Department of Mathematical Sciences

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REAL ANALYSIS AND CALCULUS (MA 111)

Week 4 / September 2023	Problem Set 6	GR, PD

Real Analysis

Applications of Rolle's, Mean Value and Taylor's Theorems

Tutorial Problems

1. Show that the following functions do not satisfy the conditions of Rolle's theorem on the indicated intervals.

   i. $f(x) = 1 - |x - 1|, \quad x \in [0, 2]$

   ii. $f(x) = 1 - (x - 1)^{\frac{2}{3}}, \quad x \in [0, 2]$

2. Verify the hypothesis and conclusion of Mean Value Theorem for the following functions on the indicated intervals.

   i. $f(x) = \dfrac{x}{x-1}, \quad x \in [2, 4]$

   ii. $f(x) = x^3 - 3x + 1, \quad x \in [1, 3]$

   iii. $f(x) = 4 - (6 - x)^{\frac{2}{3}}, \quad x \in [5, 7]$

   iv. $f(x) = \begin{cases} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ on $[-1, 1]$.

3. Prove that the equation $(x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 = 0$ has only one real root.
   [**Hint.** Use Rolle's theorem.]

4. Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

5. Let $f : [a, b] \to \mathbb{R}$ be a function such that $|f(x) - f(y)| < L\,|x - y|^{\alpha}$ for all $x, y \in [a, b]$, for some $L > 0$ and $\alpha > 1$. Prove that f is constant on $[a, b]$.

   [**Hint.** Use the definition to show that f is differentiable at any point $c \in [a, b]$ and $f'(c) = 0$.]

6. Use the MVT to prove that

$$na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a)$$

   for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that $0 < a \leq b$.
   [**Hint.** Use Mean Value Theorem to $f(x) = x^n$ on $[a, b]$.]

7. Show that $0 < \dfrac{1}{x} \log \dfrac{e^x - 1}{x} < 1$ for all $x > 0$.

   [**Hint.** Use Mean Value Theorem to $f(x) = e^x$ on $[0, x]$.]

8. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)f(b) < 0$ and $f'(x) \neq 0$ on (a, b), then show that there is a **unique** $c \in (a, b)$ such that $f(c) = 0$.

9. A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in $(0, 1)$. Prove that there is at least one point α in $(0, 1)$ for which $f''(\alpha) > 0$.

   [**Hint.** Apply Mean Value Theorem first to f on $[a, c]$ and on $[c, b]$ and produce $\alpha_1 \in (a, c)$, $\alpha_2 \in (c, b)$, then to f' on $[\alpha_1, \alpha_2]$.]

10. If $f''(x) \geq 0$ on $[a, b]$ prove that $f\left(\dfrac{\alpha_1 + \alpha_2}{2}\right) \leq \dfrac{1}{2}[f(\alpha_1) + f(\alpha_2)]$ for any two points $\alpha_1, \alpha_2 \in [a, b]$.

11. If f is differentiable on $[0, 1]$ show by Cauchy's Mean value theorem that the equation $f(1) - f(0) = \dfrac{f'(x)}{2x}$ has at least one solution in $(0, 1)$.

    [Hint. [***Cauchy's Mean value theorem***] Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

    - f and g are both continuous on $[a, b]$
    - f and g are both differentiable on (a, b)
    - $g'(x) \neq 0$ for all $x \in (a, b)$.

    Then there exist a point c in (a, b) such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.]

12. Use Taylor's theorem to prove that

    i. $\cos x \geq 1 - \dfrac{x^2}{2}$ for $-\pi < x < \pi$,

    ii. $x - \dfrac{x^2}{2} < \log(1 + x) < x$ for $x > 0$.

    iii. $\left|\log(1 + x) - \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3}\right)\right| < \dfrac{1}{4}$ for all $x \in [0, 1]$.

    [**Hint:** [General form of ***Taylor's theorem***] Let $a \in \mathbb{R}$. Let a real function f is such that $f^{(n-1)}$ is differentiable on a neighbourhood $N(a)$ of a. Then there exist a θ satisfying $0 < \theta < 1$ such that

    $$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(a + \theta(x-a))$$

    for all $x \in N(a) \setminus \{a\}$.]

13. Evaluate the following limits (using ***L'Hospital's rule***).

i. $\lim_{x\to 0} \frac{\sinh x - \sin x}{x \sin^2 x}$ [Recall $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$]

ii. $\lim_{x\to\infty} \frac{x^n}{e^x}$

iii. $\lim_{x\to 0} \frac{\log\log(1-x^2)}{\log\log\cos x}$

iv. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

v. $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$

vi. $\lim_{x\to\infty} \left(\frac{2x}{2x+1}\right)^{2x+1}$

vii. Find the values of α and β satisfying $\lim_{x\to 0} \frac{x(1-\alpha\cos x)+\beta\sin x}{x^3} = \frac{1}{3}$.

■ Assignment Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at 0, show that f is differentiable at every point $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$. In fact, show that f is infinitely differentiable. If $f'(0) = 2$, find $f^{(n)}(1)$ in terms of $f(1)$.

2. Let $f : (0,\infty) \to \mathbb{R}$ satisfies $f(xy) = f(x) + f(y)$ for all $x, y \in (0,\infty)$. If f is differentiable at 1, show that f is differentiable at every point $c \in (0,\infty)$ and $f'(c) = \frac{1}{c}f'(1)$. In fact, show that f is infinitely differentiable. If $f'(1) = 2$, find $f^{(n)}(3)$.

   [**Hint.** Use the definition of derivative to solve assignment problem 1 and 2. $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ or $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$]

3. A function f is differentiable on $[0,2]$ and $f(0) = 0$, $f(1) = 2$, $f(2) = 1$. Prove that $f'(c) = 0$ for some $c \in (0,2)$.

   [**Hint.** Apply Mean Value Theorem to f on $[0,1]$ and on $[1,2]$ and produce $\alpha_1 \in (0,1)$, $\alpha_2 \in (1,2)$. Apply Darboux's theorem to f' on $[\alpha_1, \alpha_2]$.

   Darboux Theorem

   Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then the derivative function f' has the IVP on $[a,b]$.
   That is,
   Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Let $f'(a) \neq f'(b)$. If k be a real number lying between $f'(a)$ and $f'(b)$ then there exist a point c in (a,b) such that $f'(c) = k$.

   Note that in general, f' need not be continuous for this theorem to hold.]

4. Prove that $\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\right| < \frac{1}{7!}$ for all $x \in [-1,1]$.

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