COSC341 - Assignment 3

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1 Longest Path Problem

The **Longest Path Problem** is for finding the maximum length simple path in a graph, $G(V, E)$ with $|V| = n : n \in \mathbb{N}$. The graph is undirected and all edges weights are assumed to be 1.

1.1 Simple Path

Suppose we want the **simple path** between two vertices s and t. A simple path is one which every node in the path is never visited more than once and that there is a path between two consecutive nodes.

Therefore, a simple path between two nodes $s, t \in V$ is the sequence of vertices (v_1,v_2,v_3,....,v_k) that satisfies these conditions:

- $s = v_1$ and $t = v_k$
- Each consecutive nodes (v_i, v_{i+1}) there is an edge $e = (v_i, v_{i+1}) \in E$
- No node appears more than once in the sequence.

1.2 Longest Simple Path and Decision Version

The figure below shows the longest simple path in blue, beginning in the upper left:

In the decision version of the Longest Path Problem, we input a graph G, and an integer k, and output a **yes or no** if a simplest path of **at least** length k exists in the graph.

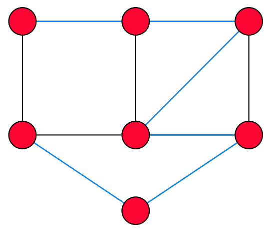

Figure 1: Longest simple path

1.3 NP

A problem is in `NP` if we can verify the solution in polynomial time, and if a corresponding certificate can be produced from the decision version of the Longest Path Problem. If we can verify this certificate in polynomial time, then the Longest Path Problem is in `NP`. The certificate is a path of at least length k with no duplicate nodes.

To check that the path has no duplicate nodes we can iterate over each node in the path for every node, $O(k^2)$ complexity. To check the length of the path we just need to count each node and check it is greater than or equal to k, this operation has a $O(k)$ complexity.

In some implementations, getting the length of a list is constant complexity $O(1)$. In most cases and the worst case we can check a certificate with is $O(n^2)$ complexity, therefore the Longest Path Problem is in `NP`.

2 NP-Complete

If a problem, X, is in `NP-Complete` and X polynomially reduces to another problem Y in `NP`, then Y is also in `NP-Complete`.

We have shown that the Longest Path Problem is in `NP`, all that is left is to show that another problem in `NP-Complete` polynomial reduces to the Longest Path Problem.

2.1 Hamiltonian Path

A **Hamiltonian path** is a sequence that visits each node **exactly** once. The **Hamiltonian Path Problem** is a known `NP-Complete` problem that asks if a Hamiltonian path exists for a given graph.

2.2 Reduction to Longest Path Problem

With any instance of the Hamiltonian Path Problem of an undirected graph of size n, we can instead ask if a longest path of size $k = n - 1$ exists.

When we set k to $n-1$, we are asserting that the longest path is equal to the Hamiltonian path.

2.3 Correctness

To prove our reduction we must show that:

$$\text{Correct decision version of Hamiltonian path}$$
$$\updownarrow$$
$$\text{Correct decision version of Longest Path}$$

That is, the instance of the Hamiltonian Path Problem exits only if a longest path of $k = n - 1$ exists. We will prove this in both directions from the definitions.

2.3.1 A Longest Path Exists

- If a longest path of size $k = n - 1$ exists, there is a simple path that passes each node exactly once.
- This simple path is a Hamiltonian path, and if a longest path does not exist there cannot be a Hamiltonian path.

2.3.2 A Hamiltonian Path Exists

- If a Hamiltonian path exists, there is a simple path of length $n - 1$.
- This is the maximum possible length of any simple path in the graph, therefore it is a longest path of size $k = n - 1$.
- If a Hamiltonian path does not exist, it is impossible for a longest path of $n - 1$ to exist, illustrated by a disconnected graph in $fig.2$

Figure 2: Graph without a Hamiltonian path, longest path on the right.

Therefore we have proved that our reduction is correct.

3 Conclusion

We have shown that the decision version of any Longest Path Problem has a certificate that is checked in polynomial time, therefore it is in `NP`. We showed a reduction from the Hamiltonian Path Problem to the Longest Path Problem, and proved the correctness of the reduction. Therefore, we have proved that the Longest Path Problem is also `NP-Complete` ■