MTH 102: Probability and Statistics

Quiz 1

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Explain your answers. Show your steps. No gadgets allowed. Approximate calculations are fine as long as the approximations are reasonable. You have 45 minutes. Good luck!

Question 1. 90 **marks** Years of data driven research has led to a probabilistic model that explains the impact of student motivation on performance in exams and vice versa. To keep it simple enough, we will characterize motivation as either high or low and exam performance as being good, fair, or poor. At the beginning of examination season a student's motivation is high with probability 0.5. A student whose motivation is high demonstrates good performance in an exam with probability 0.6 and fair performance with probability 0.3. On the other hand, a student with low motivation demonstrates good performance with probability 0.1 and fair performance with probability 0.3.

We will consider an examination season that has two exams. Researchers have tried to model the impact of performance in the first exam on motivation levels of a student, which as described above impacts the student's performance in the second exam. Good performance in the first exam leads to a high level of student motivation to prepare for the next exam with probability 0.7. Fair performance leads to high motivation with probability 0.5 and poor performance in an exam leads to high motivation with probability 0.3. Answer the following questions.

RUBRIC

10 + 10

Proper Event & Prob Define. | Tree Diagram.

1) Clearly define events (use proper notation) that correspond to motivation before the first and the second exams and performance in the two exams. Draw a tree diagram that captures student motivation at the start of examination season, performance in the first exam, the resulting level of motivation, and the performance in the second exam. For each branch, clearly state the start and end event and the associated probability. You must state both the mathematical definition of the probability using the events and also the values. For example, you must say $P[A] = 0.2$ and not just either $P[A]$ or 0.5.
2) 10 Calculate the probability that the performance in the first exam is good.
3) 10 Calculate the probability that the performance in the first exam is poor.
4) 10 Calculate the probability that the performance in the second exam is good.
5) 10 Calculate the probability that the performance in the second exam is poor.
6) 15 A student is known to have a performance of good in the second exam. What is your revised belief about the student's performance being poor in the first exam? Is performance of good in the second exam independent of poor performance in the first? Use the definition of independence of events to arrive at your answer.
7) 15 Use the definition of independence of events to show whether a good performance in the second exam is independent of a high motivation at the start of the examination season.

Question 2. 10 **marks** Show from first principles that mutually exclusive events A and B are dependent. [Hint: Define mutually exclusive events and use the definition of independent events.]

Question 1.

We have two exams. We are interested in performance at the end of each exam. We are also interested in the motivation of the student before each exam.

Let H_1 be the event that a student has high motivation at the beginning of the exam season.

Let H_2 be the event that a student has high motivation before the second exam.

Note that we could instead say:
Let H_i be the event that a student has high motivation before exam i, where $i \in \{1, 2\}$

Similarly, let G_i, F_i, and P_i be the events that the student's performance in exam i is good, fair, and poor respectively. Of course, $i \in \{1, 2\}$

We are given the following:

$P[H_1] = 0.5 \quad \therefore P[L_1] = 0.5$

Given:

$P[G_1 | H_1] = 0.6$, $P[F_1 | H_1] = 0.3$.

$\therefore P[P_1 | H_1] = 0.1$.

Also, given:

$P[G_1 | L_1] = 0.1$, $P[F_1 | L_1] = 0.3$.

$\therefore P[P_1 | L_1] = 0.6$

Since the impact of motivation on performance is the same across exams, we also have:

$P[G_2 | H_2] = 0.6$, $P[F_2 | H_2] = 0.3$, $P[P_2 | H_2] = 0.1$

$P[G_2 | L_2] = 0.1$, $P[F_2 | L_2] = 0.3$, $P[P_2 | L_2] = 0.6$

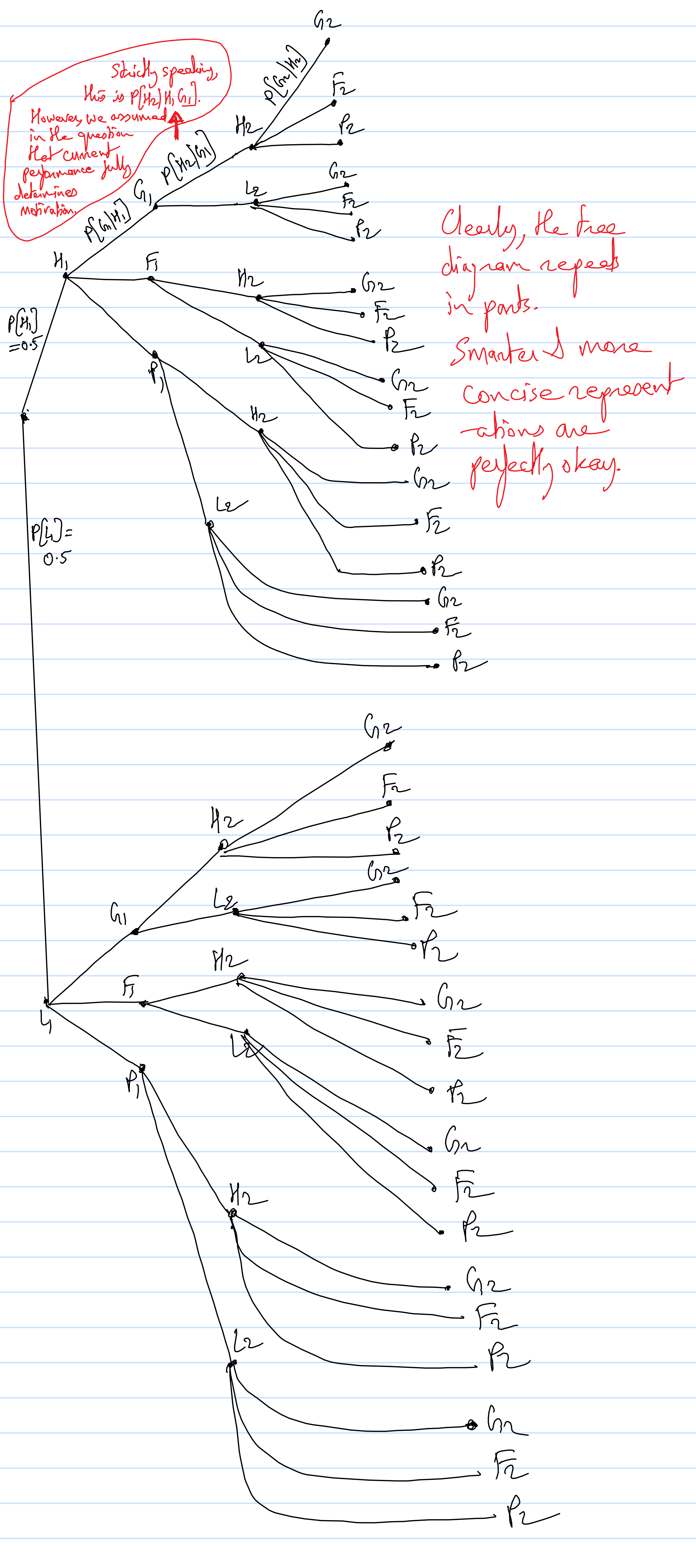
We are also given that

$P[H_2 | G_1] = 0.7 \Rightarrow P[L_2 | G_1] = 0.3$

$P[H_2 | F_1] = 0.5 \Rightarrow P[L_2 | F_1] = 0.5$

$P[H_2 | P_1] = 0.3 \Rightarrow P[L_2 | P_1] = 0.7$

Clearly, we have everything we need for our tree diagram.



(2) We want to calculate $P[G_1]$

There are two paths in the tree diagram that lead to G_1.

$S \xrightarrow{P[H_1]} H_1 \xrightarrow{P[G_1|H_1]} G_1$ [Occurs with probability $P[H_1]P[G_1|H_1]$]
Also, probability of H_1, G_1. That is $P[H_1 G_1]$.

$S \xrightarrow{P[L_1]} L_1 \xrightarrow{P[G_1|L_1]} G_1$ [$P[L_1 G_1] = P[G_1|L_1]P[L_1]$]

$\therefore$ $P[G_1]$ can be obtained by summing over the probabilities of the paths (as paths in a tree diagram are mutually exclusive). Therefore, in terms of the knowns:

$$P[G_1] = P[G_1|L_1]P[L_1] + P[G_1|H_1]P[H_1]$$

[Substitute values].

(3) As above, spot the relevant paths. Or, equivalently recognize the event space $\{H_1, L_1\}$.

$\therefore$ $P_1 = (P_1 \cap H_1) \cup (P_1 \cap L_1)$

and $P[P_1] = P[P_1 H_1] + P[P_1 L_1]$

In terms of the knowns we have

$$P[P_1] = P[P_1|H_1]P[H_1] + P[P_1|L_1]P[L_1].$$

[Substitute values].

(4) We want $P[G_2]$.

There are twelve mutually exclusive paths that end in G_2!

[List the paths]

$\boxed{\text{Consider the path:}}$

$$S \xrightarrow{P[L_1]} L_1 \xrightarrow{P[G_1|L_1]} G_1 \xrightarrow{P[H_2|G_1]} H_2 \xrightarrow{P[G_2|H_2]} G_2$$

We have everything we need to know to calculate the probability of this path!

Aside: To re-emphasize that $P[G_2|H_2]$ is in fact $P[G_2|H_2 G_1 L_1]$. Just that given our model G_2 is entirely probabilistically determined by H_2, when H_2 is given.

Consider: $P[L_1 G_1 H_2 G_2]$

$$= P[G_2 \cap (L_1 G_1 H_2)]$$

$$= P[G_2 | L_1 G_1 H_2]\, P[L_1 G_1 H_2] \quad \left[\text{By defn}\right]$$

$$= P[G_2|L_1 G_1 H_2]\, P[H_2 | L_1 G_1]\, P[L_1 G_1]$$

$$= P[G_2|L_1 G_1 H_2]\, P[H_2| L_1 G_1]\, P[G_1|L_1]\, P[L_1]$$

We implicitly assumed that performance was only dependent on motivation when motivation was given. So we wrote $P[G_2|H_2]$ on the branch that truly corresponds to $P[G_2|L_1 G_1 H_2]$.

(5) Same as (4)

(6) You have been told that event G_2 happened.
You want to calculate
$$P[P_1 | G_2].$$

$$P[P_1 | G_2] = \frac{P[P_1\, G_2]}{P[G_2]}.$$

The numerator is about the event $P_1 \cap G_2$.

P_1 could either be preceded by H_1 or L_1.

$\therefore\ P_1 G_2 = H_1 P_1 G_2 \cup L_1 P_1 G_2$

P_1 could be succeeded by either H_2 or L_2.

$\therefore\ H_1 P_1 G_2 = H_1 P_1 H_2 G_2 \cup H_1 P_1 L_2 G_2$

Similarly,
$L_1 P_1 G_2 = L_1 P_1 H_2 G_2 \cup L_1 P_1 L_2 G_2$

$$\therefore\ P_1 G_2 = H_1 P_1 H_2 G_2 \cup H_1 P_1 L_2 G_2 \cup L_1 P_1 H_2 G_2 \cup L_1 P_1 L_2 G_2.$$

Not surprisingly each of the events in the union are mutually exclusive paths in our tree diagram.
You could get them straight from the diagram. All the paths that have $P_1\, G_2$ in them. A total of 4.

$$P[P_1 G_2] = P[H_1 P_1 H_2 G_2] + P[H_1 P_1 L_2 G_2] + P[L_1 P_1 H_2 G_2] + P[L_1 P_1 L_2 G_2]$$

[Calculate the probabilities using the tree diagram].

$P[G_2]$ is straightforward. You should have calculated it in [4].

P_1 and G_2 are independent events iff $\quad P[P_1 G_2] = P[P_1]\, P[G_2]$
that is $\ P[P_1 | G_2]\, P[G_2] = P[P_1]\, P[G_2]$
or equivalently $\ P[P_1 | G_2] = P[P_1]$.

You calculated $P[P_1 | G_2]$ above.
Compare with $P[P_1]$, which you calculated in (3).
Compare $P[P_1 | G_2]$ and $P[P_1]$.
If they are equal then the events are independent. Else, they are not.

We want to test whether

$$P[G_2 | H_1] = P[G_1]$$

Other tests for independence include checking whether

$$P[G_2 H_1] = P[G_2] P[H_1]$$

Or equivalently, checking whether

$$P[H_1 | G_2] = P[H_1]$$

Of all these the second seems the easiest.

$P[G_2 H_1]$ can be gotten from the tree diagram. [Six paths].

The rest is given or you have already calculated.

Q2 A and B are mutually exclusive events. By definition, we have

$$P[A \cap B] = 0.$$

A and B are independent iff

$$P[A \cap B] = P[A]\, P[B].$$

Suppose $P[A] \neq 0$, $P[B] \neq 0$.

Clearly $P[A]\, P[B] \neq 0$.

$\therefore$ $P[A \cap B] \neq P[A]\, P[B]$.

The events are dependent.

Aside: (Okay if not done in exam)

Of course, we must consider

$P[A] = 0$, $P[B] \neq 0$,

$P[A] \neq 0$, $P[B] = 0$, and

$P[A] = 0$, $P[B] = 0$.

For either of the above conditions, the condition for independence is satisfied.

$\therefore$ Mutually exclusive events A and B are also independent when either or both events occur with 0 probability.

[Also see page 22 of the textbook (2nd Ed. by Roy Yates & Goodman)].