Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

REAL ANALYSIS & CALCULUS (MA 111)

Week 2 / September 2023 | Problem Set 4 | GR

Real Analysis

Limits (of real-valued functions)

■ Tutorial and Assignment Problems

1. Show that (using ϵ-δ *definition* or *sequential definition*)

   i. $\lim_{x\to c} \sin x = \sin c$

   ii. $\lim_{x\to c} \sqrt{x} = \sqrt{c}$, where $c > 0$

   iii. $\lim_{x\to 2} f(x) = 12$ where $f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x \neq 2 \\ 10 & \text{if } x = 2. \end{cases}$

2. Show that

   i) $\lim_{x\to 0} \sin \frac{1}{x}$

   ii) $\lim_{x\to 0} e^{\frac{1}{x}}$

   iii) $\lim_{x\to\infty} x \sin x$

   iv) $\lim_{x\to\infty} x^{1+\sin x}$

   does not exist.

3. Let $f : (0,1) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \cap \mathbb{Q} \\ 0 & \text{if } x \in (0,1) \cap \mathbb{Q}^c. \end{cases}$$

   Show that $\lim_{x\to a} f(x)$ does not exist, where $a \in [0,1]$.

4. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2-x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that

   i) $\lim_{x\to 1} f(x) = 1$

ii) $\lim_{x\to c} f(x)$ does not exist, if $c \neq 1$.

5. By finding $\lim_{x\to 0+}\left[\frac{\sin x}{x}\right]$ and $\lim_{x\to 0-}\left[\frac{\sin x}{x}\right]$, check whether $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$ exists. [Note that $[x]$ is greatest integer less than or equal to x.]

6. Find

   i. $\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+3}}$

   [Hint: To compute $\lim_{x\to\infty} f(x)$, consider $y = \frac{1}{x}$. Then $\lim_{x\to\infty} f(x) = \lim_{y\to 0+} f\left(\frac{1}{y}\right)$]

   For example, $\lim_{x\to\infty}\frac{\sin x}{x} = \lim_{y\to 0+}\frac{\sin\frac{1}{y}}{\frac{1}{y}} = \lim_{y\to 0+} y\sin\frac{1}{y} = 0.$

   ii. $\lim_{x\to\infty}\left(x - \sqrt{x^2-1}\right)$

   iii. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ [Hint: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$]

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