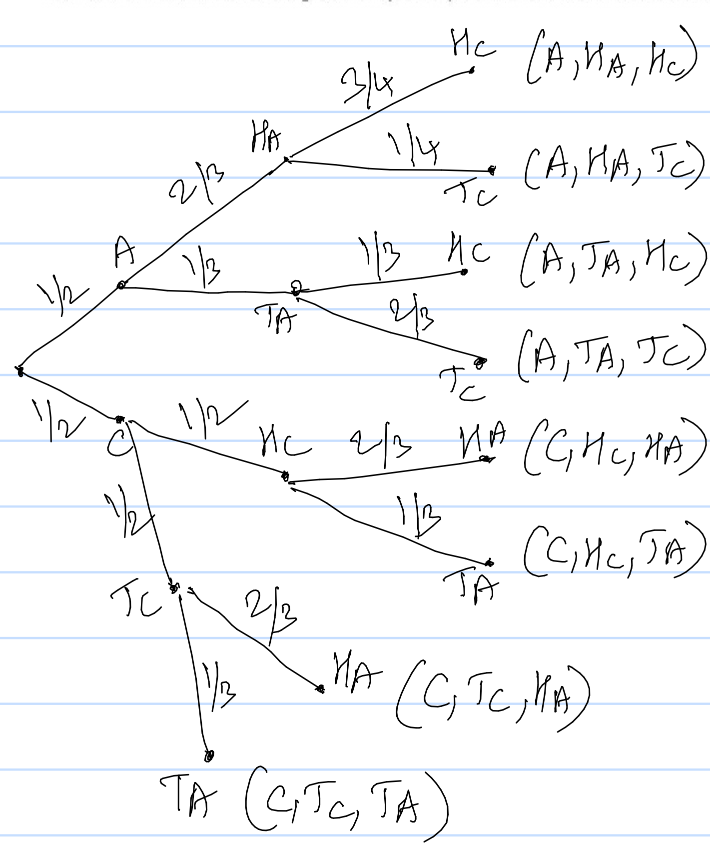
CATCHING THE COPYCAT

II. Catching the Copycat (70 marks)

There are two coins A and C, of which C is a copycat. You are heading an effort to catch the copycat coin C. Coin A has a mind of its own and tossing it gives heads with probability $2/3$. The behavior of coin C is dependent on whether it is tossed first or after coin A. If coin C is tossed first, the toss gives heads with probability $1/2$. On the other hand, if C is tossed after coin A, coin C tries to copy the outcome of the toss of coin A. Specifically, if the toss of A gave heads, the toss of C gives heads with probability $3/4$. If the toss of A gave tails, the toss of C gives heads with probability $1/3$.

You are scheduled to see the coins perform at an exhibition. The performance will proceed as follows. The two coins (identities unknown) will be brought in a jar. One of the coins will be picked randomly from the jar and tossed. Following this, the other coin in the jar will be tossed. You decide that you will choose the coin that is tossed first to be coin C in case the outcomes of the two tosses are different. Else, you will choose the second coin to be tossed to be coin C. Answer the following questions. [Hint: Tree diagrams may come in handy.]

(a) (4 marks) What is the probability that coin A is chosen for the first toss?
(b) (12 marks) What is the probability that the first coin toss gives heads and the second gives tails?
(c) (12 marks) What is the probability that the first coin toss gives tails and the second gives heads?
(d) (12 marks) What is the probability that both coin tosses give heads?
(e) (12 marks) What is the probability that both coin tosses give tails?
(f) (18 marks) What is the probability that you will choose the correct coin C?



A is the event that coin A was tossed first.
C is the event that coin C was tossed first.
H_A is the event that the toss of coin A gives Heads
T_A is the event that the toss of coin A gives Tails.
H_C is the event that the toss of coin C gives Heads
T_C is the event that the toss of coin C gives Tails.

(a) The event of interest is A.
$P[A] = 1/2$.

(b) This happens in two mutually exclusive ways. The events are $\{(A, H_A, T_C)\}$ and $\{(C, H_C, T_A)\}$
$\therefore$ The probability of the desired event is
$$P[\{(A, H_A, T_C)\}] + P[\{(C, H_C, T_A)\}]$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)$$

(c) The prob of interest is [Event: first toss tails and second heads]
$$P[\{(A, T_A, H_C)\}] + P[\{(C, T_C, H_A)\}]$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)$$

(d)
$$P[\{(A, H_A, H_C)\}] + P[\{(C, H_C, H_A)\}]$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)$$

(e)
$$P[\{(A, T_A, T_C)\}] + P[\{(C, T_C, T_A)\}]$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right).$$

(f) One of the mutually exclusive events below must occur for you to choose the correct coin C:

(C, T_C, H_A)
(C, H_C, T_A)
[Since the outcomes of the tosses are different you will choose the first coin to be coin C. These are the events where indeed the coin C is the first coin.]

(A, H_A, H_C)
(A, T_A, T_C)
[Since the outcomes are the same, you will choose the second coin to be coin C. These are the two events, which when either occurs, will have you guess correctly.]

$$P[\text{You choosing the correct coin}]$$
$$= P[\{(C, T_C, H_A)\}] + P[\{(C, H_C, T_A)\}] + P[\{(A, H_A, H_C)\}] + P[\{(A, T_A, T_C)\}]$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{9} = \frac{1}{2} + \frac{1}{9} \approx 0.6 //$$

Q) What if you always choose the second coin to be C, irrespective of the outcomes? What is the probability that you will choose the correct coin C? Is this better or worse??
Q) Can you think of a better deterministic strategy than in the question?

I. Two Coin Tosses (30 marks)

You perform an experiment that involves tossing a coin twice and noting the outcome of each toss. A coin toss leads to an outcome from the set $\{H, T\}$, where outcome H denotes heads and outcome T denotes tails. The coin tosses are independent and the probability that a toss gives heads is p. Answer the following questions.

(a) (4 marks) In terms of the given set of outcomes, write down the event that the first toss gives heads. Express the probability of the event in terms of probabilities of the outcomes it contains.
(b) (4 marks) Similar to above, write down the event that at least one toss gives heads. Express the probability of the event in terms of the probabilities of the outcomes it contains.
(c) (4 marks) You pick the outcome of any coin toss. What is the probability that the outcome is H?
(d) (4 marks) Express the probability of heads in the second toss conditioned on the knowledge that the first toss gave heads using standard notation. What is this probability as a function of p?
(e) (4 marks) Consider the probability of two heads conditioned on heads in the first toss. Express it using notation. What is this probability as a function of p?
(f) (4 marks) What is the probability that at least one heads was observed, given that the second toss gave heads?
(g) (6 marks) What is the probability of heads on the first toss, given that at least one heads was observed?

(a) The event is $\{HT, HH\}$
$$P[\{HT, HH\}] = P[\{HT\}] + P[\{HH\}] \quad [\because \text{They are mutually excl.}]$$
$$= P[H]P[T] + P[H]P[H] \quad [\text{Since tosses are independent}]$$
$$= p(1-p) + p^2$$
$$= p.$$

(b) $\{HT, TH, HH\}$
$$P[\{HT, TH, HH\}] = P[HT] + P[TH] + P[HH]$$
$$= p(1-p) + (1-p)p + p^2$$
$$= 2p - 2p^2 + p^2 = 2p - p^2.$$

(c)
$$\underbrace{\tfrac{1}{2}}_{\text{Prob you pick first outcome}} P[\{HT, HH\} \mid \text{You picked first}] + \underbrace{\tfrac{1}{2}}_{\text{prob you pick second outcome.}} P[\{TH, HH\} \mid \text{You picked 2nd}]$$

Note that what you pick and the outcomes are independent. Thus the prob of interest is
$$\tfrac{1}{2} P[\{HT, HH\}] + \tfrac{1}{2} P[\{TH, HH\}]$$
$$= \tfrac{1}{2}\left[P[\{HT\}] + P[\{TH\}] + 2P[HH]\right]$$
$$= \tfrac{1}{2}\left[2p(1-p) + 2p^2\right] = p(1-p) + p^2$$
$$= p //$$

You could also argue that each toss is heads independently of the other with probability. Since you pick a toss randomly, the probability of heads is simply p.

(d)
$$P[H \text{ in second} \mid H \text{ in first}]$$
$$= P[H \text{ in second}] \quad [\because \text{Coin tosses are independent}]$$
$$= p.$$

(e)
$$P[H \text{ in first}, H \text{ in second} \mid H \text{ in first}]$$
$$= \frac{P[H \text{ in first}, H \text{ in second}, H \text{ in first}]}{P[H \text{ in first}]}$$
$$= \frac{P[H \text{ in first}, H \text{ in second}]}{P[H \text{ in first}]}$$
$$= \frac{P[H \text{ in first}]\,P[H \text{ in second}]}{P[H \text{ in first}]} \quad [\text{Because the coin tosses are indep}]$$
$$= p.$$

(f)
$$P[\text{At least one heads} \mid \text{2nd toss gave heads}]$$
$$= \frac{P[\text{At least one heads, 2nd toss gave heads}]}{P[\text{2nd toss gave heads}]}$$
$$= \frac{P[\text{2nd toss gave heads}]}{P[\text{2nd toss gave heads}]} = 1 //$$

Does this result agree with your intuition??

(g)
$$P[\text{Heads in first} \mid \text{At least one heads}]$$
$$= \frac{P[\text{Heads in first, At least one heads}]}{P[\text{At least one heads}]}$$
$$= \frac{P[\text{Heads in first}]}{P[\{HT, TH, HH\}]}$$
$$= \frac{p}{P[\{HT\}] + P[\{TH\}] + P[\{HH\}]}$$
$$= \frac{p}{p(1-p) + (1-p)p + p^2}$$
$$= \frac{1}{2 - 2p + p} = \left(\frac{1}{2-p}\right) //$$