

Spatial Outage Capacity of Poisson Bipolar Networks

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Abstract—We introduce a new notion of capacity, termed *spatial outage capacity* (SOC), which is defined as the maximum density of concurrently active links that have a success probability greater than a predefined threshold. For Poisson bipolar networks, we provide exact analytical and approximate expressions for the density of concurrently active links satisfying an outage constraint. In the high-reliability regime, we obtain an exact closed-form expression of the SOC, which gives its asymptotic scaling behavior.

Index Terms—Bipolar network, interference, Poisson point process, SIR, spatial outage capacity, stochastic geometry

I. INTRODUCTION

A. Motivation

Stochastic geometry provides the mathematical tools to study wireless networks where node locations are modeled by a random point process. By *spatial averaging*, stochastic geometry allows us to evaluate the statistics of the wireless network such as interference distribution and average success probability. In this approach, the performance evaluation is usually done with respect to the *typical user or typical link*. Such spatial averaging often leads to tractable performance metrics for the given network parameters, which allows us to choose network parameters that optimize the network performance.

While such a macroscopic view based on spatial averaging is important, it does not give fine-grained information about the network, such as the link-wise performance characterized by the link success probability. Due to the random node locations, the success probability of each link is a random variable that depends on path loss, fading, and interferer locations. In fact, as Fig. 1 shows, for the same average success probability, depending on the network parameters, the distribution of link success probabilities in a Poisson bipolar network varies significantly. Thus the link success probability distribution is a much more comprehensive metric than the average success probability that is usually considered.

In this regard, we introduce a new notion of capacity, termed *spatial outage capacity* (SOC). The SOC is defined as the maximum density of concurrently active links that have a success probability greater than a certain threshold. In other words, the SOC is the maximum density of concurrently active links that achieve a certain *reliability* and thus represents the maximum density of reliable links. The definition of the SOC is based on the distribution of link success probabilities across the network.

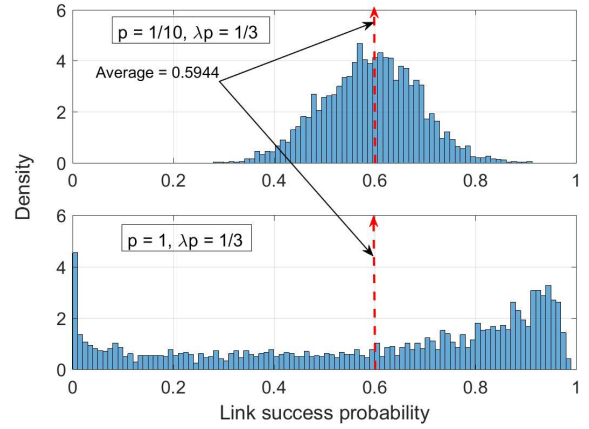


Fig. 1. The empirical probability density function of the link success probability in a Poisson bipolar network for transmit probabilities $p = 1/10$ and $p = 1$. Both cases have the same average success probability $p_s = 0.5944$, but we see a different distribution of link success probabilities for different values of the pair of density λ and transmit probability p . For $p = 1/10$, the link success probabilities lie between 0.3 and 0.9 (concentrated around their average), while for $p = 1$, they are spread more widely. The SIR threshold $\theta = -10$ dB, distance between a transmitter and its associated receiver $R = 1$, path loss exponent $\alpha = 4$, and $\lambda p = 1/3$.

Definition 1 (Spatial outage capacity). *For a stationary and ergodic point process model where λ is the density of potential transmitters, p is the fraction of nodes that are active at a time, and $\eta(\theta, x)$ is the fraction of links in each realization of the point process that have a signal-to-interference ratio (SIR) greater than θ with probability at least x , the SOC is*

$$S(\theta, x) \triangleq \sup_{\lambda, p} \lambda p \eta(\theta, x), \quad (1)$$

where $\theta \in \mathbb{R}^+$, $x \in (0, 1)$, $\lambda > 0$, and $p \in (0, 1]$.

We denote the density of concurrently active links that have a success probability greater than x (alternatively, a reliability of x or higher) as

$$\tau(\theta, x) \triangleq \lambda p \eta(\theta, x), \quad (2)$$

which results in

$$S(\theta, x) = \sup_{\lambda, p} \tau(\theta, x). \quad (3)$$

Hence $\tau(\theta, x)$ is the density of reliable links. The SOC provides a useful practical measure which tells us the maximum

number of active links per unit area a wireless network can handle at a time while guaranteeing a certain reliability. The SOC has applications in a wide range of wireless networks, such as ad hoc, D2D, M2M, and vehicular networks.

B. Background

Given the point process Φ , the link success probability is a random variable that is given as

$$P_s(\theta) \triangleq \mathbb{P}(\text{SIR} > \theta \mid \Phi), \quad (4)$$

where the conditional probability is calculated by averaging over the fading and the medium access scheme (if random) of the interferers. The probability $\eta(\theta, x)$ in (1), termed *meta distribution* of the SIR in [1], is the complementary cumulative distribution function (ccdf) of the link success probability. Thus the meta distribution is given as

$$\eta(\theta, x) \triangleq \mathbb{P}^{\text{lt}}(P_s(\theta) > x), \quad (5)$$

where $\mathbb{P}^{\text{lt}}(\cdot)$ denotes the reduced Palm probability, given that an active transmitter is present at the prescribed location, and the SIR is calculated at its associated receiver. Due to the ergodicity of the point process, given the random locations of nodes, $\eta(\theta, x)$ is also the probability that the typical link has a success probability at least x . The average success probability follows as

$$p_s(\theta) = \mathbb{P}(\text{SIR} > \theta) = \mathbb{E}^{\text{lt}}(P_s(\theta)) = \int_0^1 \eta(\theta, x) dx, \quad (6)$$

where $\mathbb{E}^{\text{lt}}(\cdot)$ denotes the expectation with respect to the reduced Palm distribution.

C. Contributions

The contributions of the paper are as follows:

- We introduce a new notion of capacity—spatial outage capacity—based on the link success probability distribution.
- For the Poisson bipolar network with ALOHA, we evaluate the density of concurrently active links satisfying an outage constraint.
- We show the trade-off between the density of active links and the fraction of reliable links.
- In the high-reliability regime where the target outage probability is close to 0, we give a closed-form expression of the SOC and prove that the SOC is achieved at $p = 1$.

D. Related Work

For Poisson bipolar networks, the success probability of the typical link $p_s(\theta)$ is calculated in [2] and [3]. The notion of the *transmission capacity* (TC) is introduced in [4], which is defined as the maximum density of successful transmissions provided the outage probability of the typical user stays below a certain threshold ϵ . While the results obtained in [4] are certainly important, the TC does not represent the actual maximum density of successful transmissions for the target outage probability, as claimed in [4], since the metric implicitly assumes that each link is typical. We illustrate the

difference between the SOC and the TC through the following example.

Example 1. For Poisson bipolar networks with ALOHA where SIR threshold $\theta = 1/10$, link distance $R = 1$, path loss exponent $\alpha = 4$, and target outage probability $\epsilon = 1/10$, the TC is 0.0608 (see [5, (4.15)]), which is achieved at $\lambda p = 0.0675$. At this value of the TC, $p_s(\theta) = 0.9$. But at $p = 1$, actually only 82% active links satisfy the 10% outage. Hence the density of links that achieve 10% outage is only 0.055. On the other hand, the SOC is 0.09227 which is the actual maximum density of concurrently active links that have an outage probability smaller than 10%, and is achieved at $\lambda = 0.23$ and $p = 1$, resulting in $p_s(\theta) = 0.6984$. Hence the maximum density of links given the 10% outage constraint is more than 50% larger than the TC.

A version of the TC based on the link success probability distribution is introduced in [6], but it does not consider a medium access control (MAC) scheme, i.e., all nodes always transmit ($p = 1$). Here, we consider the general case with $p \in (0, 1]$. The choice of p is important as it greatly affects the link success probability distribution as shown in Fig. 1. Also, the TC defined in [6] calculates the maximum density of concurrently active links subject to the constraint that a certain fraction of active links satisfy the outage constraint. Such constraint of certain fraction of active links satisfying the outage constraint is not required by our definition of the SOC, and the SOC corresponds to the actual density of active links that satisfy an outage constraint.

The meta distribution $\eta(\theta, x)$ for Poisson bipolar networks and cellular networks is studied in [1], where a closed-form expression for the moments of $P_s(\theta)$ is obtained, and an exact integral expression and simple bounds for $\eta(\theta, x)$ are provided. A key result in [1] is that, for constant transmitter density λp , as the Poisson bipolar network becomes very dense ($\lambda \rightarrow \infty$) with a very small transmit probability ($p \rightarrow 0$), all links have the same success probability, which is the average success probability $p_s(\theta)$.

II. NETWORK MODEL

We consider the Poisson bipolar network model in which the locations of transmitters form a homogeneous Poisson point process (PPP) $\Phi \subset \mathbb{R}^2$ with density λ [7, Def. 5.8]. Each transmitter has a dedicated receiver at a distance R in a uniformly random direction. In a time slot, each node in Φ independently transmits at unit power with probability p and stays silent with probability $1 - p$. Thus the active transmitters form a homogeneous PPP with density λp . We consider a standard power law path loss model with path loss exponent α . We assume that a channel is subject to independent Rayleigh fading with channel power gains as i.i.d. exponential random variables with mean 1.

We focus on the interference-limited case, where the received SIR is a key quantity of interest. The success proba-

bility $p_s(\theta)$ of the typical link depends on the SIR. From [3], [7], [8], it is known that

$$p_s(\theta) = \exp(-\lambda p C \theta^\delta), \quad (7)$$

where $C \triangleq \pi R^2 \Gamma(1+\delta) \Gamma(1-\delta)$ with $\delta \triangleq 2/\alpha$.

III. SPATIAL OUTAGE CAPACITY

A. Exact Formulation

Observe from Def. 1 that, the SOC depends on $\eta(\theta, x) = \mathbb{P}(P_s(\theta) > x)$. Let $M_b(\theta)$ denote the b th moment of $P_s(\theta)$. Then

$$M_b(\theta) \triangleq \mathbb{E}(P_s(\theta)^b). \quad (8)$$

The average success probability is $p_s(\theta) \equiv M_1(\theta)$.

From [1, Thm. 1], we can express $M_b(\theta)$ as

$$M_b(\theta) = \exp(-\lambda C \theta^\delta D_b(p, \delta)), \quad b \in \mathbb{C}, \quad (9)$$

where

$$D_b(p, \delta) \triangleq \sum_{k=1}^{\infty} \binom{b}{k} \binom{\delta-1}{k-1} p^k, \quad p, \delta \in (0, 1]. \quad (10)$$

With the finite upper limit of the sum, $D_b(p, \delta)$ in (10) becomes a polynomial which is termed *diversity polynomial* in [9]. For $b = 1$ (the first moment), $D_1(p, \delta) = p$, and we get the expression of $p_s(\theta)$ as in (7). We can also express $D_b(p, \delta)$ using the Gaussian hypergeometric function ${}_2F_1$ as

$$D_b(p, \delta) = pb {}_2F_1(1-b, 1-\delta; 2; p). \quad (11)$$

Using the Gil-Pelaez theorem [10], the exact expression of $\tau(\theta, x) = \lambda p \eta(\theta, x)$ can be obtained in integral form from that of $\eta(\theta, x)$ given in [1, Cor. 3] as

$$\tau(\theta, x) = \frac{\lambda p}{2} - \frac{\lambda p}{\pi} \int_0^\infty \frac{\sin(u \ln x + \lambda C \theta^\delta \Im(D_{ju}))}{u e^{\lambda C \theta^\delta \Re(D_{ju})}} du, \quad (12)$$

where $j \triangleq \sqrt{-1}$, $D_{ju} = D_{ju}(p, \delta)$ is given by (10), while $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of the complex number z , respectively. Note that the SOC is obtained by taking the supremum of $\tau(\theta, x)$ over λ and p .

B. Approximation with Beta Distribution

We can accurately approximate (12) in a semi-closed form using the beta distribution, which is a good approximation (and a simple one) as shown in [1]. The rationale behind such approximation is that the support of the link success probability $P_s(\theta)$ is $[0, 1]$, making the beta distribution a natural choice. With the beta distribution approximation, from [1, Sec. II.F], $\tau(\theta, x)$ is approximated as

$$\tau(\theta, x) \approx \lambda p \left(1 - I_x \left(\frac{\mu\beta}{1-\mu}, \beta \right) \right), \quad (13)$$

where $I_x(y, z) \triangleq \int_0^x t^{y-1} (1-t)^{z-1} dt / B(y, z)$ is the regularized incomplete beta function with $B(\cdot, \cdot)$ denoting beta function, $\mu = M_1$, and $\beta = (M_1 - M_2)(1 - M_1) / (M_2 - M_1^2)$.

The advantage of the beta approximation is the faster computation of $\tau(\theta, x)$ compared to the exact expression without

losing much accuracy [1, Tab. I, Fig. 4] (also see Fig. 6 of this paper). In general, it is difficult to obtain the SOC analytically due to the forms of $\tau(\theta, x)$ given in (12) and (13). But we can obtain the SOC numerically with ease. We can also gain useful insights considering some specific scenarios, on which we focus in the following two subsections of the paper.

C. Constrained SOC

1) *Constant λp* : For constant λp (or, equivalently, a fixed $p_s(\theta)$), we now study how the density of reliable links $\tau(\theta, x)$ behaves in an ultra-dense network. Given θ , R , α , and x , this case is equivalent to asking how $\tau(\theta, x)$ varies as $\lambda \rightarrow \infty$ while letting $p \rightarrow 0$ for constant transmitter density λp (constant $p_s(\theta)$).

Lemma 1 ($p \rightarrow 0$ for constant λp). *Let $\nu = \lambda p$. Then, for constant ν while letting $p \rightarrow 0$, the SOC constrained on the density of concurrent transmissions is*

$$\tilde{S}(\theta, x) = \begin{cases} \lambda p, & \text{if } x < p_s(\theta) \\ 0, & \text{if } x > p_s(\theta). \end{cases} \quad (14)$$

Proof: Applying Chebyshev's inequality to (5), for $x < p_s(\theta) = M_1$, we have

$$\eta(\theta, x) > 1 - \frac{\text{var}(P_s(\theta))}{(x - M_1)^2}, \quad (15)$$

where $\text{var}(P_s(\theta)) = M_2 - M_1^2$. From [1, Cor. 1], for constant ν , we know that

$$\lim_{\substack{p \rightarrow 0 \\ \lambda p = \nu}} \text{var}(P_s(\theta)) = 0.$$

Thus the lower bound in (15) approaches 1, leading to $\eta(\theta, x) \rightarrow 1$. This results in the SOC constrained on the density of concurrent transmissions equal to λp .

On the other hand, for $x > M_1$,

$$\eta(\theta, x) \leq \frac{\text{var}(P_s(\theta))}{(x - M_1)^2}. \quad (16)$$

As we let $p \rightarrow 0$ for constant ν , the upper bound in (16) approaches 0, leading to $\eta(\theta, x) \rightarrow 0$. This results in the SOC constrained on the density of concurrent transmissions equal to 0. ■

In fact, as $\text{var}(P_s(\theta)) \rightarrow 0$, the ccdf of $P_s(\theta)$ approaches a step function that leaps from 1 to 0 at the mean of $P_s(\theta)$, i.e., at $x = p_s(\theta)$. This behavior justifies (14).

2) $\lambda p \rightarrow 0$: For $\lambda p \rightarrow 0$, $\tau(\theta, x)$ depends linearly on λp , which we prove in the next lemma.

Lemma 2 ($\tau(\theta, x)$ as $\lambda p \rightarrow 0$). *As $\lambda p \rightarrow 0$,*

$$\tau(\theta, x) \sim \lambda p. \quad (17)$$

Proof: As $\lambda p \rightarrow 0$, M_1 approaches 1 and thus $\text{var}(P_s(\theta)) = M_1^2(M_1^{p(\delta-1)} - 1)$ approaches 0. Since $x \in (0, 1)$, we have $x < M_1$ as $\lambda p \rightarrow 0$. Using Chebyshev's inequality for $x < M_1$ as in (15) and letting $\text{var}(P_s(\theta)) \rightarrow 0$, the lower bound in (15) approaches 1, leading to $\eta(\theta, x) \rightarrow 1$. ■

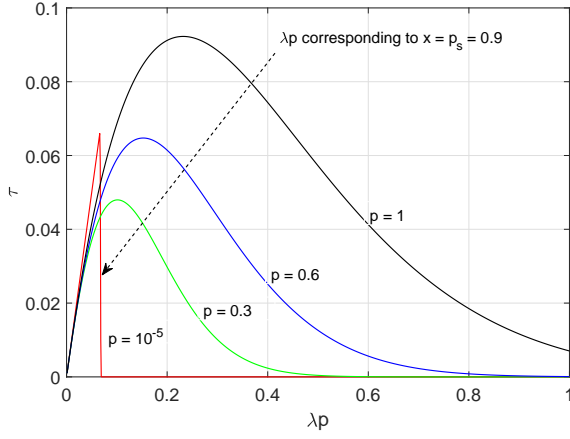


Fig. 2. The density of reliable links $\tau(\theta, x)$ given in (2) for different values of the transmit probability p for $\theta = -10$ dB, $R = 1$, $\alpha = 4$, and $x = 0.9$. Observe that the slope of $\tau(\theta, x)$ is one for small values of λp .

Lemma 2 can be understood as follows. As $\lambda p \rightarrow 0$, the density of active transmitters is very small. Thus each transmission succeeds with high probability, and $\eta(\theta, x) \rightarrow 1$ in this regime, increasing the density of reliable links linearly with λp . The case $\lambda p \rightarrow 0$ can be interpreted in two ways: 1) $\lambda \rightarrow 0$ for constant p and 2) $p \rightarrow 0$ for constant λ . Lemma 2 is valid for both cases, or any combination thereof. The case of constant p is relevant since it can be interpreted as a delay constraint: As p gets smaller, the probability that a node makes a transmission attempt in a slot reduces, increasing the delay. Since the mean delay until successful reception is larger than the mean channel access delay $1/p$, it gets large for small values of p . Thus, a delay constraint prohibits p from getting too small.

Fig. 2 illustrates Lemma 2. Also, observe that, as $p \rightarrow 0$ ($p = 10^{-5}$ in Fig. 2), $\tau(\theta, x)$ increases linearly with λp till the product λp reaches to the value that corresponds to $p_s(\theta) = x = 0.9$ and then leaps to 0. This behavior is in accordance with Lemma 1. In general, as λp increases, $\tau(\theta, x)$ increases first and then decreases after a tipping point. This is due to the two opposite effects of λp on $\tau(\theta, x)$. Because of the term λp in the expression of $\tau(\theta, x)$ (see (2)), the increase in λp increases $\tau(\theta, x)$ first, but contributes to more interference at the same time, which reduces the fraction $\eta(\theta, x)$ of links that have a reliability at least x , thereby reducing $\tau(\theta, x)$.

The contour plot in Fig. 3 visualizes the trade-off between λp and $\eta(\theta, x)$. The contour curves for small values of λp run nearly parallel to those for $\tau(\theta, x)$, indicating that $\eta(\theta, x)$ is close to 1. Specifically the contour curves for $\lambda p = 0.01$ and $\lambda p = 0.02$ match almost exactly with those for $\tau(\theta, x) = 0.01$ and $\tau(\theta, x) = 0.02$, respectively. This behavior is in accordance with Lemma 2. Conversely, for large values of λp , the decrease in $\eta(\theta, x)$ dominates $\tau(\theta, x)$. Also, notice that, for larger values of λ ($\lambda > 0.4$ for Fig. 3), $\tau(\theta, x)$ first increases and then decreases with the increase in p . This behavior is due to the following trade-off in p . For a small p , there are few

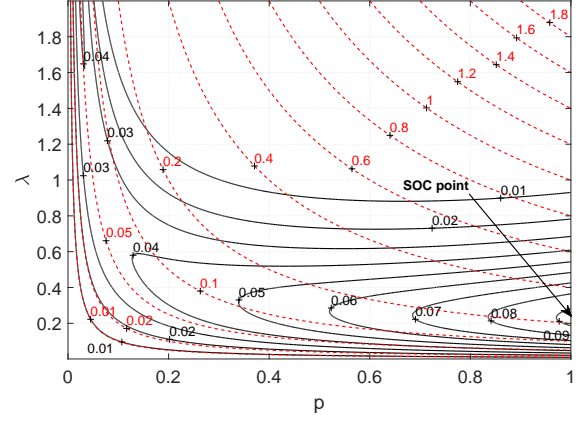


Fig. 3. Contour plots of $\tau(\theta, x)$ and the product λp for $\theta = -10$ dB, $R = 1$, $\alpha = 4$, and $x = 0.9$. The solid lines represent the contour curves for $\tau(\theta, x)$ and the dashed lines represent the contour curves for λp . The numbers in “black” and “red” indicate the contour levels for $\tau(\theta, x)$ and λp , respectively. The “SOC point” corresponds to the supremum of $\tau(\theta, x)$, and gives the SOC equal to $S(\theta, x) = 0.09227$. The values of λ and p at the SOC point are 0.23 and 1, respectively, and the corresponding average success probability is $p_s(\theta) = 0.6984$.

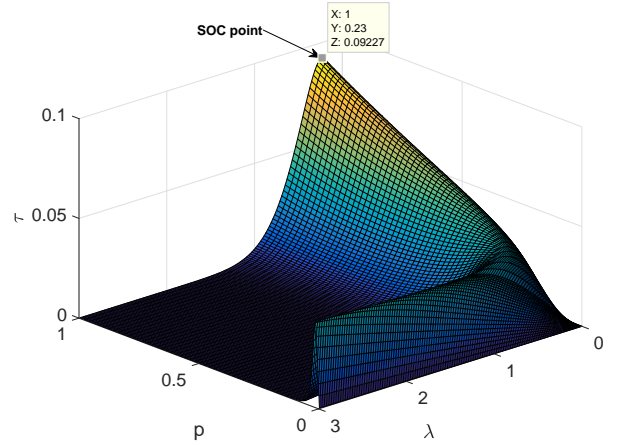


Fig. 4. Three-dimensional plot of $\tau(\theta, x)$ corresponding to the contour plot in Fig. 3.

active transmitters in the network per unit area, but a higher fraction of links are reliable. On the other hand, a large p means more active transmitters per unit area, but also a higher interference which reduces the fraction of reliable links. For $\lambda < 0.4$, the increase in the density of active transmitters dominates the increase in interference, and $\tau(\theta, x)$ increases monotonically with p . An observation for Fig. 3 is that the average success probability $p_s(\theta)$ at the SOC point is 0.6984 for 90% reliability. The three-dimensional plot corresponding to the contour plot in Fig. 3 is shown in Fig. 4.

D. High-reliability Regime

In the high-reliability regime, the reliability threshold is close to 1, *i.e.*, $x \rightarrow 1$. Alternatively, the outage probability threshold $\epsilon = 1 - x$ of a link is close to 0, *i.e.*, $\epsilon \rightarrow 0$.

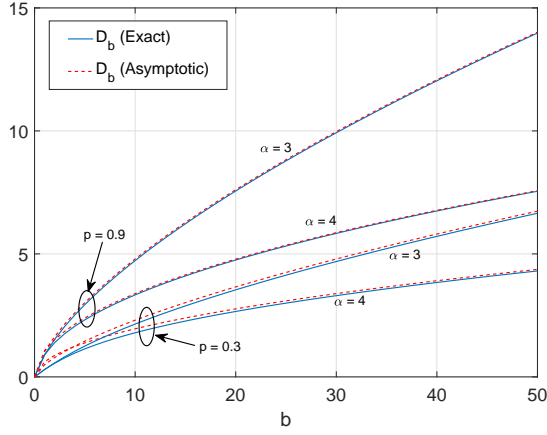


Fig. 5. The solid lines represent the exact $D_b(p, \delta)$ as in (10), while the dashed lines represent asymptotic form of $D_b(p, \delta)$ as in (19). Observe that (19) is a good approximation of (10) and is asymptotically tight.

In this section, we investigate the behavior of $\tau(\theta, x)$ and the SOC in the high-reliability regime. To this end, we first state a simplified version of de Bruijn's Tauberian theorem (see [11, Thm. 4.12.9]) which allows a convenient formulation of $\eta(\theta, 1 - \epsilon) = \mathbb{P}(P_s(\theta) > 1 - \epsilon)$ as $\epsilon \rightarrow 0$ in terms of the Laplace transform. The following simplified version of de Bruijn's Tauberian theorem suffices for our purposes.

Theorem 1 (de Bruijn's Tauberian theorem [12, Thm. 1]). *For a non-negative random variable Y , the Laplace transform $\mathbb{E}[\exp(-sY)] \sim \exp(rs^u)$ for $s \rightarrow \infty$ is equivalent to $\mathbb{P}(Y \leq \epsilon) \sim \exp(q/\epsilon^v)$ for $\epsilon \rightarrow 0$, when $1/u = 1/v + 1$ (for $u \in (0, 1)$ and $v > 0$), and the constants r and q are related as $|ur|^{1/u} = |vq|^{1/v}$.*

Theorem 2. *For $\epsilon \rightarrow 0$, the density of reliable links $\tau(\theta, 1 - \epsilon)$ satisfies*

$$\tau(\theta, 1 - \epsilon) \sim \lambda p \exp \left(- \left(\frac{\theta p}{\epsilon} \right)^\kappa \frac{(\delta \lambda C')^{\frac{\kappa}{\delta}}}{\kappa} \right), \quad \epsilon \rightarrow 0, \quad (18)$$

where $\kappa = \frac{\delta}{1-\delta} = \frac{2}{\alpha-2}$ and $C' = \pi R^2 \Gamma(1 - \delta)$.

Proof: First, note that for $b \in \mathbb{C}$,

$$D_b(p, \delta) \sim p^\delta b^\delta / \Gamma(1 + \delta), \quad |b| \rightarrow \infty, \quad (19)$$

where $D_b(p, \delta)$ is given by (10). In Fig. 5, we illustrate how quickly D_b approaches the asymptote.

Let $Y = -\ln(P_s(\theta))$. The Laplace transform of Y is $\mathbb{E}(\exp(-sY)) = \mathbb{E}(P_s(\theta)^s) = M_s(\theta)$. Using (9) and (19), we have

$$M_s(\theta) \sim \exp \left(- \frac{\lambda C' (\theta p)^\delta s^\delta}{\Gamma(1 + \delta)} \right), \quad |s| \rightarrow \infty.$$

From Thm. 1, we have $r = -\frac{\lambda C' (\theta p)^\delta}{\Gamma(1 + \delta)}$, $u = \delta$, $v = \delta/(1 - \delta) = \kappa$, and thus

$$q = \frac{1}{\kappa} (\delta \lambda C')^{\frac{\kappa}{\delta}} (\theta p)^\kappa,$$

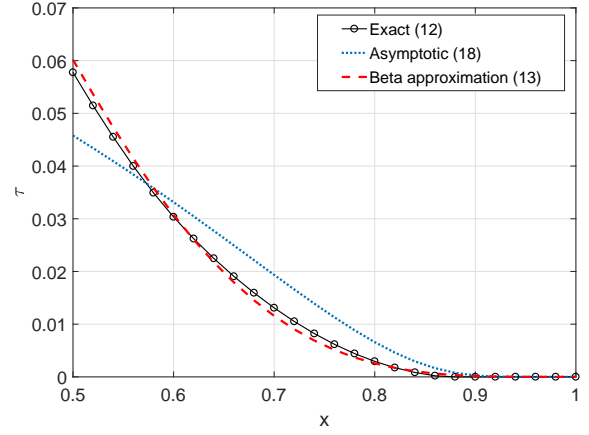


Fig. 6. The solid line with marker 'o' represents the exact expression of $\tau(\theta, x)$ as in (12), the dotted line represents the asymptotic expression of $\tau(\theta, x)$ given by (18) as $\epsilon \rightarrow 0$, and the dashed line represents the approximation by the beta distribution given by (13). Observe that the beta approximation is good. $\theta = 0$ dB, $R = 1$, $\alpha = 4$, $\lambda = 1/2$, and $p = 1/3$.

where $C' = \pi \Gamma(1 - \delta)$. Using Thm. 1, we can now write

$$\begin{aligned} \mathbb{P}(Y \leq \epsilon) &= \mathbb{P}(P_s(\theta) \geq \exp(-\epsilon)) \\ &\stackrel{(a)}{\sim} \mathbb{P}(P_s(\theta) \geq 1 - \epsilon), \quad \epsilon \rightarrow 0 \\ &= \exp \left(- \frac{(\theta p)^\kappa (\delta \lambda C')^{\frac{\kappa}{\delta}}}{\kappa \epsilon^\kappa} \right), \end{aligned} \quad (20)$$

where (a) follows from $\exp(-\epsilon) \sim 1 - \epsilon$ as $\epsilon \rightarrow 0$. Since we have

$$\tau(\theta, 1 - \epsilon) = \lambda p \mathbb{P}(P_s(\theta) > 1 - \epsilon). \quad (21)$$

the desired result in (18) follows from substituting (20) in (21). ■

For the special case of $p = 1$ (all transmitters are active), $\mathbb{P}(P_s(\theta) \geq 1 - \epsilon)$ as in (20) simplifies to

$$\mathbb{P}(P_s(\theta) \geq 1 - \epsilon) \sim \exp \left(- \frac{(\delta \lambda C' \theta^\delta)^\kappa}{\kappa \epsilon^\kappa} \right), \quad \epsilon \rightarrow 0,$$

which is in agreement with [6, Thm. 2] where it was derived in a less direct way than Thm. 2. Fig. 6 shows the tightness of (18) in the high-reliability regime and also the accuracy of the beta approximation given by (13).

We now investigate the scaling of $S(\theta, 1 - \epsilon)$ in the high-reliability regime.

Corollary 1 (SOC in high-reliability regime). *For $\epsilon \rightarrow 0$, the SOC is asymptotically equal to*

$$S(\theta, 1 - \epsilon) \sim \left(\frac{\epsilon}{\theta} \right)^\delta \frac{e^{-(1-\delta)}}{\pi R^2 \delta^\delta \Gamma(1 - \delta)}, \quad (22)$$

and it is achieved at $p = 1$.

Proof: Let us denote

$$\xi(\theta, \epsilon) = \left(\frac{\theta}{\epsilon} \right)^\kappa \frac{(\delta C')^{\kappa/\delta}}{\kappa}.$$

From (18), we can then write

$$\tau(\theta, 1 - \epsilon) \sim \lambda p \exp\left(-\lambda^{\kappa/\delta} p^{\kappa} \xi(\theta, \epsilon)\right), \quad \epsilon \rightarrow 0.$$

Thus we have

$$S(\theta, 1 - \epsilon) \sim \sup_{\lambda, p} f(\lambda, p), \quad \epsilon \rightarrow 0,$$

where $f(\lambda, p) = \lambda p \exp(-\lambda^{\kappa/\delta} p^{\kappa} \xi(\theta, \epsilon))$. First, fix $p \in (0, 1]$. As $\epsilon \rightarrow 0$, we can then write

$$\frac{\partial f}{\partial \lambda} = \underbrace{p \exp\left(-\lambda^{\kappa/\delta} p^{\kappa} \xi(\theta, \epsilon)\right)}_{>0} \left[1 - \frac{\kappa \xi(\theta, \epsilon)}{\delta} \lambda^{\kappa/\delta} p^{\kappa}\right].$$

Setting $\frac{\partial f}{\partial \lambda} = 0$, we obtain the critical point as

$$\lambda_0 = \left(\frac{\delta}{\xi(\theta, \epsilon) \kappa p^{\kappa}}\right)^{\delta/\kappa}.$$

Note that, for a fixed p , f is strictly increasing for $\lambda \in (0, \lambda_0]$ and strictly decreasing for $\lambda > \lambda_0$. Hence we have

$$\begin{aligned} S(\theta, 1 - \epsilon) &\sim \sup_p f(\lambda_0, p), \quad \epsilon \rightarrow 0, \\ &= \left(\frac{\delta}{\epsilon \kappa \xi(\theta, \epsilon)}\right)^{\delta/\kappa} \sup_p p^{1-\delta}. \end{aligned}$$

Observe that $f(\lambda_0, p)$ monotonically increases with p , and thus attains the maximum at $p = 1$. Thus the SOC is achieved at $p = 1$ and is given by (22) after simplification. ■

Remark 1. From Cor. 1, we observe that, as $\epsilon \rightarrow 0$, the exponents of θ and ϵ are the same. In the high-reliability regime, the SOC scales in ϵ similar to the TC defined in [6], while the TC defined in [4] scales linearly in ϵ (see [5, (4.29)]).

For $\alpha = 4$, the expression of SOC in (22) simplifies to

$$S(\theta, 1 - \epsilon) \sim \left(\frac{2\epsilon}{\theta e}\right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{3}{2}} R^2}, \quad \epsilon \rightarrow 0.$$

For $\alpha = 4$, Fig. 7 plots $\tau(\theta, x)$ versus λ and p for $x = 0.993$. In this case, the SOC is achieved at $p = 1$.

IV. CONCLUSIONS

The first main contribution is a new notion of capacity, termed spatial outage capacity (SOC), which is the maximum density of concurrently active intermissions while ensuring a certain reliability. The SOC gives fine-grained information about the network compared to the TC whose framework is based on the average success probability. The SOC has applications in wireless networks with strict reliability constraints.

Secondly, for Poisson bipolar networks with ALOHA, we have obtained an exact analytical expression and a simple approximation for the density τ of concurrently active links satisfying an outage constraint. The SOC can be easily calculated numerically as the supremum of τ obtained by optimizing over the density λ and the transmit probability p . When constrained on the density of concurrent transmissions, i.e., for constant λp , while letting $p \rightarrow 0$, the supremum of τ

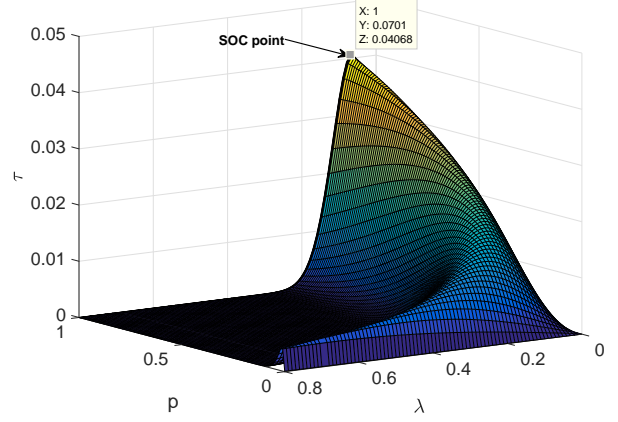


Fig. 7. Three-dimensional plot of $\tau(\theta, x)$ for $x = 0.993$, $\theta = -10$ dB, $R = 1$, and $\alpha = 4$. Observe that $p = 1$ achieves the SOC. The average success probability $p_s(\theta)$ at the SOC point is 0.8964.

is equal to the product λp if the reliability threshold is smaller than the average success probability and zero if the reliability threshold is larger than the average success probability. In the high-reliability regime where the target outage probability of a link goes to 0, we give a closed-form expression of the SOC and show that $p = 1$ achieves the SOC.

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