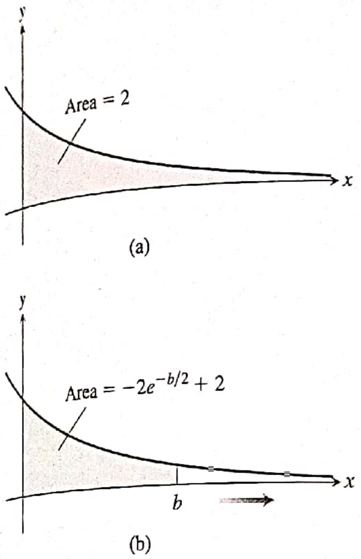

FIGURE 8.13 (a) The area in the first quadrant under the curve $y = e^{-x/2}$.
(b) The area is an improper integral of the first type.

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \ge 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.13 as an area. In that case, the area has the finite value 2. If $f \ge 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

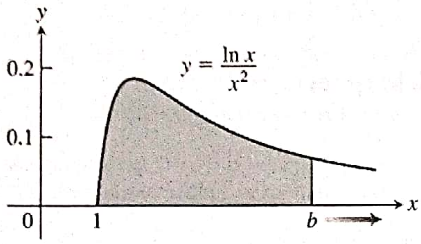

FIGURE 8.14 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.14). The area from 1 to b is

$$\begin{aligned}
\int_1^b \frac{\ln x}{x^2}\,dx &= \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx && \text{Integration by parts with } u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x \\
&= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b \\
&= -\frac{\ln b}{b} - \frac{1}{b} + 1.
\end{aligned}$$

The limit of the area as $b \to \infty$ is

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] \\
&= -\left[\lim_{b\to\infty} \frac{\ln b}{b}\right] - 0 + 1 \\
&= -\left[\lim_{b\to\infty} \frac{1/b}{1}\right] + 1 = 0 + 1 = 1. && \text{l'Hôpital's Rule}
\end{aligned}$$

Thus, the improper integral converges and the area has finite value 1. ∎

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0 \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
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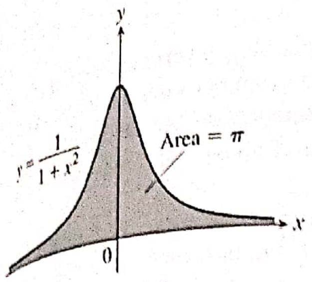


FIGURE 8.15 The area under this curve is finite (Example 2).

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x}$$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x \Big]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$ ■

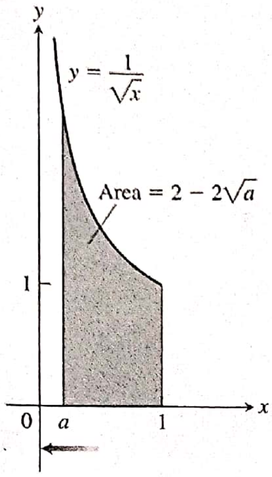


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.12b). First we find the area of the portion from a to 1 (Figure 8.16):

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

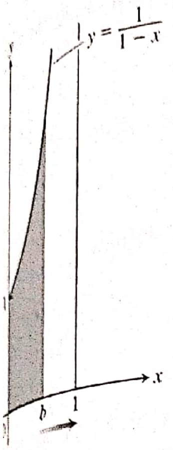
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$



GURE 8.17 The area beneath the ve and above the x-axis for $[0, 1)$ is not al number (Example 4).

Solution The integrand $f(x) = 1/(1 - x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.17). We evaluate the integral as

$$\lim_{b \to 1^-} \int_0^b \frac{1}{1 - x}\,dx = \lim_{b \to 1^-} \Big[-\ln|1 - x|\Big]_0^b$$

$$= \lim_{b \to 1^-} [-\ln(1 - b) + 0] = \infty.$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = \int_0^1 \frac{dx}{(x - 1)^{2/3}} + \int_1^3 \frac{dx}{(x - 1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\int_0^1 \frac{dx}{(x - 1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x - 1)^{2/3}}$$

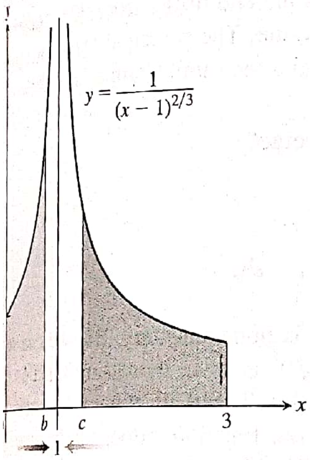
$$= \lim_{b \to 1^-} 3(x - 1)^{1/3}\Big]_0^b$$

$$= \lim_{b \to 1^-} [3(b - 1)^{1/3} + 3] = 3$$

$$\int_1^3 \frac{dx}{(x - 1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x - 1)^{2/3}}$$

$$= \lim_{c \to 1^+} 3(x - 1)^{1/3}\Big]_c^3$$

$$= \lim_{c \to 1^+} \left[3(3 - 1)^{1/3} - 3(c - 1)^{1/3}\right] = 3\sqrt[3]{2}$$



URE 8.18 Example 5 shows that the a under the curve exists (so it is a real mber).

We conclude that

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

Improper Integrals with a CAS

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral

$$\int_2^\infty \frac{x + 3}{(x - 1)(x^2 + 1)}\,dx$$

(which converges) using Maple, enter

> $f := (x + 3)/((x - 1) * (x\hat{}2 + 1));$

Then use the integration command

> int(f, x = 2..infinity);

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$In[1]{:=}\ \text{Integrate}[(x + 3)/((x - 1)(x\text{^}2 + 1)),\ \{x, 2, \text{Infinity}\}]$$

returns

$$Out[1]{=} -\frac{\pi}{2} + \text{ArcTan}[2] + \text{Log}[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

EXAMPLE 6 Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

We cannot evaluate this integral directly because it is nonelementary. But we *can* show that its limit as $b\to\infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b because the area under the curve increases as b increases. Therefore either it becomes infinite as $b\to\infty$ or it has a finite limit as $b\to\infty$. For our function it does not become infinite: For every value of $x \ge 1$, we have $e^{-x^2} \le e^{-x}$ (Figure 8.19) so that

$$\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to some finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 6. ∎

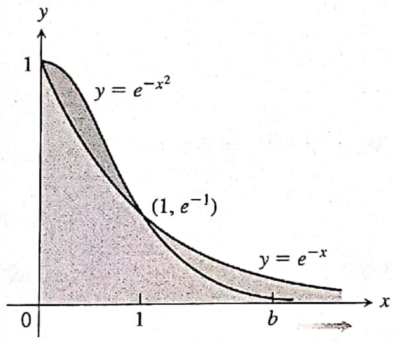


FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

The comparison of e^{-x^2} and e^{-x} in Example 6 is a special case of the following test.

THEOREM 2—Direct Comparison Test
Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.

2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

Proof The reasoning behind the argument establishing Theorem 2 is similar to that in Example 6. If $0 \le f(x) \le g(x)$ for $x \ge a$, then from Rule 7 in Theorem 2 of Section 5.3 we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 6, that

$$\int_a^\infty f(x)\,dx \quad \text{converges if} \quad \int_a^\infty g(x)\,dx \quad \text{converges.}$$

Turning this around to its contrapositive form, this says that

$$\int_a^\infty g(x)\,dx \quad \text{diverges if} \quad \int_a^\infty f(x)\,dx \quad \text{diverges.} \qquad \blacksquare$$

Although the theorem is stated for Type I improper integrals, a similar result is true for integrals of Type II as well.

HISTORICAL BIOGRAPHY
Karl Weierstrass
(1815–1897)
www.goo.gl/3RH2rO

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.} \qquad \text{Example 3}$$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.} \qquad \text{Example 3}$$

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right], \qquad 0 \le \cos x \le 1 \text{ on } \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2} \qquad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi} \quad \text{converges.} \end{aligned} \qquad \blacksquare$$

THEOREM 3—Limit Comparison Test
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either *both converge* or *both diverge*.

We omit the proof of Theorem 3, which is similar to that of Theorem 2.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

EXAMPLE 8 Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty}\frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty}\left(\frac{1}{x^2}+1\right) = 0+1 = 1,$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad ■$$

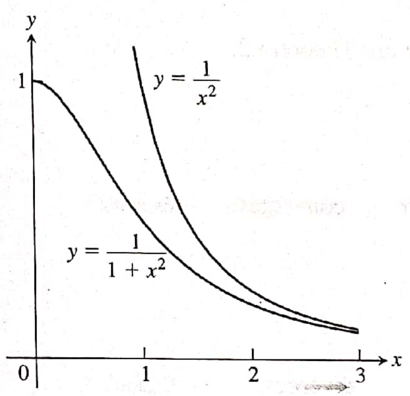


FIGURE 8.20 The functions in Example 8.

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1-e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty}(1-e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■