Practically Relevant Incompleteness

The Skill-Issue Incompleteness Theorem (SIIT pronounced like 'site') is well-known to mathematicians. It occurs when the theorem in question is likely true, but too hard to prove for the mathematician's skill-level. The state-of-the-art is to make more assumptions. Usually those assumptions make the theorem less likely to correspond to reality by restricting the possible states of reality that the theorem is likely to describe. It reduces the power of the mathematics.

It is related to the Godel Incompleteness Theorem, which states that any set of axioms that lets me prove 1+1=2 is bound to have statements at which no logical chain ends. They are related because the set of axioms that we have lets me prove 1+1=2[1], and there is no logical chain connecting me to the statements which would complete my PhD. That is why the theorem is sometimes referred to as Incompleteness for the Poor Academic (IPA). A related result is that the acronym describing a sociological affliction is also the most easily available cure for said condition.

Though thus far merely empirically meritous, we attempt a proof of IPA here. Consider the sentence "I do not have the skill to prove the IPA." If I do have the skill, I don't. If I don't have the skill, I don't. Thus, it must be true that I do not have the skill to prove the IPA. Of course, this also implies that I don't. Thus, there is at least one sentence which I lack the skill to prove. However, like the Riemann Hypothesis, most grad school admissions programs are certain that it is true.

The most commonly employed workaround is to actually get better at math, usually through arduous class-taking and work-doing. Many PhD students have wondered whether other paths exist. Thus it is of fundamental importance to those familiar with IPA whether there exists an axiomatic system where an interesting theorem[2] is provable with a sequence of steps contained in one's skill-ball. The skill-ball is the maximum number of logical operations a mathematician can do before pivoting to quantitative research.

Imagine Leibnitz's calculemus machine, which began Hilbert's grand program, for any arbitrary axiomatic system. We know it does not halt at undecidable statements, but you weren't going to prove those anyway. Simply define the skill-ball and check all valid operations from the axiomatic system. You might find one that results in the theorem-of-interest. In fact, the algorithm is embarassingly parallel![3]

Algorithm 1: Skill-Ball Constrained Proof Enumeration

Input: A: Set of Axiomatic Systems, S: statement, T: max logical steps (adjust for skill level)
Output: $a \in A$:
while *True* **do**
  a $\leftarrow$ A.next()
  $L : a \rightarrow \{$sequences of valid logical operations beginning with a $\}$
  for l in $L(a)$:
    if $l[-1] == S$: **return** (l, a)
end

One may wonder: does this program halt? Relatedly, does a program exist that would determine whether this program halts? We could define a period of time, say 1 month, after which we decide that this program does not halt for the statement S. We may wonder about a couple of related things

1. Is there more than 1 undecidable statement? That is, other than a-finite-number-of statements, is every statement decidable?

[1] for details, see Whitehead and Russell [1910–1913]
[2] interesting is determined by one's advisor of course
[3] Even with this speedup, most modern hardware cannot run this algorithm. This is due to the hardware halting problem, an equivalent formulation of the SIIT.

2. Is any statement provable in some axiomatic system? (no cheating: the statement cannot be an axiom)
3. Is any statement undecidable in every axiomatic system? (no cheating)

More practically, for any statement, can I devise a set of axioms such that the statement is more easily provable than in ZFC? The metric for 'easily provable' is the number of steps in the simplest proof. This metric is not perfect, of course, because certain steps are harder to see than others.

The answer to Question (1) is yes, and the famous example is the Continuum Hypothesis (CH). The stanford encyclopedia [Koellner, 2020] tells me that 'this independence result was quickly followed by many others' and that Gödel's large cardinal axioms were able to prove most of them except the actual CH. The answer to (2) and (3) however, seem to be unknown. The fundamental issue seems to be: what constitutes a reasonable axiom, that nonetheless improves the power of the formal system.

References

Peter Koellner. The Continuum Hypothesis. In Edward N. Zalta and Uri Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Fall 2020 edition, 2020.

Alfred North Whitehead and Bertrand Russell. *Principia Mathematica*. Cambridge University Press, 1910–1913.