MTH 102: Probability and Statistics

Quiz 1

06/07/2021

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Open book and notes. Any exchange of information related to the quiz with any other human will be deemed as cheating. Institute rules will apply. Explain your answers. Show your steps. Approximate calculations are fine as long as the approximations are reasonable. You have 60 minutes to work on the quiz, and an additional 15 minutes to upload your work as a single PDF. Good luck!

I. Seasonal RandomV

A seasonal viral infection due to a virus named RandomV is prevalent in Delhi. About 1 in 1000 people in Delhi are expected to be infected by RandomV. The virus spreads from an infected person to another person only when a sufficient amount of viral load is transmitted from the infected person to the other. A test has been designed to detect the RandomV infection. Among the infected, the test gives a positive result in 90% of them. Among healthy people (that is those who are free of infection), the test is known to come up with a positive result in 10% of them. Given the above background about RandomV, solve the following questions.

Question 1. 20 **marks** Suppose you go to a gathering of 10 people who have come together from different parts of Delhi to celebrate Delhi's history. Derive the following probabilities.

(a) You meet a randomly chosen person in the gathering. What is the probability that the person is infected with RandomV?
(b) You meet a randomly chosen pair in the gathering. What is the probability that you end up meeting a pair of infected people?
(c) You meet a randomly chosen pair in the gathering. What is the probability that you end up meeting a pair in which one or more are infected?
(d) What is the probability that two or more in the gathering of 10 are infected?

In the Google form, for this question, for every part you solve, mention the obtained answer. The PDF you upload post the end of the quiz must detail how you solved the above parts of the question.

Question 2. 40 **marks** Knowing the infectious nature of RandomV, a certain healthy person decides to limit his daily interaction with other people. However, given the person's work, he ends up meeting a total of two people on a certain day. He does not know whether the two people were infected. However, he has read that an infected person can with probability 0.1 transfer a viral load that is sufficient to infect a healthy person. Also, for a given healthy person, the outcomes of interactions with different infected people are independent of each other. The person is worried post his meetings with the two people and decides to get himself tested.

(a) Derive the probability that his test will give a positive result. [Hint: What is the probability that the person is infected?]
(b) Suppose his test gives a positive result. What is the probability that the person is infected with RandomV?

In the Google form, for this question, mention any event spaces you used. The PDF you upload post the end of the quiz must detail how you solved the above parts of the question.

Question 3. 40 **marks** Consider the person in Question 2 and his meeting two people on his work day. Now suppose that a vaccine exists to protect against RandomV. The vaccine is known to have an efficacy of 95%, which is to say that the probability that an infected person transfers a viral load that is sufficient to infect a healthy *vaccinated* person is $1/20$ of the corresponding probability for a healthy unvaccinated person. It is also known that the behavior of the test is independent of whether a person is vaccinated or not. What is the probability that the above person is vaccinated, given that his test result is positive? [*Hint* 1: Note that for events A, B, and C, $P[A, B|C] = P[A|B, C]P[B|C]$. *Hint* 2: The probability that the person is infected, given that he is vaccinated, maybe useful.].

In the Google form, for this question, mention if you used the identity $P[A, B|C] = P[A|B, C]P[B|C]$ and for which events. The PDF you upload post the end of the quiz must detail how you solved the question.

We are given:

$P[\text{A person is infected with RandomV}] = \frac{1}{1000} = 10^{-3}$.

$P[\text{Test is +ve} \mid \text{Person is infected}] = 0.9$

$P[\text{Test is +ve} \mid \text{Person is healthy}] = 0.1$.

$\Rightarrow$ $P[\text{Test is -ve} \mid \text{Person is infected}] = 0.1$ $\left[\because P[\text{Test is -ve} \mid \text{Person is infected}] + P[\text{Test is +ve} \mid \text{Person is infected}] = 1\right]$

$P[\text{Test is -ve} \mid \text{Person is healthy}] = 0.9$

(Q1) You go to a gathering of 10 people.

(a) $P[\text{The randomly chosen person you meet is infected}] = ?$

Let P_k be the event that the k^{th} person is chosen and I_k be the event that the person is infected. $P[P_k] = \frac{1}{10}$, since your choice is random.

$P[\text{The randomly chosen person you meet is infected}]$

$= P[(I_1 \cap P_1) \cup (I_2 \cap P_2) \cup (I_3 \cap P_3) \cup \cdots \cup (I_{10} \cap P_{10})]$

$= P[I_1 \cap P_1] + P[I_2 \cap P_2] + \cdots + P[I_{10} \cap P_{10}]$ (Note that the intersections are mutually exclusive)

$= P[I_1]P[P_1] + \cdots + P[I_{10}]P[P_{10}]$

$= \left(\frac{1}{1000}\right)\left(\frac{1}{10}\right) + \cdots + \left(\frac{1}{1000}\right)\left(\frac{1}{10}\right) = \frac{1}{1000}$.

3 if the fact that any person is infected with prob $\frac{1}{1000}$ is stated correctly.

+2 if the final answer is correct.

(b) $P[\text{A randomly chosen pair of people is infected}]$

$= P[\text{One in the pair is infected, The other in the pair is infected}]$

3 if the product of probabilities of the correct events is used

$= P[\text{One in the pair is infected}]\, P[\text{Other in the pair is infected}]$

[We can make the assumption that the event that one is infected is independent of whether the other is infected, as people have come from different parts of Delhi]

$= (10^{-3})(10^{-3}) = 10^{-6}$

However, you have $^{10}C_2$ pairs and any pair is chosen randomly, i.e., with prob $\frac{1}{^{10}C_2}$.

Thus the prob of interest, that is the

$P[\text{You meet a pair of infected people}] = {}^{10}C_2 \left[\left(\frac{1}{^{10}C_2}\right) 10^{-6}\right] = 10^{-6}$

2 if this final prob is correct. That is mutually exclusive events are counted correctly.

(c) In a pair, $P[\text{One or more are infected}]$

$= P[\text{One is infected} \cup \text{Other is infected}]$

$= P[\text{One is infected}] + P[\text{Other is infected}] - P[\text{One is infected, Other is infected}]$

$= 10^{-3} + 10^{-3} - 10^{-6} = \boxed{2(10^{-3}) - 10^{-6}}$

3 for correctly handling the event one or more are infected

You can also calculate the above probability by recognizing that the complement of the event "one or more in a pair are infected" is "none in the pair is infected".

$\therefore P[\text{One or more are infected}] = 1 - P[\text{none is infected}]$

$= 1 - (1 - 10^{-3})(1 - 10^{-3})$

$= 1 - [1 - 2(10^{-3}) + 10^{-6}] = 2(10^{-3}) - 10^{-6}$

Thus our probability of interest

$P[\text{You end up meeting a pair in which one or more are infected}]$

$= 2(10^{-3}) - 10^{-6}$

2 for counting correctly

(d) $P[\text{Two or more in the gathering are infected}]$

$= 1 - P[\text{0 or 1 in the gathering are infected}]$

$= 1 - (P[\text{0 are infected}] + P[\text{1 is infected}])$

$= 1 - \left((1-10^{-3})^{10} + {}^{10}C_1\, 10^{-3} (1-10^{-3})^9\right)$

5 for correctly interpreting the event.

Question 2. 40 marks Knowing the infectious nature of RandomV, a certain healthy person decides to limit his daily interaction with other people. However, given the person's work, he ends up meeting a total of two people on a certain day. He does not know whether the two people were infected. However, he has read that an infected person can with probability 0.1 transfer a viral load that is sufficient to infect a healthy person. Also, for a given healthy person, the outcomes of interactions with different infected people are independent of each other. The person is worried post his meetings with the two people and decides to get himself tested.

(a) Derive the probability that his test will give a positive result. [Hint: What is the probability that the person is infected?]

(b) Suppose his test gives a positive result. What is the probability that the person is infected with RandomV?

(a) P[Test is +ve]

= P[Test is +ve, Person is infected post meetings]
+ P[Test is +ve, Person is Healthy post meetings]

To make life simpler, let's use some notation.
Let I be the event that the person is infected post meetings.
Let H be the event that the person is healthy post meetings.

$P[\text{Test is +ve}] = P[\text{Test is +ve} | I]\, P[I] + P[\text{Test is +ve} | H]\, P[H]$

Recasting the probability of interest in terms of the knowns
10

We know about the test that
$P[\text{Test is +ve} | I] = 0.9$.
Also, $P[\text{Test is +ve} | H] = 0.1$.

See the very beginning of the question & this solution.

We must calculate P[I].

$I = \underbrace{\{\text{First person met transfers sufficient viral load}\}}_{\text{Event } T_1} \cup \underbrace{\{\text{2nd person met transfers sufficient viral load}\}}_{\text{Event } T_2}$

Writing I and P[I]
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$I = T_1 \cup T_2$

$P[I] = P[T_1] + P[T_2] - P[T_1 \cap T_2]$

T_1 & T_2 are independent events.
Therefore $P[T_1 \cap T_2] = P[T_1]\, P[T_2]$

$P[T_1] = P[\text{First met person is infected and transfers a sufficient load}]$

$= P[\text{Transfers a sufficient load} | \text{First met person is infected}]\; P[\text{First met person is infected}]$

$= (0.1)(10^{-3}) = 10^{-4}$

Calculating $P[T_1]$ and $P[T_2]$
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Similarly, $P[T_2] = 10^{-4}$

$\therefore P[I] = 2(10^{-4}) - (10^{-4})(10^{-4})$
$= 2(10^{-4}) - 10^{-8}$

(b) P[Infected | Test is +ve]

$$= \frac{P[\text{Test is +ve} | \text{Infected}]\, P[\text{Infected}]}{\left(P[\text{Test is +ve} | \text{Infected}]\, P[\text{Infected}] + P[\text{Test is +ve} | \text{Not Infected}]\, P[\text{Not Infected}]\right)}$$

$$= \frac{(0.9)\, P[\text{Infected}]}{(0.9)\, P[\text{Infected}] + (0.1)\, P[\text{Not Infected}]}$$

Correctly expressing the prob of interest using law of Total Prob
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Now what is P[Infected]?
Is it the prior probability 1/1000 or is it P[I] that we calculated in part (a)?

Given our knowledge of the person's interaction and his being healthy before the two meetings:

$P[\text{Infected}] = P[I]$.

$P[\text{Not Infected}] = 1 - P[I]$.

Correctly identify P[Infected]
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Question 3. 40 **marks** Consider the person in Question 2 and his meeting two people on his work day. Now suppose that a vaccine exists to protect against RandomV. The vaccine is known to have an efficacy of 95%, which is to say that the probability that an infected person transfers a viral load that is sufficient to infect a healthy *vaccinated* person is 1/20 of the corresponding probability for a healthy unvaccinated person. It is also known that the behavior of the test is independent of whether a person is vaccinated or not. What is the probability that the above person is vaccinated, given that his test result is positive? [*Hint* 1: Note that for events A, B, and C, $P[A, B|C] = P[A|B, C]P[B|C]$. *Hint* 2: The probability that the person is infected, given that he is vaccinated, maybe useful.].

$P[\text{An infected person transfers a viral load sufficient to infect a healthy vaccinated person}] = \left(\frac{1}{20}\right)(0.1) = 0.005 = 5\times 10^{-3}$

$P[\text{Person is vaccinated} \mid \text{Test result is +ve}] = ?$

Let V be the event that the person is vaccinated.

$$V = (V\cap I)\cup(V\cap I^c)$$

$P[V \mid \text{Test is +ve}] = ?$

Consider:

$$P[\text{Test is +ve} \mid V] = P[\text{Test is +ve}, I \mid V] + P[\text{Test is +ve}, I^c \mid V]$$
$$= P[\text{Test is +ve} \mid I, V]\,P[I \mid V] + P[\text{Test is +ve} \mid I^c, V]\,P[I^c \mid V]$$

Since the test's efficacy is not impacted by vaccination, we can simplify the above and write

$$P[\text{Test is +ve} \mid V] = P[\text{Test is +ve} \mid I]\,P[I \mid V] + P[\text{Test is +ve} \mid I^c]\,P[I^c \mid V]$$

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Our probability of interest is $P[V \mid \text{Test is +ve}]$

$$= \frac{P[\text{Test is +ve} \mid V]\,P[V]}{P[\text{Test is +ve} \mid V]\,P[V] + P[\text{Test is +ve} \mid V^c]\,P[V^c]}$$

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$P[\text{Test is +ve} \mid V] = (0.9)\,P[I \mid V] + (0.1)\,P[I^c \mid V]$

$P[\text{Test is +ve} \mid V^c]$ can similarly be derived to be

$$P[\text{Test is +ve} \mid V^c] = (0.9)\,P[I \mid V^c] + (0.1)\,P[I^c \mid V^c]$$

$P[I \mid V^c]$ is the prob of getting infected we calculated in part (b).

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$P[I \mid V]$ can be calculated in a manner we calculated the prob of getting infected in (b).

We have

$$P[I \mid V] = P[T_1 \cup T_2 \mid V]$$
$$= P[T_1 \mid V] + P[T_2 \mid V] - P[T_1, T_2 \mid V]$$
$$= P[T_1 \mid V] + P[T_2 \mid V] - P[T_1 \mid V]\,P[T_2 \mid V]$$

(Since getting a significant viral load from the first person is independent of whether one gets it from the second person)

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$$= (0.1)\left(\frac{1}{20}\right)(10^{-3}) + (0.1)\left(\frac{1}{20}\right)10^{-3} - (0.1)^2\left(\frac{1}{20}\right)^2 10^{-6}$$

To calculate $P[V \mid \text{Test is +ve}]$ we have everything we need other than $P[V]$, which is a prior prob that the person is vaccinated.

Unfortunately this is missing from the question. If you chose any valid prob for $P[V]$ you are good.

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WRONG WAY OF SOLVING QUESTION 1 AND WHY.

We are given:

$P[\text{A person is infected with Random V}] = \frac{1}{1000} = 10^{-3}$

$P[\text{Test is +ve} \mid \text{Person is infected}] = 0.9$

$P[\text{Test is +ve} \mid \text{Person is healthy}] = 0.1$

$\Rightarrow P[\text{Test is -ve} \mid \text{Person is infected}] = 0.1$ $\left[\because P[\text{Test is -ve} \mid \text{Person is infected}] + P[\text{Test is +ve} \mid \text{Person is infected}] = 1\right]$

$P[\text{Test is -ve} \mid \text{Person is healthy}] = 0.9$

(Q1) You go to a gathering of 10 people.

(a) $P[\text{The randomly chosen person you meet is infected}]$

$10_{C_1} P[\text{A person is infected}] = \frac{10}{1000} = \frac{1}{100}$.

What did we do here? Suppose I_k is the event that the k^{th} person in the gathering is infected. We calculated $P[I_1] + P[I_2] + \dots + P[I_{10}]$. → This isn't what we want. What we want is the probability that the person we select is infected. We are selecting a person randomly, which is to say that we are selecting any person with probability $\frac{1}{10}$. → Suppose P_k is the event that the k^{th} person is selected. $P[P_k] = \frac{1}{10}$, $k = 1, 2, \dots, 10$. Let I_k be the event that the k^{th} person is infected. That an infected person is selected is the event $(P_1 \cap I_1) \cup (P_2 \cap I_2) \cup \dots \cup (P_{10} \cap I_{10})$ → These intersections are mutually exclusive. The prob of our event of interest is

$P[P_1 \cap I_1] + P[P_2 \cap I_2] + \dots + P[I_{10} \cap P_{10}]$

$= 10\left(\frac{1}{10} \cdot \frac{1}{1000}\right) = \frac{1}{1000}$.

Of course, you can more simply state that there are → ten people and I am randomly selecting one. This I can do in 10 ways. Any selection is made with prob $\frac{1}{10}$ & is infected with prob $\frac{1}{1000}$. Thus the prob I select an infected person is $10\left(\frac{1}{10} \cdot \frac{1}{1000}\right) = \frac{1}{1000}$.

(b) $P[\text{A randomly chosen pair of people is infected}]$

$= P[\text{One in the pair is infected, The other in the pair is infected}]$

$= P[\text{One in the pair is infected}] \, P[\text{Other in the pair is infected}]$

We can make the assumption that the event that one is infected is independent of whether the other is infected, as people have come from different parts of Delhi

(5) if the product of probabilities of the correct events is used

$= (10^{-3})(10^{-3}) = 10^{-6}$

However, you can pick a pair in 10_{C_2} ways.

Thus the prob of interest, that is the

$P[\text{You meet a pair of infected people}] = 10_{C_2} 10^{-6} = 45 \times 10^{-6}$

For reasons same as in part (a), this is incorrect. Any pair is selected with probability $\left(\frac{1}{10_{C_2}}\right)$ and → is infected with probability 10^{-6} (as calculated). Since we have 10_{C_2} pairs, the prob of interest is $10_{C_2}\left(\frac{1}{10_{C_2}} \times 10^{-6}\right) = 10^{-6}$.

(c) In a pair, $P[\text{One or more are infected}]$

$= P[\text{One is infected} \cup \text{Other is infected}]$

$= P[\text{One is infected}] + P[\text{Other is infected}] - P[\text{One is infected, Other is infected}]$

$= 10^{-3} + 10^{-3} - 10^{-6} = \boxed{2(10^{-3}) - 10^{-6}}$

(3) for correctly handling the event one or more are infected

You can also calculate the above probability by recognizing that the complement of the event "one or more in a pair are infected" is "none in the pair is infected".

$\therefore P[\text{One or more are infected}] = 1 - P[\text{none is infected}]$

$= 1 - (1 - 10^{-3})(1 - 10^{-3})$

$= 1 - [1 - 2(10^{-3}) + 10^{-6}] = 2(10^{-3}) - 10^{-6}$.

Again we have 10_{C_2} ways of choosing a pair.

(2) for counting correctly

Thus our probability of interest

$P[\text{You end up meeting a pair in which one or more are infected}]$

$= 10_{C_2}\left[2(10^{-3}) - 10^{-6}\right]$

This is the wrong way of counting. While there are 10_{C_2} ways of choosing a pair, any pair is chosen with prob $1/10_{C_2}$. The prob of the event of interest in part (c) is infact $10_{C_2}\left(\frac{1}{10_{C_2}}\right)\left(2(10^{-3}) - 10^{-6}\right) = 2(10^{-3}) - 10^{-6}$

(d) $P[\text{Two or more in the gathering are infected}]$

$= 1 - P[\text{0 or 1 in the gathering are infected}]$

$= 1 - \left(P[\text{0 are infected}] + P[\text{1 is infected}]\right)$

$= 1 - \left((1 - 10^{-3})^{10} + 10_{C_1} 10^{-3} (1 - 10^{-3})^9\right)$

(5) for correctly interpreting the event.