① (i) Let $c > 0$ be any real number.

Let $\epsilon > 0$. We have

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}|$$

$$= \frac{|\sqrt{x} - \sqrt{c}|\,|\sqrt{x} + \sqrt{c}|}{|\sqrt{x} + \sqrt{c}|}$$

$$= \frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} \leq \frac{|x-c|}{|\sqrt{c}|} = \frac{|x-c|}{\sqrt{c}}$$

Let $\delta = \sqrt{c} \cdot \epsilon$, then whenever

$0 < |x - c| < \delta$, we have

$$|x - c| < \sqrt{c} \cdot \epsilon$$

$$\Rightarrow \frac{|x-c|}{\sqrt{c}} < \epsilon$$

$$\therefore |f(x) - f(c)| \leq \frac{|x-c|}{\sqrt{c}} < \epsilon.$$

This show that f is continuous at c.

(ii) Clearly if $x \neq 2$, then

$$f(x) = \frac{x^3 - 8}{x - 2} = x^2 + 2x + 4$$

being a polynomial in x is continuous.

Let $x = 2$, $\epsilon > 0$.

$$|f(x) - f(2)| = \left| \frac{x^3 - 8}{x - 2} - 12 \right|$$

$$= |x^2 + 2x + 4 - 12| \qquad (x \neq 2)$$

$\therefore\ |f(x) - f(2)| = |x^2 + 2x - 8| = |(x-2)^2 + 6(x-2)|$

$\leq |x-2|^2 + 6|x-2|$

If we choose $\delta > 0$ such that $\delta^2 + 6\delta \leq \varepsilon$

$0 < |x-2| < \delta \Rightarrow x \neq 2$ and

$|f(x) - f(2)| \leq |x-2|^2 + 6|x-2| < \delta^2 + 6\delta \leq \varepsilon$

Remark: Note that we can take $\delta = \dfrac{-6 \pm \sqrt{36 + 4\varepsilon}}{2}$,

that is $\delta = -3 + \sqrt{9 + \varepsilon}$ then we shall have

$\delta > 0$ and $\delta^2 + 6\delta = \varepsilon$.

2. $f(x) = [x]$, $0 < x < 2$

Note that $f(1) = 1$.

Show that

$\lim_{x \to 1-} f(x) \neq 1 = f(1)$

[Note that here $\lim_{x \to 1-} f(x) = 0$]

$\therefore$ ~~f~~ As $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$.

$\therefore$ $\lim_{x \to 1} f(x)$ does NOT exist.

$\therefore f$ is NOT continuous at $x = 1$.

(3) Let $x_0 \in \mathbb{R}$ be any point.

Take $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\}$ is a sequence of rational numbers converging to x_0, and $\{y_n\}$ is a sequence of irrational numbers converging to x_0.

Then $f(x_n) = 1 \longrightarrow 1$ as $n \to \infty$

and $f(y_n) = 0 \longrightarrow 0$ as $n \to \infty$.

If f happens to be continuous at x_0 then we must have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n) = f(x_0)$$

$\Rightarrow \quad 1 = 0$ a contradiction.

$\therefore$ f is not continuous at x_0. Since x_0 is an arbitrary real number, f is nowhere continuous.

④ Let $\epsilon > 0$.

$$|f(x) - f(1)| = \begin{cases} |x-1| & \text{if } x \in \mathbb{Q} \\ |2-x-1| & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\therefore$ $|f(x) - f(1)| = |x-1|$ for all $x \in \mathbb{R}$, as

$|2-x-1| = |1-x| = |-(x-1)| = |x-1|$.

Taking $\delta = \epsilon$, we see that whenever

$|x-1| < \delta$ we have $|f(x) - f(1)| = |x-1| < \delta = \epsilon$.

This shows that f is continuous at $x = 1$.

Now let $x_0 \neq 1$ be any point.

Let $\{x_n\}$ be a sequence of rational numbers, and $\{y_n\}$ be a sequence of irrational number both converging to x_0. Then

$$f(x_n) = x_n \longrightarrow x_0$$

and $f(y_n) = 2 - y_n \longrightarrow 2 - x_0$.

If f is continuous at x_0 then we must have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(x_0).$$

$\Rightarrow$ $x_0 = 2 - x_0$ $\Rightarrow$ $x_0 = 1$ which is a contradiction.

$\therefore$ f cannot be continuous at any point $x_0 \neq 1$.

Pramod

(5) Given f is continuous. Let $x_0 \in \mathbb{R}$ be any point.

Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. (using the continuity of f at x_0)

But $\Big| |f|(x) - |f|(x_0) \Big| = \Big| |f(x)| - |f(x_0)| \Big|$

$\leq |f(x) - f(x_0)|$ (Reverse triangle inequality, $\big||x| - |y|\big| \leq |x - y|$)

$< \varepsilon$

whenever $|x - x_0| < \delta$.

This shows that $|f|$ is continuous at x_0. ~~Since~~ Thus we have shown that if f is continuous at x_0 then so is $|f|$. Proved

<u>Converse is not true!</u>

Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Then by a logic similar to problem no. 3, f is nowhere continuous. But

$|f|(x) = 1$ for all $x \in \mathbb{R}$, which is

everywhere continuous.

$\therefore$ $|f|$ continuous does not necessarily imply that f will be continuous.

(6) $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$.

Take $x=y=0$.

$$f(0+0) = f(0)+f(0)$$
$$\Rightarrow f(0)=0 \quad \text{——— (i)}$$

Let $x_n \to x$.

$$\Rightarrow x_n - x \to 0.$$
$$\Rightarrow x_n - x + c \to c$$

$\because$ f is continuous at c, we have

$$f(x_n - x + c) \longrightarrow f(c)$$
$$\Rightarrow f(x_n - x) + f(c) \longrightarrow f(c)$$
$$\Rightarrow f(x_n) + f(-x) \longrightarrow 0$$
$$\Rightarrow f(x_n) \longrightarrow -f(-x). \quad \text{——— (ii)}$$

Also taking $y=-x$ in $f(x+y) = f(x)+f(y)$ we get

$$f(x-x) = f(x)+f(-x)$$
$$\Rightarrow f(0) = f(x)+f(-x)$$
$$\Rightarrow 0 = f(x)+f(-x) \quad \text{(using (i))}$$
$$\Rightarrow -f(-x) = f(x)$$

$\therefore$ By (ii) we get

$$f(x_n) \longrightarrow f(x)$$

$\therefore$ f is continuous at x.

Since x is arbitrary f is continuous everywhere.

(7) We have $f(x+y) = f(x)\cdot f(y) \ \forall x, y \in \mathbb{R}$. —— (*)

Take $x = y = 0$,

$$f(0+0) = f(0)\cdot f(0)$$
$$\Rightarrow f(0)\,(f(0)-1) = 0$$
$$\Rightarrow f(0) = 0 \quad \text{or} \quad f(0) = 1.$$

Case I $\quad f(0) = 0$.

Taking $y = 0$ in (*) we get

$$f(x+0) = f(x)\cdot f(0)$$
$$\Rightarrow f(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

$\therefore$ f is a constant function, hence continuous.

Case II $\quad f(0) = 1$.

Taking $y = -x$ in (*) we get

$$f(x-x) = f(x)\cdot f(-x)$$
$$\Rightarrow f(0) = f(x)\cdot f(-x)$$
$$\Rightarrow f(-x) = \frac{1}{f(x)} \cdot \text{ for all } x.$$

~~Since f is conti~~

Let $x_n \to x$.

$$\Rightarrow x_n - x \to 0$$

$\because$ f is continuous at 0, we have

$$f(x_n - x) \to f(0)$$
$$\Rightarrow f(x_n)\cdot f(-x) \to 1$$
$$\Rightarrow f(x_n) \to \frac{1}{f(-x)} = f(x)$$

Hence f is continuous at x. Since x is arbitrary f is continuous everywhere.

(8) Recall that $f: \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if for each sequence $\{x_n\}$ converging to x, the sequence $\{f(x_n)\}$ converges to $f(x)$.

Also recall that for every irrational number x_0, there is a sequence of rational numbers $\{x_n\}$ that converges to x_0.

So, given any irration number x, let us choose a sequence $\{x_n\}$ of rational numbers such that $x_n \to x$.

Since f is continuous, $f(x_n) \to f(x)$

But $f(x_n) = 0$ for all n, as each x_n is rational.

$\therefore$ $\{f(x_n)\}$ is the constant sequence $\{0\}$, whose limit is 0.

As we know that a sequence in $\mathbb{R}$ has atmost one limit, we must have $f(x) = 0$. Proved

(9) Let us define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = f(x) - g(x).$$

Then h is continuous on $[a,b]$ as f and g are continuous on $[a,b]$.

Also, $h(a) = f(a) - g(a) < 0$.

$h(b) = f(b) - g(b) > 0$

$\therefore$ $h(a) < 0 < h(b)$.

$\therefore$ By IVT, there exists $c \in (a,b)$ such that

$h(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\Rightarrow f(c) = g(c)$. Proved

Let $[a,b] = [0, \pi/2]$,

$f : [a,b] \to \mathbb{R}$ be the function $f(x) = x^2$

$g : [a,b] \to \mathbb{R}$ be the function $g(x) = \cos x$

Then $f(0) = 0 < 1 = \cos 0 = g(0)$.

$f(\pi/2) = \pi^2/4 > 0 = \cos \pi/2 = g(\pi/2)$.

So by the above result there exists $c \in [0, \pi/2)$ such that $f(c) = g(c)$

i.e. $c^2 = \cos c$ Proved

(10) $f : [0,2] \to \mathbb{R}$ is continuous.

Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = f(x) + f(x+1).$$

The g is continuous, being sum of two continuous functions.

Now $g(0) = f(0) - f(1)$

and
$$g(1) = f(1) - f(2)$$
$$= f(1) - f(0)$$
$$= -(f(0) - f(1))$$

If $g(0)=0$ then $f(0) = f(1)$, so take $c=0$.

If $g(1) = 0$ then $f(0) = f(1)$, so again take $c=0$.

If $g(0) \neq 0$ and $g(1) \neq 0$ then $g(0) \cdot g(1) < 0$ being of opposite signs. Since 0 is an intermediate value of $g(0)$ and $g(1)$, by the IVT, there exists $c \in [0,1]$ such that

$$g(c) = 0$$
$$\Rightarrow f(c) - f(c+1) = 0$$
$$\Rightarrow f(c) = f(c+1) \quad \underline{\text{proved}}$$

(11) Let $u = \log(x+1)$ and $v = x^3$ in the Leibnitz formula

$$D^n(uv) = D^n u \cdot v + n D^{n-1} u \cdot Dv + {}^nC_2 D^{n-2} u \cdot D^2 v + \cdots + {}^nC_r D^{n-r} u \cdot D^r v + \cdots + u \cdot D^n v.$$

Then we have

$$u = \log(x+1)$$

$$D^n u = \frac{(-1)^{n-1} \cdot (n-1)!}{(x+1)^n} \qquad v = x^3$$

$$D^{n-1} u = \frac{(-1)^{n-2} \cdot (n-2)!}{(x+1)^{n-1}} \qquad Dv = 3x^2$$

$$D^{n-2} u = \frac{(-1)^{n-3} \cdot (n-3)!}{(x+1)^{n-2}} \qquad D^2 v = 6x$$

$$D^{n-3} u = \frac{(-1)^{n-4} \cdot (n-4)!}{(x+1)^{n-3}} \qquad D^3 v = 6$$

$$D^k v = 0 \text{ for } k \geq 4.$$

$$\therefore D^n(\log(x+1) \cdot x^3) = \frac{(-1)^{n-1} \cdot (n-1)!}{(x+1)^n} \cdot x^3 + n \cdot \frac{(-1)^{n-2} (n-2)!}{(x+1)^{n-1}} \cdot 3x^2$$

$$+ \frac{n(\cancel{n+1})}{2} \cdot \frac{(-1)^{n-3} (n-3)!}{(x+1)^{n-2}} \cdot 6x +$$

$$\frac{n(n+1)(n+2)}{6} \cdot \frac{(-1)^{n-3} \cdot (n-4)!}{(x+1)^{n-3}} \cdot 6$$

$$\therefore \; D^n\left(x^3 \log(x+1)\right)$$

$$= \frac{(-1)^{n-1} \cdot (n-4)!}{(x+1)^{n-3}} \left[\frac{(n-1)(n-2)(n-3)x^3}{(x+1)^3} - \frac{3n(n-2)(n-3)x^2}{(x+1)^2} + \frac{3n(n+1)(n-3)x}{x+1} - n(n+1)(n+2) \right]$$

Ans

(12) we need to find the n^{th} derivative of

$x^n(1-x)^n$, (replace y by $1-x$ as $x+y=1$).

we have $D^r(x^n) = \frac{n!}{(n-r)!} x^{n-r}$ $\quad r \le n$.

$$D^r(y^n) = D^r(1-x)^n = \frac{(-1)^n . n!}{(n-r)!}(1-x)^{n-r}$$

$$= \frac{(-1)^n\, n!}{(n-r)!} y^{n-r}; \quad r \le n$$

By Leibnitz rule,

$D(x^n y^n)$

$$= D^n(x^n) y^n + {}^nC_1 D^{n-1}(x^n) D(y^n) + {}^nC_2 D^{n-2}(x^n) \cdot D^2(y^n) + \cdots + x^n \cdot D^n(y^n)$$

$$= n! \cdot y^n - {}^nC_1 \cdot \frac{n!}{1!} x \cdot \frac{n!}{(n-1)!} y^{n-1} + {}^nC_2 \cdot \frac{n!}{2!} x^2 \cdot \frac{n!}{(n-4)!} y^{n-2} - + \cdots + (-1)^n x^n . n!$$

$$= n! \left[y^n - ({}^nC_1)^2 x y^{n-1} + ({}^nC_2)^2 x^2 y^{n-2} + \cdots + (-1)^n x^n \right]$$

Ans

(13) $y = (x+\sqrt{1+x^2})^m$

$$y' = m(x+\sqrt{1+x^2})^{m-1} \cdot \left(1+\frac{2x}{2\sqrt{1+x^2}}\right) \quad \left(\text{Differentiate w.r.t. } x\right)$$

$$= \frac{m\cdot(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}}$$

$$\Rightarrow \sqrt{1+x^2}\, y' = m\cdot y \text{ ——— (i)}$$

$$\Rightarrow \frac{2x}{2\sqrt{1+x^2}} y' + \sqrt{1+x^2}\, y'' = my' \quad \left(\text{Differentiate w.r.t. } x\right)$$

$$\Rightarrow xy' + (1+x^2)y'' = m\sqrt{1+x^2}\, y'$$

$$\Rightarrow (1+x^2)y'' + xy' - m^2 y = 0 \quad (\text{Use (i)})$$

Differentiate n-times w.r.t. x applying Leibnitz rule,

$$\left[(1+x^2)\, y^{(n+2)} + {}^nC_1 \cdot y^{(n+1)} \cdot 2x + {}^nC_2\, y^{(n)} \cdot 2\right] + \left[x y^{(n+1)} + n y^{(n)}\right] - m^2 y^{(n)} = 0$$

$$\Rightarrow (1+x^2)y^{(n+2)} + (2n+1)x\, y^{(n+1)} + (n^2-m^2)\, y^{(n)} = 0.$$

Putting $x=0$, we get

$$y^{(n+2)}(0) = (m^2-n^2)\, y^{(n)}(0).$$

Putting $n = 1, 2, 3, \dots, n-2$, and observing that $y(0)=1$,

$y'(0) = m$, $y''(0) = m^2$ we find,

$$y^{(3)}(0) = (m^2-1^2)\, y'(0) = m(m^2-1^2)$$

$$y^{(4)}(0) = (m^2-2^2)\, y''(0) = m^2(m^2-2^2)$$

$$y^{(5)}(0) = (m^2-3^2)\, y^{(3)}(0) = m(m^2-1^2)(m^2-3^2)$$

$$y^{(6)}(0) = (m^2-4^2)\, y^{(4)}(0) = m^2(m^2-2^2)(m^2-4^2) \dots$$