* Limit of a function.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D.

A real number l is said to be a limit of f at c if corresponding to a pre-assigned $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{for all } x \in (c-\delta, c+\delta) - \{c\}.$$

or $0 < |x-c| < \delta$

symbol : $\lim\limits_{x \to c} f(x) = l$.

Theorem: Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. Then f can have at most one limit point at c.

Example: Show that $\lim\limits_{x \to 1} (5x-3) = 2$.

$\Rightarrow$ Let $\varepsilon > 0$ be given.

(depends on ε)

We have to find a suitable $\delta > 0$ so that

$$0 < |x-1| < \delta \text{ implies } |(5x-3)-2| < \varepsilon.$$

Note that $|(5x-3)-2| < \varepsilon$

if $|5(x-1)| < \varepsilon$

that if $|x-1| < \frac{\varepsilon}{5}$

choose $\delta = \frac{\varepsilon}{5}$. Then $0 < |x-1| < \delta$ implies

$|f(x) - 2| < \varepsilon$, where $f(x) = 5x-3$

$\therefore \lim\limits_{x \to 1} f(x) = 2$

▨

* Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Then c is called a limit point of D if there is a sequence $\{x_n\}$ in $D - \{c\}$ such that $x_n \to c$.

* Sequential definition.

Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$.

Then $\lim_{x \to c} f(x) = l$ ~~iff~~ if and only if

for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 1 \text{ if } x \text{ is rational}$$
$$= 0 \text{ if } x \text{ is irrational}$$

Let $c \in \mathbb{R}$. Show that $\lim_{x \to c} f(x)$ does NOT exist.

$\Rightarrow$ Let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to c$ and, $\{y_n\}$ be a sequence of irrational number such that $y_n \to c$.

But $f(x_n) \to 1$ and $f(y_n) \to 0$.

Thus there can ~~not~~ be no $l \in \mathbb{R}$ such that $f(x) \to l$ as $x \to c$.

▨

* Limit Theorem for functions.

Let $D \subseteq \mathbb{R}$ and c be a limit point of D. Also, let $l, m \in \mathbb{R}$ and $f, g: D \to \mathbb{R}$ be functions such that

$$\lim_{x \to c} f(x) = l \text{ and } \lim_{x \to c} g(x) = m. \text{ Then}$$

(i) $\lim_{x \to c} (f+g)(x) = l + m$, (ii) $\lim_{x \to c} (rf)(x) = rl$, for $r \in \mathbb{R}$

(iii) $\lim_{x \to c} (fg)(x) = lm$.

(iv) if $l \neq 0$, then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$; also c is a limit point of $\{x \in D : 0 < |x-c| < \delta\}$ and

$$\lim_{x \to c} \left(\frac{1}{f}\right)(x) = \frac{1}{l} .$$

Theorem: Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$. Let c be a limit point of D.

i) If f is bounded on some deleted neighbourhood of c

and ii) $\lim_{x \to c} g(x) = 0$,

then $\lim_{x \to c} (f \cdot g)(x) = 0$

Example. As i) $\sin \frac{1}{x^2}$ is bounded in any deleted neighbourhood of 0,

ii) $\lim_{x \to 0} x = 0$,

$\Rightarrow$ $\lim_{x \to 0} x \sin \frac{1}{x^2} = 0$.

Theorem: [Sandwich theorem]

Let $D \subset \mathbb{R}$ and c be a limit point of D. Let $f, g, h : D \to \mathbb{R}$ be functions.

If i) $f(x) \leq g(x) \leq h(x)$ for all $x \in D - \{c\}$ and

ii) $\lim_{x \to c} f(x) = l = \lim_{x \to c} h(x)$,

Then $\lim_{x \to c} g(x) = l$.

Example: Show that $\lim_{x \to 0} x \cos\frac{1}{x} = 0$.

Let $f(x) = \cos\frac{1}{x}$, $x \in D = \mathbb{R} - \{0\}$. { Note that 0 is a limit point of D.

Then we have $-1 \leq f(x) \leq 1$ for all $x \in D$

This implies, $-x \leq x f(x) \leq x$ for all $x > 0$

and $x \leq x f(x) \leq -x$ for all $x < 0$.

Therefore $-|x| \leq x f(x) \leq |x|$ for all $x \in D$.

Since $\lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x|$, by

Sandwich theorem $\lim_{x \to 0}$ ~~$x \cos\frac{1}{x} = 0$~~ $x f(x) = 0$

that is $\lim_{x \to 0} x \cos\frac{1}{x} = 0$. $\boxtimes$

Definition. [Left hand limit]

Let $D \subset \mathbb{R}$ and c be a limit point of $D \cap (-\infty, c)$ ~~D~~. Let $f : D \to \mathbb{R}$ be a function.

f is said to have a <u>left hand limit</u> $l \in \mathbb{R}$ at c if for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in D$ ~~such that~~ with $c - \delta < x < c$. ~~$x \in D$~~

We denote $\lim_{x \to c-} f(x) = l$.

[Right hand limit]

Let c be a limit point of $D \cap (c, \infty)$. f is said to have right hand limit $l \in \mathbb{R}$ at c if for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in D$ satisfying $c < x < c + \delta$.

Notation: $\lim_{x \to c+} f(x) = l$.

Example : i) $f(x) = \operatorname{sgn} x$, $x \in \mathbb{R}$.

that is
$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

To find $\lim_{x \to c+} f(x)$, consider $f(x)$ on the set $D \cap (c, \infty)$.

For $\lim_{x \to c-} f(x)$, need to know $f(x)$ on $D \cap (-\infty, c)$.

Examine if $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ exist.

$\Rightarrow$ Here $D = \mathbb{R}$. and $c = 0$.

For $\lim_{x \to 0+} f(x)$.

Note that $D \cap (0, \infty) = (0, \infty)$ and 0 is a limit point of $(0, \infty)$.

Also, $f(x) = 1$ for all $x \in (0, \infty)$

This implies $\lim_{x \to 0+} f(x) = 1$

Similarly, $D \cap (-\infty, 0) = (-\infty, 0)$ and 0 is a limit point of $(-\infty, 0)$.

$f(x) = -1$ for all $x \in (-\infty, 0)$

therefore, $\lim_{x \to 0-} f(x) = -1$ ☒

In the above example both the LHL and RHL exists but NOT equal.

ii) Let $f(x) = \sin \frac{1}{x}$, $x \in \mathbb{R} - \{0\}$

To check if $\lim_{x \to 0+} \sin \frac{1}{x}$ exist.

Here $D = \mathbb{R} - \{0\}$ and $D \cap (0, \infty) = (0, \infty)$

(ii) 0 is a limit point of $(0,\infty)$

~~Let~~ Consider the sequences $\{x_n\}$ and $\{y_n\}$ where

$$x_n = \frac{1}{n\pi}, \quad \forall n \in \mathbb{N},$$

and

$$y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \quad \forall n \in \mathbb{N}.$$

Note that both the sequences $\{x_n\}, \{y_n\}$ are in $(0,\infty)$.

$$x_n \longrightarrow 0 \quad \text{and} \quad y_n \longrightarrow 0$$

$$\text{but } f(x_n) \longrightarrow 0, \quad f(y_n) \longrightarrow 1$$

Therefore $\lim\limits_{x\to 0+} f(x)$ does NOT exist.

Similarly, considering two sequences $\{u_n\}, \{v_n\}$ from $(-\infty, 0)$ such that

$$u_n \longrightarrow 0 \quad \text{and} \quad v_n \longrightarrow 0 \quad \text{but}$$

$$f(u_n) \longrightarrow 0 \quad \text{and} \quad f(v_n) \longrightarrow -1,$$

it can be concluded that $\lim\limits_{x\to 0-} f(x)$ does NOT exist.

iii) Let $f(x) = e^{\frac{1}{x}}$, $x \in \mathbb{R} - \{0\} = D$.

Then show that $\lim\limits_{x\to 0+} f(x)$ does NOT exist,

but $\lim\limits_{x\to 0-} f(x) = 0$

$\Rightarrow$ Left as exercise. [Hint. to show $\lim\limits_{x\to 0-} f(x) = 0$, use $0 < t < e^t$ for $t > 0$]