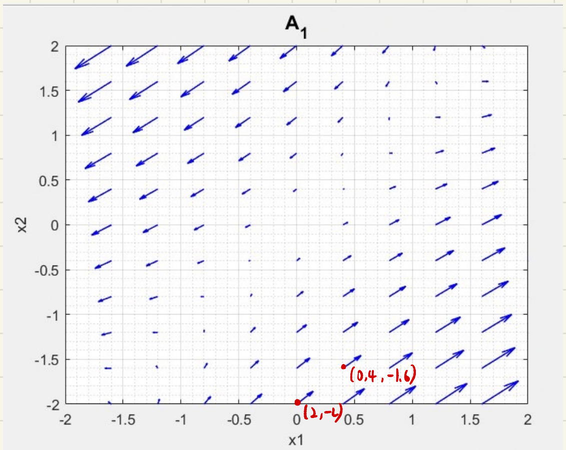
Problem 1.



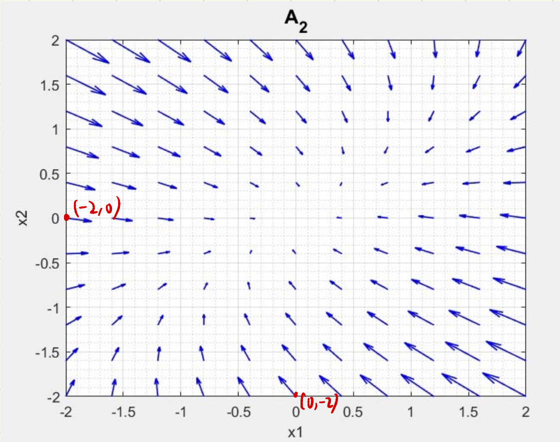
$(x_1, x_2) = (0, -2)$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}3 & -2\\ 2 & -2\end{bmatrix}\begin{bmatrix}0\\ -2\end{bmatrix} = \begin{bmatrix}4\\ 4\end{bmatrix}$$

$(x_1, x_2) = (0.4, -1.6)$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}3 & -2\\ 2 & -2\end{bmatrix}\begin{bmatrix}0.4\\ -1.6\end{bmatrix} = \begin{bmatrix}4.4\\ 4.0\end{bmatrix}$$

$\Rightarrow$ directions all correct



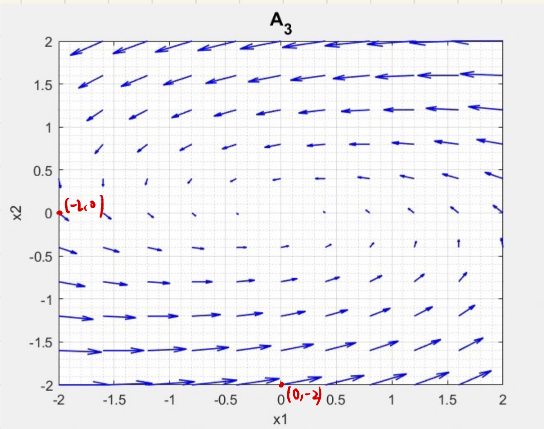
$(x_1, x_2) = (0, -2)$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{3}{4} & \frac{1}{2}\\ \frac{1}{8} & -\frac{3}{4}\end{bmatrix}\begin{bmatrix}0\\ -2\end{bmatrix} = \begin{bmatrix}-1\\ \frac{3}{2}\end{bmatrix}$$

$(x_1, x_2) = (-2, 0)$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{3}{4} & \frac{1}{2}\\ \frac{1}{8} & -\frac{3}{4}\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\ -\frac{1}{4}\end{bmatrix}$$

$\Rightarrow$ directions all correct



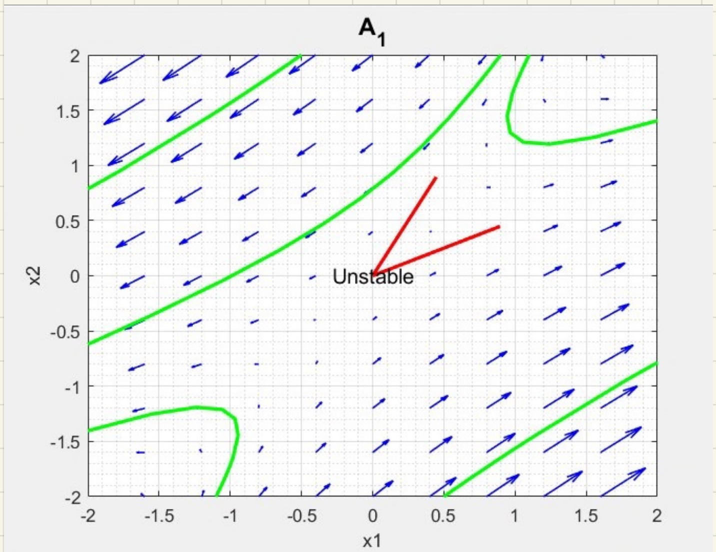
$(x_1, x_2) = (0, -2)$

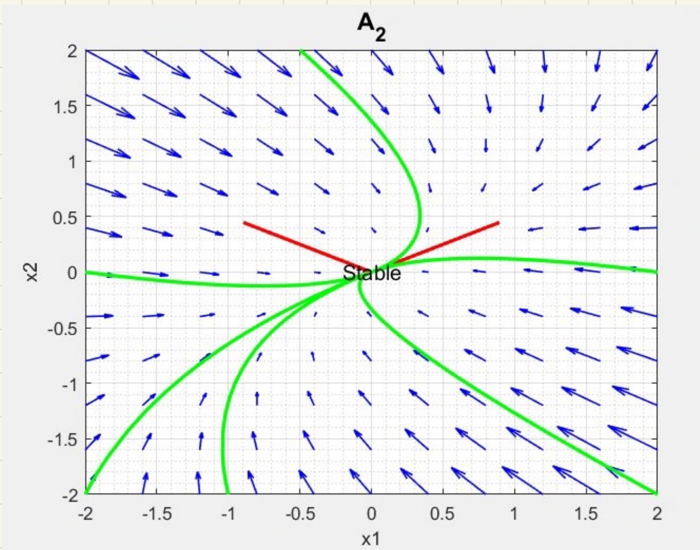
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -4\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ -2\end{bmatrix} = \begin{bmatrix}8\\ 2\end{bmatrix}$$

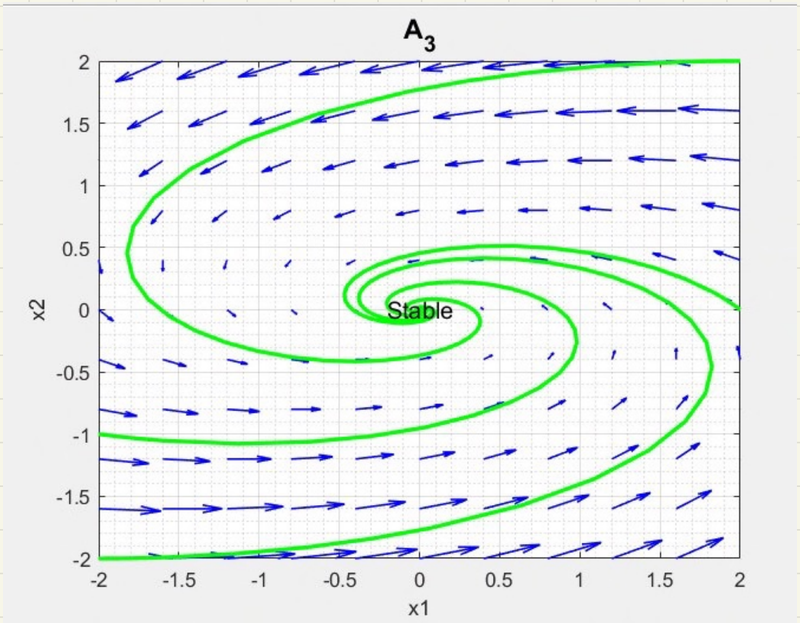
$(x_1, x_2) = (-2, 0)$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -4\\ 1 & -1\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix} = \begin{bmatrix}2\\ 2\end{bmatrix}$$

$\Rightarrow$ directions all correct







$$A_3 = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

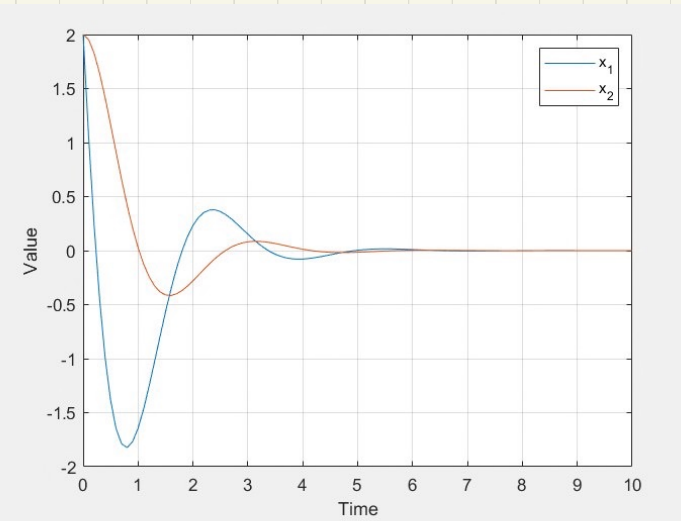
$$\det(\lambda I - A) = 0 \Rightarrow (\lambda+1)^2 + 4 = 0 \quad \lambda = -1 \pm 2i$$

$$(\lambda_1 I - A) e_1 = \begin{bmatrix} 2i & 4 \\ -1 & 2i \end{bmatrix} e_1 = 0 \Rightarrow e_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

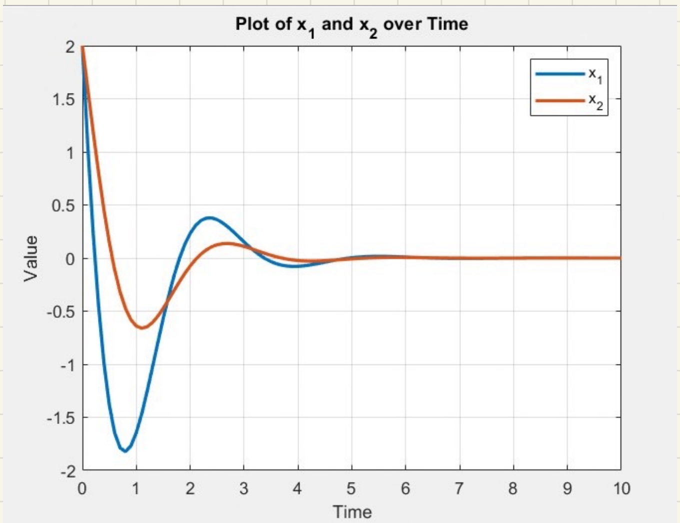
$$(\lambda_2 I - A) e_2 = \begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} e_2 = 0 \Rightarrow e_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

$$e^{A_3 t} = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{(-1-2i)t} \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-t}\cos 2t & 2e^{-t}\sin 2t \\ -\frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$$

$$X(t) = e^{A_3 t} x_0 = \begin{bmatrix} 2e^{-t}\cos 2t + 4e^{-t}\sin 2t \\ -e^{-t}\sin 2t + 2e^{-t}\cos 2t \end{bmatrix}$$



using sample code



using spectral decomposition

Problem 2.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 + \delta \\ 3x_1 - x_2 \end{bmatrix}$$

case 1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ 3x_1 - x_2 \end{bmatrix}$$

find eq. point:

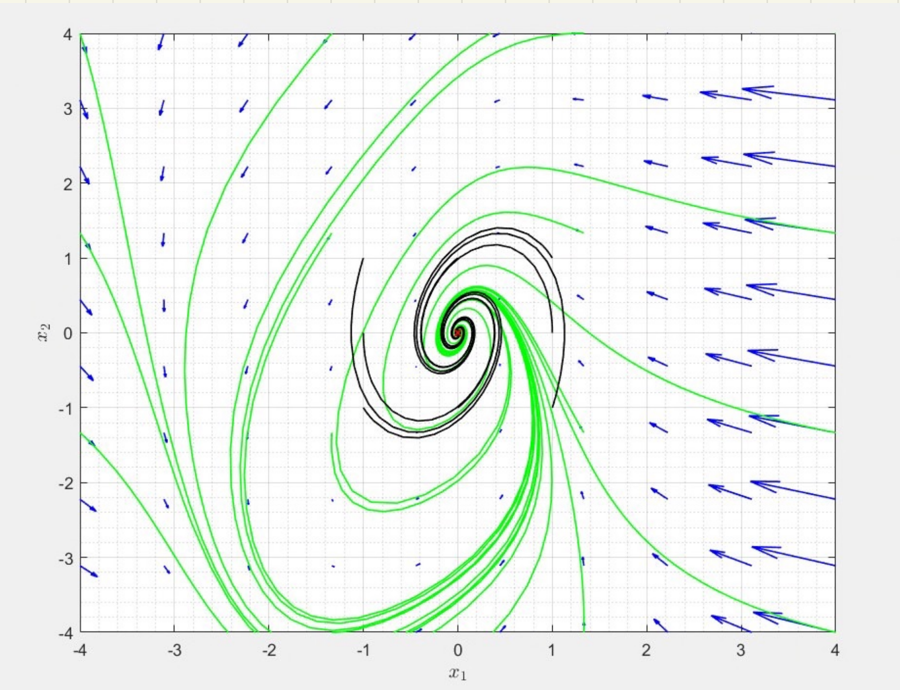
$$\begin{cases} -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 = 0 \\ 3x_1 - x_2 = 0 \end{cases} \Rightarrow \text{eq. point is } x^* = [0, 0]^T$$

Jacobian linearization:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -3x_1 - \frac{3}{2}x_1^2 & -1 \\ 3 & -1 \end{bmatrix} \xrightarrow{\text{plug } [0,0]^T} \begin{bmatrix} 0 & -1 \\ 3 & -1 \end{bmatrix}$$

$\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$ the real part of eigenvalues are negative

$\Rightarrow$ stable node



case 2.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2-\frac{3}{2}x_1^2-\frac{1}{2}x_1^3-1.0376\\ 3x_1-x_2\end{bmatrix}$$

find eq. point:

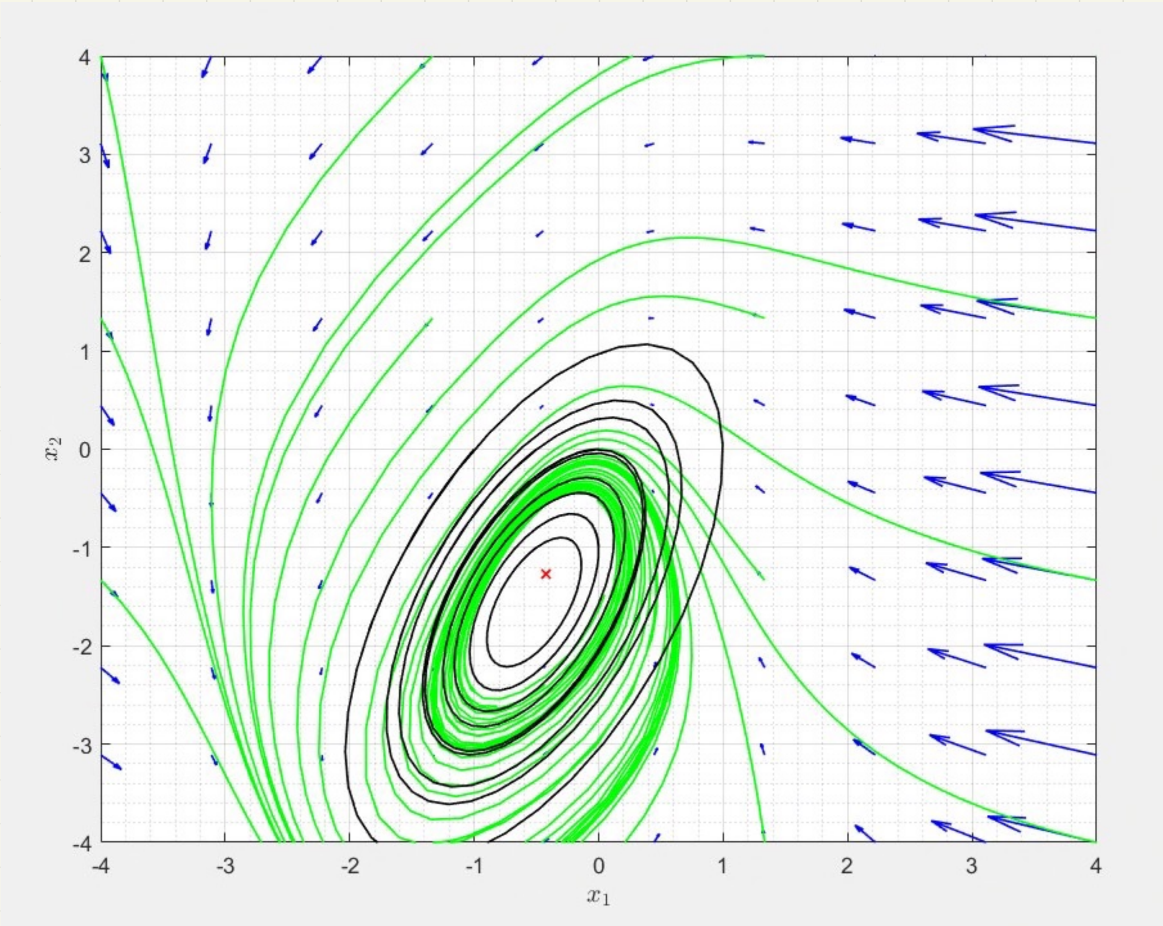
$$\begin{cases}-x_2-\frac{3}{2}x_1^2-\frac{1}{2}x_1^3-1.0376=0\\ 3x_1-x_2=0\end{cases} \Rightarrow \text{eq. point is } x^* = [-0.422, -1.266]$$

Jacobian linearization:

$$\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}-3x_1-\frac{3}{2}x_1^2 & -1\\ 3 & -1\end{bmatrix} \xrightarrow{\text{plug } x^*} \begin{bmatrix}0.999 & -1\\ 3 & -1\end{bmatrix}$$

$\Rightarrow \lambda_{1,2} = -0.0005 \pm 1.4146i \Rightarrow$ the real part of $\lambda_{1,2}$ are negative

$\Rightarrow$ stable node



Problem 3.

$x_1 = \phi \quad x_2 = x \quad x_3 = \dot{\phi} \quad x_4 = \dot{x}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{mgL}{I_G + mL^2}\sin x_1 - \frac{mL}{I_G + mL^2}\dot{x}_4 \cos x_1 + \frac{\tau}{I_G + mL^2} \\ \frac{mL}{M+m} x_3^2 \sin x_1 - \frac{mL}{M+m}\dot{x}_3 \cos x_1 + f \end{bmatrix}$$

$$= \begin{bmatrix} x_3 \\ x_4 \\ 24.53 \sin x_1 - 2.5\dot{x}_4 \cos x_1 \\ 0.086 x_3^2 \sin x_1 - 0.086 \dot{x}_3 \cos x_1 \end{bmatrix}$$

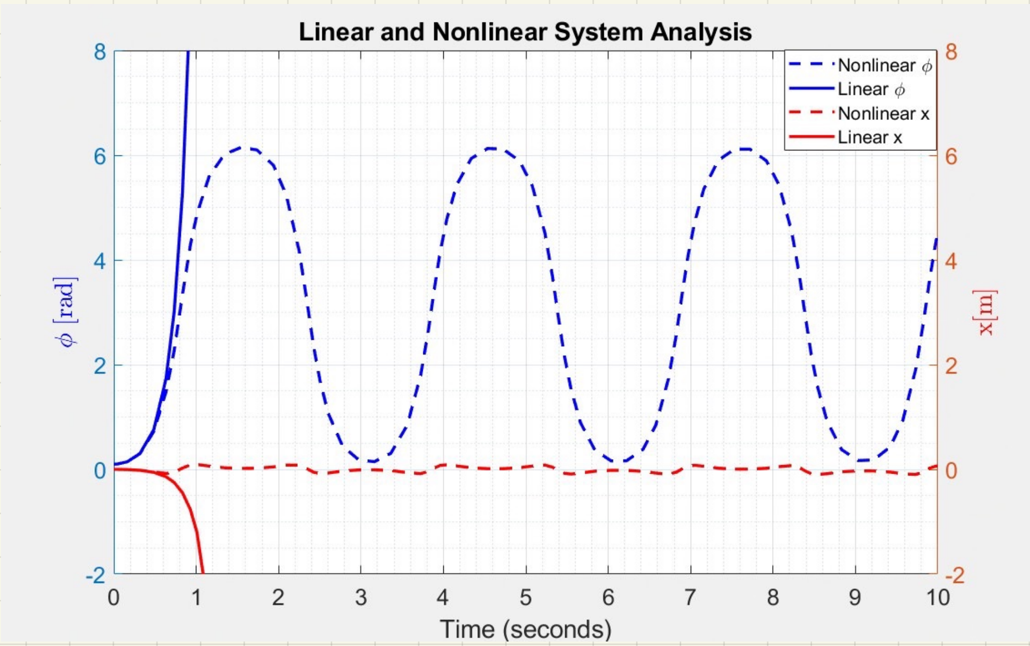
Linearization:

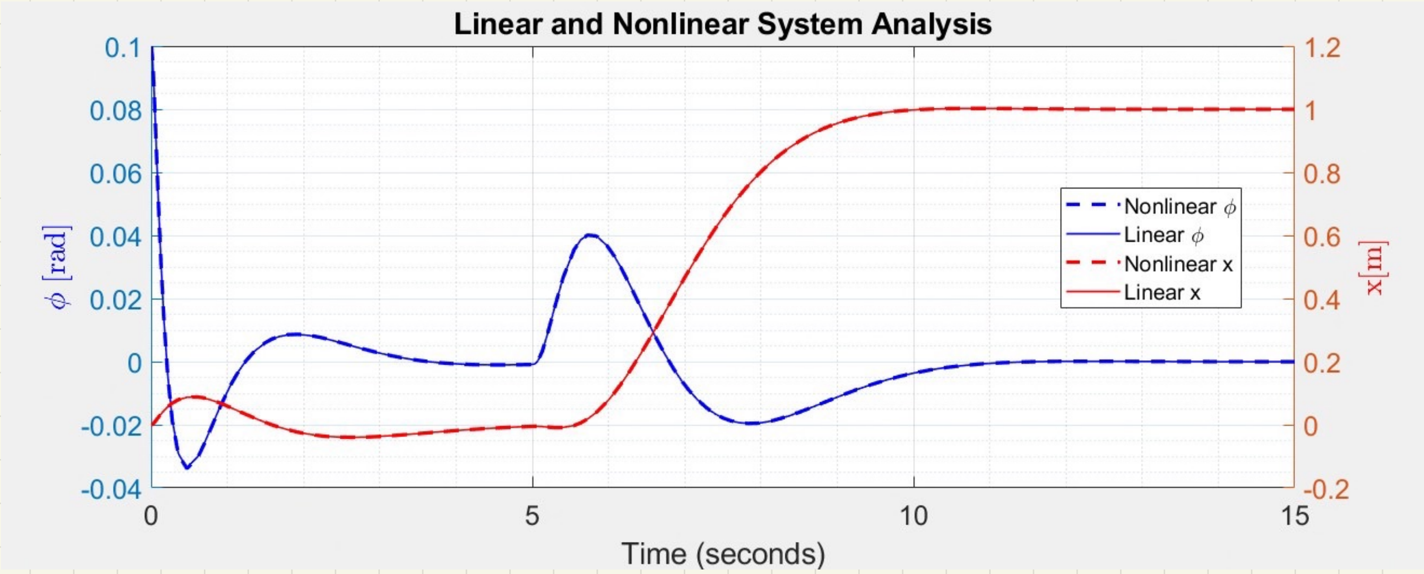
$\sin x_1 \approx x_1$

$\cos x_1 \approx 1$

$x_3^2 \approx 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 24.53 x_1 - 2.5\dot{x}_4 \\ -0.086\dot{x}_3 \end{bmatrix}$$




Linear and Nonlinear System Analysis
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
ϕ [rad]
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
x[m]
0
5
10
15
Time (seconds)
Nonlinear ϕ
Linear ϕ
Nonlinear x
Linear x