Linear Integral Equations

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These notes are based on chapters from [1]. They are meant as a summary, and generally exclude rigorous/long proofs or demonstrations, pointing the reader (me) to the correct place in the text.

0.1 Chapter 6 - Potential Theory

Concerned in this section with *harmonic functions*. Green's theorems are essential to their study.

Green's First Theorem. Let D be a bounded domain of class C^1, and let ν denote the outerward unit normal to a boundary ∂D directed to the exterior of D. Then for $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$:

$$\int_D (u\Delta v + \operatorname{grad} u \cdot \operatorname{grad} v) dx = \int_{\partial D} u \frac{\partial v}{\partial \nu} ds \tag{1}$$

This theorem relates a volume integral through D to a surface integral over its boundary. It's essentially a generalisation of the integration by parts formula.

Using **Gauss' divergence theorem**

$$\int_D \div A dx = \int_{\partial D} \nu \cdot A ds \tag{2}$$

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where $A \in C^1(\bar{D})$ is a vector field defined by $A := u \operatorname{grad} v$, with the identity,

$$\operatorname{div}(u \operatorname{grad} v) = \operatorname{grad} u \cdot \operatorname{grad} v + u \operatorname{div} \operatorname{grad} v$$

We substitute this identity to find (0.1). For **Green's second theorem**,

$$\int_D (u\Delta v - v\Delta u)dx = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) \tag{3}$$

For $u, v \in C^2(\bar{D})$. We interchange u and v in the identity, apply it to the divergence theorem, and subtract the quantities.

Divergence theorem basically relates the flux of a vector field through a closed surface, to its divergence in the enclosed volume.

For $v \in C^2(\bar{D})$, harmonic in D, Then

$$\int_{\partial D} \frac{\partial v}{\partial \nu} ds = 0 \tag{4}$$

This follows from choosing $u = 1$ in (0.1). The following formula known as **Green's Formula** is incredibly important. It allows us to calculate a harmonic function $u \in C^2(\bar{D})$ from its boundary values/derivatives (Cauchy data) as well as the fundamental solution,

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu} \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \;\; x \in D \tag{5}$$

See p. 77 of [1] for a proof. Harmonic functions are analytic, ie each harmonic function has a local power series expansion.

Well posedness of BVP - a unique solution depending continuously on the given boundary data.