Theorem: Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$.

(i) If $0 \le l < 1$ then $\lim_{n\to\infty} u_n = 0$,

ii) if $l > 1$ then u_n is divergent (Actually $u_n \to \infty$ as $n \to \infty$)

Note: If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$,

no definite conclusion can be made about the nature of the sequence.

For example, i) if $u_n = \frac{n+1}{n}$ then $\frac{u_{n+1}}{u_n} = \frac{n+2}{n+1} \cdot \frac{n}{n+1}$

$$= \frac{n^2+2n}{n^2+2n+1}$$

$$= \frac{1+\frac{2}{n}}{1+\frac{2}{n}+\frac{1}{n^2}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim \left(1+\frac{1}{n}\right) = 1.$

ii) if $u_n = \frac{1}{n}$ then $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

~~Also~~ ~~On~~ But in this case, $\lim_{n\to\infty} u_n = 0$.

Theorem: Let $\{u_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l.$$

i) If $0 \le l < 1$ then $\lim u_n = 0$

ii) if $l > 1$ then $u_n \longrightarrow \infty$ as $n \to \infty$.

Note: If $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$, no definite conclusion can be made about the nature of the sequence $\{u_n\}$.

For example. i) if $u_n = \frac{n+1}{n}$ then $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = 1$

and $\lim_{n\to\infty} u_n = 1$,

ii) if $u_n = \frac{n+1}{2n}$ then $\lim (u_n)^{\frac{1}{n}} = 1$

and $\lim u_n = \frac{1}{2}$.

* Monotone Sequence.

A real sequence $\{x_n\}$ is said to be a monotone increasing sequence if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. $x_n \uparrow$

A real sequence $\{x_n\}$ is said to be a monotone decreasing sequence if $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$. $x_n \downarrow$

Example: i) Let $f(n) = 2^n$, $n \geq 1$

Then $f(n+1) > f(n)$ for all $n \in \mathbb{N}$.

$\therefore$ The sequence $\{f(n)\}$ is a monotone increasing sequence. It is strictly monotone.

ii) Let $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ $\quad n \geq 1$

Then $f(n+1) - f(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0 \quad \forall n \in \mathbb{N}$.

$\therefore \{f(n)\}$ monotone increasing.

iii) The sequence $\{(-2)^n\}$ is neither a monotone increasing sequence, nor a monotone decreasing sequence.

A real sequence $\{f(n)\}$ is said to be a monotone sequence if it is either a monotone increasing or a monotone decreasing sequence.

<u>Theorem</u>: A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound.

<u>Proof</u>: Let $\{x_n\}$ be a monotone increasing sequence bounded above and let M be its least upper bound.

then i) $x_n \leq M$ for all $n \in \mathbb{N}$ and

ii) for a pre-assigned $\varepsilon > 0$, there exists a natural number k such that

$$x_k > M - \varepsilon$$

Since $\{x_n\}$ is a monotone increasing sequence,

$$M - \varepsilon < x_k \leq x_{k+1} \leq x_{k+2} \leq \cdots \leq M.$$

That is, $M - \varepsilon < x_n < M + \varepsilon$ for all $n \geq k$.

This shows that the sequence $\{x_n\}$ is convergent and

$$\lim_{n \to \infty} x_n = M.$$

<u>Theorem</u>: A monotone decreasing sequence, if bounded below, is convergent and converges to the greatest lower bound.

The proof is similar.

$\boxed{\text{Example:}}$ The sequence $\{(1+\frac{1}{n})^n\}$ is convergent.

Proof: Let $u_n = \left(1+\frac{1}{n}\right)^n$.

Then $u_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$.

Let us consider $(n+1)$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, 1+\frac{1}{n}, \ldots \ldots \ldots 1+\frac{1}{n}$ (n times) and 1.

Applying A.M. > G.M., we have

$$\frac{n(1+\frac{1}{n})+1}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$$

or, $$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

ie. $u_{n+1} > u_n \quad \forall n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a monotone increasing sequence.

Now $$u_n = 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{2}{n}\cdot\frac{1}{n}$$

$$< 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} \quad \text{for all } n>2$$

We have $n! > 2^{n-1}$ for all $n>2$. Utilising this

$$1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}, \text{ for } n>2.$$

Also $$1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+2\left[1-\left(\frac{1}{2}\right)^n\right] < 3 \text{ for all } n\in\mathbb{N}.$$

It follows that $u_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence bounded above, is convergent.

The limit of the sequence is denoted by e.

Since $u_1 = 2$, it follows that $2 < u_n < 3$ for all $n \in \mathbb{N}$.

1. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

- $x_{n+1} - x_n = \frac{1}{n+1} > 0$

$\Rightarrow x_{n+1} > x_n$ for all $n \in \mathbb{N}$

$\therefore$ the sequence ~~is~~ $\{x_n\}$ is monotonically increasing.

Now, $x_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n}$$

$$= 1 + \frac{n}{2}$$

So, $x_n \geq 1 + \frac{n}{2}$ for all n.

As $\left\{1 + \frac{n}{2}\right\}$ is an unbounded above sequence, $\{x_n\}$ is also unbounded.

$\therefore$ $\{x_n\}$ is NOT convergent.

* A monotone increasing sequence is convergent iff bounded above.

(2) Let $u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$ for all $n \in \mathbb{N}$.

We shall show that $\{u_n\}$ is convergent.

let $x_n = u_{2n-1}$ and $y_n = u_{2n}$ for all $n \in \mathbb{N}$.

then $x_{n+1} - x_n = u_{2n+1} - u_{2n-1} = \frac{1}{2n+1} - \frac{1}{2n} \leq 0$

$\Rightarrow x_{n+1} \leq x_n \;\; \forall n \in \mathbb{N}$

$$y_{n+1} - y_n = u_{2n+2} - u_{2n} = \frac{1}{2n+1} - \frac{1}{2n+2} \geq 0$$

$y_{n+1} \geq y_n \;\; \forall n \in \mathbb{N}$.

$\therefore$ the sequence $\{x_n\}$ is monotonically decreasing
and " " $\{y_n\}$ " " increasing.

$$u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\cdot\frac{1}{n}$$

$$\Rightarrow u_n = 1 - \underbrace{\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + (-1)^n \cdot \frac{1}{n}\right)}_{\text{positive}}$$

≤ 1

Also $u_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\quad\right)_{n^{th}\text{ term}}$

$\geq \frac{1}{2}$

$\therefore \frac{1}{2} \leq u_n \leq 1$.

As $\{u_n\}$ is bounded below by $\frac{1}{2}$, $\{x_n\}$ is bounded below by $\frac{1}{2}$
Since, $\{u_n\}$ " " above by 1, $\{y_n\}$ " " above by 1.

$\therefore$ both $\{x_n\}$ and $\{y_n\}$ are convergent to the same limit.
$\Rightarrow \{u_n\}$ is convergent. $\left[x_n - y_n = \frac{1}{2n} \to 0 \text{ as } n \to \infty \;\; \Rightarrow x = y\right]$

* Subsequence

Let $\{u_n\}$ be a real sequence and $\{r_n\}$ be a strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$

Then the sequence $\{u_{r_n}\}$ is said to be a subsequence of the sequence $\{u_n\}$.

The elements of the subsequence $\{u_{r_n}\}$ are $u_{r_1}, u_{r_2}, \ldots u_{r_n}, \ldots$

Example: i) Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.

Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots\}$

$= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\}$

ii) Let $u_n = \frac{1}{n}$ and $r_n = 2n-1$ for all $n \in \mathbb{N}$.

Then $\{u_{r_n}\} = \{u_1, u_3, u_5, \ldots\}$

$= \{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$

iii) Let $u_n = 1 + \frac{1}{n}$ and $r_n = n^2$ for all $n \in \mathbb{N}$.

Then $\{u_{r_n}\} = \{\cancel{1+\frac{1}{n}}\ 1+1, 1+\frac{1}{2^2}, 1+\frac{1}{3^2}, \ldots\}$ is a subsequence of $\{1+\frac{1}{n}\}$.

$\boxed{\text{Theorem}}$: If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converges to l.

Proof: Let $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{u_{r_n}\}$ is subsequence of the sequence $\{u_n\}$.

Let $\epsilon > 0$ be given. Since $\lim u_n = l$, there exists a natural number k such that $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of real numbers, there exists a natural number k_0 such that $r_n > k$, $\forall\, n \geq k_0$.

$\therefore\ l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k_0$.

Since ϵ is arbitrary, $\lim u_{r_n} = l$.

1) Prove that $\lim\left(1+\frac{1}{2n}\right)^n = \sqrt{e}$

let $u_n = \left(1+\frac{1}{n}\right)^n$,

$v_n = \left(1+\frac{1}{2n}\right)^{2n}$ and $w_n = \left(1+\frac{1}{2n}\right)^n$

for all $n \in \mathbb{N}$.

Note that $\lim u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ and $\lim v_n = e$

Now $w_n = \sqrt{v_n}$ for all $n \in \mathbb{N}$.

$\therefore \lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2) Prove that the sequence $\{(-1)^n\}$ is divergent.

let $u_n = (-1)^n$ for all $n \in \mathbb{N}$.

let $v_n = u_{2n}$ and $w_n = u_{2n-1}$, $\forall n \in \mathbb{N}$.

Then $\{v_n\}$ is ~~the~~ a subsequence of $\{u_n\}$ with $\lim v_n = 1$,

$\{w_n\}$ is a subsequence of $\{u_n\}$ with $\lim w_n = -1$.

Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is divergent.