Problem set-6 Solutions

(1) (i) $f(x) = 1 - |x-1| \qquad x \in [0,2]$

$$\Rightarrow f(x) = \begin{cases} 2-x & 1 \le x \le 2 \\ x & 0 \le x \le 1 \end{cases}$$

Then we see that f is not differentiable at $x=1$, hence does not satisfy conditions of Rolle's theorem.

(ii) $f(x) = 1-(x-1)^{2/3} \qquad x \in [0,2]$

$f'(x) = -\frac{2}{3}(x-1)^{-1/3}$

clearly f' does not exist at $x=1$, as we get division by zero.

(2) (i) $f(x) = \frac{x}{x-1} \qquad x \in [2,4]$

- f is continuous being a rational function with denominator $x-1 \neq 0$ on $[2,4]$.
- f is also differentiable on $[2,4]$ as

$$f'(x) = \frac{(x-1)-x}{(x-1)^2} = -\frac{1}{(x-1)^2} \quad \text{on } [2,4].$$

Now $\frac{f(4)-f(2)}{4-2} = f'(c)$

$$\Rightarrow \frac{4/3 - 2}{2} = -\frac{1}{(c-1)^2}$$

$$\Rightarrow (c-1)^2 = 3$$

$$\Rightarrow c-1 = \pm\sqrt{3}$$

$$\Rightarrow c = 1 \pm \sqrt{3}$$

Now $c = 1+\sqrt{3} \in [2,4]$.

$\therefore f'(1+\sqrt{3}) = \frac{f(4)-f(2)}{4-2}$. So the conclusion holds true.

Similarly solve (ii) — (iv).

(3) Let $f(x) = (x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3$

We have $f(1) = -1-8-27 < 0$

$f(4) = 27+8+1 > 0$

$\therefore$ By intermediate value theorem $\exists$ $c \in [1,4]$ such that $f(c) = 0$. (Note that f is continuous on $[1,4]$ as it is a polynomial)

$\therefore$ c is a root of the given equation. If possible, let c' be another root of $f(x)$. Then

$$f(c) = f(c') = 0$$

Since f is differentiable every where, by Rolle's theorem

$f'(d) = 0$ for some d lying between c and c'.

But $f'(d) = 3(d-1)^2 + 3(d-2)^2 + 3(d-3)^2 + 3(d-4)^2 > 0$

$\therefore$ we get a contradiction.

Hence c is the only root.

Let $f(x) = e^x \sin x + 1$.

Let $f(a) = f(b) = 0$. Then by Rolle's theorem

$f'(c) = 0$ for some $c \in (a, b)$.

$\Rightarrow e^c \sin c + e^c \cos c = 0$.

$\Rightarrow \sin c + \cos c = 0$ [$\because e^c$ is never zero].

$\Rightarrow \cos c (1 + \tan c) = 0$. ⊗

$\Rightarrow 1 + \tan c = 0$ because

$\cos c = 0 \Rightarrow \sin c = \pm 1$

$\therefore \sin c + \cos c \neq 0$, a contradiction to ⊗.

$\therefore c$ is a root of $1 + \tan x = 0$. Proved

(5) Let $c \in (a,b)$.

$$0 \le \left|\frac{f(x)-f(c)}{x-c}\right| \le \frac{L|x-c|^{\alpha}}{|x-c|} = L|x-c|^{\alpha-1}$$

$\because$ $\alpha>1$, $\alpha-1>0$

$\therefore$ $$\lim_{x\to c} (x-c)^{\alpha-1} = 0.$$

$\therefore$ By Sandwich theorem

$$\lim_{x\to c}\left|\frac{f(x)-f(c)}{x-c}\right| = 0$$

$\Rightarrow$ $f'(c) = 0$.

$\Rightarrow$ f is constant on (a,b).

Moreover the condition implies that f is continuous on $[a,b]$.

It follows that f is constant on $[a,b]$.

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(6) Let $f(x) = x^n$ on $[a,b]$.

Then f is continuous on $[a,b]$ and differentiable on (a,b). By LMVT,

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for some } c \in (a,b).$$

$\Rightarrow$ $b^n - a^n = n.c^{n-1}(b-a)$

$\because$ $c \in (a,b)$ and $a>0$, $a^{n-1} \le c^{n-1} \le b^{n-1}$

$\therefore$ $n(b-a)a^{n-1} \le n(b-a)c^{n-1} \le n(b-a)b^{n-1}$

$\Rightarrow$ $n(b-a)a^{n-1} \le b^n - a^n \le n(b-a)b^{n-1}$ proved

(7) Let $x > 0$. We consider the function

$$f(y) = e^y \quad \text{on} \quad [0, x]$$

Then f is continuous on $[0, x]$ and differentiable on $(0, x)$.

$\therefore$ By LMVT,

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \quad \text{for some} \quad c \in (0, x)$$

$$\Rightarrow \quad \frac{e^x - 1}{x} = e^c$$

$$\Rightarrow \quad \log \frac{e^x - 1}{x} = c$$

$$\Rightarrow \quad \frac{1}{x} \log \frac{e^x - 1}{x} = \frac{c}{x}$$

$\because$ $c \in (0, x)$, $\quad 0 < c < x \quad \Rightarrow \quad 0 < \frac{c}{x} < 1$

$$\therefore \quad 0 < \frac{1}{x} \log \frac{e^x - 1}{x} < 1. \quad \text{proved.}$$

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(8) $\because$ $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), by ~~Rolle's theorem~~ IVT

$$f(c) = 0 \quad \text{for some} \quad c \in (a, b)$$

as 0 lies between $f(a)$ and $f(b)$, since $f(a) \cdot f(b) < 0$.

Let c' be another number in (a, b) such that $f(c') = 0$.

Then $\quad f(c) = f(c') = 0$

Since, the conditions of Rolle's theorem is satisfied on $[c, c']$ (if $c < c'$) or $[c', c]$ (if $c' < c$), so there exists d between c and c' such that $f'(d) = 0$, which is a contradiction as $f'(x) \neq 0 \; \forall x$ in $[a, b]$. $\therefore$ c is unique.

(9) Typo: c in $(0,1)$: Replace by c in (a,b). (B)

Applying MVT on $[a,c]$ we get

$$\frac{f(c)-f(a)}{c-a} = f'(d_1) \quad \text{for some } d_1 \in (a,c).$$

$$\Rightarrow f(c) = (c-a)\cdot f'(d_1), \quad d_1 \in (a,c)$$

Applying MVT on $[c,b]$ we get,

$$\frac{f(b)-f(c)}{b-c} = f'(d_2) \quad \text{for some } d_2 \in (c,b).$$

$$\Rightarrow f(c) = -(b-c)\, f'(d_2), \quad d_2 \in (c,b).$$

clearly $d_1 < d_2$. Applying MVT to f' on $[d_1, d_2]$,

$$\frac{f'(d_2)-f'(d_1)}{d_2-d_1} = f''(e) \quad \text{for some } e \in [d_1, d_2]$$

$$\Rightarrow f''(e) = \frac{1}{d_2-d_1}\left[-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}\right]$$

$$= -\frac{f(c)}{d_2-d_1}\left[\frac{1}{b-c} + \frac{1}{c-a}\right]$$

$\because$ $a < d_1 < c < d_2 < b$ and $d_1 < e < d_2$ and $f(c) < 0$,

we see that $f''(e) > 0$. Proved

(10) Let us apply MVT on $[\alpha_1, \frac{\alpha_1+\alpha_2}{2}]$ and $[\frac{\alpha_1+\alpha_2}{2}, \alpha_2]$ respectively to the function $f(x)$ to get

$$\frac{f\left(\frac{\alpha_1+\alpha_2}{2}\right) - f(\alpha_1)}{\frac{\alpha_2-\alpha_1}{2}} = f'(c_1)$$

$$\frac{f(\alpha_2) - f\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\frac{\alpha_2-\alpha_1}{2}} = f'(c_2)$$

for some $c_1 \in \left(\alpha_1, \frac{\alpha_1+\alpha_2}{2}\right)$ and $c_2 \in \left(\frac{\alpha_1+\alpha_2}{2}, \alpha_2\right)$.

Now, applying MVT to $f'(x)$ on $[c_1, c_2]$ we get,

$$\frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = f''(\alpha), \quad \alpha \in (c_1, c_2)$$

$$\Rightarrow \quad f'(c_2) - f'(c_1) = f''(\alpha).(c_2 - c_1) \geqslant 0$$

$$\Rightarrow \quad f'(c_2) \geqslant f'(c_1)$$

$$\Rightarrow \quad f(\alpha_2) - f\left(\frac{\alpha_1+\alpha_2}{2}\right) \geqslant f\left(\frac{\alpha_1+\alpha_2}{2}\right) - f(\alpha_1)$$

$$\Rightarrow \quad f(\alpha_2) + f(\alpha_1) \geqslant 2 \cdot f\left(\frac{\alpha_1+\alpha_2}{2}\right)$$

$$\Rightarrow \quad f\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{2}\left[f(\alpha_1) + f(\alpha_2)\right]$$ proved

(11) Let $g(x) = x^2$ on $[0,1]$.

Then both f, g are continuous on $[0,1]$ and differentiable on $(0,1)$. Therefore by CMVT,

$$\frac{f(1)-f(0)}{g(1)-g(0)} = \frac{f'(c)}{g'(c)} \quad \text{for some } c \in (0,1)$$

$$\Rightarrow \frac{f(1)-f(0)}{1-0} = \frac{f'(c)}{2c}$$

$$\Rightarrow f(1)-f(0) = \frac{f'(c)}{2c}.$$

Hence $c \in (0,1)$ is a solution of the given equation.

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(12) (i) Let $f(x) = \cos x$. $f'(x) = -\sin x$ $f''(x) = -\cos x$

$f'''(x) = \sin x$.

Applying Taylor's theorem we get

$$\cos x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi)}{3!}x^3$$

for some $\xi \in (0, x)$.

$$\Rightarrow \cos x = 1 - \frac{x^2}{2} + \frac{x^3}{3!}\sin x$$

~~Now on $[-\pi, 0)$ we have $\sin x$~~

Now we have, $\sin x \leq 0$ on $[-\pi, 0)$

$x^3 < 0$

$\therefore x^3 \sin x > 0$ on $[-\pi, 0]$

And $x^3, \sin x \geqslant 0$ on $[0, \pi]$

$\therefore x^3 \sin x \geqslant 0$ on $[0, \pi]$

$\therefore \quad \dfrac{x^3 \sin x}{3!} \geq 0 \quad$ on $[-\pi, \pi]$

$\therefore \quad \cos x \geq 1 - \dfrac{x^2}{2} \quad$ on $[-\pi, \pi]$. proved

Similarly solve (ii) and (iii).

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(13) (i) $\displaystyle \lim_{x \to 0} \frac{\sinh x - \sin x}{x \sin^2 x} \quad \left(\frac{0}{0} \text{ form}\right)$

$\displaystyle = \lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin 2x + \sin^2 x} \quad \left(\frac{0}{0} \text{ form}\right)$

$\displaystyle = \lim_{x \to 0} \frac{\sinh x + \sin x}{2x \cos 2x + \sin 2x + \sin 2x} \quad \left(\frac{0}{0} \text{ form}\right)$

$\displaystyle = \lim_{x \to 0} \frac{\cosh x + \cos x}{-4x \sin 2x + 6 \cos 2x} \quad \left(\text{Not } \frac{0}{0} \text{ form}\right)$

$\displaystyle = \frac{1+1}{0+6} = \frac{2}{6} = \frac{1}{3}$ Ans

(ii) $\displaystyle \lim_{x \to \infty} \frac{x^n}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

$\displaystyle = \lim_{x \to \infty} \frac{n x^{n-1}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

$\vdots$

$\displaystyle = \lim_{x \to \infty} \frac{n!}{e^x} \quad$ (differentiating n times)

$= 0.$

(iii) $\displaystyle \lim_{x \to 0} \frac{\log \log (1-x^2)}{\log \log \cos x} \quad \left(\frac{-\infty}{-\infty} \text{ form}\right)$

$\displaystyle = \lim_{x \to 0} \frac{\frac{1}{\log(1-x^2)} \cdot \frac{1}{1-x^2} \cdot (-2x)}{\frac{1}{\log \cos x} \cdot \frac{1}{\cos x} \cdot (-\sin x)}$

$$= \lim_{x \to 0} \frac{2x \cos x \cdot \log(\cos x)}{\sin x \cdot (1-x^2)\log(1-x^2)}$$

$$= \lim_{x \to 0} \frac{2x \cos x}{\sin x} \cdot \lim_{x \to 0} \frac{\log(\cos x)}{(1-x^2)\log(1-x^2)}$$

$$= 2 \lim_{x \to 0} \frac{\cos x}{\frac{\sin x}{x}} \cdot \lim_{x \to 0} \frac{\log(\cos x)}{(1-x^2)\log(1-x^2)}$$

$$= 2 \cdot \frac{1}{1} \cdot \lim_{x \to 0} \frac{\log(\cos x)}{(1-x^2)\log(1-x^2)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= 2 \cdot \lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{\cancel{(1-x^2)} \cdot \frac{-2x}{\cancel{1-x^2}} + (-2x) \cdot \log(1-x^2)}$$

$$= \cancel{2} \cdot \lim_{x \to 0} \frac{\sin x}{\cos x \cdot x\left(1 + \log(1-x^2)\right)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x \cdot \left(1 + \log(1-x^2)\right)}$$

$$= 1 \cdot \frac{1}{1 \cdot 1} = 1. \quad \underline{\text{Ans}}$$

(iv) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$ $\qquad (1^{\infty} \text{ form})$

Let $A = \left(\frac{\sin x}{x}\right)^{1/x}$

$\Rightarrow \log A = \frac{1}{x} \log \frac{\sin x}{x}$

$\Rightarrow \lim_{x \to 0} \log A = \lim_{x \to 0} \frac{\log(\sin x / x)}{x}$ $\qquad \left(\frac{0}{0} \text{ form}\right)$

$$\Rightarrow \lim_{x\to 0} \log A = \lim_{x\to 0} \frac{\frac{x}{\sin x}\cdot(x\cos x - \sin x)}{1}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\cdot \lim_{x\to 0}\left(x\cos x - \sin x\right)$$

$$= 1\cdot(0-0) = 0.$$

$$\therefore \lim_{x\to 0} A = e^0 = 1.$$

(V) $\lim_{x\to \pi/2} (\sin x)^{\tan x}$ $\quad$ (1^∞ form)

Let $A = (\sin x)^{\tan x}$

$$\Rightarrow \log A = \tan x.\log(\sin x) = \frac{\sin x \log(\sin x)}{\cos x}$$

$$\therefore \lim_{x\to \pi/2} \log A = \lim_{x\to \pi/2} \frac{\sin x\cdot\log(\sin x)}{\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to \pi/2} \frac{\sin x\cdot\frac{\cos x}{\sin x} + \cos x\log(\sin x)}{-\sin x}$$

$$= \frac{0+0}{-1} = 0.$$

$$\therefore \lim_{x\to \pi/2} A = e^0 = 1.$$