Department of Mathematical Sciences

Rajiv Gandhi Institute Of Petroleum Technology, Jais

REAL ANALYSIS & CALCULUS (MA 111)

Week 2 / August 2023	Problem Set 2	GR

Real Analysis

Real sequences

Tutorial Problems

1. Using the definition of limit, show that $\lim\limits_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0.$

2. Find $\lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right).$

3. (i) Give two examples of sequences of rational numbers that converge to irrational numbers.

   (ii) Give two examples of sequences of irrational numbers that converge to rational numbers.

   (iii) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n+v_n\}$ is convergent.

   (iv) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_nv_n\}$ is convergent.

4. Use Sandwich theorem to prove that

   (i) $\lim\,(2^n+3^n)^{\frac{1}{n}}=3,$

   (ii) $\lim\left(\sqrt{n+1}-\sqrt{n}\right)=0.$

5. Show that

   (i) $\lim\sqrt[n]{n+1}=1,$

   (ii) $\lim\sqrt[n+1]{n}=1,$

   (iii) $\lim\frac{(n+1)^{2n}}{(n^2+1)^n}=e^2,$

   (iv) $\lim\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\right\}^{n^2}=e^6.$

   [Hint. Use $\lim n^{\frac{1}{n}}=1,\ \ \lim\left(1+\frac{1}{n}\right)^n=e$. If $x_n>0$ and $\lim x_n=x>0$ for all $n\in\mathbb{N}$ and $\lim y_n=y$, then $\lim\,(x_n)^{y_n}=x^y$.]

6. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \geq 1$. Show that

   (i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$;

   (ii) the sequence $\{u_n\}$ is monotone decreasing if $0 < u_1 > 3$.

   Find $\lim u_n$.

7. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

   [Hint. Monotone increasing and bounded above implies convergent.]

8. If $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)$ for all $n \geq 1$. Prove that the sequence $\{x_n\}$ converges to 3.

   [Hint. Monotone decreasing and bounded below implies convergent.]

9. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

   (i) Prove that the sub-sequences $\{u_{2n+1}\}$ and $\{u_{2n}\}$ converges to a common limit.

   (ii) Find $\lim u_n$.

10. Establish the convergence and find the limits of the following sequences

    (i) $\left(1 + \frac{1}{3n+1}\right)^n$,

    (ii) $\left(1 + \frac{1}{n^2+2}\right)^{n^2}$.

    [Hint. Approach through sub-sequence.]

11. Let $\{u_n\}$ be a bounded sequence and $\lim v_n = 0$. Prove that $\lim u_n v_n = 0$. Utilise this to prove that

    (i) $\lim \frac{\sin n}{n} = 0$.

    (ii) $\lim \frac{(-1)^n n}{n^2+1} = 0$.

12. Prove that

    (i) $\lim n^{\frac{1}{n}} = 1$.

    (ii) $\lim \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

    [Hint. Use the **Theorem**: Let $\{u_n\} > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \ell$ (finite or infinite). Then $\lim \sqrt[n]{u_n} = \ell$]

■ Assignment Problems

1. Prove that the sequence $\{u_n\}$ defined by

   (i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1+3u_2}{4}$,

   (ii) $0 < u_1 < u_2$ and $u_{n+2} = \frac{u_{n+1}+2u_n}{3}$ for $n \geq 1$, converges to $\frac{2u_1+3u_2}{5}$

   [**Hint.** Observe that $u_3 - u_2 = \left(-\frac{1}{3}\right)(u_2 - u_1), \ldots, u_n - u_{n-1} = \left(-\frac{1}{3}\right)^{n-2}(u_2 - u_1)$. Add all these equations and get $u_n - u_1 = \frac{3}{4}(u_2 - u_1)\left[1 - \left(-\frac{1}{3}\right)^{n-1}\right]$]

2. Prove that the sequence $\{u_n\}$ defined by

   (i) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to the limit $\sqrt[3]{u_1u_2^2}$,

   (ii) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to the limit $\frac{3}{\left(\frac{1}{u_1}+\frac{2}{u_2}\right)}$.

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