Problem 1

case 1:

$$M\ddot{q} + Kq = 0$$

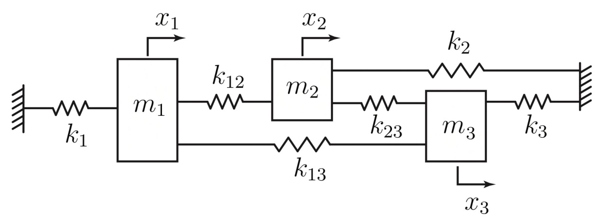


Figure 1: 3-DOF MCK system.

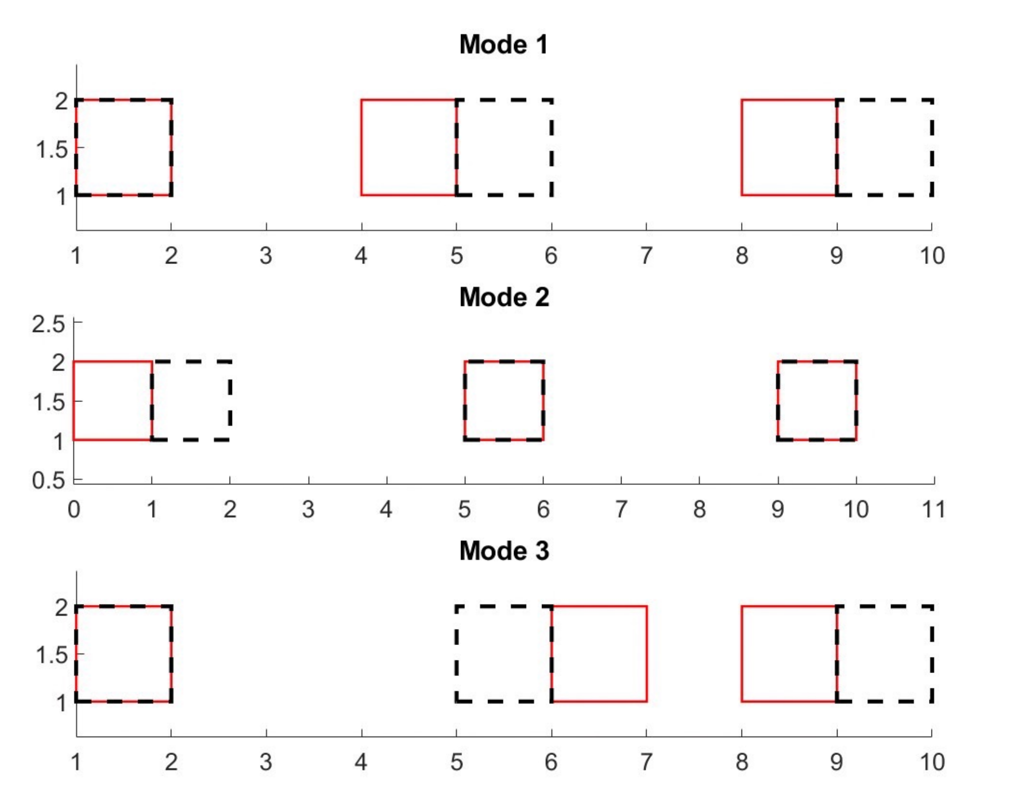
$$K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} + \begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} k_{13} & 0 & -k_{13} \\ 0 & 0 & 0 \\ -k_{13} & 0 & k_{13} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} \end{bmatrix}$$

$$= \begin{bmatrix} k_1+k_{12}+k_{13} & -k_{12} & -k_{13} \\ -k_{12} & k_2+k_{12}+k_{23} & -k_{23} \\ -k_{13} & -k_{23} & k_3+k_{13}+k_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \#$$

$$(\omega_1, v_1) = (0, [0, -1, -1]^T)$$

$$(\omega_2, v_2) = (1, [-1, 0, 0]^T)$$

$$(\omega_3, v_3) = (1.414, [0, 1, -1]^T)$$



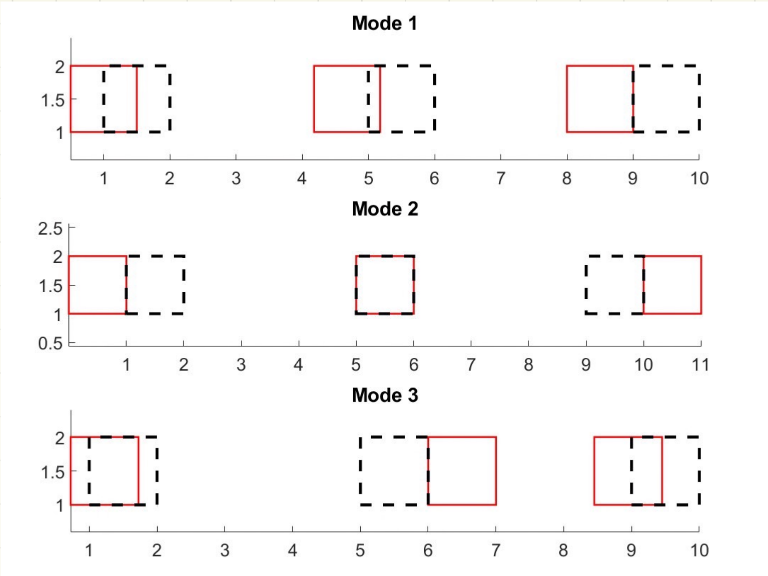
case 2:

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$(\omega_1, v_1) = (0, [-0.41, -1, -0.82]^T)$

$(\omega_2, v_2) = (1, [-1, 0, 1]^T)$

$(\omega_3, v_3) = (1.492, [-0.41, 1, -0.82]^T)$

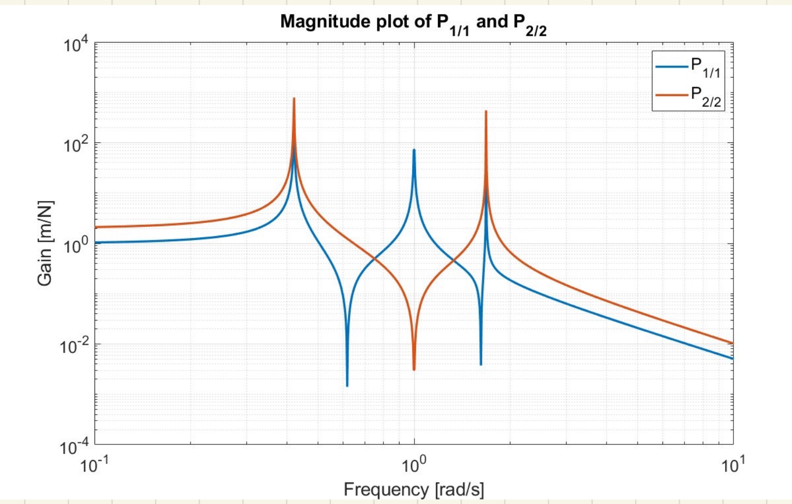


when apply force input u on m_1, $H = [1,0,0]$ $B = [1,0,0]^T$

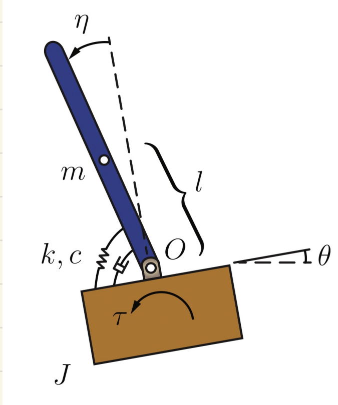
$$P_{1/1}(s) = [1,0,0](Ms^2+K)^{-1}[1,0,0]^T = \frac{0.5s^4+1.5s^2+0.5}{s^6+4s^4+3.5s^2+0.5}$$

when apply force input u on m_1, $H = [0,1,0]$ $B = [0,1,0]^T$

$$P_{2/2}(s) = [0,1,0](Ms^2+K)^{-1}[0,1,0]^T = \frac{s^4+2s^2+1}{s^6+4s^4+3.5s^2+0.5}$$

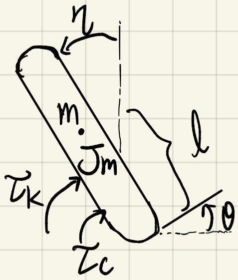


Problem 1.



Newton law and force balance:

FBD for mass m:

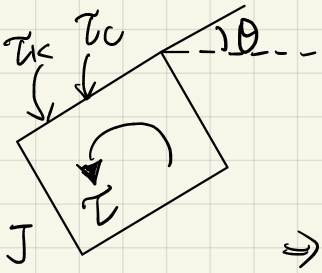


$$\Sigma \tau_m = 0$$

$$(J_m + ml^2)\ddot{\eta} + \tau_k + \tau_c = 0$$

$$\Rightarrow (J_m + ml^2)\ddot{\eta} + k(\eta - \theta) + C(\dot{\eta} - \dot{\theta}) = 0 \; \#$$

FBD for main body:



$$\tau = J\ddot{\theta} + \tau_k + \tau_c$$

$$\Rightarrow \tau = J\ddot{\theta} - k(\eta - \theta) - C(\dot{\eta} - \dot{\theta})$$

$$\Rightarrow \tau = J\ddot{\theta} + k(\theta - \eta) + C(\dot{\theta} - \dot{\eta}) \; \#$$

Lagrange's equation

general form: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$

$$T = \frac{1}{2}(J_m + ml^2)\dot{\eta}^2 + \frac{1}{2}J\dot{\theta}^2$$

$$D = \frac{1}{2}C(\dot{\eta} - \dot{\theta})^2$$

$$V = \frac{1}{2}k(\eta - \theta)^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \begin{bmatrix} J\ddot{\theta} \\ (J_m + ml^2)\ddot{\eta} \end{bmatrix} \qquad \frac{\partial T}{\partial q_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial D}{\partial \dot{q}_i} = \begin{bmatrix} c\dot{\theta} - c\dot{\eta} \\ c\dot{\eta} - c\dot{\theta} \end{bmatrix} \qquad \frac{\partial V}{\partial q_i} = \begin{bmatrix} k\theta - k\eta \\ k\eta - k\theta \end{bmatrix}$$

$$\begin{bmatrix} J\ddot{\theta} \\ (J_m + ml^2)\ddot{\eta} \end{bmatrix} + \begin{bmatrix} c\dot{\theta} - c\dot{\eta} \\ c\dot{\eta} - c\dot{\theta} \end{bmatrix} + \begin{bmatrix} k\theta - k\eta \\ k\eta - k\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau$$

$$\Rightarrow \begin{bmatrix} J\ddot{\theta} + C(\dot{\theta} - \dot{\eta}) + k(\theta - \eta) \\ (J_m + ml^2)\ddot{\eta} + C(\dot{\eta} - \dot{\theta}) + k(\eta - \theta) \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \#$$

$M\ddot{q} + C\dot{q} + Kq = Bu$, $q = [\theta, \eta]^T$, $u = \tau \in R$

$$\Rightarrow \begin{bmatrix} J & 0 \\ 0 & (J_m + ml^2) \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \theta \\ \eta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau \#$$

displacement and velocity feedback $u = -C_d\theta - C_v\dot{\theta}$

modify stiffness and damping matrix:

$$\begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \rightarrow \begin{bmatrix} c + C_v & -c \\ -c & c \end{bmatrix} \qquad \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \rightarrow \begin{bmatrix} k + C_d & -k \\ -k & k \end{bmatrix}$$

$$eig(K) = \det\left(\begin{bmatrix} k + C_d - \lambda & -k \\ -k & k - \lambda \end{bmatrix}\right) = 0 \Rightarrow (k + C_d - \lambda_k)(k - \lambda_k) - k^2 = 0$$

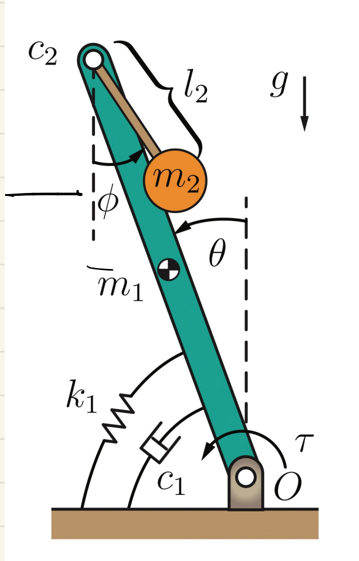
$$\Rightarrow \lambda_k^2 - (2k + C_d)\lambda_k + C_d k = 0 \qquad \lambda_k = \frac{(2k + C_d) \pm \sqrt{(2k + C_d)^2 - 4C_d}}{2} > 0 \Rightarrow K_{mod} > 0$$

$$eig(C) = \det\left(\begin{bmatrix} C + C_v - \lambda_c & -C \\ -C & C - \lambda_c \end{bmatrix}\right) = 0 \Rightarrow (C + C_v - \lambda_c)(C - \lambda_c) - C^2 = 0$$

$$\Rightarrow \lambda_c^2 - (2C + C_v)\lambda_c - C_v C = 0 \qquad \lambda_c = \frac{(2C + C_v) \pm \sqrt{(2C + C_v)^2 - 4C_v}}{2} > 0 \Rightarrow C_{mod} > 0$$

Since $K_{mod} > 0$ and $C_{mod} > 0$, this closed-loop system is stable.

Problem 3.



ground joint $O(x,y) = (0,0)$

the location of C.M. of rod $(x_1, y_1) = (-\frac{1}{2}l_1\sin\theta, \frac{1}{2}l_1\cos\theta)$

the location of C.M. of TMD $(x_2, y_2) = (-l_1\sin\theta + l_2\sin\phi, \; l_1\cos\theta - l_2\cos\phi)$

since $\theta, \phi \approx 0$, $\cos\theta \approx 1$ $\sin\theta \approx \theta$

simplify $(x_1, y_1) = (-\frac{1}{2}l_1\theta, \frac{1}{2}l_1)$

$(x_2, y_2) = (-l_1\theta + l_2\phi, \; l_1 - l_2)$

General form of Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

the moment of inertial at the C.M of the rod is $\frac{1}{3}m(\frac{1}{2}l)^2 = \frac{1}{12}ml_1^2$

$$T = \frac{1}{2}\cdot(\frac{1}{12}ml_1^2)\dot{\theta}^2 + \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{6}m_1l_1^2\dot{\theta}^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 - 2l_1l_2\dot{\theta}\dot{\phi})$$

$$D = \frac{1}{2}c_1\dot{\theta}^2 + \frac{1}{2}c_2(\dot{\theta} - \dot{\phi})^2 = \frac{1}{2}c_1\dot{\theta}^2 + \frac{1}{2}c_2\dot{\theta}^2 + \frac{1}{2}c_2\dot{\phi}^2 - c_2\dot{\theta}\dot{\phi}$$

$$V = \frac{1}{2}k\theta^2 + \frac{1}{2}m_1gl_1\cos\theta + m_2g(l_1\cos\theta - l_2\cos\phi)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \begin{bmatrix} \frac{1}{3}m_1l_1^2\ddot{\theta} + m_2l_1^2\ddot{\theta} - m_2l_1l_2\ddot{\phi} \\ m_2l_2^2\ddot{\phi} - m_2l_1l_2\ddot{\theta} \end{bmatrix}$$

$$\frac{\partial T}{\partial q_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial D}{\partial \dot{q}_i} = \begin{bmatrix} (c_1 + c_2)\dot{\theta} - c_2\dot{\phi} \\ c_2\dot{\phi} - c_2\dot{\theta} \end{bmatrix}$$

$$\frac{\partial V}{\partial q_i} = \begin{bmatrix} k\theta - \frac{1}{2}m_1 g l_1 \theta - m_2 g l_1 \theta \\ m_2 g l_2 \phi \end{bmatrix}$$

$$M\ddot{q} + C\dot{q} + Kq = Bu$$

$$\begin{bmatrix} \frac{1}{3}m_1 l_1^2 + m_2 l_1^2 & -m_2 l_1 l_2 \\ -m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_2 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} k - \frac{1}{2}m_1 g l_1 - m_2 g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}\begin{bmatrix} \theta \\ \phi \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau$$

Find $P_0(s)$, $(m_2, l_2, C_2) = (0, 0, 0)$

$$\frac{1}{3}m_1 l_1^2 s^2 \theta(s) + C_1 s\,\theta(s) + \left(k - \frac{1}{2}m_1 g l_1\right)\theta(s) = \tau$$

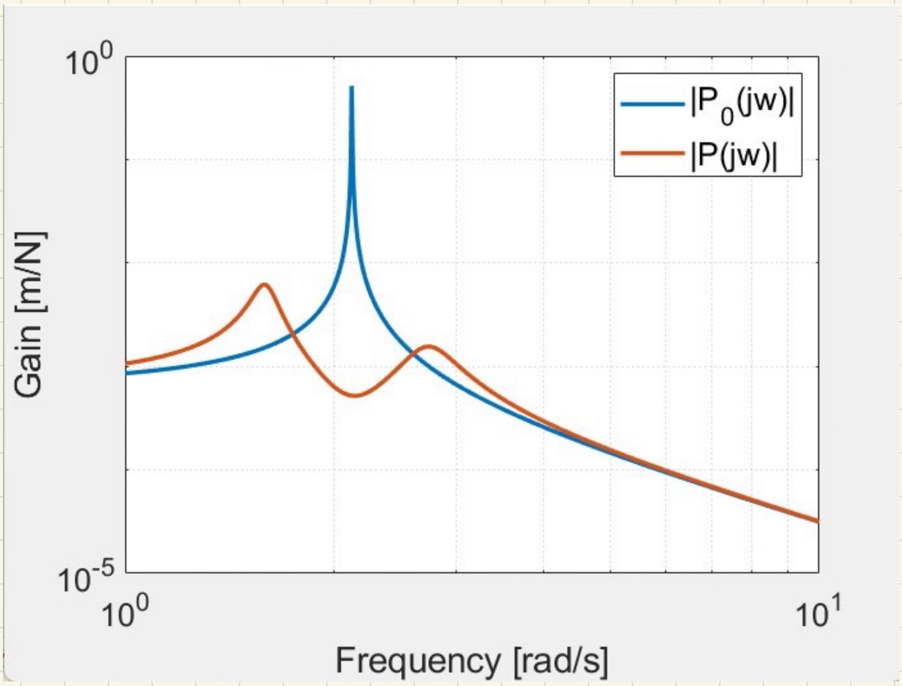
$$P_0(s) = \frac{1}{\frac{1}{3}m_1 l_1^2 s^2 + C_1 s + \left(k - \frac{1}{2}m_1 g l_1\right)} \quad \#$$

$$P(s) = H(Ms^2 + Cs + K)^{-1} B$$

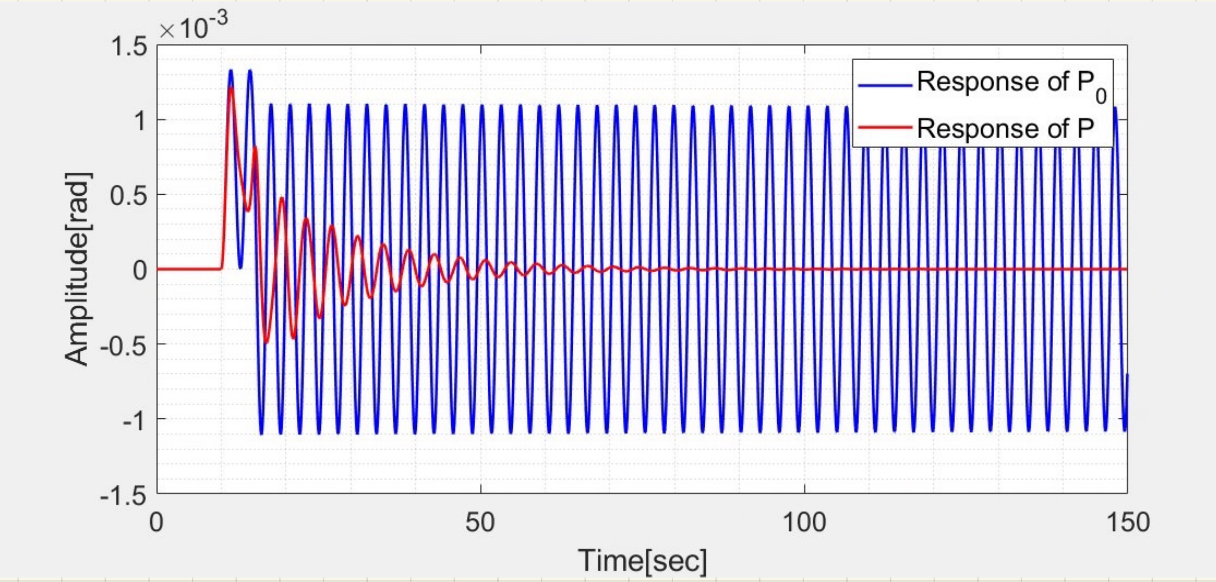
$$= [1, 0]\begin{bmatrix} \left(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2\right)s^2 + (C_1 + C_2)s + \left(k - \frac{1}{2}m_1 g l_1 - m_2 g l_1\right) & (-m_2 l_1 l_2)s^2 - C_2 s \\ (-m_2 l_1 l_2)s^2 - C_2 s & m_2 l_2^2 s^2 + C_2 s + m_2 g l_2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P(s) = \frac{m_2 l_2^2 s^2 + C_2 s + m_2 g l_2}{\begin{gathered}\left(\frac{1}{3}m_1 l_1^2 m_2 l_2^2\right)s^4 + \left(\frac{1}{3}m_1 l_1^2 C_2 + m_2 l_1^2 C_2 + m_2 l_2^2 C_1 + m_2 l_2^2 C_2 - 2m_2 l_1 l_2 C_2\right)s^3 + \\ \left(\frac{1}{3}m_1 l_1^2 m_2 g l_2 + m_2^2 l_1^2 g l_2 + m_2 l_2^2\left(k - \frac{1}{2}m_1 g l_1 - m_2 g l_1\right) + C_1 C_2\right)s^2 + \\ \left(C_2\left(k - \frac{1}{2}m_1 g l_1 - m_2 g l_1\right) + C_1 m_2 g l_2 + C_2 m_2 g l_2\right)s + \left(m_2 g l_2\left(k - \frac{1}{2}m_1 g l_1 - m_2 g l_1\right)\right)\end{gathered}}$$

太多了吧……



10^0
10^{-5}
10^0
10^1
Gain [m/N]
Frequency [rad/s]
$|P_0(jw)|$
$|P(jw)|$



$\times 10^{-3}$
1.5
1
0.5
0
-0.5
-1
-1.5
0
50
100
150
Amplitude[rad]
Time[sec]
Response of P_0
Response of P