Theory

April 29, 2016

1 Imaging Projections Theory

This page attempts to gather the key maths combining snapshots, facets, w-reprojection and Sault-style image transformations in order to minimse the image transformation cost while retaining image quality.

This is under the assumption that for imaging, we are going to do the following steps:

1. Rotate the phase centre overhead in order to minimize w-values
2. Use image-based shifts in order to shift the original phase centre into the middle again
3. Use Sault's uv-translation in order to roughly correct projection skews
4. Use Cornwell's w-projection in gridding to deal with remaining w offsets
5. Do reprojection in order to correct the remaining image skews

2 Phase Centre Rotation

The basic visibility equation for a sky sphere is given by (following roughly "Sythesis Imaging, Course Notes from NRAO Summer School", 1985):

$$V(r_1, r_2) = \int I(s) e^{-2\pi i s \cdot \frac{r_1 - r_2}{\lambda}} \, d\Omega$$

We normally use visibilities relative to a certain phase centre $s = s_0 + \sigma$ such that:

$$\begin{aligned} V(r_1, r_2) &= e^{-2\pi i s_0 \cdot \frac{r_1 - r_2}{\lambda}} \int I(s) e^{-2\pi i \sigma \cdot \frac{r_1 - r_2}{\lambda}} \, d\Omega \\ V_{s_0}(r_1, r_2) &= \int I(s) e^{-2\pi i \sigma \cdot \frac{r_1 - r_2}{\lambda}} = e^{2\pi i s_0 \cdot \frac{r_1 - r_2}{\lambda}} V(r_1, r_2) \end{aligned}$$

We can easily transform visibilites concerning a phase centre s_0 to a new phase centre $s_0' = R s_0$ for some rotation matrix R:

$$\begin{aligned} V_{s_0'}(r_1, r_2) &= e^{2\pi i s_0' \cdot \frac{r_1 - r_2}{\lambda}} V(r_1, r_2) \\ &= e^{2\pi i (R s_0 - s_0) \cdot \frac{r_1 - r_2}{\lambda}} V_{s_0}(r_1, r_2) \end{aligned}$$

Because we can choose R arbitrarily at this step, we will minimise $s_0' \cdot (r_1 - r_2) = R s_0 \cdot (r_1 - r_2)$ for the entirety of the baselines. This will yield us minimal w values for w-projection later.

In practice, this means that we optimise the phase centre to be perpindicular to most baselines. Of course, due to the arrangement of the antennas on the earth's surface, the optimal choice for s_0' will most likely be the zenith direction at the telescope's centre.

3 Projection

Let us now rotated all coordinates with a rotation matrix S such that $Ss'_0 = (0,0,1)$. We assign $SR(r_1 - r_2)/\lambda = (u,v,w)$ for the baseline coordinates concerning the new phase centre, and $(l,m,n) = SRs = SR\sigma + (0,0,1)$ for the sky directional cosines relative to the new phase centre.

If we replace the parameters to V and I respectively, we get the following form:

$$V_{s'_0}(u,v,w) = \int I_{s'_0}(l,m) e^{-2\pi i[ul+vm+w(n-1)]}\, d\Omega$$

Now let us project the sky coordinates to a tangent plane of the unity sphere at s'_0 with coordinates $(l,m,1)$. This gives us the visibility equation from Cornwell at al "The noncoplanar baselines effect in radio interferometry: The W-projection algorithm", 2008:

$$V_{s'_0}(u,v,w) = \int \frac{I_{s'_0}(l,m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i\left[ul+vm+w(\sqrt{1-l^2-m^2}-1)\right]}\, dl\, dm$$

3.1 w-projection

At this point, we would like to do an FFT and recover the I function in order to construct an image of the sky. This can be accomplished using Tim Cornwell's w-reprojection algorithm (from the paper cited above), which reformulates the above formula as:

$$\begin{aligned} V_{s'_0}(u,v,w) &= \int \frac{I_{s'_0}(l,m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i w(\sqrt{1-l^2-m^2}-1)} e^{-2\pi i(ul+vm)}\, dl\, dm \\ &= \int \frac{I_{s'_0}(l,m)}{\sqrt{1-l^2-m^2}} G_{s'_0}(l,m,w) e^{-2\pi i(ul+vm)}\, dl\, dm \end{aligned}$$

at which point we can use FFT laws to show that:

$$\begin{aligned} V_{s'_0}(u,v,w) &= \tilde{G}_{s'_0}(u,v,w) * V_{s'_0}(u,v,w=0) \\ \Leftrightarrow \quad V_{s'_0}(u,v,w=0) &= \overline{\tilde{G}_{s'_0}(u,v,w)} * V_{s'_0}(u,v,w) \end{aligned}$$

As can be shown, low values of w cause the convolution function $\tilde{G}_{s'_0}(u,v,w)$ to drop to zero faster. This is the reason we insisted on minimising w before: If we keep the convolution operator small, computing the convolution is relatively inexpensive.

Once we have obtained the convolution result $V_{s'_0}(u,v,w=0)$, we can then easily determine the intensity distribution using the Fourier transformation:

$$I_{s'_0}(l,m) = \sqrt{1-l^2-m^2}\, \tilde{V}_{s'_0}(l,m,w=0)$$

However, this will yield us an image around $(l,m) = (0,0)$, which images the sky at s'_0. This - as we established - probably points towards the zenith, which does not need to be the direction we are interested in. So instead, we need to shift our desired viewing direction into view. Easiest way would be by shifting the phase centre again, but we do not want to change our w values, so we need to find another way.

4 Image-plane rotation

We want to recover an image that is as close as possible to what we would have gotten if we had chosen yet another phase centre s''_0. This will likely be close to our original phase centre s_0, but does not have to be equal - e.g. for faceting we might want an extra shift. So instead, we will now transform the image itself as described by Sault et al in "An approach to interferometric mosaicing", 1996.

Let us assume that we have yet another rotation matrix R' such that $s''_0 = R's'_0 = R'Rs$. When we originally projected, we mapped sky coordinates s to the plane such that $(l,m,n) = SRs$. If we had done this for our "ideal" projection, we would have chosen a different rotation matrix S' such that $S's_0'' = S'R's_0' = (0,0,1)$. This would have given us image coordinates $(l', m', n') = S'R'Rs$ for the new intensity function $I_{s_0''}(l', m')$.

If we now try to express (l, m, n) in terms of (l', m', n'), we get:

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = SRs = SR'^{-1}S'^{-1} \begin{pmatrix} l' \\ m' \\ n' \end{pmatrix}$$

let us assign names to the fields of $SR'^{-1}S'^{-1}$:

$$SR'^{-1}S'^{-1} = T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

We know that $n' = \sqrt{1 - l'^2 - m'^2}$ and n is equally redundant, so we can now obtain:

$$\begin{aligned} l &= t_{11}l' + t_{12}m' + t_{13}\sqrt{1 - l'^2 - m'^2} \\ m &= t_{21}l' + t_{22}m' + t_{23}\sqrt{1 - l'^2 - m'^2} \end{aligned}$$

So at this point we know that we could derive our desired intensity image $I_{s_0''}$ from $I_{s_0'}$ as follows:

$$I_{s_0''}(l', m') = I_{s_0'}(t_{11}l' + t_{12}m' + t_{13}\sqrt{1 - l'^2 - m'^2}, t_{21}l' + t_{22}m' + t_{23}\sqrt{1 - l'^2 - m'^2})$$

however we cannot actually exploit this, because we are still limited by the fact that the FFT will only give us the intensity for low values of l and m. Instead, we ought to approximate this function.

4.1 Shift

Let us call the two big constant shift $\Delta l = t_{13}$ and $\Delta m = t_{23}$. Note that we have only $\sqrt{1 - l'^2 - m'^2} \approx 1$, but we'll correct for that later. For the moment, let us attempt a first approximation of $I_{s_0''}(l', m')$, namely

$$I^{shift}(l', m') = I(l' + \Delta l, m' + \Delta m)$$

If we put this into our w-projection formula above we obtain:

$$\begin{aligned} V_{s_0'}(u, v, w) &= \int \frac{I^{shift}(l', m')}{\sqrt{1 - (l' + \Delta l)^2 - (m' + \Delta m)^2}} G_{s_0'}(l' + \Delta l, m' + \Delta m, w) e^{-2\pi i (u(l' + \Delta l) + v(m' + \Delta m))} \, dl' \, dm' \\ &= e^{-2\pi i (u \Delta l + v \Delta m)} \int \frac{I^{shift}(l', m')}{\sqrt{1 - (l' + \Delta l)^2 - (m' + \Delta m)^2}} G_{s_0'}(l' + \Delta l, m' + \Delta m, w) e^{-2\pi i (u l' + v m')} \, dl' \, dm' \\ &= e^{-2\pi i (u \Delta l + v \Delta m)} \left(\tilde{G}_{s_0'}^{shift}(u, v, w) * V_{s_0'}^{shift}(u, v, w = 0) \right) \end{aligned}$$

which is w-reprojection for the "shifted" visibility function

$$V_{s_0'}^{shift}(u, v, w) = e^{2\pi i (u \Delta l + v \Delta m)} V_{s_0'}(u, v, w)$$

Note that we have a shift in the parameters to G as well - G^{shift} is moved by $(\Delta l, \Delta m)$ relative to G and therefore we also have $\tilde{G}^{shift} \neq \tilde{G}$. Yet we can easily convert these using, again, simple FFT laws:

$$
\begin{aligned}
\tilde{G}^{shift}_{s'_0}(u,v,w) &= \int G^{shift}_{s'_0}(l,m,w)e^{-2\pi i(ul+vm)}\,dl\,dm \\
&= \int G_{s'_0}(l+\Delta l, m+\Delta m, w)e^{-2\pi i(ul+vm)}\,dl\,dm \\
&= e^{2\pi i(u\Delta l+v\Delta m)}\int G_{s'_0}(l,m,w)e^{-2\pi i(ul+vm)}\,dl\,dm \\
&= e^{2\pi i(u\Delta l+v\Delta m)}\tilde{G}_{s'_0}(u,v,w)
\end{aligned}
$$

Which can be inverted and used to determine $V^{shift}_{s'_0}(l,m,w=0)$, which yields the shifted image:

$$
I^{shift}_{s'_0}(l,m) = \sqrt{1-(l+\Delta l)^2-(m+\Delta m)^2}\,\tilde{V}^{shift}_{s'_0}(l,m,w=0)
$$

So we have three changes due to the shift - a phase rotation to determine $V^{shift}_{s'_0}(u,v,w)$, an equivalent rotation of the convolution function $\tilde{G}^{shift}_{s'_0}(u,v,w)$ and finally a (slightly) different multiplier to determine the final intensity.

4.2 Transform

At this point, we should find that the shifted intensity function $I^{shift}_{s'_0}$ images s''_0 at $(l',m')=(0,0)$ despite the fact that our phase tracking centre is still s'_0. However, as our coordinate transformation shows, this still means that we might get skewed results for $l'\neq 0$ or $m'\neq 0$. Therefore, let us derive an even closer approximation to the ideal intensity function $I^{shift}_{s''_0}$ that takes the linear translation of l' and m' into account.

Our new approximation function shall be:

$$
\begin{aligned}
I^{trans}_{s'_0}(l',m') &= I_{s'_0}(t_{11}l'+t_{12}m'+\Delta l, t_{21}l'+t_{22}m'+\Delta m) \\
&= I^{shift}_{s'_0}(t_{11}l'+t_{12}m', t_{21}l'+t_{22}m') \\
&= I^{shift}_{s'_0}\,(T_{2\times 2}\sigma_2)
\end{aligned}
$$

With $T_{2\times 2}$ the upper-left 2×2 sub-matrix of T as well as $\sigma_2=(l,m)$. We will also use $\Delta\sigma_2=(\Delta l,\Delta m)$ for brevity. This yields the visibility equation:

$$
V^{shift}_{s'_0}(u,v,w) = \int \frac{I^{trans}_{s'_0}(\sigma_2)}{\sqrt{1-(T_{2\times 2}\sigma_2+\Delta\sigma_2)^2}} G^{trans}_{s'_0}(\sigma_2,w)e^{-2\pi i(u,v)\cdot T_{2\times 2}\sigma_2}\,d\sigma_2
$$

where analogous to $I^{trans}_{s'_0}$ we have $G^{trans}_{s'_0}(\sigma_2,w) = G^{shift}_{s'_0}(T_{2\times 2}\sigma_2,w)$. This time, we cannot simply factor the difference out of the integral. Yet notice that we can pull $T_{2\times 2}$ to the side of (u,v) instead:

$$
\begin{pmatrix} u \\ v \end{pmatrix}\cdot T_{2\times 2}\sigma_2 = T^T_{2\times 2}\begin{pmatrix} u \\ v \end{pmatrix}\cdot\sigma_2
$$

So let us replace $T^T_{2\times 2}(u,v)=(u',v')$ to again obtain the equivalent of w-projection:

$$
\begin{aligned}
V^{trans}_{s'_0}(u',v',w) &= \int \frac{I^{trans}_{s'_0}(\sigma_2)}{\sqrt{1-(T_{2\times 2}\sigma_2+\Delta\sigma_2)^2}} G^{trans}_{s'_0}(\sigma_2,w)e^{-2\pi i(u',v')\cdot\sigma_2}\,d\sigma_2 \\
&= \tilde{G}^{trans}_{s'_0}(u',v',w) * V^{trans}_{s'_0}(u',v',w=0)
\end{aligned}
$$

with

$$V^{trans}_{s'_0}(T^T_{2\times 2}(u,v),w) = V^{shift}_{s_0}(u,v,w) \quad \text{or equivalently} \quad V^{trans}_{s'_0}(u',v',w) = V^{shift}_{s_0}\left((T^T_{2\times 2})^{-1}(u',v'),w\right)$$

So this means that we transform the u and v parameters of V^{shift}. Note that we are not actually changing the values as we did in the step before. Also note that w was not changed, and therefore the complexity of the w-reprojection has stayed the same.

This leaves the convolution function. In order to determine $\tilde{G}^{trans}_{s'_0}(u',v',w)$, we have to take the coordinate scale into account. This is however yet again a straightforward FFT transformation:

$$\begin{aligned}
\tilde{G}^{trans}_{s'_0}(u,v,w) &= \int G^{trans}_{s'_0}(\sigma_2,w)e^{-2\pi i(u,v)\cdot\sigma_2}\,d\sigma_2 \\
&= \int G^{shift}_{s'_0}(T_{2\times 2}\sigma_2,w)e^{-2\pi i(u,v)\cdot\sigma_2}\,d\sigma_2 \\
&= \frac{1}{|\det(T_{2\times 2})|}\int G^{shift}_{s'_0}(\sigma_2,w)e^{-2\pi i\left(T^T_{2\times 2}\right)^{-1}(u,v)\cdot\sigma_2}\,d\sigma_2 \\
&= \frac{1}{|\det(T_{2\times 2})|}\tilde{G}^{shift}_{s'_0}\left(\left(T^T_{2\times 2}\right)^{-1}(u,v),w\right)
\end{aligned}$$

Note that we know that $|\det(T_{2\times 2})| < 1$ and therefore $\det\left(\left(T^T_{2\times 2}\right)^{-1}\right) > 1$, which means $G^{trans}_{s'_0}$ is actually smaller than $G^{shift}_{s'_0}$. Therefore this does not increase the computational complexity of w-projection. This is because we are stretching the image plane, which means compressing the uv-plane.

At this point, we can determine the transformed intensity function by:

$$\begin{aligned}
I^{trans}_{s'_0}(l,m) &= \sqrt{1-(T_{2\times 2}\sigma_2+\Delta\sigma_2)^2}\,\tilde{V}^{trans}_{s'_0}(l,m,w=0) \\
&= \sqrt{1-(T_{2\times 2}\sigma_2+\Delta\sigma_2)^2}\frac{1}{|\det(T_{2\times 2})|}\,\tilde{V}^{shift}_{s'_0}(l,m,w=0)
\end{aligned}$$

Notice that we need to divide the fourier transform result by the determinant due to fact that $V^{trans}_{s'_0}$ has different scaling in (u,v) than $V^{shift}_{s'_0}$.

5 Reprojection

The transformed intensity function comes fairly close to the ideal. In fact, we now have:

$$\begin{aligned}
I_{s''_0}(l',m') &= I_{s'_0}\left(t_{11}l'+t_{12}m'+t_{13}\sqrt{1-l'^2-m'^2}, t_{21}l'+t_{22}m'+t_{23}\sqrt{1-l'^2-m'^2}\right) \\
&= I^{trans}_{s'_0}\left(l'+t_{13}\left(\sqrt{1-l'^2-m'^2}-1\right), m'+t_{23}\left(\sqrt{1-l'^2-m'^2}-1\right)\right)
\end{aligned}$$

Which means that the farther we get from the central point, the more our image skews in one direction (the image location of the original phase centre). See Sault's paper for a geometric interpretation of this.

However, this is a fairly local transformation that we can easily undo. In fact, consider that we restrict our "field of view" to $|(l',m')| < \theta$. Then we can show that we can equally restrict the range at which we have to query $I^{trans}_{s'_0}$:

$$\begin{aligned}
\left|\begin{pmatrix} l'+t_{13}\left(\sqrt{1-l'^2-m'^2}-1\right) \\ m'+t_{23}\left(\sqrt{1-l'^2-m'^2}-1\right)\end{pmatrix}\right| &= \left|\sigma_2+\left(\sqrt{1-l'^2-m'^2}-1\right)\begin{pmatrix}t_{13}\\t_{23}\end{pmatrix}\right| \\
&< \theta+\left(1-\sqrt{1-\theta}\right)\left|\begin{pmatrix}t_{13}\\t_{23}\end{pmatrix}\right|
\end{aligned}$$

Which can not be very large, given that the transform $|(t_{13},t_{23})| < 1$ and the facet size $\theta \ll 1$.

6 Element reception patterns

Actually our visibility function has an additional direction-dependent term for the reception pattern (primary beam) of the involved antennas:

$$V(r_1, r_2) = \int I(s)A(s)e^{-2\pi i s \cdot \frac{r_1 - r_2}{\lambda}} \, d\Omega$$

This pattern is typically centered at the original phase centre s_0, which would be a straightforward convolution in (l, m) had we not moved the phase centre. However, our transformation make this more complicated, so we most likely need to correct the A kernel as well...

TODO

7 Anti-aliasing kernel

So far we have always worked with continous functions, which can be exactly shifted and transformed as described. However, the whole point of this is that we could use grids, which can be efficiently transformed into images using FFT algorithms. This however introduces a new problem: By sampling the visibility plane in a pixel pattern, we introduce "aliasing": We cannot distinguish sources any more that are exactly our chosen image size apart. Therefore, the FFT might "alias" sources from outside our chosen field of view onto the image plane.

Therefore, we additionally add an anti-aliasing convolution before the FFT that surpresses sources from outside our chosen field of view. As this is concerned with problems that come fairly late in the pipeline, the kernel probably does not need extra transformations.

TODO