Derivation Type 1:

Starting with the complete log-likelihood for derivations: plug into the equation at bottom of page 9 (of emh.pdf notes), using the general derivations from the top of page 9 to get to the plug-in point:

$L_C(\theta|\mathbf{Y}) = \prod_{i=1}^{n} L_{C_i}(\theta|X_i, Z_i)$

$L_C(\theta|\mathbf{Y}) = \prod_{i=1}^{n}\prod_{k=1}^{K} [Pr(X_i, Z_i = k)]^{1\{Z_i=k\}}$

$l_C(\theta|\mathbf{Y}) = \sum_{i=1}^{n}\sum_{k=1}^{K} 1\{Z_i = k\} LnPr(X_i, Z_i = k)$

$E\ [l_C(\theta|\mathbf{Y})|\mathbf{X};\theta^m] = \sum_{i=1}^{n}\sum_{k=1}^{K} E\ [1\{Z_i = k\}|\mathbf{X};\theta^m] LnPr(X_i, Z_i = k) = \sum_{i=1}^{n}\sum_{k=1}^{K} Pr(Z_i = k|X_i;\theta^m) LnPr(X_i, Z_i = k)$

$\sum_{i=1}^{N}\sum_{k\in K} Pr(Z_i = k|X_i;\theta^m)\cdot Ln(Pr(X_i \cap Z_i = k))$

Where

AA $\rightarrow 1\{X_i = A\} \rightarrow \frac{P_A^2}{P_A^2+2P_AP_O} \rightarrow LnP_A^2$

AO $\rightarrow 1\{X_i = A\} \rightarrow \frac{2P_AP_O}{P_A^2+2P_AP_O} \rightarrow Ln2P_AP_O$

BB $\rightarrow 1\{X_i = B\} \rightarrow \frac{P_B^2}{P_B^2+2P_BP_O} \rightarrow LnP_B^2$

BO $\rightarrow 1\{X_i = B\} \rightarrow \frac{2P_BP_O}{P_B^2+2P_BP_O} \rightarrow Ln2P_BP_O$

AB $\rightarrow 1\{X_i = AB\} \rightarrow 1 \rightarrow Ln2P_AP_B$

O $\rightarrow 1\{X_i = O\} \rightarrow 1 \rightarrow LnP_O^2$

$Q = n_A(\frac{2P_A^{2^m}}{P_A^{2^m}+2P_A^mP_O^m} LnP_A + \frac{2P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m} Ln2P_AP_O) + n_B(\frac{2P_B^{2^m}}{P_B^{2^m}+2P_B^mP_O^m} LnP_B + \frac{2P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m} Ln2P_BP_O)$
$+n_{AB}(Ln2P_AP_B) + n_O(2LnP_O) + \lambda(1 - P_A - P_B - P_O)$

$\frac{\partial Q}{\partial P_A} = n_A((\frac{2P_A^{2^m}}{P_A^{2^m}+2P_A^mP_O^m}(\frac{1}{P_A}) + \frac{2P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}(\frac{1}{P_A})) + n_{AB}(\frac{1}{P_A}) - \lambda = 0$

$= \frac{2n_A}{P_A}(\frac{P_A^{2^m}+P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}) + \frac{n_{AB}}{P_A} = \lambda$

$= \frac{1}{\lambda}[2n_A(\frac{P_A^{2^m}+P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}) + n_{AB}] = \hat{P_A}$

$\frac{\partial Q}{\partial P_B} = \frac{2n_B}{P_B}([\frac{P_B^{2^m}}{P_B^{2^m}+2P_B^mP_O^m}] + [\frac{P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}]) + \frac{n_{AB}}{P_B} - \lambda = 0$

$= \frac{1}{\lambda}[2n_B(\frac{P_B^{2^m}+P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}) + n_{AB}] = \hat{P_B}$

$\frac{\partial Q}{\partial P_O} = \frac{2n_A}{P_O}([\frac{P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}]) + \frac{2n_B}{P_O}([\frac{P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}]) + \frac{2n_O}{P_O} - \lambda = 0$

$= \frac{1}{\lambda}[2n_A[\frac{P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}] + 2n_B[\frac{P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}] + 2n_O] = \hat{P_O}$

$P_A + P_B + P_O = 1$

$[2n_A[\frac{P_A^{2^m}+P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}] + n_{AB} + 2n_B[\frac{P_B^{2^m}+P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}] + n_{AB} + 2n_A[\frac{P_A^mP_O^m}{P_A^{2^m}+2P_A^mP_O^m}] + 2n_B[\frac{P_B^mP_O^m}{P_B^{2^m}+2P_B^mP_O^m}] + 2n_O] = \lambda$

$[2n_B + 2n_{AB} + 2n_A + 2n_O] = \lambda$

$2[n_B + n_{AB} + n_A + n_O] = \lambda$

$2N = \lambda$

Derivation Type 2:

$L_C(P_A, P_B, P_O|\bar{Y}) = \binom{N}{n_{AA}n_{AB}n_{AO}n_{BB}n_{BO}n_{OO}} P_A^{2n_{AA}} 2(P_AP_O)^{n_{AO}} 2(P_AP_B)^{n_{AB}} P_B^{2n_{BB}} 2(P_BP_O)^{n_{BO}} P_O^{2n_{OO}}$

$l_C = Ln\binom{N}{n_{AA}n_{AB}n_{AO}n_{BB}n_{BO}n_{OO}}$
$+2n_{AA}LnP_A + Ln2 + n_{AO}Ln(P_AP_O) + Ln2 + n_{AB}Ln(P_AP_B) + 2n_{BB}LnP_B + Ln2 + n_{BO}Ln(P_BP_O) + 2n_{OO}LnP_O$

$n_{AO}Ln(P_AP_O) \rightarrow n_{AO}(LnP_A + LnP_O) \rightarrow n_{AO}LnP_A + n_{AO}LnP_O$

$n_{AB}Ln(P_AP_B) \rightarrow n_{AB}(LnP_A + LnP_B) \rightarrow n_{AB}LnP_A + n_{AB}LnP_B$

$n_{BO}Ln(P_BP_O) \rightarrow n_{BO}(LnP_B + LnP_O) \rightarrow n_{BO}LnP_B + n_{BO}LnP_O$

$Q(P_A, P_B, P_O; P_A^m, P_B^m, P_O^m; Y) = E[l_C(P_A, P_B, P_O|Y)|\bar{X}; P_A^m, P_B^m, P_O^m]$
$= E[Ln\binom{N}{n_{AA}n_{AB}n_{AO}n_{BB}n_{BO}n_{OO}}$
$+2n_{AA}LnP_A + n_{AO}Ln(P_AP_O) + n_{AB}Ln(P_AP_B) + 2n_{BB}LnP_B + n_{BO}Ln(P_BP_O) + 2n_{OO}LnP_O|\bar{X}; P_A^m, P_B^m, P_O^m]$

$= E[Ln\binom{N}{n_{AA}n_{AB}n_{AO}n_{BB}n_{BO}n_{OO}}|\bar{X}; P_A^m, P_B^m, P_O^m]$
$+E[2n_{AA}LnP_A|\bar{X}; P_A^m, P_B^m, P_O^m] + E[Ln2|\bar{X}; P_A^m, P_B^m, P_O^m] + E[n_{AO}LnP_A|\bar{X}; P_A^m, P_B^m, P_O^m]$
$+E[n_{AO}LnP_O|\bar{X}; P_A^m, P_B^m, P_O^m] + ...$

$= C + 2LnP_A(E[n_{AA}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + LnP_A(E[n_{AO}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + LnP_O(E[n_{AO}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + LnP_A(E[n_{AB}|\bar{X}; P_A^m, P_B^m, P_O^m]$
$+C + LnP_B(E[n_{AB}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + 2LnP_B(E[n_{BB}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + LnP_B(E[n_{BO}|\bar{X}; P_A^m, P_B^m, P_O^m] + C + LnP_O(E[n_{BO}|\bar{X}; P_A^m, P_B^m, P_O^m]$
$+C + 2LnP_O(E[n_{OO}|\bar{X}; P_A^m, P_B^m, P_O^m]$

$Q = 2E(n_{AA}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_A^{m+1} + E(n_{AO}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_A^{m+1} + E(n_{AB}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_A^{m+1}$
$+2E(n_{BB}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_B^{m+1} + E(n_{BO}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_B^{m+1} + E(n_{AB}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_B^{m+1}$
$+2E(n_{OO}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_O^{m+1} + E(n_{AO}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_O^{m+1}$
$+E(n_{BO}|\bar{X}; P_A^m, P_B^m, P_O^m)LnP_O^{m+1} - \lambda(1 - P_A^{m+1} - P_B^{m+1} - P_O^{m+1})$

Q is now a function of $P_A^{m+1}, P_B^{m+1}, P_O^{m+1}$; $Q = f(P_A^{m+1}, P_B^{m+1}, P_O^{m+1})$, take the partial derivatives of $P_A^{m+1}, P_B^{m+1}, P_O^{m+1}$ and set them equal to zero to maximize: $\frac{\partial Q}{\partial P_A^{m+1}} = 0$, $\frac{\partial Q}{\partial P_B^{m+1}} = 0$, $\frac{\partial Q}{\partial P_O^{m+1}} = 0$

Need the expectations, find the expectation values of: $n_{AA}, n_{AO}, n_{BB}, n_{BO}, n_{AB}, n_{OO}$, using observed data and current iterations value of: P_A^m, P_B^m, P_O^m

$E(n_{AA}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_A Pr(AA_{geno}|A_{pheno}) = n_A \frac{Pr(AA_{geno} \cap A_{pheno})}{Pr(A_{pheno})} = n_A \frac{P_A^{m^2}}{P_A^{m^2} + 2P_A^m P_O^m}$
$E(n_{AO}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_A Pr(AO_{geno}|A_{pheno}) = n_A \frac{Pr(AO_{geno} \cap A_{pheno})}{Pr(A_{pheno})} = n_A \frac{2P_A P_O}{P_A^{m^2} + 2P_A^m P_O^m}$
$E(n_{BB}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_B Pr(BB_{geno}|B_{pheno}) = n_B \frac{Pr(BB_{geno} \cap B_{pheno})}{Pr(B_{pheno})} = n_B \frac{P_B^{m^2}}{P_B^{m^2} + 2P_B^m P_O^m}$
$E(n_{BO}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_B Pr(BO_{geno}|B_{pheno}) = n_B \frac{Pr(BO_{geno} \cap B_{pheno})}{Pr(B_{pheno})} = n_B \frac{2P_B P_O}{P_B^{m^2} + 2P_B^m P_O^m}$
$E(n_{AB}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_{AB_{pheno}}$
$E(n_{OO}|\bar{X}; P_A^m, P_B^m, P_O^m) = n_{O_{pheno}}$

For: $\hat{P}_A^m$
$\frac{\partial Q}{\partial P_A^m}$
$= (\frac{1}{P_A^m})\ [2E(n_{AA}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] - \lambda = 0$
$= (\frac{1}{P_A^m})\ [2E(n_{AA}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \lambda$
$= (\frac{1}{\lambda})\ [2E(n_{AA}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \hat{P}_A^m$
$= (\frac{1}{\lambda})\ [2(n_A \frac{P_A^{m^2}}{P_A^{m^2} + 2P_A^m P_O^m}) + (n_A \frac{2P_A^m P_O^m}{P_A^{m^2} + 2P_A^m P_O^m}) + (n_{AB_{pheno}})] = \hat{P}_A^m$

For: $\hat{P}_B^m$
$\frac{\partial Q}{\partial P_B^m}$
$= (\frac{1}{P_B^m})\ [2E(n_{BB}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] - \lambda = 0$
$= (\frac{1}{P_B^m})\ [2E(n_{BB}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \lambda$
$= (\frac{1}{\lambda})\ [2E(n_{BB}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AB}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \hat{P}_B^m$
$= (\frac{1}{\lambda})\ [2(n_B \frac{P_B^{m^2}}{P_B^{m^2} + 2P_B^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{P_B^{m^2} + 2P_B^m P_O^m}) + (n_{AB_{pheno}})] = \hat{P}_B^m$

For: $\hat{P}_O^m$
$\frac{\partial Q}{\partial P_O^m}$
$= (\frac{1}{P_O^m})\ [2E(n_{OO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m)] - \lambda = 0$
$= (\frac{1}{P_O^m})\ [2E(n_{OO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \lambda$
$= (\frac{1}{\lambda})\ [2E(n_{OO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{AO}|\bar{X}, P_A^m, P_B^m, P_O^m) + E(n_{BO}|\bar{X}, P_A^m, P_B^m, P_O^m)] = \hat{P}_O^m$
$= (\frac{1}{\lambda})\ [2(n_{O_{pheno}}) + (n_A \frac{2P_A^m P_O^m}{P_A^{m^2} + 2P_A^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{P_B^{m^2} + 2P_B^m P_O^m}] = \hat{P}_O^m$

$\hat{P}_A^m + \hat{P}_B^m + \hat{P}_O^m = 1$

$(\frac{1}{\lambda})\ [2(n_A \frac{{P_A^m}^2}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_{AB_{pheno}})]$
$+(\frac{1}{\lambda})\ [2(n_B \frac{{P_B^m}^2}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_{AB_{pheno}})]$
$+(\frac{1}{\lambda})\ [2(n_{O_{pheno}}) + (n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m})] = 1$

$(\frac{1}{\lambda})\ [[2(n_A \frac{{P_A^m}^2}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_{AB_{pheno}})]$
$+[2(n_B \frac{{P_B^m}^2}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_{AB_{pheno}})]\ +[2(n_{O_{pheno}}) + (n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m})]] = 1$

$(\frac{1}{\lambda})\ [2(n_A \frac{{P_A^m}^2}{{P_A^m}^2 + 2P_A^m P_O^m}) + 2(n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + 2(n_{AB_{pheno}})\ +2(n_B \frac{{P_B^m}^2}{{P_B^m}^2 + 2P_B^m P_O^m})$
$+2(n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m}) + 2(n_{O_{pheno}})] = 1$

$2[(n_A \frac{{P_A^m}^2}{{P_A^m}^2 + P_A^m P_O^m}) + (n_A \frac{2P_A^m P_O^m}{{P_A^m}^2 + 2P_A^m P_O^m}) + (n_{AB_{pheno}})\ +(n_B \frac{{P_B^m}^2}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_B \frac{2P_B^m P_O^m}{{P_B^m}^2 + 2P_B^m P_O^m}) + (n_{O_{pheno}})] = \lambda$

$\lambda = 2N$

Derivation Type 3:
A $\rightarrow$ 1
B $\rightarrow$ 2
AB $\rightarrow$ 3
O $\rightarrow$ 4

$$Q(P, P^m) = \sum_{i=1}^{N} \sum_{k \in K} Pr(Z_i = k | X_i; P^m) \cdot Ln(Pr(X_i, Z_i = k))$$

$\sum_{i=1}^{N} 1\{X_i = A\} ln(Pr(X_i = A, Z_i = AA) Pr(X_i = A, Z_i = AO)) +$
$1\{X_i = B\} ln(Pr(X_i = B, Z_i = BB) Pr(X_i = B, Z_i = BO)) +$
$1\{X_i = AB\} ln(Pr(X_i = AB, Z_i = AB) +\ 1\{X_i = O\} ln(Pr(X_i = O, Z_i = O)$

$\sum_{i=1}^{n} 1\{X_i = 1\}\ [\frac{P_A^m}{2P_A^m(1 - P_A^m - P_B^m)} ln(P_A^2)\ +(1 - \frac{P_A^m}{2P_A^m(1 - P_A^m - P_B^m)}) ln(2P_A(1 - P_A - P_B))]$
$\rightarrow \sum_{i=1}^{n} 1\{X_i = 1\}\ \frac{1}{2(1 - P_A^m - P_B^m)} Ln(P_A^2)\ +(1 - \frac{1}{2(1 - P_A^m - P_B^m)}) ln(2P_A(1 - P_A - P_B))]$

$\sum_{i=1}^{n} 1\{X_i = 2\}\ [\frac{P_B^m}{2P_B^m(1 - P_A^m - P_B^m)} ln(P_B^2)\ +(1 - \frac{P_B^m}{2P_B^m(1 - P_A^m - P_B^m)}) ln(2P_B(1 - P_A - P_B))]$
$\rightarrow \sum_{i=1}^{n} 1\{X_i = 2\}\ \frac{1}{2(1 - P_A^m - P_B^m)} Ln(P_B^2)\ +(1 - \frac{1}{2(1 - P_A^m - P_B^m)}) ln(2P_B(1 - P_A - P_B))]$

$$\sum_{i=1}^{n} 1\{X_i = 3\}\ [Ln(2(P_A P_B))]$$

$$\sum_{i=1}^{n} 1\{X_i = 4\}\ [Ln(1 - P_A - P_B)^2]$$