CLASSICAL MECHANICS

[For M.Sc. (Physics), B.Sc. (Honours), B.E., Net, GATE and Other Competitive Examinations]

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CHAPTER 2

Lagrangian Dynamics

2.1. INTRODUCTION

In the previous chapter, we have discussed Newtonian approach to deal with the motion of particles. Application of Newton's laws of motion needs the specification of all forces acting on the body at all instants of time. In practical situations, when the constraint forces are present, the application of the Newtonian approach may be a difficult task. The greatest disadvantage with the Newtonian procedure is that the mechanical problems are always tried to resolve geometrically rather than analytically. In case of constrained motion, the determination of all the unimportant reaction forces is a great nuisance in the Newtonian approach. In order to overcome the difficulties, posed by the Newtonian scheme in solving the problems of constrained motion, methods have been developed by D'Alembert, Lagrange, Hamilton and others. The techniques of Lagrange and Hamilton use generalized coordinates which have been discussed and used in the present and next chapters. In the Lagrangian formulation, the generalized coordinates used are position and velocity, resulting in the second order linear differential equations, while in the Hamiltonian formulation, the generalized coordinates being position and momentum result in the first order linear differential equations.

2.2. BASIC CONCEPTS

We discuss some basic concepts regarding the motion of particles.

(1) Coordinate Systems : The fundamental concept involved in the motion of a particle (or system) is the position coordinate and how it is changing with time. The position of a particle is represented by choosing a coordinate system. In the cartesian or rectangular coordinate system, the position vector **r** of a particle is defined in terms x, y and z coordinates. In a two dimensional motion, rectangular coordinates (x, y) or polar coordinates (r, θ) can represent the position of the particle [Fig. 2.1(a)]. They are related as

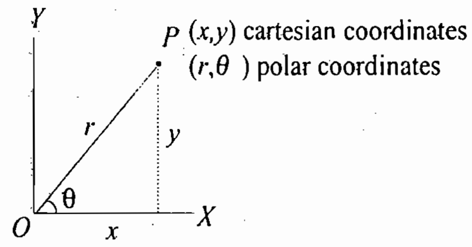


Fig. 2.1(a) : Rectangular and polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta;\ r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}.$$

In three dimensions, the cylindrical coordinates (ρ, θ, z) and the spherical coordinates (r, θ, ϕ) of the position of a particle are related to the cartesian coordinates (x, y, z) as follows :

For cylindrical and cartisian coordinates [Fig. 2.1(b)] :

$$x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = z;\quad \rho = \sqrt{x^2 + y^2},\quad \theta = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{\rho}$$

For spherical and cartesian coordinates [Fig. 2.1(c)]:

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$;

$$r = (x^2 + y^2 + z^2)^{1/2},\ \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\ \phi = \tan^{-1}\frac{y}{x}$$

We may represent, for example, the relationships for spherical and cartesian coordinates as follows :

$$x = x(r,\theta,\phi),\ y = y(r,\theta,\phi),\ z = z(r,\theta)$$

or $\quad \mathbf{r} = \mathbf{r}(r, \theta, \phi)$

If we include the time variable also, then

$$\mathbf{r} = \mathbf{r}(r, \theta, \phi, t),$$

In general, we may represent the coordinates by q_1, q_2, q_3, having the relationships with the cartesian coordinates as

$$x = x(q_1, q_2, q_3, t),\quad y = y(q_1, q_2, q_3, t),$$

$$z = z(q_1, q_2, q_3, t)$$

or $$\mathbf{r} = \mathbf{r}(q_1, q_2, q_3, t) \quad ...(1)$$

In fact, these are the transformation equations from a general system to the cartesian coordinate system.

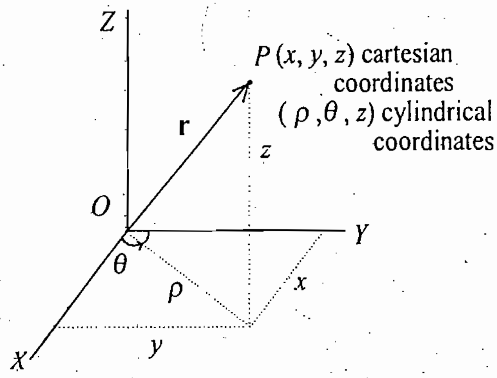


Fig. 2.1(b) : Cartesian and cylindrical coordinates

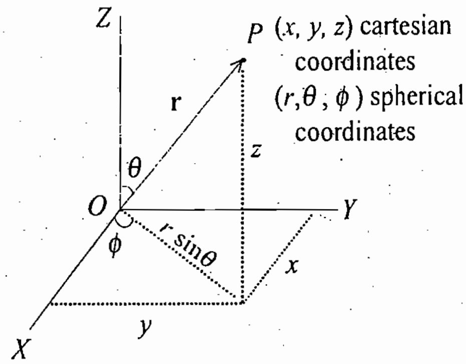


Fig. 2.1(c) : Cartesian and spherical coordinates

(2) Degrees of Freedom — *Configuration Space* : The minimum number of independent variables or coordinates required to specify the position of a dynamical system, consisting of one or more particles, is called the number of **degrees of freedom** of the system. For example, the motion of a particle, moving freely in space, can be described by a set of three coordinates *e.g.*, (x, y, z) and hence the number of degrees of freedom, possessed by the particle, is three. A system of two particles, moving freely in space, requires two sets of three coordinates [*e.g.*, (x_1, y_1, z_1) and (x_2, y_2, z_2)] *i.e.*, six coordinates to specify its position. Thus the system has six degrees of freedom. If a system consists of N particles, moving freely in space, we need $3N$ independent coordinates to describe its position. Hence the number of degrees of freedom of the system is $3N$.

The configuration of the system of N particles, moving freely in space, may be represented by the position of a single point in $3N$ dimensional space, which is called **configuration space** of the system. The configuration space for a system of one freely moving particle is 3-dimensional and for a system of two freely moving particles, it is six dimensional. In the later case, the configuration of the system of the two particles can be represented by the position of a single point with six coordinates in six dimensional space. This system has six degrees of freedom and its configuration space is six dimensional.

The number of coordinates, needed to specify a dynamical system, becomes smaller, when the constraints (which we describe below) are present in the system. Hence ***the degrees of freedom of a dynamical system is defined as the minimum number of independent coordinates (or variables) required to specify the system compatible with the constraints.*** If there are n independent variables, say $q_1, q_2, ..., q_n$ and n constants $C_1, C_2, ..., C_n$ such that

$$\sum_{i=1}^{n} C_i\, dq_i = 0 \quad ...(2)$$

at any position of the system, then we must have

$$C_1 = C_2 = ... = C_n = 0$$

2.3. CONSTRAINTS

Often the motion of a particle or system of particles is restricted by one or more conditions. ***The limitations on the motion of a system are called constraints and the motion is said to be constrained motion*** .

2.3.1. Holonomic constraints

Constraints limit the motion of a system and the number of independent coordinates, needed to describe the motion, is reduced. For example, if a particle is allowed to move on the circumference of a circle, then only one coordinate $q_1 = \theta$ is sufficient to describe the motion, because the radius (a) of the circle remains the same. If **r** is the position vector of the particle at any angular coordinate θ relative to the centre of the circle, then

$$|\mathbf{r}| = a \quad \text{or} \quad r - a = 0 \qquad ...(3)$$

Eq. (3) expresses the constraint for a particle in circular motion. Similarly in the case of a particle, moving on the surface of a sphere, the correct coordinates are spherical coordinates r, θ and ϕ, where θ and ϕ only vary. Therefore $q_1 = \theta$ and $q_2 = \phi$ are the two independent coordinates for the problem, because the constraint is that the radius of the sphere (a) is constant (*i.e.*, $|\mathbf{r}| = a$). Since in the circular motion of the particle, one independent coordinate θ is needed, the number of degrees of freedom of the system is 1. For the particle, constrained to move on the surface of the sphere, two independent coordinates specify its motion and hence the degrees of freedom is 2.

Suppose the constraints are present in the system of N particles. If the constraints are expressed in the form of equations of the form

$$f(\mathbf{r}_1, \mathbf{r}_2, ..., t) = 0 \qquad ...(4)$$

then they are called ***holonomic constraints***. Let there be m number of such equations to describe the constraints in the N particle system. Now, we may use these equations to eliminate m of the $3N$ coordinates and we need only n independent coordinates to describe the motion, given by

$$n = 3N - m$$

The system is said to have n or $3N - m$ degrees of freedom. The elimination of the dependent coordinates can be expressed by introducing $n = 3N - m$ independent variables $q_1, q_2, ..., q_n$. These are referred as ***generalized coordinates***.

Superfluous Coordinates : The idea of degrees of freedom makes it clear that when we are using, say rectangular cartesian coordinates, we have several ***redundant*** or ***superfluous*** coordinates, if there are holonomic constraints. This redundance and non-independence of the coordinates makes the problem complicated and this difficulty is resolved by using the generalized coordinates. For example, let us consider a body be thrown vertically upward with an initial velocity v_0. The body will move in a straight line. In cartesian coordinates, the motion will be represented as

$$x = 0, \quad y = v_0 t - \frac{1}{2} g t^2, \quad z = 0$$

where X and Z axes are horizontal and Y-axis is in vertical direction. At different values of the time t, only y coordinate varies and x and z coordinates remain the same. Therefore x and z coordinates are ***superfluous coordinates***. In conclusion, we need only one coordinate $q = y$ to describe the vertical motion.

2.3.2. Nonholonomic constraints

The constraints which are not expressible in the form of eq. (4) are called ***nonholonomic***. For example, the motion of a particle, placed on the surface of a sphere of radius a, will be described by

$$|\mathbf{r}| \geq a \text{ or } r - a \geq 0$$

in a gravitational field, where **r** is the position vector of the particle relative to the centre of the sphere. The particle will first slide down the surface and then fall off. The gas molecules in a container are constrained to move inside it and the related constraint is another example of nonholonomic constraints. If the gas container is in spherical shape with radius a and **r** is the position vector of a molecule, then the condition of constraint for the motion of molecules can be expressed as

$$|\mathbf{r}| \leq a \text{ or } r - a \leq 0 .$$

It is to be mentioned that in holonomic constraints, each coordinate can vary independently of the other. In a nonholonomic system, all the coordinates cannot vary independently and hence the number of degrees of freedom of the system is less than the minimum number of coordinates needed to specify the configuration of the system. We shall in general consider the holonomic systems.

Constraints are further described as (i) **rheonomous** and (ii) **scleronomous**. In the former, the equations of constraint contain the time as an explicit variable, while in the later they are not explicitly dependent on time. Constraints may also be classified as (i) **conservative** and (ii) **dissipative**. In case of conservative constraints, total mechanical energy of the system is conserved during the constrained motion and the constraint forces do not do any work. In dissipative constraints, the constraint forces do work and the total mechanical energy is not conserved. Time-dependent or rheonomic constraints are generally dissipative.

2.3.3. Some more Examples of Holonomic and Non-holonomic constraints

Besides the above examples of holonomic and nonholonomic constraints, we are giving below some important examples :

(1) **Rigid body** (***Holonomic constraint***) : In case of the motion of a rigid body, the distance between any two particles of the body remains fixed and do not change with time. If $\mathbf{r}_i$ and $\mathbf{r}_j$ are the position vectors of the ith and jth particles, then the distance between them can be expressed by the condition

$$| \mathbf{r}_i - \mathbf{r}_j | = C_{ij} \text{ (constant)}$$

If (x_i, y_i, z_i) and (x_j, y_j, z_j) are the cartesian coordinates (components of position vector) of the two particles, then the constraints will be expressed as

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = C_{ij}^2$$

The constraint is *holonomic* and *scleronomic*.

(2) **Simple pendulum with rigid support** (***Holonomic constraint***) : In case of a simple pendulum with rigid support, the constraint is that during the motion, the distance (l) of the bob (particle) from the point of suspension remains constant with time. Thus if **r** is the position vector of the particle relative to the point of suspension, then the condition of constraint can be expressed as

$$| \mathbf{r} | = l \text{ (constant)}$$

This is also an example of *holonomic* and *scleronomic* constraint. If the motion is confined to move in a vertical plane, only one coordinate θ, the angular displacement, is sufficient to describe the motion.

(3) **Rolling disc** (***Non-holonomic constraint***) : The nomenclature '*holonomic*' constraints comes from the word '*holos*' which means '*integer*' in Greek and '*whole*' or '*integrable*' in Latin languages. A system is said to be non-holonomic if it corresponds to non-integrable differential equations of constraints. Such constraints cannot be expressed in the form of eq.(4). Obviously holonomic system has integrable differential

equations of constraints, expressible in the form (4). In order to explain this, let us consider a disc rolling on a rough horizontal X-Y plane with the condition of constraint is that the plane of the disc is always vertical. We choose the coordinates x, y for the centre of the disc, ϕ for the angle of rotation about the axis of the disc and θ for the angle between the axis of the disc and X-axis [Fig. 2.2].

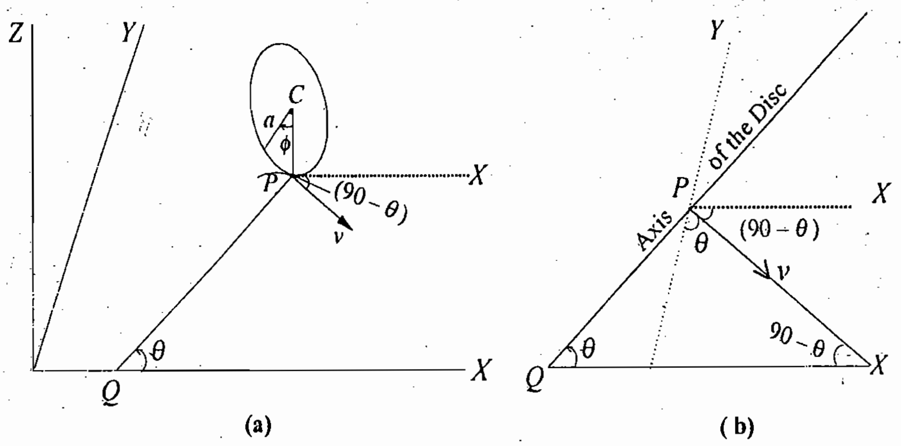


Fig. 2.2 : Vertical disc rolling on a horizontal XY-plane

If a is the radius of the disc, the constraint that the axis of the disc is perpendicular to the vertical Z-direction, gives the velocity v of the disc with magnitude

$$v = a\dot{\phi} = a\frac{d\phi}{dt}$$

As the direction of the velocity is perpendicular to the axis of the disc, the components of the velocity along X-axis and Y-axis are

$$v_x = \frac{dx}{dt} = v\sin\theta\ , \quad v_y = \frac{dy}{dt} = -\,v\cos\theta$$

Therefore, $$\frac{dx}{dt} = a\frac{d\phi}{dt}\sin\theta \quad \text{and} \quad \frac{dy}{dt} = -\,a\frac{d\phi}{dt}\cos\theta$$

or $$dx - a\sin\theta\, d\phi = 0 \quad \text{and} \quad dy + a\cos\theta\, d\phi = 0 \qquad \text{...(5)}$$

None of the equations, given by (5), can be integrated without solving the entire problem. Thus the constraint cannot be put in the form (4) and hence the constraint is nonholonomic.

2.3.4. Forces of Constraint

Constraints are always related to forces which restrict the motion of the system. These forces are called ***forces of constraint.*** For example, the reaction force on a sliding particle on the surface of a sphere is the force of constraint. In case of a rigid body , the inertial forces of action and reaction between any two particles are the forces of constraint. Constraint force in a simple pendulum is the tension in the string. When a bead slides on a wire, the reaction force exerted by the wire on the bead is the force of constraint. These forces of constraint are elastic in nature and usually appear at the surface of contact because the motion due to external applied forces' is hindered by the contact. However, Newton has not given any prescription to calculate these forces of constraint.

Usually the constraint forces act in a direction perpendicular to the surface of constraints while the motion of the object is parallel to the surface. In such cases, the work done by the forces of constraint is

These constraints are termed as *workless* and may be called as ***ideal constraints***. For example, when a particle slides on a frictionless horizontal surface, the force of constraint is normal to the surface. There are examples, where the constraint force does work. When a body slides on a frictional surface, the work is done by the force of constraint (frictional force) for real displacements.

By definition, the external or applied forces are all known forces. In the solution of dynamical problems either we have to determine all the forces of constraints or we should eliminate them from final equations. If we want to use Newton's form of equations, the forces of constraints are to be determined. We discuss below the difficulties, introduced by such an approach and how to remove them.

2.3.5. Difficulties introduced by the Constraints and their Removal

Two types of difficulties are introduced by constraints in the solution of mechanical problems :

(1) Let us consider a system of N interacting particles. The force on the ith particle is given by

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_{j=1}^{N} \mathbf{F}_{ij}$$

where $\mathbf{F}_i^e$ stands for an external force and $\mathbf{F}_{ij}$ is the internal (constraint) force on the ith particle due to jth particle. The equation of motion of the ith particle, in view of Newton's second law, is

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i^e + \sum_{j=1}^{N} \mathbf{F}_{ij} \qquad \text{...(6)}$$

where $i = 1, 2, ..., N$. Thus eq. (6) represents a set of N equations. The coordinates $\mathbf{r}_i$ are connected by equations of constraints of the form :

$$f(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, t) = 0$$

Hence the coordinates $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N$ of various particles are no longer all independent. This means that N equations represented by (6) are not all independent and therefore, the equations of motion are to be written again taking into consideration the equations of the constraints.

(2) The second difficulty introduced by the constraints is that several times the constraint forces are not known initially and they are among the unknowns of the problem. In fact, these unknown constraint forces are to be obtained from the solution of the problem which we are seeking and thus introduce complications in obtaining the solution. For example, if a bead is moving on a wire, the force (of constraint) which the wire exerts on the bead is not known in the beginning of the problem.

In case of holonomic constraints, as discussed already, the first difficulty is solved by introducing $n = 3N - m$ generalized coordinates, where m is the number of equations of constraints in N particle system. In order to remove the second difficulty, namely the forces of constraint are not known initially, we formulate the mechanics in such a way that the forces of constraint disappear. We require then only the known applied forces. Such an approach is due to D'Alembert which uses the ideas of virtual displacement and virtual work.

Ex. 1. *Determine the number of degrees of freedom in the following cases* : (1) *A particle moving on the circumference of a circle.* (2) *Five particles moving freely in a plane.* (3) *Two particles connected by a rigid rod moving freely in a plane.* (4) *A rigid body moving freely in three dimensional space.* (5) *A rigid body moving in space with one point fixed.* (6) *A rigid body rotating about a fixed axis in space.* (7) *The bob of simple pendulum oscillating in a plane.* (8) *The bob of a conical pendulum.* (9) *Dumbbell moving in space.*

Solution : (1) The constraint is $x^2 + y^2 = a^2$ or $r = a$ (constant). Hence in cartesian coordinates one variable x or y and in polar coordinates one variable θ could suffice to describe the motion on the circle. Hence the degree of freedom is 1.

(2) Each free particle needs two coordinates to specify its position in a plane. Hence 5 free particles will need 10 coordinates and therefore the number of degrees of freedom of the system is 10.

(3) Degrees of freedom = $2 \times 2 - 1 = 3$, because two particles in a plane will need $[(x_1, y_1)]$ and $(x_2, y_2)]$ 4 coordinates and there is one constraint equation for the distance l and between the two particles:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2.$$

(4) A rigid body is a system of particles in which the distance between any two particles remain fixed throughout the motion. Let us consider three non-collinear particles P_1, P_2, P_3 of a rigid body [Fig. 10.1]. As each particle has 3 degrees of freedom and there are three constraints of the form

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = r_{12}^2 \text{ (constraint)}$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{23}^2$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = r_{13}^2,$$

Hence the degrees of freedom for these particles are $3 \times 3 - 3 = 6$. The consideration of any other particle in the body needs three coordinates, say $P_i(x_i, y_i, z_i)$, and obviously there are three equations of constraints because the distances of P_i from P_1, P_2, P_3 are fixed. Hence any other particle will not add any new degree of freedom to the six degrees of freedom of the three-particle system of the body. Thus a rigid body moving freely in a three dimensional space has 6 degrees of freedom.

(5) One point of the rigid body is fixed, say the particle at the origin. Hence for this particle, $x_1 = 0$, $y_1 = 0$, $z_1 = 0$. The constraint equations for other particles are

$$x_2^2 + y_2^2 + z_2^2 = r_2^2, \quad x_3^2 + y_3^2 + z_3^2 = r_3^2$$

and $$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{23}^2 \text{ (constant).}$$

Hence the degrees of freedom for the system are $3 \times 3 - 6 = 3$.

(6) A rigid body, rotating about a fixed axis, has one degree of freedom, because relative to the origin, say on the fixed axis, taken as Z-axis, z = constant and $x^2 + y^2 = r_0^2$ for a particle, where r_0 is the radius of the circle about the fixed axis.

(7) The bob of an oscillating simple pendulum has one degree of freedom, because the motion is described by one θ coordinate with the constraint $|r| = l$ or in cartesian coordinates by x or y with the constraint $x^2 + y^2 = l^2$.

(8) The bob of a conical pendulum has 2 degrees of freedom as the constraint is $x^2 + y^2 + z^2 = l^2$ relative to the centre of suspension.

(9) In a dumbbel two heavy particles are connected by a rigid massless rod. The system has $3 \times 2 - 1 =$ 5 degrees of freedom, because the distance $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = \text{constant}]$ between the two particles is fixed.

Ex. 2. *In the following cases, discuss whether the constraint is holonomic or nonholonomic. Specify the constraint force also :*

(1) *Motion of a body on an inclined plane under gravity .*

(2) *A bead on a circular wire .*

(3) *A particle moving on an ellipsoid under the influence of gravity .*

(4) *A pendulum with variable length .*

Solution : (1) When a body moves on an inclined plane, it is constrained to move on the inclined plane surface. Hence the constraint is holonomic. The force of constraint is the reaction of the plane, acting normal to the inclined surface.

(2) The constraint is that the bead remains at a constant distance a, the radius of the circular wire and can be expressed as $r = a$. Hence the constraint is holonomic. The force of constraint is the reaction of the wire, acting on the bead.

(3) The constraint is nonholonomic, because the particle after reaching a certain point will leave the ellipsoid. Force of constraint is the reaction force of the ellipsoid surface on the particle.

(4) The constraint is holonomic and rheonomic, because the constraint equation is $|\mathbf{r}| = l\,(t)$. The constraint force is the tension in the string.

Ex. 3. *Show that the constraints in a rigid body are conservative .*

Solution : The distance between any two particles i and j of a rigid body can be expressed as

$$r_{ij}^2 = |\mathbf{r}_i - \mathbf{r}_j|^2 = \text{constant.}$$

Therefore,
$$d\,(r_{ij}^2) = 0 = d\,[(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_i - \mathbf{r}_j)]$$
$$= (\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) + d\,(\mathbf{r}_i - \mathbf{r}_j)\cdot(r_i - r_j)$$
$$= 2\,(\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) = 0$$

i.e., $(\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) = 0$

According to Newton's third law, the force $\mathbf{F}_{ij}$ on the ith particle due to jth particle is equal and opposite to the force on the jth particle due to ith particle *i.e.*,

$$\mathbf{F}_{ij} = -\,\mathbf{F}_{ji}$$

Now the work done

$$W = \sum_{\substack{i,j\\ i\neq j}} \int \mathbf{F}_{ij}\cdot d\mathbf{r}_i = \sum_{\substack{i,j\\ i>j}} \int \left(\mathbf{F}_{ij}\cdot d\mathbf{r}_i + \mathbf{F}_{ji}\cdot d\mathbf{r}_j\right)$$

$$= \sum_{\substack{i,j\\ i\neq j}} \int \mathbf{F}_{ij}\cdot d\left(\mathbf{r}_i - \mathbf{r}_j\right) = \sum_{\substack{i,j\\ i>j}} \int C_{ij}\left(\mathbf{r}_i - \mathbf{r}_j\right)\cdot d\left(\mathbf{r}_i - \mathbf{r}_j\right) = 0$$

because the internal force $\mathbf{F}_{ij}$ is considered parallel to the line joining ith and jth particles of the form : $\mathbf{F}_{ij} = C_{ij}(\mathbf{r}_i - \mathbf{r}_j)$, where C_{ij} are constants.

Hence the work done by constraint forces in a rigid body is zero and consequently the constraint is conservative in nature.

2.4. GENERALIZED COORDINATES

The name generalized coordinates is given to a set of independent coordinates sufficient in number to describe completely the state of configuration of a dynamical system. These coordinates are denoted as

$$q_1, q_2, q_3, \ldots, q_k, \ldots, q_n \qquad \ldots(7)$$

where n is the total number of generalized coordinates. In fact, these are the minimum number of coordinates, needed to describe the motion of the system. For example, for a particle constrained to move on the circumference of a circle only one generalized coordinate $q_1 = \theta$ is sufficient and two generalized coordinates

$q_1 = \theta$, and $q_2 = \phi$ for a particle moving on the surface of a sphere. The generalized coordinates for a system of N particles, constrained by m equations, are $n = 3N - m$. It is not necessary that these coordinates should be rectangular, spherical or cylindrical. In fact, the quantities like length, (length)^2, angle, energy or a dimensionless quantity may be used as generalized coordinates but they should completely describe the state of the system. Further these n generalized coordinates are not restricted by any constraint.

For a system of N particles, if x_i, y_i, z_i are the cartesian coordinates of the ith particle, then these coordinates in terms of the generalized coordinates q_k can be expressed as

$$\begin{aligned} x_i &= x_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \\ y_i &= y_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \\ z_i &= z_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \end{aligned} \qquad \ldots(8\text{ a})$$

or in general the position vector $\mathbf{r}_i(x_i, y_i, z_i)$ of the ith particle is

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \qquad \ldots(8\text{ b})$$

Eqs. (8a) or (8b) give the transformation equations. It may be mentioned that the generalized coordinates may be the cartesian coordinates.

One should note that the system is said to be rheonomic, when there is an explicit time dependence in some or all of the functions defined by eq. (8). If there is not the explicit time dependence, the system is called scleronomic and t is not written in the functional dependence, *i.e.*,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_k, \ldots, q_n) \qquad \ldots(9)$$

2.5. PRINCIPLE OF VIRTUAL WORK

In order to investigate the properties of a system, we can imagine arbitrary instantaneous change in the position vectors of the particles of the system *e.g.*, virtual displacements. An infinitesimal virtual displacement of ith particle of a system of N particles is denoted by $\delta\mathbf{r}_i$. This is the displacement of position coordinates only and does not involve *variation of time i.e.*,

$$\delta\mathbf{r}_i = \delta\mathbf{r}_i(q_1, q_2, \ldots, q_n) \qquad \ldots(10)$$

Suppose the system is in equilibrium, then the total force on any particle is zero *i.e.*,

$$\mathbf{F}_i = 0, \qquad i = 1, 2, \ldots, N$$

The virtual work of the force $\mathbf{F}_i$ in the virtual displacement $\delta\mathbf{r}_i$ will also be zero *i.e.*,

$$\delta W_i = \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$$

Similarly, the sum of virtual work for all the particles must vanish *i.e.*,

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \qquad \ldots(11)$$

This result represents the ***principle of virtual work*** which states that ***the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium***.

The total force $\mathbf{F}_i$ on the ith particle can be expressed as

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i$$

where $\mathbf{F}_i^a$ is the applied force and $\mathbf{f}_i$ is the force of constraint.

Hence eq. (11) assumes the form

$$\sum_{i=1}^{N} \mathbf{F}_i^a \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

We restrict ourselves to the systems where the virtual work of the forces of constraints is zero, *e.g.*, in case of a rigid body. Then

$$\sum_{i=1}^{N} \mathbf{F}_i^a \cdot \delta \mathbf{r}_i = 0 \qquad ...(12)$$

i.e., for equilibrium of a system, the virtual work of applied forces is zero. We see that the principle of virtual work deals with the statics of a system of particles. However, we want a principle to deal with the general motion of the system and such a principle was developed by D'Alembert.

2.6. D'ALEMBERT'S PRINCIPLE

According to Newton's second law of motion, the force acting on the ith particle is given by

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} = \dot{\mathbf{p}}_i$$

This can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0, \qquad i = 1, 2, ..., N$$

These equations mean that any particle in the system is in equilibrium under a force, which is equal to the actual force $\mathbf{F}_i$ plus a reversed effective force $\dot{\mathbf{p}}_i$. Therefore, for virtual displacements $\delta \mathbf{r}_i$,

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

But $\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i$, then

$$\sum_{i=1}^{N} \left(\mathbf{F}_i^a - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

Again, we restrict ourselves to the systems for which the virtual work of the constraints is zero, *i.e.*, $\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$. Then

$$\sum_{i=1}^{N} \left(\mathbf{F}_i^a - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i = 0 \qquad ...(13)$$

This is known as ***D'Alembert's principle***. Since the forces of constraints do not appear in the equation and hence now we can drop the superscript. Therefore, the D'Alembert's principle may be written as

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \qquad ...(14)$$

Ex. 1. *Two heavy particles of weights W_1 and W_2 are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radius R, the axis of which is horizontal* [*Fig.* 2.3]. *Find the condition of equilibrium of the system by applying the principle of virtual work.*

Solution : According to the principle of virtual work

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

Here, $i = 1, 2$ and therefore

$$W_1 \sin\theta\, \delta r_1 + W_2 \sin\phi\, \delta r_2 = 0$$

But $\delta r_1 = R\, d\theta$, $\delta r_2 = R\, d\phi$

$$\therefore \quad W_1 \sin\theta\, \delta\theta + W_2 \sin\phi\, \delta\phi = 0$$

But $\theta + \phi =$ constant

Therefore $\quad \delta\theta + \delta\phi = 0$ or $\delta\phi = -\delta\theta$

Thus $\quad (W_1 \sin\theta - W_2 \sin\phi)\, \delta\theta = 0$

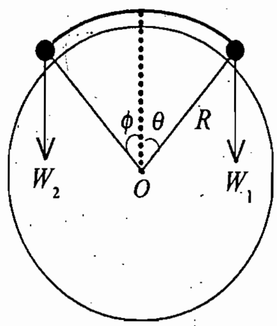


Fig. 2.3

The system is in equilibrium, hence the following condition is to be satisfied ($\delta\theta \neq 0$) :

$$W_1 \sin\theta - W_2 \sin\phi = 0 \quad \text{or} \quad \frac{W_1}{W_2} = \frac{\sin\phi}{\sin\theta}.$$

Ex. 2. *An inextensible string of negligible mass hanging over a smooth peg B [Fig. 2.4] connects one mass m_1 on a frictionless inclined plane of angle θ to another mass m_2. Using D'Alembert's principle, prove that the masses will be in equilibrium, if* $\sin\theta = \frac{m_2}{m_1}$.

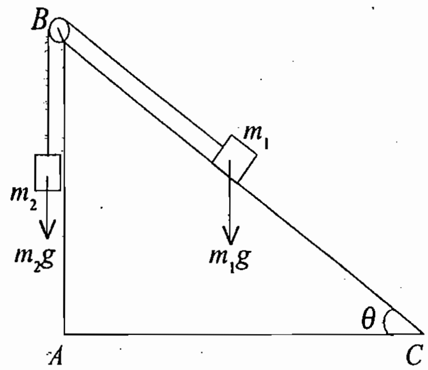


Fig. 2.4

Solution : According to D'Alembert's principle

$$\sum_{i=1}^{2} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of m_1 and m_2 relative to B.

$$(m_1\mathbf{g} - m_1\ddot{\mathbf{r}}_1) \cdot \delta\mathbf{r}_1 + (m_2\mathbf{g} - m_2\ddot{\mathbf{r}}_2) \cdot \delta\mathbf{r}_2 = 0$$

or $$(m_1 g \sin\theta - m_1 \ddot{r}_1)\, \delta r_1 + (m_2 g - m_2 \ddot{r}_2)\, \delta r_2 = 0 \qquad ...(i)$$

But the string is inextensible,

$r_1 + r_2 =$ a constant or $\delta r_1 + \delta r_2 = 0$, *i.e.*, $\delta r_2 = -\delta r_1$.

Also $\quad \ddot{r}_1 + \ddot{r}_2 = 0$ or $\ddot{r}_2 = -\ddot{r}_1$

Hence eq. (i) takes the form

$$(m_1 g \sin\theta - m_2 g)\, \delta r_1 - (m_1 + m_2)\, \ddot{r}_1\, \delta r_1 = 0$$

The system is in equilibrium, hence $\ddot{r}_1 = 0$. Further dividing by $\delta r_1 \neq 0$, we obtain

$$m_1 g \sin\theta - m_2 g = 0 \quad \text{or} \quad \sin\theta = \frac{m_2}{m_1}$$

Note : *Work Ex. 2 for the incline having the coefficient of friction μ and prove that the equilibrium condition is*

$$\sin\theta - \mu\cos\theta = \frac{m_2}{m_1}.$$

2.7. LAGRANGE'S EQUATIONS FROM D'ALEMBERT'S PRINCIPLE

Consider a system of N particles. The transformation equations for the position vectors of the particles are

$$\mathbf{r}_i = \mathbf{r}_i (q_1, q_2, ..., q_k, ..., q_n, t) \quad ...(15)$$

where t is the time and q_k $(k = 1, 2, ..., n)$ are the generalized coordinates.

Differentiating eq. (15) with respect to t, we obtain the velocity of the ith particle, *i.e.*,

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \mathbf{r}_i}{\partial q_2}\frac{dq_2}{dt} + ... + \frac{\partial \mathbf{r}_i}{\partial q_k}\frac{dq_k}{dt} + ... + \frac{\partial \mathbf{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \mathbf{r}_i}{\partial t}$$

or

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \quad ...(16)$$

where $\dot{q}_k$ are the **generalized velocities.**

The virtual displacement is given by

$$\delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\delta q_2 + ... + \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k + ... + \frac{\partial \mathbf{r}_i}{\partial q_n}\delta q_n$$

or

$$\delta \mathbf{r}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k \quad ...(17)$$

since by definition the virtual displacements do not depend on time.

According to D'Alembert's principle,

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \quad ...(18)$$

Here

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{n} G_k\, \delta q_k \quad ...(19)$$

where

$$G_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right] \quad ...(20)$$

are called the components of **generalized force** associated with the generalized coordinates q_k. This may be mentioned that as the dimensions of the generalized coordinates need not be those of length, similarly the generalized force components G_k may have dimensions different than those of force. However, the dimensions of $G_k\, \delta q_k$ are those of work. For example, if δq_k has the dimensions of length, G_k will have the dimensions of force; if δq_k has the dimensions of angle (θ), G_k will have the dimensions of torque (τ).

Further

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\left[\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right]\delta q_k \quad ...(21)$$

Now

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\right] \quad ...(22)$$

It is easy to prove that

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{r}_i}{dt}\right) = \frac{\partial \mathbf{v}_i}{\partial q_k} \quad \text{...(23 a)*}$$

and

$$\frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} \quad \text{...(23 b)**}$$

Therefore, eq. (22) is

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left[m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k}\right] - m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial q_k}\right] \quad \text{...(24)}$$

Substituting in (21), we get

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k}\right) - m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(\mathbf{v}_i \bullet \mathbf{v}_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\frac{1}{2}m_i(\mathbf{v}_i \bullet \mathbf{v}_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k; \quad \text{...(25)}$$

---

* Here $$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \sum_{j=1}^{n}\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \mathbf{r}_i}{\partial t\, \partial q_k} \quad \text{...}(i)$$

which has been obtained by treating $\partial \mathbf{r}_i/\partial q_k$ as a single quantity being the function of the generalized coordinates q_j and time t.

But $$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_{j=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}$$

and its partial derivative with respect to q_k is

$$\frac{\partial \mathbf{v}_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{r}_i}{dt}\right) = \sum_{j=1}^{n}\frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \quad \text{...}(ii)$$

From eqs.(i) and (ii)

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial \mathbf{v}_i}{\partial q_k} \quad \text{...(23}a\text{)}$$

** $$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\sum_{j=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}\right] = \frac{\partial \mathbf{r}_i}{\partial q_j}\delta_{jk} = \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \text{...(23 }b\text{)}$$

as the constraints are holonomic and $\dfrac{\partial \dot{q}_j}{\partial \dot{q}_k} = \delta_{jk}$ is kronecker delta which is 1 for $j = k$ and zero for $j \neq k$.

where $\sum\limits_{i}\frac{1}{2}m_i(\mathbf{v}_i \bullet \mathbf{v}_i) = \sum\limits_{i}\frac{1}{2}m_i v_i^2 = T$ is the **kinetic energy** of the system.

Substituting for $\sum\limits_{i}\mathbf{F}_i \bullet \delta\mathbf{r}_i$ from (19) and $\sum\limits_{i}\dot{\mathbf{p}}_i \bullet \delta\mathbf{r}_i$ from (25) in eq.(18), the D'Alembert's principle becomes

$$\sum_{k=1}^{n}\left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right\} - G_k\right]\delta q_k = 0 \qquad ...(26)$$

As the constraints are holonomic, it means that any virtual displacement δq_k is independent of δq_j. Therefore, the coefficient in the square bracket for each δq_k must be zero, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k = 0 \text{ or } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad ...(27)$$

This represents the ***general form of Lagrange's equations.***

For a **conservative system**, the force is derivable from a scalar potential V:

$$\mathbf{F}_i = \nabla_i V = -\hat{\mathbf{i}}\frac{\partial V}{\partial x_i} - \hat{\mathbf{j}}\frac{\partial V}{\partial y_i} - \hat{\mathbf{k}}\frac{\partial V}{\partial z_i} \qquad ...(28)$$

Hence from eq. (20), the generalized force components are

$$G_k = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \qquad ...(29)$$

Clearly the right hand side of the above equation is the partial derivative of $-V$ with respect to q_k, *i.e.*,

$$G_k = -\frac{\partial V}{\partial q_k} \qquad ...(30)$$

Thus eq.(27) assumes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} \qquad ...(31)$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad ...(32)$$

Since the scalar potential V is the function of generalized coordinates q_k only not depending on generalized velocities, we can write eq. (32) as

$$\frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_k}\right] - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad ...(33)$$

We define a new function given by

$$L = T - V \qquad ...(34)$$

which is called the ***Lagrangian*** of the system. Thus, eq. (33) takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad ...(35)$$

for $k = 1, 2, ..., n$.

These equations are known as **Lagrange's equations** for conservative system. They are n in number and there is one equation for each generalized coordinate. In order to solve these equations, we must know the Lagrangian function $L = T - V$ in the appropriate generalized coordinates.

2.8. PROCEDURE FOR FORMATION OF LAGRANGE'S EQUATIONS

The Lagrangian function L of a system is given by

$$L = T - V \qquad \text{...(36)}$$

In order to form L, kinetic energy T and potential energy V are to be written in generalized coordinates. This is then substituted in the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{...(37)}$$

to obtain the equations of motion of the system. This involves first to find the partial derivatives of L, *i.e.*, $\partial L/\partial q_k$ and $\partial L/\partial \dot{q}_k$ and then to put their values in eq. (37).

Kinetic Energy in Generalized Coordinates : The transformation equations (15) and (16) are used to transform T from cartesian coordinates to generalized coordinates. Therefore

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i \frac{1}{2}\left(\sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}\right)^2$$

or

$$T = M_0 + \sum_k M_k \dot{q}_k + \frac{1}{2}\sum_{k,l} M_{kl}\dot{q}_k \dot{q}_l \qquad \text{...(38)}$$

where

$$M_0 = \sum_k \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t}\right)^2, \quad M_k = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

and

$$M_{kl} = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l}$$

Thus we see from (38) that in the expression for kinetic energy, first term is independent of generalized velocities, while second and third terms are linear and quadratic in generalized velocities respectively.

For scleronomic systems, the transformation equations do not contain time explicitly. So that

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k$$

Therefore,

$$T = \sum_i \frac{1}{2} m v_i^2 = \frac{1}{2}\sum_{k,l} M_{kl}\dot{q}_k \dot{q}_l \qquad \text{...(39)}$$

In such a case, the expression for T is a homogeneous quadratic form in generalized velocities.

Ex. 1. ***Newton's equation of motion from Lagrange's equations*** : *Consider the motion of a particle of mass m. Using cartesian coordinates as generalized coordinates, deduce Newton's equation of motion from Lagrange's equations.*

Solution : The general form of the Lagrange's equations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad \text{...(}i\text{)}$$

Here, $q_1 = x$, $q_2 = y$, $q_3 = z$ and generalized force components are $G_1 = F_x$, $G_2 = F_y$, $G_3 = F_z$. The kinetic energy T is

$$T = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right]$$

For x-coordinate, eq.(i) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \qquad \text{...(ii)}$$

But $\quad \dfrac{\partial T}{\partial x} = 0$ and $\dfrac{\partial T}{\partial \dot{x}} = m\dot{x}$

Substituting in eq. (*ii*), we get

$$\frac{d}{dt}(m\dot{x}) = F_x \quad \text{or} \quad F_x = \frac{dp_x}{dt}$$

where $p_x = m\dot{x}$ is the x-component of momentum. Similarly, we can obtain

$$F_y = \frac{dp_y}{dt} \quad \text{and} \quad F_z = \frac{dp_z}{dt}$$

Thus $\quad \mathbf{F} = \dfrac{d\mathbf{p}}{dt} \qquad$ ...(*iii*)

which is Newton's equation of motion.

Ex. 2. *Simple Pendulum* : *Obtain the equation of motion of a simple pendulum by using Lagrangian method and hence deduce the formula for its time period for small amplitude oscillations.*

(Agra 1999, 91; Garwal 98, 97; Kanpur 2003)

Solution : Let θ be the angular displacement of the simple pendulum from the equilibrium position. If l be the effective length of the pendulum and m be the mass of the bob, then the displacement along arc $OA = s$ is given by

$$s = l\theta \qquad \left[\text{because } \theta = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{l}\right]$$

Kinetic energy $\quad T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \quad \left[\because v = \frac{ds}{dt} = \frac{d(l\theta)}{dt} = \frac{l\,d\theta}{dt} = l\dot{\theta}\right]$

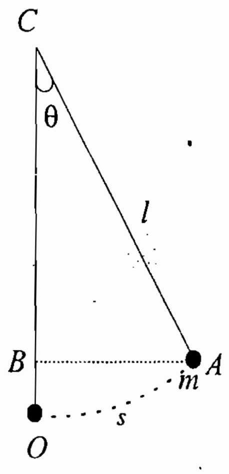


Fig. 2.5 : Simple pendulum

If the potential energy of the system, when the bob is at O, is zero, then the potential energy, when the bob is at A, is given by

$$V = mg\,(OB) = mg\,(OC - BC) = mg\,(l - l\cos\theta) = mgl\,(1 - \cos\theta)$$

Hence $\quad L = T - V$, or $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1 - \cos\theta)$

Now, $\quad \dfrac{\partial L}{\partial \theta} = -mgl\sin\theta$ and $\dfrac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

Substituting these values in the Lagrange's equation (here there is only one generalized coordinate $q_1 = \theta$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

we get

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] + mgl \sin\theta = 0 \text{ or } ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This represents the equation of motion of a simples pendulum.

For small amplitude oscillations, $\sin\theta \cong \theta$, and therefore the equation of motion of a simple pendulum is

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This represents a *simple harmonic motion* of period, given by

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Ex. 3. ***Atwood's Machine*** : *Obtain the equation of motion of a system of two masses, connected by an inextensible string passing over a small smooth pulley.* **(Mumbai 2002; Agra 1989, 96)**

Solution : The Atwood's machine is an example of a conservative system with holonomic constraint. The pulley is small, massless and frictionless. Let the two masses be m_1 and m_2 which are connected by an inextensible string of length l. Suppose x be the variable vertical distance from the pulley to the mass m_1. Then mass m_2 will be at a distance $l - x$ from the pulley [Fig. 2.6]

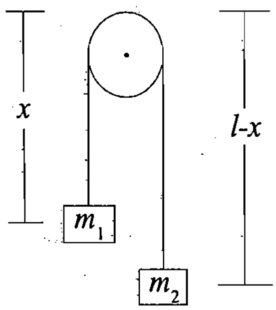


Fig. 2.6 : Atwood's machine

Thus there is only one independent coordinate x. The velocities of the two masses are $v_1 = \frac{dx}{dt} = \dot{x}$ and $v_2 = \frac{d(l-x)}{dt} = -\dot{x}$.

Therefore, $$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Potential energy of the system with reference to the pulley is

$$V = -m_1 gx - m_2 g(l - x)$$

Thus the Lagrangian is

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 gx + m_2 g(l - x)$$

Now, $$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} \text{ and } \frac{\partial L}{\partial x} = (m_1 - m_2)g$$

Here the generalized coordinate is $q = x$. Now Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \quad \text{or} \quad (m_1+m_2)\,\ddot{x}-(m_1-m_2)\,g=0$$

or
$$\ddot{x}=\frac{(m_1-m_2)}{m_1+m_2}g$$

which is the desired equation of motion.

If $m_1 > m_2$, the mass m_1 descends with constant acceleration and if $m_1 < m_2$, the mass m_1 ascends with constant acceleration. It is to be noted that the tension in the rope, the force of constraint, is not seen anywhere in the Lagrangian formulation.

Ex. 4 . *In Ex. 3, calculate the acceleration of the system, if the pulley is a disc of radius R and moment of inertia I about an axis through its centre and perpendicular to its plane.*

Solution : Angular velocity of the pulley $\omega=\frac{v}{R}=\frac{\dot{x}}{R}$

Rotational kinetic energy of the pulley $=\frac{1}{2}I\omega^2=\frac{1}{2}I\frac{\dot{x}^2}{R^2}$

where $\quad v=\dot{x}=|v_1|=|v_2|$

$$\therefore \quad T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}I\frac{\dot{x}^2}{R^2}$$

Also, $\quad V=-m_1 gx-m_2 g\,(l-x).$

Therefore, $\quad L=\frac{1}{2}\left(m_1+m_2+\frac{I}{R^2}\right)\dot{x}^2+(m_1-m_2)\,gx+m_2\,gl$

The Lagrange's equation is

$$\left(m_1+m_2+\frac{I}{R^2}\right)\ddot{x}-[m_1-m_2]\,g=0$$

whence
$$\ddot{x}=\frac{[m_1-m_2]\,g}{m_1+m_2+\dfrac{I}{R^2}}$$

Equation of motion of Ex. 3 will be obtained for $I=0$.

Ex. 5. *Compound Pendulum* : *Use Lagrange's equations to find the equation of motion of a compound pendulum in a vertical plane about a fixed horizontal axis. Hence find the period of small amplitude oscillations of the compound pendulum.* **(Agra 1999, 97, 93)**

Solution : Let the compound pendulum be suspended from S with C as centre of mass. It is oscillating in the vertical plane which is the plane of the paper.

Moment of inertia of the pendulum about the axis of rotation through S is given by

$$I=I_c+Ml^2=M(K^2+l^2)$$

where M is the mass of the pendulum, $I_c=MK^2$ (K = radius of gyration) about a parallel axis through C and l the distance between centre of suspension and centre of mass.

If θ is the instantaneous angle which SC makes with the vertical axis through O, then the kinetic energy of the oscillating system is

$$T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} M\left(K^2 + l^2\right)\dot{\theta}^2$$

Potential energy with respect to horizontal plane through S is

$$V = -\,Mgl \cos \theta$$

Lagrangian $\quad L = T - V$

or $\quad L = \frac{1}{2} M\left(K^2 + l^2\right)\dot{\theta}^2 + Mgl \cos\theta$

Now, $\quad \frac{\partial L}{\partial \theta} = -Mgl \sin\theta$ and $\frac{\partial L}{\partial \dot{\theta}} = M(K^2 + l^2)\,\dot{\theta}$

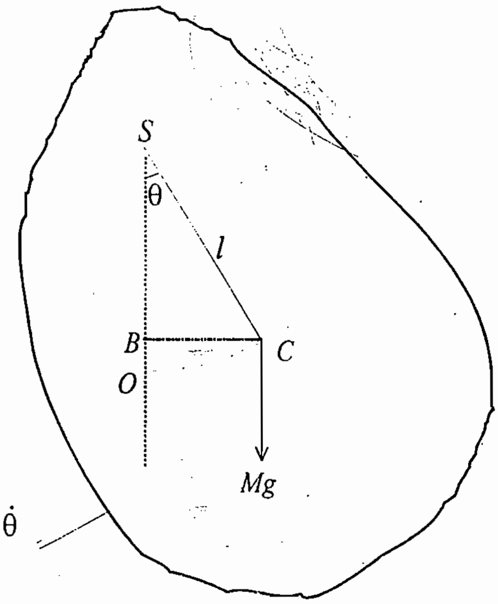


Fig. 2.7 : Compound pendulum

Lagrange's equation in θ coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Therefore, $\quad \frac{1}{2} M\left(K^2 + l^2\right)\ddot{\theta} + Mgl \sin\theta = 0$ or $\ddot{\theta} + \frac{gl}{K^2 + l^2}\sin\theta = 0$

This is the equation of motion of the compound pendulum. If θ is small, $\sin\theta \cong \theta$ and then

$$\ddot{\theta} + \frac{gl}{K^2 + l^2}\theta = 0$$

This equation represents a simple harmonic motion whose period is given by

$$T = 2\pi \sqrt{\frac{K^2 + l^2}{lg}} = 2\pi \sqrt{\frac{\frac{K^2}{l} + l}{g}}.$$

Ex. 6. ***Radial and Tangential Components of a Force :*** *Consider the motion of a particle of mass m moving in a plane. Using the plane polar coordinates (r, θ) as generalized coordinates, deduce expressions for the components of generalized force. What are radial and tangential components of the force ?*

Solution : For the motion of a particle in a plane, the cartesian and polar coordinates are related as

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Hence $\quad \dot{x} = \dot{r} \cos\theta - r\dot{\theta} \sin\theta$ and $\dot{y} = \dot{r} \sin\theta + r\dot{\theta} \cos\theta$

Therefore, $\quad T = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2)$

Here, there are two generalized coordinates, *i.e.*, $q_1 = r$ and $q_2 = \theta$.

Now, $\quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2,\ \frac{\partial T}{\partial \dot{r}} = m\dot{r},\ \frac{\partial T}{\partial \theta} = 0$ and $\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$

Corresponding to two generalized coordinates, there are two Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = G_r \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = G_\theta$$

or $$m\ddot{r} - mr\dot{\theta}^2 = G_r \quad \text{and} \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = G_\theta$$

We can express the components of the generalized force in terms of the radial and tangential components of the force. From the definition of the generalized force, we have

$$G_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} \quad \text{and} \quad G_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \quad \left[\text{as } G_k = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k}\right]$$

But $$\mathbf{r} = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$$

Hence $$\frac{\partial \mathbf{r}}{\partial r} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

and $$\frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\,\hat{\mathbf{i}} + r\cos\theta\,\hat{\mathbf{j}} = r\hat{\boldsymbol{\theta}}$$

where $\hat{\theta}$ = unit vector perpendicular to $\hat{\mathbf{r}}$.

Therefore, $$G_r = \mathbf{F} \cdot \hat{\mathbf{r}} = F_r \quad \text{or} \quad F_r = m\ddot{r} - mr\dot{\theta}^2 \qquad \text{...}(i)$$

and $$G_\theta = \mathbf{F} \cdot r\hat{\boldsymbol{\theta}} = r\,\mathbf{F} \cdot \hat{\boldsymbol{\theta}} = rF_\theta \quad \text{or} \quad rF_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right) \qquad \text{...}(ii)$$

Note that in eq. (ii), $mr^2\dot{\theta} = mvr = J$, the angular momentum and its time derivative is just the applied torque (rF_θ). This is the torque equation, *i.e.*, rate of change of angular momentum is equal to the applied torque.

Thus the radial and tangential components of the force are

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad \text{and} \quad F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \qquad \text{...}(iii)$$

Ex. 7. *Langrange's equation for L-C circuit* : *Find Lagrange's equation of motion for an electrical circuit comprising an inductance L and capacitance C. The capacitor is charged to q coulombs and current flowing in the circuit is i amperes.* **(Agra 2003, 1999, 1992)**

Solution : Let us consider an electrical circuit, containing inductance L and capacitance C. We want to find Lagrange's equation of motion for the L-C circuit, when the charge on the condenser is q and the current flowing in the circuit is i.

The magnetic energy $\frac{1}{2}Li^2$ in an electrical circuit is analogous to the kinetic energy $\frac{1}{2}mv^2$ in a mechanical system, where we can think inductance L as charge inertia similar to mass inertia and $i = \frac{dq}{dt}$ as $v = \frac{dx}{dt}$; charge q is playing the role of displacement. The electrical potential energy of the circuit is $V = q^2/2C$. Hence the Lagrangian of the L-C circuit is

$$L = T - V = \tfrac{1}{2}Li^2 - \frac{q^2}{2C} \quad \text{or} \quad L = \tfrac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}$$

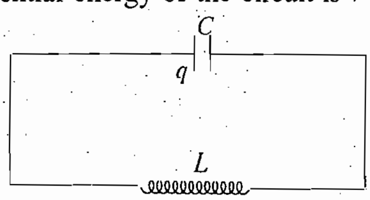


Fig. 2.8 : L-C Circuit

Taking q as the generalized coordinate, the Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Here $\qquad \dfrac{\partial L}{\partial \dot{q}}=L\dot{q}$ and $\dfrac{\partial L}{\partial q}=-\dfrac{q}{C}.$

Therefore, $\qquad \dfrac{d}{dt}[L\dot{q}]+\dfrac{q}{C}=0$ or $L\dfrac{d^2q}{dt^2}+\dfrac{q}{C}=0$ or $\dfrac{d^2q}{dt^2}+\dfrac{q}{LC}=0$

This is the differential equation for L-C circuit, having frequency $\nu=\dfrac{1}{2\pi\sqrt{LC}}$.

Ex. 8. ***Motion under Central Force :*** *Derive equations of motion for a particle moving under central force. What is the form of the equations, when the particle is moving under an attractive inverse square law force $(F=-k/r^2)$.* **(Rohilkhand 1998; Agra 1991)**

Solution : When a particle is moving under central force, then the force is conservative and the motion is in a plane.

Let (r, θ) be the plane coordinates of the particle of mass m.

Kinetic energy $\qquad T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$

Lagrangian $\qquad L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r)$

where $V(r)$ is the potential energy in the central force field.

Now, $\qquad \dfrac{\partial L}{\partial r}=mr\dot{\theta}^2-\dfrac{\partial V}{\partial r},\ \dfrac{\partial L}{\partial \dot{r}}=m\dot{r}$ and $\dfrac{\partial L}{\partial \theta}=0,\ \dfrac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$

Hence equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

or $$m\ddot{r}-mr\dot{\theta}^2+\frac{\partial V}{\partial r}=0 \quad \text{and} \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \qquad ...(i)$$

For attractive inverse square law force $F=-\partial V/\partial r=-k/r^2$, we have equations of motion as

$$m\ddot{r}-mr\dot{\theta}^2+\frac{k}{r^2}=0 \qquad ...(ii\ a)$$

and $$\frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \text{ or } r\ddot{\theta}+2\dot{r}\dot{\theta}=0 \qquad ...(ii\ b)$$

2.9. LAGRANGE'S EQUATIONS IN PRESENCE OF NON-CONSERVATIVE FORCES

When the forces acting on the system consist of non-conservative forces ($\mathbf{f}_i$) in addition to the conservative forces ($\mathbf{F}_i$), then the components of generalized force can be written as [using eq. (20)];

$$G_k = \sum_{i=1}^{N}[\mathbf{F}_i + \mathbf{f}_i]\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} + \sum_i \mathbf{f}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} \quad \text{or} \quad G_k = -\frac{\partial V}{\partial q_k} + G_k' \qquad ...(40)$$

where $G_k' = \sum \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$ are the components of generalized non-potential force resulting from non-conservative forces and $\sum \mathbf{f}_i \cdot \frac{\partial \bar{\mathbf{r}}_i}{\partial q_k} = -\frac{\partial V}{\partial q_k}$ for conservative part [eq. (30)].

Here V is the scalar potential for conservative forces. In such a case, eq. (35) assumes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = G_k' \qquad ...(41)$$

where $L = T - V$.

Eqs. (41) represent the Lagrange's equations in the presence of non-conservative forces.

An example of the non-conservative force is the presence of frictional force, acting on the system. If the frictional force is proportional to the velocity of a particle, then

$$\mathbf{f}_i = -k_i \mathbf{v}_i \qquad ...(42)$$

where k_i is the constant of proportionality for the movement of the ith particle.

We may derive such frictional forces from a function of the form

$$R = \frac{1}{2}\sum_i k_i \mathbf{v}_i^2 = \frac{1}{2}\sum_i k_i\left(v_{xi}^2 + v_{yi}^2 + v_{zi}^2\right) \qquad ...(43)$$

This is known as ***Rayleigh's dissipation function.*** Obviously

$$f_{xi} = -\frac{\partial R}{\partial v_{xi}} = -k_i v_{xi}.$$

Thus $\qquad \mathbf{f}_i = -k_i \mathbf{v}_i = -\nabla_v R \qquad ...(44)$

Hence the component of the generalized force due to the force of friction is given by

$$G_k' = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \nabla_v R \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \nabla_v R \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}$$

$$= -\sum_i \left[\frac{\partial R}{\partial v_{xi}}\frac{\partial v_{xi}}{\partial \dot{q}_k} + \frac{\partial R}{\partial v_{yi}}\frac{\partial v_{yi}}{\partial \dot{q}_k} + \frac{\partial R}{\partial v_{zi}}\frac{\partial v_{zi}}{\partial \dot{q}_k}\right] = -\frac{\partial R}{\partial \dot{q}_k} \qquad ...(45)$$

Thus Lagrange's equation (41) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = -\frac{\partial R}{\partial \dot{q}_k}$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0 \qquad ...(46)$$

It can be proved that the Rayleigh's dissipation function R is equal to one half of the rate of dissipation energy against friction. The work done against friction is

$$dW = -\sum_i \mathbf{f}_i \cdot d\mathbf{r}_i = -\sum_i \mathbf{f}_i \cdot \mathbf{v}_i\, dt = [\sum_i k_i v_i^2]\, dt$$

whence $$\frac{dW}{dt} = \sum_i k_i v_i^2 = 2R \qquad ...(47)$$

This gives the physical interpretation of the Rayleigh's dissipation function.

2.10. GENERALIZED POTENTIAL — *Lagrangian for a Charged Particle Moving in an Electromagnetic Field (Gyroscopic Forces)*

In general, the Lagrange's equations can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad ...(48)$$

For a conservative system, $G_k = -\dfrac{\partial V}{\partial q_k}$ and then the Lagrange's equations in the usual form are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{with} \quad L = T - V \qquad ...(49)$$

However, Lagrange's equations can be put in the form (49), provided the generalized forces are obtained from a function $U(q_k, \dot{q}_k)$, given by

$$G_k = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) \qquad ...(50)$$

In such a case, $L = T - U$ ...(51)

where $U(q_k, \dot{q}_k)$ is called ***velocity dependent potential*** or ***generalized potential.*** This type of case occurs in case of a charge moving in an electromagnetic field.

In S.I. system, two of the Maxwell's field equations are

$$\text{div}\,\mathbf{B} = 0 \quad \text{and} \quad \text{curl}\,\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or $$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad ...(52)$$

where **E** and **B** are electric field and magnetic field vectors respectively.

The force acting on a charge q, moving with velocity **v** in an electric field **E** and magnetic induction **B** is given by

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad ...(53)$$

Since $\nabla \cdot \mathbf{B} = 0$ in eq. (52) and hence **B** can be expressed as curl of a vector *i.e.*,

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad ...(54)$$

where **A** is called the ***magnetic vector potential.*** Substituting for **B** from (54) into the second equation of (52), we get

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\nabla \times \mathbf{A} = 0 \quad \text{or} \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \qquad ...(55)$$

Hence we can express the vector quantity $\left(\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}\right)$ as the gradient of a scalar function ϕ, *i.e.*,

$$\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}=-\nabla\phi \quad \text{or} \quad \mathbf{E}=-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t} \qquad ...(56)$$

Substituting for $\mathbf{E}$ from (56) in (53), we obtain

$$\mathbf{F}=q\left(-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}+\mathbf{v}\times\nabla\times\mathbf{A}\right) \qquad ...(57)$$

The terms in eq. (57) can be written in a more convenient form.

Let us consider the x-component. Since $\nabla\phi=\hat{\mathbf{i}}\frac{\partial\phi}{\partial x}+\hat{\mathbf{j}}\frac{\partial\phi}{\partial y}+\hat{\mathbf{z}}\frac{\partial\phi}{\partial z}$, x-component of $\nabla\phi$ is $\frac{\partial\phi}{\partial x}$. Also,

$$(\mathbf{v}\times\nabla\times\mathbf{A})_x=v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

We add and subtract the term $v_x\partial A_x/\partial x$. Then

$$(\mathbf{v}\times\nabla\times\mathbf{A})_x=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}-v_x\frac{\partial A_x}{\partial x}-v_y\frac{\partial A_x}{\partial y}-v_z\frac{\partial A_x}{\partial z} \qquad ...(58)$$

However, $$\frac{dA_x}{dt}=\frac{\partial A_x}{\partial x}\frac{dx}{dt}+\frac{\partial A_x}{\partial y}\frac{dy}{dt}+\frac{\partial A_x}{\partial z}\frac{dz}{dt}+\frac{\partial A_x}{\partial t}=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}$$

whence $$v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}=\frac{dA_x}{dt}-\frac{\partial A_x}{\partial t} \qquad ...(59)$$

Further $$\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})=\frac{\partial}{\partial x}\left(v_xA_x+v_yA_y+v_zA_z\right)$$

$$=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x} \qquad ...(60)$$

Substituting from (59) and (60) in (58), we get

$$(\nabla\times\nabla\times\mathbf{A})_x=\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t} \qquad ...(61)$$

Hence from eq. (57), the x-component of the force $\mathbf{F}$ is

$$F_x=q\left(-\frac{\partial\phi}{\partial x}-\frac{\partial A_x}{\partial t}+\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}\right)=q\left(-\frac{\partial}{\partial x}(\phi-\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}\right) \qquad ...(62)$$

Since $$\frac{\partial}{\partial v_x}(\mathbf{v}\cdot\mathbf{A})=\frac{\partial}{\partial v_x}\left(v_xA_x+v_yA_y+v_zA_z\right)=A_x$$

and scalar potential ϕ is independent of v_x, we have

$$-\frac{dA_x}{dt}=\frac{d}{dt}\frac{\partial}{\partial v_x}(\phi-\mathbf{v}\cdot\mathbf{A})$$

Therefore $F_x = q\left[-\frac{\partial}{\partial x}(\phi - \mathbf{v}\cdot\mathbf{A}) + \frac{d}{dt}\{\frac{\partial}{\partial v_x}(\phi - \mathbf{v}\cdot\mathbf{A})\}\right]$ ...(63)

We define a ***generalized potential*** U, given by

$$U = q\,(\phi - \mathbf{v}\cdot\mathbf{A}) \qquad ...(64)$$

which is a ***velocity dependent potential*** in the sense of eq. (50). Therefore, eq. (63) takes the form

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \qquad ...(65)$$

The Lagrange's equations (48) in this case take the form

$$(q_k = x,\ \dot{q}_k = \dot{x} = v_x \text{ and } G_k = F_x)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_x}\right) - \frac{\partial T}{\partial x} = F_x \qquad ...(66)$$

Substituting F_x from (66) in (65), we get the Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(T-U)\right) - \frac{\partial}{\partial x}(T-U) = 0$$

or $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad ...(67)$$

where $$L = T - U = T - q\phi + q\,\mathbf{v}\cdot\mathbf{A} \qquad ...(68)$$

Eq. (68) gives the Lagrangian for a charged particle moving in an electromagnetic field.

Note : In Gaussian C.G.S. system **B** is to be replaced by **B**/c in eqs. (52) and (53), where c is the speed of light. Therefore the expression for generalized potential is obtained to be $U = q\phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})$.

Gyroscopic Forces

All the velocity dependent forces, which do not consume power, are called gyroscopic forces. If a charge q is moving with velocity **v** in a magnetic field **B**, then the force acting on the particle

$$\mathbf{F} = q\,(\mathbf{v}\times\mathbf{B})$$

is gyroscopic in nature.

For such a force the power consumed happens to be zero, i.e.,

$$P = \mathbf{F}\bullet\mathbf{v} = q\,(\mathbf{v}\times\mathbf{B})\bullet\mathbf{v} = q\mathbf{v}\bullet(\mathbf{v}\times\mathbf{B}) = q(\mathbf{v}\times\mathbf{v})\bullet\mathbf{B} = 0$$

because for a scalar triple product $(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C} = \mathbf{C}\bullet(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\times\mathbf{A})\bullet\mathbf{B}$.

Thus the velocity dependent magnetic force, given in eq. (53) is an example of gyroscopic force. A gyroscopic force can be incorporated in a generalised potential U similar to the one due to magnetic force, given in eq. (64) with the Lagrangian L [eq. (68)] and Lagrange's equation (67).

2.11. Hamilton's Principle and Lagrange's Equations

In Sec. 2.7, we have used the D'Alembert's principle to deduce Lagrange's equations. This principle uses the idea of virtual work and stems from Newton's second law of motion. These Langrange's equations can be derived by an entirely different way, namely Hamilton's variational principle.

Hamilton's principle : *This principle states that for a conservative holonomic system, its motion from time t_1 to time t_2 is such that the line integral* (known as **action** or **action integral**)

$$S = \int_{t_1}^{t_2} L\, dt \qquad ...(69)$$

with $L = T - V$ has stationary (extremum) value for the correct path of the motion.

The quantity S is called as ***Hamilton's principal function***. The principle may be expressed as

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad ...(70)$$

where δ is the variation symbol.

Lagrange's equation from Hamilton's principle : The Lagrangian L is a function of generalized coordinates q_k's and generalized velocities $\dot{q}_k$'s and time t, *i.e.*,

$$L = L(q_1, q_2, ..., q_k, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_n, t)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be written as

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \qquad ...(71)$$

Integrating both sides from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt$$

But in view of the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

Therefore,
$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = 0 \qquad ...(72)$$

where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$.

Integrating by parts the second term on the left hand side of eq. (72), we get

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = \sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k\, dt \qquad ...(73)$$

At the end points of the path at the times t_1 and t_2, the coordinates must have definite values $q_k(t_1)$ and $q_k(t_2)$ respectively, *i.e.*, $\delta q_k(t_1) = \delta q_k(t_2) = 0$ (Fig. 2.9) and hence

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$$

Therefore, eq. (72) takes the form

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k\, dt = 0$$

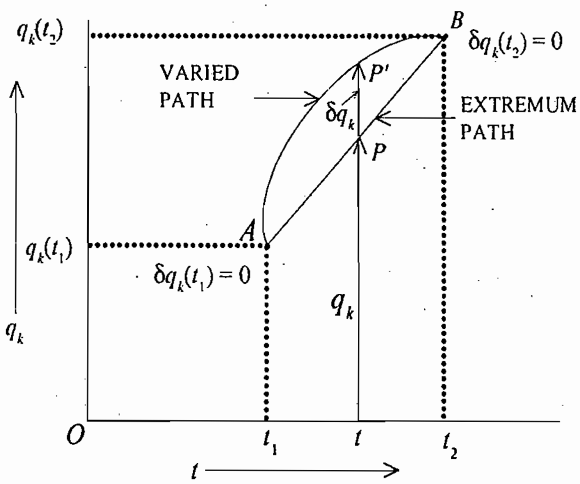


Fig. 2.9 : δ-variation - extremum path

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \delta q_k \, dt = 0 \qquad ...(74)$$

For holonomic system, the generalized coordinates δq_k are independent of each other. Therefore, the coefficient of each δq_k must vanish, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \qquad ...(75)$$

where $k = 1, 2, ..., n$ are the generalized coordinates.

Eqs. (75) are the *Lagrange's equations of motion.*

2.12. SUPERIORITY OF LAGRANGIAN MECHANICS OVER NEWTONIAN APPROACH

In the Newtonian mechanics, the equations of motion involve vector quantities like force, momentum etc. which increase complexity in solving the problems. This approach also cannot avoid constraints present in a problem. These forces of constraints, if not known, make the solution of the problem difficult and even if they are known, the use of rectangular or other commonly used coordinates may make the solution of the problem to be impossible. These drawbacks are removed in the Lagrangian mechanics, where the technique involves scalars, like potential and kinetic energies, instead of vectors. The use of generalized coordinates in the Lagrangian formulation often allows automatically for the constraints. In this formulation, the difficulty in solving the problems is many times much reduced, when any quantity like momentum or (length)^2 is taken as a generalized coordinate instead of rectangular or commonly used coordinates. Further the form of the Lagrange's equations of motion remains invariant under any generalized coordinate transformation.

2.13. GAUGE INVARIANCE OF THE LAGRANGIAN

If L is a Lagrangian for a system of n degrees of freedom, satisfying Lagrange's equations, it can be shown that

$$L' = L + \frac{dF}{dt} \qquad ...(76)$$

also satisfies Lagrange's equations, where F is an arbitrary function :

$$F = F(q_1, q_2, \ldots, q_n, t) \qquad \ldots(77)$$

Such a function is called **Gauge function** for the Lagrangian.

Proof : Time derivative of the function F is

$$\frac{dF}{dt} = \sum_{k=1}^{n} \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t} \qquad \ldots(78)$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \ldots(79)$$

If $L' = L + \frac{dF}{dt}$ satisfies (79), then

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(L + \frac{dF}{dt}\right) = 0 \qquad \ldots(80)$$

Subtracting (79) from (80), we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right)\right] - \frac{\partial L}{\partial q_k}\left(\frac{dF}{dt}\right) = 0 \qquad \ldots(81)$$

If we prove L.H.S. of eq. (81) to be equal to zero, then L' will satisfy the Lagrange's equations.

Now, $\text{L.H.S.} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right)$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right)\right] - \frac{\partial}{\partial q_k}\left[\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \dot{q}_l \frac{\partial}{\partial q_l}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial \dot{q}_k}\right] - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l - \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial q_k}$$

Here, $\frac{\partial F}{\partial \dot{q}_k} = 0$, $\frac{\partial \dot{q}_l}{\partial q_k} = 0$ and $\frac{\partial \dot{q}_l}{\partial \dot{q}_k} = \delta_{lk}$

Hence $\text{L.H.S.} = \frac{d}{dt}\left(\frac{\partial F}{\partial q_k}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$

$$= \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t} + \sum_l \frac{\partial F}{\partial q_l}\dot{q}_l\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= 0$$

Thus $\quad L' = L + \frac{dF}{dt}$ satisfies the Langrange's equations.

2.14. SYMMETRY PROPERTIES OF SPACE AND TIME AND CONSERVATION LAWS

When we consider the motion of a free particle or a closed system in an inertial frame, the space is assumed to be **homogeneous** and **isotropic** and the time to be **homogeneous**. By a closed system we mean a system, not acted by any external force.

The space is said to be homogeneous, if the physical properties of a closed system are not affected by an arbitrary displacement of the origin of the frame of reference. This means that in order to describe the state of motion of a closed system, any point in space is equivalent to any other point of the space.

The space is said to be isotropic, if the physical properties of closed system are not changed for arbitrary rotation about the origin of the frame of reference. Therefore, for the description of a closed system, every direction in space is equivalent and any direction for the Cartesian axes can be used.

The time is said to be homogeneous, if the physical properties of a closed system are not affected by an arbitrary displacement of the origin of time. Hence any moment of time can be taken to describe a closed system.

The homogeneity and isotropy of space and homogeneity of time imply the invariance of the physical properties of a closed system under certain operations, known as *symmetry operations*. These operations leave the configuration and states of motion unchanged. The homogeneity of space correspond to an arbitrary translation (symmery operation), isotropy of space to an arbitrary rotation and homogeneity of time to an arbitrary shifting of the time or time-translation.

We can describe a closed system by its Lagrangian. This Lagrangian must be invariant under the operations of translation and rotation in space and time-shifting. These symmetry operations on the Lagrangian have very important consequences. Each symmetry operation results in a conservation law, representing a physical quantity or an integral of motion to be conserved. This physical quantity is additive, *i.e.*, the value of the physical quantity for the entire system is the sum of its values for different parts of the system.

Thus every symmetry in the Lagrangian corresponds to a conservation law. Homogeneity of space results in the conservation law of linear momentum, isotropy of space in the conservation law of angular momentum and homogeneity of time in the conservation law of energy. These conservation laws have been obtained in the following discussion.

(1) Homogeneity of Space and Conservation of Linear Momentum : The homogeneity of space implies that the Lagrangian of a closed system is not changed by an arbitrary translation of all the particles of the system. In Carterian coordinates, a small arbitrary translation of the coordinate of the i^{th} particle can be written as

$$\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i \text{ or } \mathbf{r}_i \to \mathbf{r}_i + \boldsymbol{\varepsilon}$$

where $\delta\mathbf{r}_i = \boldsymbol{\varepsilon}$ is constant small translation for each particle.

Now corresponding to this change in coordinate, the change δL in L is

$$\delta L = \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \cdot \boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon} \cdot \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \tag{82}$$

However for any arbitrary translation $\boldsymbol{\varepsilon}$, $\delta L = 0$. This means

$$\sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0 \tag{83}$$

Langrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) - \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

Hence for all particles of the system

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) - \sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

Using (83), $$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = 0 \quad \text{or} \quad \frac{d}{dt}\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = 0 \qquad \text{...(84)}$$

But $$L = T - V = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - V(\mathbf{r}_1, \ldots\ldots, \mathbf{r}_N)$$

$\therefore$ $$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i = \mathbf{p}_i, \text{ linear momentum of } i^{\text{th}} \text{ particle,} \qquad \text{...(85)}$$

Hence, from (84),

$$\frac{d}{dt}\left(\sum_i \mathbf{p}_i\right) = 0 \quad \text{or} \quad \sum_i \mathbf{p}_i = \text{Constant} \qquad \text{...(86)}$$

where $\sum_i \mathbf{p}_i = \mathbf{P}$ is the total linear momentum of the system.

Thus, *the total linear momentum of a closed system is conserved due to the homogeneity of the space.*

(2) Isotropy of Space and Conservation of Angular Momentum : Due to the isotropy of space, the Lagrangian of a closed system remains unchanged under arbitrary rotation. Let us consider an arbitrary infinitesimal rotation of the system about some direction, say Z-direction. Therefore the position vector $\mathbf{r}_i$ due to rotation $\delta\theta$ will change by (Fig. 2.10)

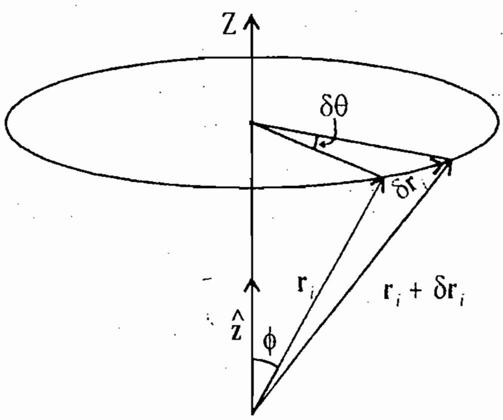


Fig. 2.10

$$\delta \mathbf{r}_i = \delta\theta\, \hat{\mathbf{z}} \times \mathbf{r}_i$$

$$(\because\ |\delta \mathbf{r}_i| = r_i \sin\phi\, \delta\theta = |\hat{\mathbf{z}} \times \mathbf{r}_i|\delta\theta\ ) \qquad \text{...(87)}$$

The change in velocity vector $\mathbf{v}_i$ due to arbitrary rotation $\delta\theta$ is

$$\delta \mathbf{v}_i = \delta\theta\, \hat{\mathbf{z}} \times \mathbf{v}_i \qquad \text{...(88)}$$

Now, $$L = L\ (\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \ldots\ldots)$$

Hence $$\delta L = \sum_i \left(\frac{\partial L}{\partial \mathbf{r}_i}\cdot \delta \mathbf{r}_i + \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot \delta \dot{\mathbf{r}}_1\right)$$

We use $$\dot{\mathbf{r}}_1 = v_i \text{ and } \frac{\partial L}{\partial \mathbf{r}_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = \dot{\mathbf{p}}_i$$

Therefore, $$\delta L = \sum_i (\dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i + \mathbf{p}_i \cdot \delta \mathbf{v}_i) = \delta\theta \sum_i \left[\dot{\mathbf{p}}_i \cdot (\hat{\mathbf{z}} \times \mathbf{r}_i) + \mathbf{p}_i \cdot (\hat{\mathbf{z}} \times \mathbf{v}_i)\right]$$

$$= \delta\theta \sum_i \left[\frac{d}{dt}(\mathbf{r} \times \mathbf{p}_i) \cdot \hat{\mathbf{z}}\right] \quad \text{(using the property of scalar triple product)}$$

$$= \delta\theta\,\hat{\mathbf{z}} \cdot \frac{d}{dt}\left[\sum_i (\mathbf{r}_i \times \mathbf{p}_i)\right] \qquad ...(89)$$

As $\delta\theta$ is arbitrary and $\delta L = 0$ for arbitrary rotation, we obtain

$$\frac{d}{dt}\left[\sum_i (\mathbf{r}_i \times \mathbf{p}_i)\right] = 0 \;\text{ or }\; \sum_i (\mathbf{r}_i \times \mathbf{p}_i) = \text{Constant} \qquad ...(90)$$

where $\sum_i (\mathbf{r}_i \times \mathbf{p}_i) = \mathbf{J}$ is the total angular momentum of the system.

Thus *the total angular momentum of a closed system is conserved due to the isotropy of space.*

(3) Homogeneity of Time and Conservation of Energy : The homogeneity of time implies that the Lagrangian is invariant under time-translation similar to space-translation.

For arbitrary small time-translation δt, the change in Langragian is

$$\delta L = \frac{\partial L}{\partial t}\delta t \qquad ...(91)$$

But for $t \to t + \delta t$, $\delta L = 0$. Hence for arbitrary δt,

$$\frac{\partial L}{\partial t} = 0 \qquad ...(92)$$

i.e., L does not depend on time t explicitly.

Thus, $$L = L\,(\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \ldots\ldots) \qquad ...(93)$$

Hence, $$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial \mathbf{r}_i}\cdot\dot{\mathbf{r}}_i + \sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\ddot{\mathbf{r}}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right)\cdot\dot{\mathbf{r}}_i + \sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\ddot{\mathbf{r}}_i = \frac{d}{dt}\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right)$$

Thus $$\frac{d}{dt}\left[\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L\right] = 0 \text{ or } \left[\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L\right]\; \sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L = \text{Constant} \qquad ...(94)$$

But $$L = T - V = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - V(r_1, r_2, \ldots.)$$

Hence, $$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i \text{ and } \sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i^2 = 2T \qquad ...(95)$$

Therefore, from (94)

$$2T - L = \text{Constant} \quad \text{or} \quad T + V = \text{Constant} \qquad ...(96)$$

Thus *the total energy is conserved for a closed system due to homogeneity of time.*

2.15. INVARIANCE UNDER GALILEAN TRANSFORMATION

Consider two inertial frames S and S'. The frame S' is moving with constant velocity $\mathbf{v}_0$ relative to frame S. If $\mathbf{r}_i$ and $\mathbf{r}_i'$ are the position vectors of i^{th} particle in frames S and S' respectively, then the two frames are connected by Galilean transformation with the transformation equations given by

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{v}_0\, t \qquad \text{[eq. (9), Chapter 1]} \;...(97)$$

with the implicit assumption $t' = t$.

If v and v_i' be the velocities of the particle in two frames, then

$$\mathbf{v}_i' = \mathbf{v}_i - \mathbf{v}_0 \qquad \text{[eq. (10), Chapter 2] ...(98)}$$

Suppose the particle is moving under the action of external field force due to ordinary potential V. The Lagrangian of the particle in frame S is given by

$$L = \frac{1}{2}mv_i^2 - V \qquad \text{...(99)}$$

where V is the potential function in S. This V is normally a function of difference of position vectors of two particles $\mathbf{r}_2 - \mathbf{r}_1$ and this V will remain the same in S' frame, because $\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1$ from eq. (97).

The Lagrangian L' in frame S' is given by

$$L' = \frac{1}{2}mv'^2 - V = \frac{1}{2}m|\mathbf{v}_i - \mathbf{v}_0|^2 - V = \frac{1}{2}mv_i^2 - V - m\mathbf{v}_i \cdot \mathbf{v}_0 + \frac{1}{2}mv_0^2$$

Thus
$$L' = L + \frac{d}{dt}\left(\frac{1}{2}mv_0^2 t - m\mathbf{v}_0 \cdot \mathbf{r}_i\right) = L + \frac{dF(\mathbf{r}_i, t)}{dt} \qquad \text{...(99)}$$

where
$$F(\mathbf{r}_i, t) = \frac{1}{2}mv_0^2 t - m\mathbf{v}_0 \cdot \mathbf{r}_i \qquad \text{...(100)}$$

Hence through the gauge function $F(\mathbf{r}_i, t)$ both L and L' must satisfy the same Lagrange equations of motion [see Sec. 2.13]. Thus the from of Lagrange equation retain the same form in S' frame *i.e.*, **the Lagrange equations are invariant under Galilean transformation.**

Further from (97) and (98)

$$\mathbf{r}_i - \mathbf{r}_i' = (\mathbf{v}_i - \mathbf{v}_i')\,t \ \text{ or } \ \mathbf{r}_i - \mathbf{v}_i t = \mathbf{r}_i' - \mathbf{v}_i' t$$

Hence for the entire system,

$$\sum_i m_i(\mathbf{r}_i - \mathbf{v}_i t) = \sum_i m_i(\mathbf{r}_i' - \mathbf{v}_i' t) \qquad \text{...(101)}$$

But
$$\sum_i m_i \mathbf{r}_i = M\mathbf{R} \ \text{ and } \ \sum_i m_i \mathbf{v}_i = \mathbf{P} \qquad \text{...(102)}$$

where $M = \sum_i m_i$, total mass of the system, $\mathbf{R}$ the position vector of the centre of mass and $\mathbf{P}$ the total linear momentum of the system.

Thus
$$M\mathbf{R} - \mathbf{P}\,t = M\,\mathbf{R}' - \mathbf{P}'\,t \qquad \text{...(103)}$$

In other words, $M\mathbf{R} - \mathbf{P}\,t$ is a constant of motion which is in fact obtained because of the Galilean invariance of Newton's equations of motion.

Some More Solved Examples

Ex. 1. ***Motion under gravity.*** *Write down the Lagrange's equation of motion for a particle of mass m falling freely under gravity near the surface of earth.* **(Rohilkhand, 1997)**

Solution : If X and Y axes are taken on the surface of the earth and Z-axis vertically upward, then the kinetic energy of the freely falling particle of mass m is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Its potential energy $V = mgz$

Therefore, Lagrangian $L = T - V$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

For $q_k = x, y, z$

$$\frac{dL}{d\dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}, \quad \frac{dL}{dx} = 0 = \frac{\partial L}{\partial y} \text{ and } \frac{dL}{dz} = mg$$

Hence Lagrange's equations are

$$\frac{d}{dt}(m\dot{x}) = 0, \quad \frac{d}{dt}(m\dot{y}) = 0, \quad \frac{d}{dt}(m\dot{z}) + mg = 0$$

or $$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} + g = 0$$

Note : The above equations hold good for the case of projectile as well as for a particle falling freely vertically under gravity.

In the later case,

$$T = \frac{1}{2}m\dot{z}^2, \; V = mgz \text{ and } L = \frac{1}{2}m\dot{z}^2 - mgz$$

In such a case, the Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \text{ or } m\ddot{z} + mg = 0 \text{ or } \ddot{z} + g = 0$$

Ex. 2. *A point mass moves in a vertical plane along a given curve in a gravitational field. The equation of motion in parametric form is*

$$x = x(s), \quad z = z(s)$$

Write down the Lagrange's equations. **(Rohilkhand 1996)**

Solution : Here $\dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\cdot\frac{ds}{dt} = x'\dot{s}\left(x' = \frac{dx}{ds}, \dot{s} = \frac{ds}{dt}\right)$ and $\dot{z} = \frac{dz}{dt} = \frac{dx}{ds}\frac{ds}{dt} = z'\dot{s}\left(z' = \frac{dz}{ds}\right)$

Kinetic energy $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2}\left(x'^2 + z'^2\right)\dot{s}^2$

Potential energy $V = mgz$

where Z-axis is assumed to be vertical upward from the earth.

Therefore, $L = T - V = \frac{1}{2}m\left(x'^2 + z'^2\right)\dot{s}^2 - mgz$

Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

Here $$\frac{\partial L}{\partial \dot{s}} = \frac{1}{2}\left(x'^2+z'^2\right)2\dot{s} = \left(x'^2+z'^2\right)\dot{s}$$

and $$\frac{\partial L}{\partial s} = \left(\frac{1}{2}m\dot{s}^2(2x'\,x''+2z'\,z''\right) - mg\,\frac{\partial z}{\partial s} = m\dot{s}^2(x'\,x''+z'\,z'') - mgz'$$

Hence $$\frac{d}{dt}\left[m\left(x'^2+z'^2\right)\dot{s}\right] - m\dot{s}^2(x'\,x''+z'\,z'') + mgz' = 0$$

This is the desired Lagrange's equation.

Ex. 3. *Fig. 2.11 (a) shows an inclined plane of mass m_1. It is sliding on a horizontal smooth surface and a body of mass m_2 is sliding on its smooth inclined surface. Derive the equations of motion of the body and the inclined plane.*

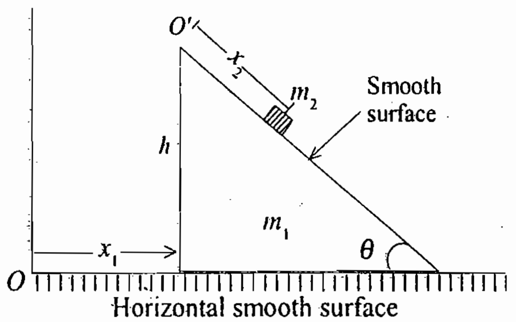


Fig. 2.11 (a)

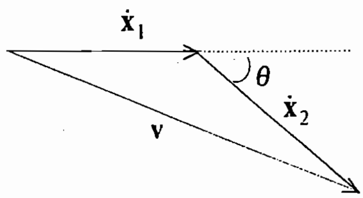


Fig. 2.11 (b)

Solution : Here m_1 slides on the horizontal smooth surface and m_2 slides on the smooth inclined plane of mass m_1. Thus the system has two degrees of freedom and hence we need two generalized coordinates. Let x_1 and x_2 represent the displacements of m_1 and m_2 from O and O' respectively.

Velocity of m_1 with respect to $O = \dot{\mathbf{x}}_1$

Velocity of m_2 with respect to $O' = \dot{\mathbf{x}}_2$

Velocity of m_2 with respect to $O = \mathbf{v} = \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2$ [Fig. 2.10 (*b*)]

or $$v^2 = \dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2\cos\theta$$

Kinetic energy of the whole system as observed from O is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2 v^2$$

$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2\cos\theta\right)$$

Potential energy of the system is due to the position of the mass m_2 (with respect to horizontal smooth surface) only.

Hence,

$$V = m_2 g\,(\,h - x_2\sin\theta\,)$$

$$\therefore \quad L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2\cos\theta\right) - m_2 g\left(h - x_2\sin\theta\right) \quad ...(i)$$

Lagrange's equations for x_1 and x_2 coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$\therefore$ $$m_1\ddot{x}_1 + m_2(\ddot{x}_1 + \ddot{x}_2 \cos\theta) = 0 \qquad \text{...}(ii)$$

and $$m_2(\ddot{x}_2 + \ddot{x}_1 \cos\theta) - m_2 g \sin\theta = 0 \qquad \text{...}(iii)$$

Solving eqs. (*ii*) and (*iii*), we get

$$\ddot{x}_1 = \frac{-g\sin\theta\cos\theta}{\dfrac{m_1+m_2}{m_2} - \cos^2\theta} \qquad \text{...}(iv)$$

and $$\ddot{x}_2 = \frac{g\sin\theta}{1-\dfrac{m_2\cos^2\theta}{m_1+m_2}} \qquad \text{...}(v)$$

Eqs. (*iv*) and (*v*) are the equations of motion of the inclined plane and sliding body respectively.

Ex. 4. *A particle of mass m moves on a plane in the field of force given by (in polar coordinates)*

$$\bar{F} = -kr\cos\theta\ \hat{\mathbf{r}}$$

where k is constant and $\hat{\mathbf{r}}$ *is the radial unit vector.*

(*a*) *Will the angular momentum of the particle about the origin be conserved ? Justify your statement.*

(*b*) *Obtain the differential equation of the orbit of the particle.* **(Agra 1995)**

Solution : $$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$\frac{\partial T}{\partial \theta} = 0,\quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta},\quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2 \quad \text{and} \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

(*a*) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = G_\theta.$$

Since there is no transverse force, $G_\theta = 0$. Therefore, $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$. Hence the angular momentum about the origin is conserved.

(b) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = G_r \quad \text{or} \quad m\ddot{r} - mr\dot{\theta}^2 = -kr\cos\theta$$

which is the differential equation of motion of the orbit of the particle.

Ex. 5. *A cylinder of radius a and mass m rolls down an inclined plane making an angle* θ *with the horizontal. Set up the Lagrangian and find the equation of motion.*

Solution : Let the cylinder start to roll from O so that $x = a\phi$ (Fig. 2.12) and hence $\dot{x} = a\dot{\phi}$.

Now, $$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{ma^2}{2}\frac{\dot{x}^2}{a^2} = \frac{3}{4}m\dot{x}^2$$

(because $I = \frac{ma^2}{2}$ for cylinder and $\omega = \dot{\phi} = \frac{\dot{x}}{a}$.)

and $\quad V = mg\,(s-x)\sin\theta + mg\,a\cos\theta$

$\therefore \quad L = \frac{3}{4} m\dot{x}^2 - mg\,(s-x)\sin\theta - mg\,a\cos\theta$

where s is the lenght of the inclined plane.

$\therefore$ Equation of motion is

$$\frac{3}{2} m\ddot{x} - mg\sin\theta = 0.$$

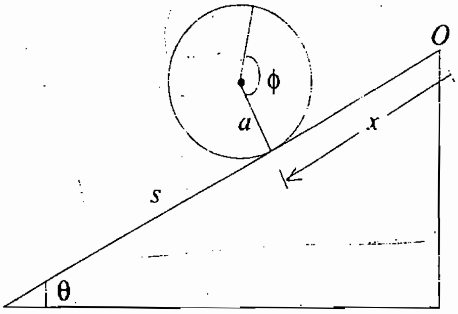


Fig. 2.12

Ex. 6. *A bead slides on a smooth rod which is rotating about one end in a vertical plane with uniform angular velocity* ω [Fig. 2.13] . *Show that the equation of motion is* $m\ddot{r} = mr\omega^2 - mg\sin\omega t$.

Solution : $\quad T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$, and $V = mgy = mgr\sin\theta$

$\therefore \quad L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta$

Here, $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\sin\theta$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

From Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0.$$

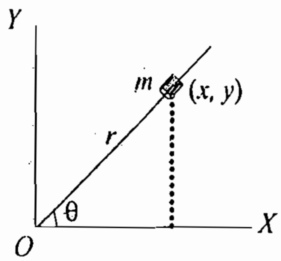


Fig. 2.13

we have $\quad m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0$ or $m\ddot{r} - mr\omega^2 + mg\sin\omega t = 0$

where $\quad \dot{\theta} = \omega$ and $\theta = \omega t$.

Ex. 7. *A pendulum of mass m is attached to a block of mass M. The block slides on a horizontal frictionless surface* (Fig. 2.14). *Find the Lagrangian and equation of motion of the pendulum. For small amplitude oscillations, derive an expression for periodic time.*

Solution : Let at any time t the coordinates of M and m be $(x_1, 0)$ and (x_2, y_2) respectively.

Here, $\quad x_2 = x_1 + l\sin\theta$ and $y_2 = -l\cos\theta$

$$T = \frac{1}{2} M\dot{x}_1^2 + \frac{1}{2} m\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2} M\dot{x}_1^2 + \frac{1}{2} m\left(\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta\right)$$

(because $\dot{x}_2 = \dot{x}_1 + l\cos\theta\,\dot{\theta}$ and $\dot{y}_2 = l\sin\theta\,\dot{\theta}$)

$V = -mgl\cos\theta$

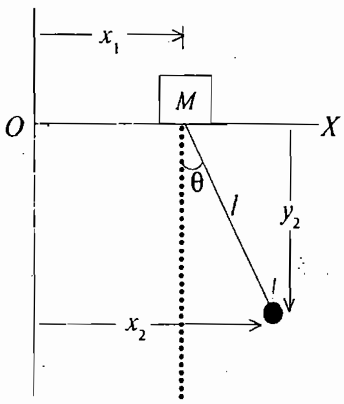


Fig. 2.14

Hence, $\quad L = T - V = \frac{1}{2}(M+m)\dot{x}_1^2 + \frac{1}{2} ml^2\dot{\theta}^2 + ml\dot{x}_1\dot{\theta}\cos\theta + mgl\cos\theta$

Here, $\quad \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial \dot{x}_1} = (M+m)\dot{x}_1 + ml\dot{\theta}\cos\theta$,

$$\frac{\partial L}{\partial \theta} = ml\left(\dot{x}_1\dot{\theta} + g\right)(-\sin\theta) \text{ and } \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\,\dot{x}_1\cos\theta$$

Equation of motion in θ is

$$ml^2\ddot{\theta} + m_1\ddot{x}_1\cos\theta + ml(-\sin\theta)\dot{\theta}\,\dot{x}_1 - ml(-\sin\theta)\,\dot{x}_1\,\dot{\theta} + mgl\sin\theta = 0$$

or $$ml^2\ddot{\theta} + ml\cos\theta\,\ddot{x}_1 + mgl\sin\theta = 0$$

If θ is small, $\sin\theta \cong \theta$ and also $\cos\theta \cong 1$, then

$$ml^2\ddot{\theta} + ml\ddot{x}_1 + mgl\theta = 0$$

or $$\ddot{\theta} + \frac{\ddot{x}_1}{l} + \frac{g}{l}\theta = 0 \qquad ...(i)$$

Equation of motion in x_1 is

$$(M + m)\ddot{x}_1 + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

For small θ, ($\cos\theta \cong 1$, $\sin\theta \cong \theta$ and $\dot{\theta}^2\theta$ is negligible)

$$(M + m)\ddot{x}_1 + ml\ddot{\theta} = 0 \qquad ...(ii)$$

From equations (*i*) and (*ii*), we have

$$\ddot{\theta} - \frac{m\ddot{\theta}}{M+m} + \frac{g}{l}\theta = 0$$

Hence $$\ddot{\theta} = -\left[\frac{M+m}{M}\right]\frac{g}{l}\theta \qquad ...(iii)$$

This is the equation of simple harmonic motion whose period is given by

$$T = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{M}{M+m}}$$

Ex. 8. *In a* ***spherical pendulum*****, *a small bob (particle) of mass m is constrained to move on a smooth spherical surface, say of radius R, R being the length of the pendulum (Fig. 2.15). Set up the Lagrangian for the spherical pendulum and obtain the equation of motion.* **(Rohilkhand 1994)**

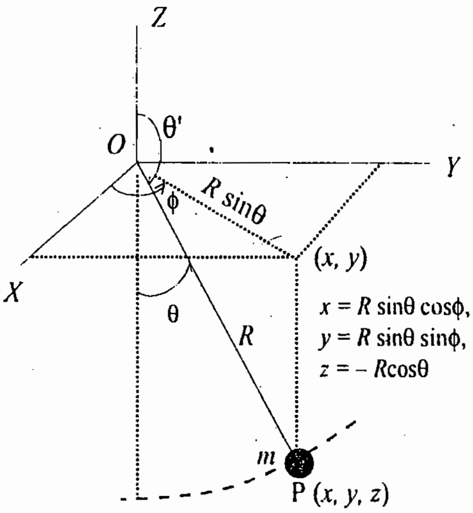


Fig. 2.15

Solution : The constraint of motion is holonomic and the constraint equation is

$$x^2 + y^2 + z^2 - R^2 = 0$$

We take θ and ϕ as the generalized coordinates. The cartesian coordinates of the bob P are

$$x = R\sin\theta'\cos\phi,\ y = R\sin\theta'\sin\phi, \text{ and } z = R\cos\theta'$$

or $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, and $z = -R\cos\theta$ (as $\theta' = \pi - \theta$)

Now $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\cos\theta$

* If spherical pendulum moves in a vertical plane, it constitutes a simple pendulum.

Equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0$$

i.e., $$mR^2\,\ddot{\theta}-mR^2\dot{\phi}^2\sin\theta\cos\theta+mg\,R\sin\theta=0 \text{ and } \frac{d}{dt}(mR^2\dot{\phi}\sin^2\theta)=0$$

or $$\ddot{\theta}-\tfrac{1}{2}\sin 2\theta\dot{\phi}+\frac{g}{R}\sin\theta=0 \text{ and } mR^2\sin^2\theta\,\dot{\phi}=\text{constant.}$$

These are the equations of motion for spherical pendulum.

Ex. 9. *A particle moves in a plane under the influence of a force, acting towards a centre of force, whose magnitude is*

$$F=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2\ddot{r}\,r}{c^2}\right)$$

where r is the distance of the particle from the centre of force. Find the generalized potential that will result in such a force and from that the Lagrangian for the motion in a plane. **(Rohilkhand 1986)**

Solution : For velocity dependent potential

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q}=G_k$$

where $G_k=-\dfrac{\partial U}{\partial q_k}+\dfrac{d}{dt}\left(\dfrac{\partial U}{\partial \dot{q}_k}\right)$ is the generalized force and $U(q_k, \dot{q}_k)$ is the generalized potential. The generalized force for $q_k = r$ is

$$G_r=-\frac{\partial U}{\partial r}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right)$$

Here $$G_r=F=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2\ddot{r}\,r}{c^2}\right)=\frac{1}{r^2}-\frac{\dot{r}^2}{c^2r^2}+\frac{2\ddot{r}}{c^2r}=\frac{1}{r^2}+\frac{\dot{r}^2}{c^2r^2}+\frac{2\ddot{r}}{c^2r}-\frac{2\dot{r}^2}{c^2r^2}$$

$$=-\frac{\partial}{\partial r}\left(\frac{1}{r}+\frac{\dot{r}^2}{c^2r}\right)+\frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{r}+\frac{\dot{r}^2}{c^2r}\right)=-\frac{\partial U}{\partial r}+\frac{d}{dt}\frac{\partial U}{\partial \dot{r}}$$

where $U=\dfrac{1}{r}\left(1+\dfrac{\dot{r}^2}{c^2}\right)$. This is the expression for the generalized potential.

Questions

1. (*a*) What are constraints ? Classify the constraints with some examples.

(Agra 2004, 1998, 95; Kanpur 98; Garwal 98, 93)

(*b*) What type of difficulties arise due to the constraints in the solution of mechanical problems and how these are removed ? **(Agra 1998, 93)** (*c*) Write a note on 'holonomic and non-holonomic constraints'with two examples of each type. **(Kanpur 1997; Garwal 99; Gorakhpur 95; Agra 2002)**

2. What do you understand by holonomic and nonholonomic constraints ? Obtain differential equations of constraints in case of a disc of radius R, rolling on the horizontal xy plane and constrained to move so that plane of the disc is always vertical. **(Kanpur 1996)**

3. Write down the generalized coordinates for a simple pendulum and exptain why cartesian coordinates are not suitable here. **(Gorakhpur 1995)**

4. What are generalized coordinates and generalized velocities ? Set up the Lagrangian for a spherical pendulum. **(Ruhelkhand 1994)**

5. (*a*) State and prove D 'Alembert's principle. **(Garwal 1996)**

(*b*) What is D 'Alembert's principle ? Give its one application. **(Kanpur 1997)**

(*c*) Derive Lagrange's equations from D 'Alemberts principle. **(Kanpur 2001)**

6. What is D 'Alembert's principle ? Derive Lagrange's equations of motion from it for conservation system. How will the result be modified for non-conservative system ? **(Agra 2001, 2000; Meerut 2001; Garwal 1999; Bundelkeband 97)**

7. (*a*) Discuss the superiority of Lagrangian approach over Newtonian approach. **(Rohilkhand 1994)**

(*b*) Define Lagrangian function for conservative and non-conservative systems.

8. Explain what is meant by generalized coordinates, holonomic constraints and the principle of virtual work. Obtain the *D* 'Alembert's principle in generalized coordinates and use it to obtain the Lagrange's equations of motion for a holonomic conservative system. **(Agra 1991, 89, 87, 86)**

9. Obtain Lagrange's equations and show that these can be written as $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$, **(Kanpur 1997)**

10. Derive Lagrangian expression for a charged particle in an electromagnetic field. **(Agra 2001, 1999, 95)**

11. Define Raylish's dissipation function for frictional forces, which are proportional to velocities and obtian Lagrange's equations. Also give a physical interpretation of this function. **(Garwal 1995)**

12. What is Hamilton's principle ? Derive Lagrange's equation of motion from it. Find the Lagrangian equation of moton for a L-C circuit and also deduce the time period. **(Agra 1995)**

13. What is Hamilton's principle ? Derive Lagrange's equation with its helps for a conservative system. Derive equation of motion for a particle moving under central force. **(Agra 2002, 1999; Rohilkhand, 96; Merrut, 95)**

14. Set up the Lagrangian and obtain the Lagrange's equation for a simple pendulum. Deduce the formula for its time period. **(Agra 1994, 91)**

15. Prove that if the transformation equations are given by

$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, q_2, \ldots\, q_n)$$

which do not involve time explicitly, then the kinetic energy can be written as

$$T = \sum_{\alpha=1}^{n} \sum_{\beta-1}^{n} C_{\alpha\beta}\, \dot{q}_\alpha \dot{q}_\beta$$

where $C_{\alpha\beta}$ are functions of q_α.

16. Write the Lagrangian and equation of motion for the following systems :

(*a*) A mass m is suspended to a spring of force constant k and allowed to swing vertically.

(*b*) A uniform rod of mass m and length a, pivoted at a distance l from the centre of mass, swings in a vertical plane.

Ans. (*a*) $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 ; m\ddot{x} + kx = 0$

(*b*) $L = \frac{1}{2} I\dot{\theta}^2 - mgl\,(1 - \cos\theta);\ I\ddot{\theta} + mgl\sin\theta = 0$

where $I = \dfrac{ma^2}{12} + m\left(\dfrac{l}{2}\right)^2 = \dfrac{m}{4}\left(\dfrac{a^2}{3} + l^2\right).$

17. The force on a particle of mass m and charge e, moving with a velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, is given by

$\mathbf{F} = \mathbf{e}\left(\mathbf{E} + \dfrac{\mathbf{v} \times \mathbf{B}}{\mathbf{c}}\right)$, where c is the speed of light.

If the fields are expressed by the relations : $\mathbf{E} = -\nabla\phi - \dfrac{1}{c}\dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$, ϕ and $\mathbf{A}$ being the scalar and vector potentials respectively, prove that the Lagrangian for the charged particle is

$$L = \frac{1}{2}mv^2 + \frac{e}{c}(\mathbf{A}\,.\,\mathbf{v}) - e\phi$$ **(Garwal 1996)**

Problems

[SET- I]

1. Determine the number of degrees of freedom in the following cases :

(1) A particle moving on a space curve, (2) 4 particles moving freely in space, (3) 4 particles moving freely in a plane, (4) three particles connected by three rigid massless rods, (5) two particles moving on a space curve and having constant distance between them, (6) a rigid body moving parallel to a fixed plane surface, (7) a rigid body having two points fixed.

Ans : (1) 1, (2) 12, (3) 8, (4) 6, (5) 1, (6) 3, (7) 1.

2. Determine the number of degrees of freedom for a massless rod, moving freely in space with a particle which is constrained to move on the rod.

Ans : 4.

3. Two particles are connected by a rod of variable length $l = f(t)$. What is the nature of the constraint ?

Ans : The constraint is $|\mathbf{r}_1 - \mathbf{r}_2|^2 = f^2(t)$ which is holonomic and rheonomic.

4. A lever ABC has weights W_1 and W_2 at distances l_1 and l_2 from the fixed support B (Fig. 2.16).

Apply the principle of virtual work to prove that the condition of the equilibrium is $\dfrac{W_1}{W_2} = \dfrac{l_1}{l_2}$

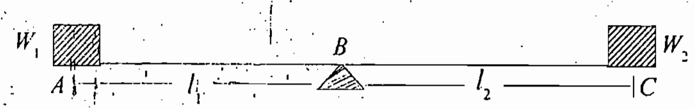


Fig. 2.16

5. Use D'Alembert's principle to determine the equation of motion of a simple pendulum.

   Ans : $\ddot{\theta} + (g/l)\,\theta = 0.$

6. An incline that makes an angle α with the horizontal is given a horizontal acceleration a in the vertical plane of the incline so that the sliding of a frictionless block on the incline is prevented. Apply D'Alembert's principle to obtain the value of a.

   Ans : $a = g \tan \alpha$.

7. A ladder slides down a smooth wall and smooth floor [Fig. 2.17]. Set up the Lagrangian for the system and deduce the equation of motion.

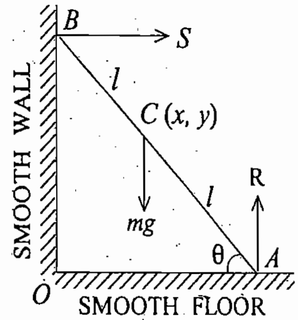


Fig. 2.17

   Ans : $L = \frac{1}{2} m\,(l^2 + K^2)\dot{\theta}^2 - mgl \sin\theta;\ \ \ddot{\theta} = lg \cos\theta / (l^2 + K^2)$,

   where K is the radius of gyration.

   [**Hint :** $T = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\omega^2$.

   Here, $x = a \cos\theta$, $y = a \sin\theta$, $I = mK^2$ and $\omega = \dot{\theta}$]

8. (*a*) Two point masses m are connected by a rod of length $2a$, the centre of which moves on a circle of radius r. Write down kinetic energy in generalized coordinates.

   Ans : $m\,(r^2\dot{\theta}^2 + a^2\dot{\phi}^2)$.

   (b) Obtain the Lagrangian of a particle moving in a force free field in spherical coordinate and cylindrical coordinate systems.

   Ans : $\frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$; $\frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$.

9. Two particles of masses m_1 and m_2 are located on a frictionless double incline and connected by an inextensible massless string passing over a smooth peg (Fig. 2.18). Use the principle of virtual work to show that for equilibrium, we must have

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{m_2}{m_1}$$

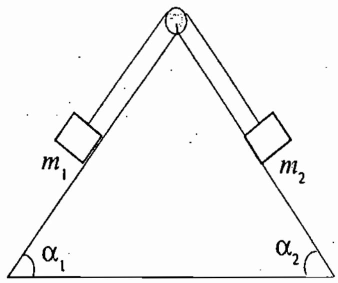


Fig. 2.18

   where α_1 and α_2 are the angles of the incline.

   Apply D'Alembert's principle to describe the motion. **(Garwal 1992)**

   Ans : $a = \dfrac{m_1 g \sin\alpha_1 - m_2 g \sin\alpha_2}{m_1 + m_2}$, where for $m_1 g \sin\alpha_1 > m_2 g \sin\alpha_2$, the particle 1 goes down and particle 2 goes up.

10. A bead is sliding on a uniform rotating rod in a force-free space, find its equation of motion.

    Ans : $m\ddot{r} - mr\omega^2 = 0$.

11. A block of mass m is pulled up as the mass M moves down as shown in Fig. 2.19. The coefficient of friction between the incline and m is μ. Find the acceleration of m and M. Assume the pulley P as frictionless.

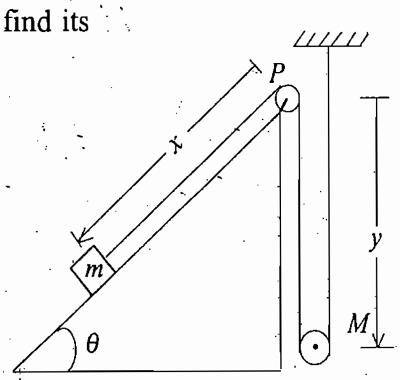


Fig. 2.19

    Ans : $\ddot{x} = (-\frac{1}{2} Mg + mg \sin\theta - \mu\, mg \cos\theta)/(m + M/4)$;

    $\ddot{y} = \ddot{x}/2$.

12. A bead is constrained to move along a smooth conical spiral [Fig. 2.20] defined by coordinates ρ, ϕ and z, related as

$\rho = az$ and $\phi = -bz$

where a and b are constants. Gravity force is acting in the negative z-direction. Set up the Lagrangian for the system.

Ans : $L = \frac{1}{2} m \dot{z}^2 (a^2 + a^2 b^2 z^2 + 1) - mgz.$

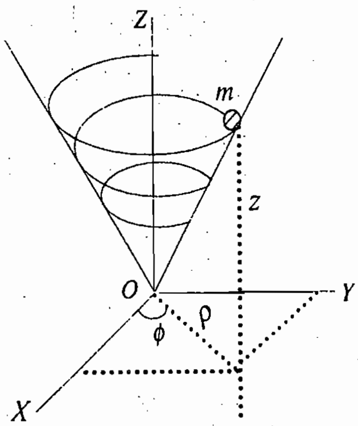


Fig. 2.20

13. Discuss the motion of a particle of mass m moving on the surface of a cone of half angle ϕ and subject to gravitational force only, as shown in Fig. 2.21.

Ans : Equations of motion are

$\ddot{r} - r\dot{\theta}^2 \sin^2 \phi + g \cos \phi \sin \phi = 0$; $J_z = mr^2 \dot{\theta}$ = constant.

[**Hint :** $L = \frac{1}{2} m (\dot{r}^2 \operatorname{cosec}^2 \phi + r^2 \dot{\theta}^2) - mgr \cot \phi$]

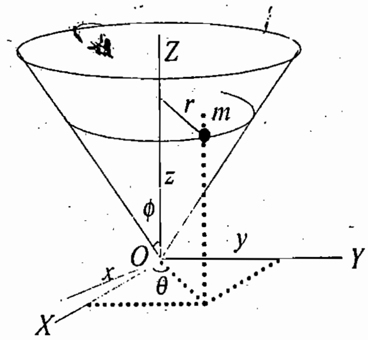
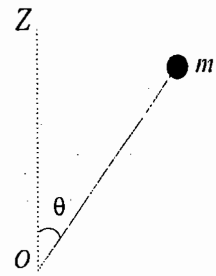


Fig. 2.21

14. In an inverted pendulum, particle of mass m is attached to a rigid massless rod of length l [Fig. 2.22]. If the vertical motion of the point O is represented by the equation $z = a \sin \omega t$, set up the Lagrangian and obtain the equation of motion.

Ans : $L = \frac{1}{2} ml^2 \dot{\theta}^2 - m (g - a\omega^2 \sin \omega t)\, l \cos \theta$;

$$\ddot{\theta} - \left(\frac{g}{l} - \frac{\omega^2 a}{l} \sin \omega t\right) \sin \theta = 0.$$



[**Hint :** $T = \frac{1}{2} ml^2 \dot{\theta}^2$; $V = mg'l \cos \theta$, where $g' = g - \ddot{z} = g - \omega^2 a \sin \omega t$.]

Fig. 2.22

15. A particle of mass m is free to slide on a smooth helical wire whose position in cylindrical coordinates is represented as $\rho = a$ and $z = b\phi$. The particle is released from rest at $\rho = a$, $\phi = 0$ and $z = 0$. Discuss the motion of the particle.

Ans : $z = gb^2 t^2 / [2(a^2 + b^2)]$.

[**Hint :** $T = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$

Equations of constraints are $\rho - a = 0$ and $z - b\phi = 0$

Hence there is only one generalized coordinate. Now, $T = m (a^2 + b^2) \dot{z}^2 / 2b^2$

Generalized force $G_z = mg$; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = G_z$ or $m (a^2 + b^2) \ddot{z} / b^2 = mg$,

$\therefore \qquad z = gb^2 t^2 / [2(a^2 + b^2)]$.

16. A small bead of mass M is initially at rest on a horizontal wire and is attached to a point on the wire by a massless spring of spring constant k and unstretched length a. A mass m is freely suspended from the bead at the end of a wire of length $2b$. For the displacement shown in Fig. 2.23, obtain the Lagrangian.

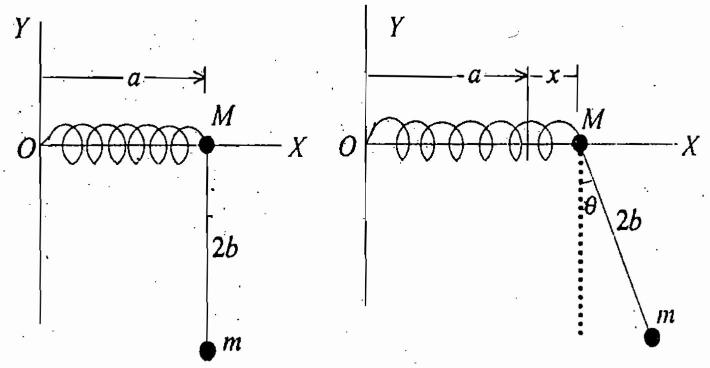


Fig. 2.23

Ans : $L=\frac{1}{2}(m+M)\dot{x}^2+2mb\dot{\theta}(b\dot{\theta}+\dot{x}\cos\theta)-\frac{1}{2}kx^2+2b\,mg\cos\theta.$

[**Hint :** In the displaced position, coordinate of M, $x'=a+x$ and coordinate of m, $x''=a+x+2b\sin\theta$, $y=-2b\cos\theta$].

17. A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's equations to show that the path of the projectile is parabola. **(Rohilkhand 1999)**

18. A particle of mass m can move in a frictionless thin circular tube of radius r [Fig. 2.24]. If the tube rotates with an angular velocity ω about a vertical diameter, deduce the differential equation of motion of the particle.

Ans : $\ddot{\theta}-\omega^2\sin\theta\cos\theta-(g/r)\sin\theta=0.$

[**Hint :** $L=\frac{1}{2}mr^2(\dot{\theta}^2+\omega^2\sin^2\theta)-mgr\cos\theta$].

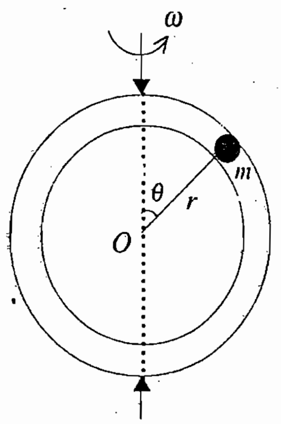


Fig. 2.24

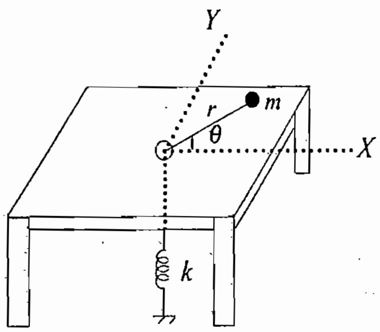


Fig. 2.25

19. Using Lagrangian formulation, find the equation of motion of a particle of mass m, constrained to move on a smooth horizontal table under the action of a spring of force constant k [Fig. 2.25]. In the system, a string attached to the particle passes through a hole in the table and is connected to the spring. Assume the spring is unstretched, when m is at the hole.

Ans : $m\ddot{r}-mr\dot{\theta}^2+kr=0$ and $\frac{d}{dt}(mr^2\dot{\theta})=0.$

[**Hint :** $T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$ and $V=\frac{1}{2}kr^2$]

20. A particle of mass m_2 moves on a vertical axis in the system shown in Fig. 2.26 and the whole system rotates about this axis with a constant angular velocity ω under the action of gravity. Set up the Lagrangian for the system.

Ans : $L = m_1 l^2 (\dot{\theta}^2 + \sin^2\theta\ \omega^2) + 2\, m_2 l^2 \sin^2\theta\ \dot{\theta}^2$
$+ 2\,(m_1 + m_2)\, gl \cos\theta$.

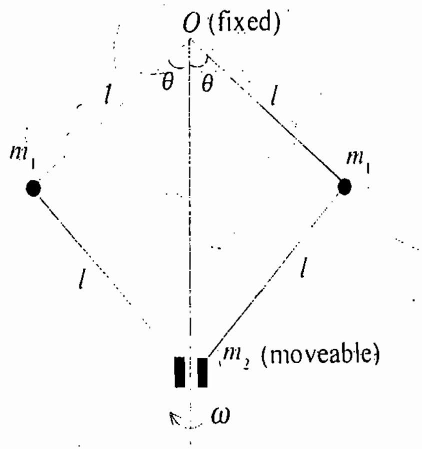


Fig. 2.26

21. Fig. 2.27 shows a mass m resting on a smooth table between two firm supports A and B and controlled by two massless springs of force constants C_1 and C_2. Set up the Lagrangian of the system and deduce the equation of motion.

Ans : $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2}(C_1 + C_2)\, x^2;\ m\ddot{x} + (C_1 + C_2)\, x = 0$.

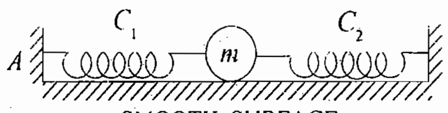


Fig. 2.27

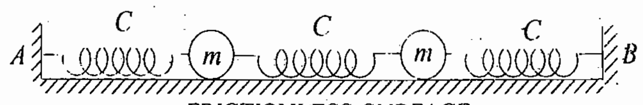


Fig. 2.28

22. Two equal masses are connected by springs having each force constant C [Fig. 2.28]. The masses are free to slide on a frictionless table AB. The walls are at A and B to which the ends of the springs are fixed. Set up the Lagrangian and deduce the equations of motion of the vibrating system.

Ans : $L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} C x_1^2 - \frac{1}{2} C x_2^2 - \frac{1}{2} C (x_1 - x_2)^2;\ m\ddot{x}_1 = C(x_2 - 2x_1),\ \ m\ddot{x}_2 = C(x_1 - 2x_2)$.

[**Hint :** Lagrange's equations are $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] - \frac{\partial L}{\partial x_1} = 0$ and $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_2}\right] - \frac{\partial L}{\partial x_2} = 0$].

23. A particle is constrained to move in a plane under the influence of an attraction towards the origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction. Find (*i*) the Lagrangian, and (*ii*) the equations of motion. **(Agra 1999)**

Ans : (i) $L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} kr^2$ (ii) $m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r} = 0$; $\frac{d}{dt}(mr^2\dot{\theta}) = k'$;

where $F_r = -kr$ and $F_\theta = \frac{k'}{r}$

[SET- II]

1. A pendulum bob of radius r is rolling on a circular track of radius R [Fig. 2.29]. Set up the Lagrangian, derive the equation of motion and compare its period of small oscillations with that of a simple pendulum of string length $(R - r)$.

Ans : $L = \frac{1}{2} m\, (R-r)^2 \dot{\theta}^2 + \frac{I}{2}\frac{(R-r)^2}{r^2}\dot{\theta}^2 - mg\,(R-r)\,(1 - \cos\theta)$;

$$\ddot{\theta} + \frac{5}{7}\frac{g}{(R-r)}\theta = 0; \sqrt{1.4} : 1]$$

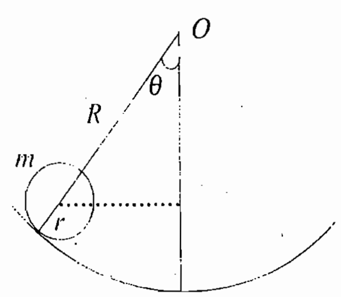


Fig. 2.29

[Hint : $v = r\omega = (R-r)\dot{\theta}$; $T = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2 = \frac{1}{2} m \frac{(R-r)^2}{r^2} \dot{\theta}^2$; $V = mg(R-r)(1-\cos\theta)$]

2. A solid homogeneous cylinder of radius r rolls without slipping on the inside of a stationary large cylinder of radius R. Find the equation of motion. What is the period of small oscillations about the stable equilibrium position ?

Ans : $\frac{3}{2}(R-r)\ddot{\theta} + g\theta = 0$, $T = 2\pi\sqrt{\frac{3(R-r)}{2g}}$

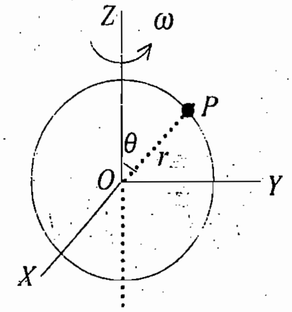


Fig. 2.30

3. (a) A bead of mass m slides on a smooth uniform circular wire of radius r which is rotating with a constant angular velocity ω about a fixed vertical diameter [Fig. 2.30]. Set up the Lagrangian and find the equation of motion of the bead.

Ans : $L = \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{1}{2} mr^2 \omega^2 \sin^2\theta - mgr\cos\theta$; $\ddot{\theta} - \frac{1}{2}\omega^2 \sin 2\theta - \frac{g}{r}\sin\theta = 0$

(b) In the above problem, if the bead is released with no vertical velocity from a point on the level of the centre of the circular wire, show that it will not reach the lowest point if $\omega > \sqrt{2g/r}$.

4. A bead of mass m can slide freely on a smooth circular wire of radius a. The wire is rotating anticlockwise in a horizontal plane with an angular velocity ω about an axis through O [Fig. 2.31]. Show that the motion of the bead is simple harmonic about the rotating line OA with a period $T = 2\pi/\omega$.

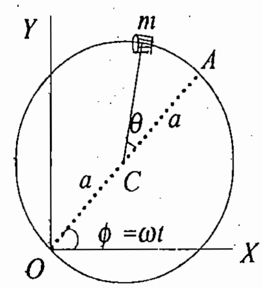


Fig. 2.31

[Hint : $x = a\cos\omega t + a\cos(\omega t + \theta)$; $y = a\sin\omega t + a\sin(\omega t + \theta)$;

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} ma^2[\omega^2 + (\dot{\theta} + \omega)^2 + 2\omega(\dot{\theta} + \omega)\cos\theta \,;$$

Lagrange's equation is $ma^2(\ddot{\theta} - \omega\dot{\theta}\sin\theta) + ma^2\omega(\dot{\theta} + \omega)\sin\theta = 0$, whence for small θ, $\ddot{\theta} + \omega^2\theta = 0$].

5. The point of support of a simple pendulum of length l and mass m is moved along a vertical line according to the equation

$$y = y(t)$$

The motion of the pendulum is restricted to a vertical plane. Show that the kinetic energy of the pendulum is given by

$$T = \frac{1}{2} m(l\dot{\theta})^2 + \frac{1}{2} m\dot{y}^2 + ml\dot{y}\dot{\theta}\sin\theta$$

If the potential energy is given by $V = mgy - mgl\cos\theta$, derive the equation of motion for the variable θ.

6. A simple pendulum of mass m whose point of support (a) moves uniformly on a vertical circle with constant angular frequency ω [Fig. 2.32], (b) oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos\omega t$, (c) oscillates vertically according to the law $y = a\cos\omega t$. Set up the Lagrangian in the three cases.

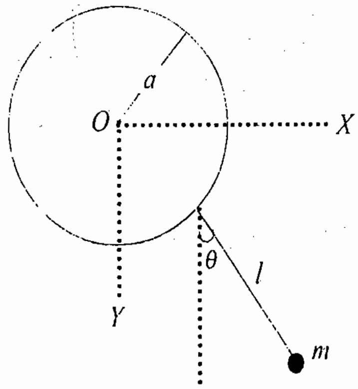


Fig. 2.32

Ans : (*a*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\,\omega^2 \sin(\theta - \omega t) + mgl \cos\theta$,

where the terms depending only on time have been omitted together with the total time derivatives of $mla\,\omega \cos(\theta - \omega t)$.

[**Hint :** $x = a \cos \omega t + l \sin\theta$, $y = -a \sin \omega t + l \cos\theta$]

(*b*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\omega^2 \cos \omega t \sin\theta + mgl \cos\theta$

[**Hint :** $x = a \cos \omega t + l \sin\theta$, $y = l \cos\theta$].

(*c*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\,\omega^2 \cos \omega t \cos\theta + mgl \cos\theta$.]

7. A mass m_2 hangs at one end of a string which passes over a fixed frictionless non-rotating pulley. At the other end of the string there is a non-rotating pulley of mass m_1 over which there is a string carrying masses m_1' and m_2' [Fig. 2.33]. Set up the Lagrangian of the system and find the acceleration of the mass m_2.

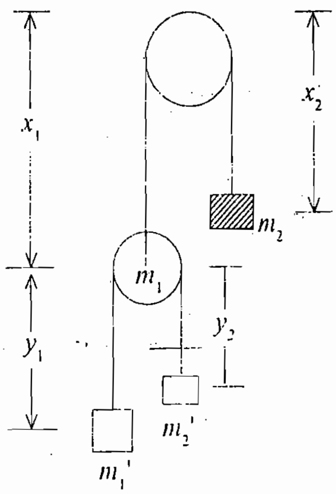


Fig. 2.33

Ans : $\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1'(\dot{x}_1 + \dot{y}_1)^2 + \frac{1}{2} m_2'(\dot{x}_1 - \dot{y}_1)^2 + m_1 g x_1$

$+ m_2 g(l_1 - x_1) + m_1' g(x_1 + y_1) + m_2' g(x_1 + l_2 - y_1)$, where l_1 and l_2 are the lengths of upper and lower (in-extensible) strings;

$$a = \frac{(m_2 - m_1)(m_2' + m_1') - 4m_1' m_2'}{(m_1 + m_2)(m_1' + m_2') + 4m_1' m_2'}.$$

8. A sphere of radius r and mass m rests on the top of a fixed rough sphere of radius R. The first sphere is slightly displaced so that it rolls without slipping down the second sphere. Find out the equation of motion of the rolling sphere.

Ans : $\ddot{\theta} - \frac{5g}{7(r+R)} \sin\theta = 0$, where θ is the angle between vertical and the line joining the centres of two spheres at an instant.

9. A system consists of two equal masses m fixed at the ends of a light rod PQ of length $2l$. The middle point C of this rod is attached to the end of a light rod OC of length a. The rod OC is mounted in such a way that it can move freely in a horizontal plane, while PQ is mounted so that it can rotate freely in a vertical plane through OC [Fig. 2.34]. Set up the Lagrangian and the equation of motion of this syatem, placed in a uniform gravitational field. (This system is called Thompson-Tait pendulum).

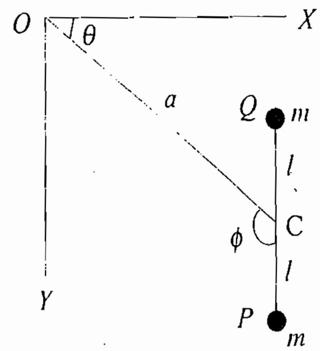


Fig. 2.34

Ans : $L = ml^2 \dot{\phi}^2 + m(a^2 + l^2 \cos^2\phi)\dot{\theta}^2$; $\ddot{\phi} + \dfrac{p_\theta^2 \sin\phi \cos\phi}{4m^2(a^2 + l^2 \cos^2\phi)^2} = 0$,

where $p_\theta = 2m\dot{\theta}(a^2 + l^2 \cos\phi)$.

10. A bead slides on a wire in the shape of a cycloid described by the equations :

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta), \text{ where } 0 \le \theta \le 2\pi.$$

Find (a) the Lagrangian function, and (b) the equations of motion. Neglect the friction between the bead and wire. **(Agra 1998)**

Ans : $L = ma^2\dot{\theta}^2(1-\cos\theta) - mga(1+\cos\theta)$; $\ddot{\theta}(1-\cos\theta) + \frac{1}{2}\sin\theta\dot{\theta} - \frac{g}{2a}\sin\theta = 0.$

Objective Type Questions

1. A particle is constrained to move along the inner surface of a fixed hemispherical bowl. The number of degrees of freedom of the particle is
(*a*) one (*b*) two
(*c*) three (*d*) four **(GATE 1996)**
Ans : (*b*).

2. A rigid body moving freely in space has degrees of freedom
(*a*) 3 (*b*) 6
(*c*) 9 (*d*) 4
Ans : (*b*)

3. Constraint in a rigid body is
(*a*) holonomic (*b*) nonholonomic
(*c*) scleronomic (*d*) rheonomic.
Ans : (*a*), (*c*).

4. Generalized coordinates
(*a*) depend on each other. (*b*) are independent of each other.
(*c*) are necessarily spherical coordinates. (*d*) may be cartesian coordinates.
Ans : (*b*), (*d*).

5. The constraints on a bead on a uniformly rotating wire in a force free space is
(*a*) Rheonomous (*b*) Scleronomous
(*c*) (*a*) and (*b*) both (*d*) None of these **(Kanpur 2003)**
Ans : (*a*).

6. If the generalized coordinate is angle θ, the corresponding generalized force has the dimensions of
(*a*) force (*b*) momentum
(*c*) torque (*d*) energy
Ans : (*c*).

7. If a generalized coordinate has the dimentions of velocity, generalized velocity has the dimensions of
(*a*) displacement (*b*) velocity
(*c*) acceleration (*d*) force
Ans : (*c*).

8. The Lagrangian for a charged particle in an electromagnetic field is
(*a*) $L = T + q\phi + q(\mathbf{v}\cdot\mathbf{A})$ (*b*) $L = T - q\phi - q(\mathbf{v}\cdot\mathbf{A})$
(*c*) $L = T - q\phi + q(\mathbf{v}\cdot\mathbf{A})$ (*d*) $L = T + q\phi - q(\mathbf{v}\cdot\mathbf{A})$
where T is the kinetic energy and ϕ and **A** are magnetic scalar and vector potentials.
Ans. (*c*).

9. A mass m is connected on either side with a spring each of spring constants k_1 and k_2. The free ends of springs are tied to rigid supports. The displacement of the mass is x from equilibrium position. Which one of the following is TRUE ?

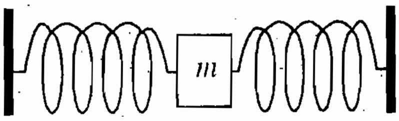


Fig. 2.35

(a) The force acting on the mass is $-(k_1 k_2)^{1/2} x$.

(b) The angular momentum of the mass is zero about the equilibrium point and its Langrangian is $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1 + k_2)x^2$.

(c) The total energy of the system is $\frac{1}{2}m\dot{x}^2$.

(d) The angluar momentum of the mass is $mx\dot{x}$ and the Lagrangian of the system is $\frac{m}{2}\dot{x}^2 + \frac{1}{2}(k_1 + k_2)x^2$. **(Gate 2004)**

Ans. (b)

10. The homogeneity of time leads to the law of conservation of

(a) linear momentum (b) angular momentum

(c) energy (d) parity. **(Gate 2002)**

Ans. (c)

Short Answer Questions

1. Discuss the D'Alemberts principle. **(Agra 2004, 03)**
2. What do you mean by degrees of freedom ?
3. What are holonomic and non-holonomic constraints ? **(Agra 2002; Kanpur 2002)**
4. Show that the work done by constraint forces in a rigid body is zero. **(Kanpur 2001)**
5. What are generalized coordinates ? What is the advantage of using them ? **(Agra 2004, 02)**
6. Write the Lagrange's equations in presence of non-consecutive forces.
7. For a non-conservative system obtain Lagrange's equations. **(Kanpur 2002)**
8. Write the Lagrangian and equation of motion for a mass M suspended by a spring of force constant k and allowed to swing vertically. **(Kanpur 2003, Rohilkhand 1994)**

   [**Ans.** $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$; $m\ddot{x} + kx = 0$]
9. Deduce the Lagrange equation of motion for L - C circuit **(Agra 2003)**

   [**Ans.** $L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$]
10. What is Hamilton's's principle ? **(Agra 2004; Kanpur 2001)**
11. Fill in the blanks :

    (i) The number of independent coordinates required to describe a system is called......................... **(Agra 2004)**

    (ii) Generalized coordinates are defined to be any quantities by means of which....................... **(Agra 2004)**

    [**Ans.** (i) Generalized coordinates (ii) we describe the state of configuration of a system]