Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

REAL ANALYSIS & CALCULUS (MA 111)

Week 1 / September 2023 | Problem Set 3 | GR

Real Analysis

(Infinite) Series

■ Tutorial and Assignment Problems

1. If $\sum u_n$ be a convergent series of positive real numbers prove that $\sum \dfrac{u_n}{n}$ is convergent.

2. If $\sum u_n$ be a convergent series of positive real numbers prove that $\sum \dfrac{u_n}{1+u_n}$ is convergent.

3. If $\sum u_n$ be a series of positive real numbers and $v_n = \frac{u_1+u_2+u_3+\cdots+u_n}{n}$, prove that $\sum v_n$ is divergent.

4. Test the convergence of the following series.

   (i) $\dfrac{1}{1+2} + \dfrac{1}{1+2^2} + \dfrac{1}{1+2^3} + \cdots,$

   (ii) $\sin\dfrac{\pi}{2} + \sin\dfrac{\pi}{4} + \sin\dfrac{\pi}{6} + \cdots,$

   (iii) $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + \cdots,$

   (iv) $\dfrac{1}{1+2^{-1}} + \dfrac{1}{1+2^{-2}} + \dfrac{1}{1+2^{-3}} + \cdots,$

   (v) $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n,$

   (vi) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n},$

   (vii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{1}{n},$

   (viii) $\dfrac{1}{\log 2} + \dfrac{1}{\log 3} + \dfrac{1}{\log 3} + \cdots$ [Hint. $\log(1+x) < x$ for $x > 0$.]

   (ix) $1 + \dfrac{1}{2}\cdot\dfrac{1}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{1}{5} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdot\dfrac{1}{7} + \cdots$ [Hint. Use Raabe's Test.]

   (x) $\left(\dfrac{1}{2}\right)^{\ln 1} + \left(\dfrac{1}{2}\right)^{\ln 2} + \left(\dfrac{1}{2}\right)^{\ln 3} + \cdots$ [Hint. Use Logarithmic Test.]

(xi) $\frac{1}{3} + \left(\frac{1}{3}\right)^{1+\frac{1}{2}} + \left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}} + \cdots$ [Hint. Use Logarithmic Test.]

[Hint. **Logarithmic Test**: Let $\sum u_n$ be a series of positive real numbers and $n \ln\left(\frac{u_n}{u_{n+1}}\right) = \ell$. Then $\ell > 1 \implies \sum u_n$ is convergent, $\ell < 1 \implies \sum u_n$ is divergent.]

5. Use **Cauchy's condensation test** [Let $\{f(n)\}$ be a monotone decreasing sequence of positive real numbers and a be a positive integer > 1. Then the series $\sum_{n=2}^{\infty} f(n)$ and $\sum_{n=2}^{\infty} a^n f(a^n)$ converge of diverge together] to discuss the convergence of the following series

   (i) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ $\quad p > 0$

   (ii) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

   (iii) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^2}$

   (iv) $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

   [Hint. Use $\{\ln n\}$ is monotone increasing sequence.]

6. Prove that the following series are convergent.

   (i) $1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots$

   (ii) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$

   (iii) $\frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \cdots$

   (iv) $\sum_{1}^{\infty} \frac{\sin nx}{n^2}$, where x is a fixed real number.

   [**Hint**. A series $\sum_{n=1}^{\infty} u_n$ is said be *absolutely convergent* if $\sum_{n=1}^{\infty} |u_n|$ is convergent. If a series is *absolutely convergent*, then the series is convergent.
   (absolutely convergent $\implies$ convergent.)]

7. Show that the following series are conditionally convergent.

   (i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

(ii) $\sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \sin\frac{\pi}{6} - \cdots$

(iii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \cdots$

[**Definition**. A series $\sum u_n$ is said be *conditionally convergent* if $\sum u_n$ is convergent but $\sum |u_n|$ is NOT convergent.]

8. Show that the following series are convergent.

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(ii) $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \cdots$

[Hint. Use **Leibnitz's Test :** If $\{u_n\}$ be a sequence of positive real numbers which is a monotone decreasing and converging to 0, then the alternating series

$$u_1 - u_2 + u_3 - u_4 + \cdots \quad \text{is convergent. }]$$

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