Department of Mathematical Sciences

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REAL ANALYSIS AND CALCULUS (MA 111)

Week 1 / October 2023 Problem Set 7 GR,PD,DB

Real Analysis

Riemann Integration

Notations : Set of Real Numbers ($\mathbb{R}$); Set of Natural Numbers ($\mathbb{N}$); sup is *supremum* or *least upper bound*; inf is *infimum* or *greatest lower bound*; $\mathcal{P}[a,b]$ is the collection of all partitions of $[a,b]$; $B(m,n)$ is Beta(m,n); $\Gamma(n)$ is Gamma(n).

1. Use finite approximations to estimate the area under the graph of the function using

   (i) a lower sum with two rectangles of equal width.

   (ii) a lower sum with four rectangles of equal width.

   (iii) an upper sum with two rectangles of equal width.

   (iv) an upper sum with four rectangles of equal width.

   of the function $f(x) = x^3$ between $x = 0$ and $x = 1$. What is the approximate average value of the function in each case?

2. Take the partition $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$ of $[0,1]$ and verify the inequality

$$m(b-a) \le L(P_n, f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P_n, f) \le M(b-a)$$

   for the function f on $[a,b]$, where $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$. We know that $\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$ and $\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\}$.

   (i) $f(x) = x^2, \quad x \in [0,1]$,

   (ii) $f(x) = x^3, \quad x \in [0,1]$.

   (Here $a = 0$, $b = 1$.) Show that $\sup\{L(P_n, f) : n \in \mathbb{N}\} = \inf\{U(P_n, f) : n \in \mathbb{N}\}$ for the above function f. Deduce that f is integrable on $[0,1]$.

3. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

4. Let $f : [a,b] \to \mathbb{R}$ be bounded and monotone increasing on $[a,b]$. If P_n be the partition of $[a,b]$ dividing into n sub-intervals of equal length, prove that

$$\int_a^b f \;\le\; U(P_n, f) \;\le\; \int_a^b f + \frac{b-a}{n}\left[f(b) - f(a)\right].$$

Consider the sequence of partition $\{P_n\}$ and deduce that $\lim_{n\to\infty} U(P_n, f) = \int_a^b f$. Utilise this result to evaluate

$$(i) \quad \int_0^1 x^2\, dx \qquad (ii) \quad \int_0^1 e^x\, dx.$$

[**Hint:** $U(P_n, f) - L(P_n, f) = \frac{b-a}{n}[f(b) - f(a)], \quad L(P_n, f) \le \int_a^b f \le U(P_n, f)$.]

5. A function f is defined on $[0, 1]$ by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ be rational} \\ x^3, & \text{if } x \text{ be irrational.} \end{cases}$$

Find $\overline{\int_0^1} f$ and $\underline{\int_0^1} f$. Then, show that f is not integrable on $[0, 1]$.

[**Hint:** Calculate $U(P_n, f)$ and $L(P_n, f)$ corresponding to the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ of $[0, 1]$. Since $\|P_n\| = \frac{1}{n} \to 0$ as $n \to \infty$, $\lim_{n\to\infty} U(P_n, f) = \overline{\int_0^1} f$ and $\lim_{n\to\infty} L(P_n, f) = \underline{\int_0^1} f$.]

6. Let $f(x) = x[x],\ x \in [0, 3]$. Show that f is integrable on $[0, 3]$. Evaluate $\int_0^3 f$.

7. A function f is defined on $I = [0, 10]$ by $f(x) = \begin{cases} 0, & \text{if } x \in I \cap \mathbb{Z} \\ 1, & \text{if } x \in I - \mathbb{Z}. \end{cases}$ Show that f is integrable on $[0, 10]$. Evaluate $\int_0^{10} f$. [Here $\mathbb{Z}$ denotes the set of all integers.]

8. A function f is defined on $[0, 1]$ by $f(0) = 0$,

$$f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \text{ for } n = 0, 1, 2, \ldots.$$

Prove that f is integrable on $[0, 1]$ and $\int_0^1 f = \frac{2}{3}$.

[**Hint:** f is monotonic increasing and bounded on $[0, 1]$.]

9. A function f is continuous for all $x \ge 0$ and $f(x) \ne 0$ for all $x > 0$. If $\{f(x)\}^2 = 2\int_0^x f(t)\, dt$, prove that $f(x) = x$ for all $x \ge 0$.

[**Hint:** $f(0) = 0$ and $f'(x) = 1$ for all $x > 0$. Use Lagranges mean value theorem to f on $[0, x]$.]

10. A function f is defined on $[0, 3]$ by $f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 < x \le 2 \\ 2, & 2 < x \le 3. \end{cases}$

Let $F(x) = \int_0^x f(t)\, dt,\ x \in [0, 3]$. Find F. Show that F is continuous on $[0, 3]$.

11. A function f is defined on $[0, 3]$ by $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 < x \le 2 \\ x - 1, & 2 < x \le 3. \end{cases}$

Show that f is integrable on $[0,3]$.

Let $F(x) = \int_0^x f(t)\,dt$, $x \in [0,3]$. Find F. Show that $F'(x) = f(x),\ \ x \in [0,3]$.

12. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$. (**Warning:** $f(x)$ takes both positive and negative values in the given interval)

13. Find F' where F is defined on $[1,\infty)$ by

$$(i)\ F(x) = \int_x^{e^x} \sqrt{1+t^2}dt \qquad (ii)\ F(x) = \int_x^{x^2} \sin\sqrt{t}\,dt$$

[**Hint:** (***Leibniz Rule for Integrals***)
$F(x) = \int_{u(x)}^{v(x)} f(t)\,dt \implies F'(x) = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$]

14. Prove that

(i) $\displaystyle\lim_{x\to 2} \frac{\int_2^x e^{\sqrt{1+t^2}}dt}{x-2} = e^{\sqrt{5}}$

(ii) $\displaystyle\lim_{x\to 0} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3} = \frac{2}{3}$.

Improper Integration

1. Examine the convergence of the improper integrals:

   (i) $\displaystyle\int_0^1 \frac{\log(1-x)}{\sqrt{x}}\,dx,$

   (ii) $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx,$

   (iii) $\displaystyle\int_0^\pi \frac{\tan x}{x}\,dx,$

   (iv) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

   (v) $\displaystyle\int_0^{\frac{\pi}{2}} \log\sin x\,dx.$

   [**Note:** $\lim\limits_{x\to 0+} x^r \log x = 0$, for all $r > 0$.]

2. Examine the convergence of the improper integrals:

   (i) $\displaystyle\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx,$

   (ii) $\displaystyle\int_0^\infty \frac{1}{x\log x}\,dx,$

   (iii) $\displaystyle\int_0^\infty \left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right)dx,$

   (iv) $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2},$

   (v) $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta.$

3. Show that the following improper integrals are absolutely convergent:

   (i) $\displaystyle\int_0^1 \frac{\cos\frac{1}{x}}{\sqrt{x}}\,dx,$

   (ii) $\displaystyle\int_0^\infty \frac{\cos x}{(1+x)^2}\,dx,$

   (iii) $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx, \quad (a > 0).$

4. Discuss the convergence of the improper integral $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx.$

5. Show that $B(m+1,n) = \frac{m}{m+n}B(m,n)$, $m > 0, n > 0$.

6. Prove that
$$\int_0^\infty e^{-kt}t^{n-1}\,dt = \int_1^\infty \frac{(\log y)^{n-1}}{y^{k+1}}\,dy = \frac{\Gamma(n)}{k^n}.$$

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