MTH 102: Probability and Statistics

Quiz 1

20/05/2022

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No books, notes, or devices are allowed. Just a pen and eraser. Any exchange of information related to the quiz with a human or machine will be deemed as cheating. Institute rules will apply. Explain your answers. Show your steps. Approximate calculations are fine as long as the approximations are reasonable. You have 50 minutes.

Question 1. 20 **marks** There are 300 students in a section. The names of the students are placed in a bin. Assume all students have unique names. Each student is assigned a randomly chosen name from the bin.

Assume that a name chosen from the bin is placed back in it before the next assignment.

(a) Calculate the probability that any student is assigned the true name.

(b) Calculate the probability that every student is assigned the true name.

Now assume that a name chosen from the bin is removed from the bin before the next assignment.

(aa) Suppose a student is randomly chosen after all students have been assigned names chosen from the bin. Calculate the probability that the student was assigned the true name.

(bb) Suppose two students were randomly chosen post assignment of names to all. Calculate the probability that both were assigned their true names.

(cc) Is the event that a student is assigned the true name independent of the names assigned to other students? Explain your answer.

(dd) Calculate the probability that all students were assigned their true names.

Question 2. 80 **marks** A randomly chosen person in a city claims to be healthy (event H_1) with probability 0.75 and unwell (event U_1) otherwise. The city puts a person through the following treatment for a month. A person who claims to be unwell is given a placebo (event B) with probability 0.25 and a get-healthy drug (event D) with probability 0.75. On the other hand, a person who claims to be healthy is given a placebo with probability 0.75 and a get-healthy drug otherwise.

After a month of treatment as described above, the person is subject to a diagnostic test and also asked whether the person feels healthy or unwell. Independent of the person's initial claim about being healthy or unwell, the test reports with probability 0.75 a person who was on a placebo to be unwell (define R_U as the event that test reports unwell) and reports with probability 0.75 a person who was on the get-healthy drug to be healthy (R_H is the event that the test reports healthy).

The diagnostic test has the following characteristics. It reports as unwell, a person who claims to be healthy post treatment (event H_2), with probability 0.1. It reports as unwell, a person who claims to be unwell (event U_2), with probability 0.9.

(a) (5 marks) Calculate the probability that a person is given the placebo.

(b) (5 marks) Calculate the probability that the test reports a person to be healthy.

(c) (15 marks) Calculate the probability that at the end of the treatment, a person whose test report says unwell, claims to be healthy.

(d) (15 marks) Calculate the probability that at the end of the treatment, a person whose test report says healthy, claims to be healthy.

(e) (20 marks) Calculate the probability that at the end of the treatment a person claims to be healthy.

(f) (20 marks) Suppose at the end of the treatment the person claims to be unwell. Calculate the probability that the person claimed to be healthy before the treatment?

Question 1. 20 **marks** There are 300 students in a section. The names of the students are placed in a bin. Assume all students have unique names. Each student is assigned a randomly chosen name from the bin.

Assume that a name chosen from the bin is placed back in it before the next assignment.

(a) Calculate the probability that any student is assigned the true name. 2/20

(b) Calculate the probability that every student is assigned the true name. 2/20

Now assume that a name chosen from the bin is removed from the bin before the next assignment.

(aa) Suppose a student is randomly chosen after all students have been assigned names chosen from the bin. Calculate the probability that the student was assigned the true name. 4/20

(bb) Suppose two students were randomly chosen post assignment of names to all. Calculate the probability that both were assigned their true names. 4/20

(cc) Is the event that a student is assigned the true name independent of the names assigned to other students? Explain your answer. 4/20

(dd) Calculate the probability that all students were assigned their true names. 4/20.

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[Marks for steps. Marks for the correct answer only if how the answer was arrived at is clear]

(a) P[Any student is assigned the true name]

$= \frac{1}{300}$.

This is easy to see as assignments are made "with replacement." That is, for any student the name is chosen from all 300 names.

(b) P[Every student is assigned the true name]

$= \left(\frac{1}{300}\right)^{300}$.

[Note that, given the "with replacement" assumption, name assignments are independent]

Now we assume that names are assigned without replacement.

(aa) P[A randomly chosen student was assigned the true name] =

$$\frac{|\underline{299}}{|\underline{300}} = \frac{1}{300},$$

where $|\underline{299}$ is the number of ways in which names may be assigned with the student assigned the true name and $|\underline{300}$ is the total no. of ways names can be assigned.

An alternate way of calculating the probability is

P[No other student was assigned the student's true name & the student was assigned the true name]

$$= \left(\frac{299}{300}\right)\left(\frac{298}{299}\right)\left(\frac{297}{298}\right)\cdots\cdots\left(\frac{1}{2}\right)$$

$$= \frac{1}{300},$$

where $\left(\frac{299}{300}\right)$ is the probability that the first assignment amongst the other 299 students is not the chosen student's true name, that is it can be done in 299 out of 300 ways.

$\left(\frac{299-k+1}{299-k+2}\right)$, $k = 1, 2, \ldots, 299$,

is the probability that the k^{th} assignment amongst the other 299 students is not the chosen student's true name.

Another way to think about the probability is as follows:

Suppose the chosen student was assigned the k^{th} drawn name, when the assignment was carried out.

P[Student is assigned the true name | student was assigned k^{th} drawn name]

= P[First (k-1) assignments use a name other than our chosen student's true name & the student is assigned the true name]

$$= \frac{299}{300} \times \frac{298}{299} \times \cdots \times \frac{299-k+1}{299-k+2} \times \frac{1}{299-k+1}$$

$$= \frac{1}{300}.$$

Further note that k can take the values 1, 2, .., 300 with equal probability.

(bb) After all students have been assigned names, we choose two randomly.

P[The two chosen were assigned their true names]

$$= \frac{|\underline{298}}{|\underline{300}} = \frac{1}{(300)(299)},$$

where $|\underline{298}$ is the no. of ways in which names can be assigned such that the two are assigned their true names.

You could approach this part in other ways too, as we did in (aa).

(cc) P[Student is assigned a true name] $= \frac{1}{300}$.

We calculated this in (aa).

P[Another student is assigned the first student's true name] $= \frac{1}{300}$. This should be easy to see & can be calculated as we did in (aa).

P[Student is assigned the true name, Another student is assigned our student's true name]

$= 0.$

Clearly, given the definition of independent events, a student isn't assigned the true name independent of names assigned to other students.

(dd) There is only one way in which all students are assigned their true names.

The probability is $\frac{1}{|\underline{300}}$.

Question 2. 80 **marks** A randomly chosen person in a city claims to be healthy (event H_1) with probability 0.75 and unwell (event U_1) otherwise. The city puts a person through the following treatment for a month. A person who claims to be unwell is given a placebo (event B) with probability 0.25 and a get-healthy drug (event D) with probability 0.75. On the other hand, a person who claims to be healthy is given a placebo with probability 0.75 and a get-healthy drug otherwise.

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The diagnostic test has the following characteristics. It reports as unwell, a person who claims to be healthy post treatment (event H_2), with probability 0.1. It reports as unwell, a person who claims to be unwell (event U_2), with probability 0.9.

(a) (5 marks) Calculate the probability that a person is given the placebo.

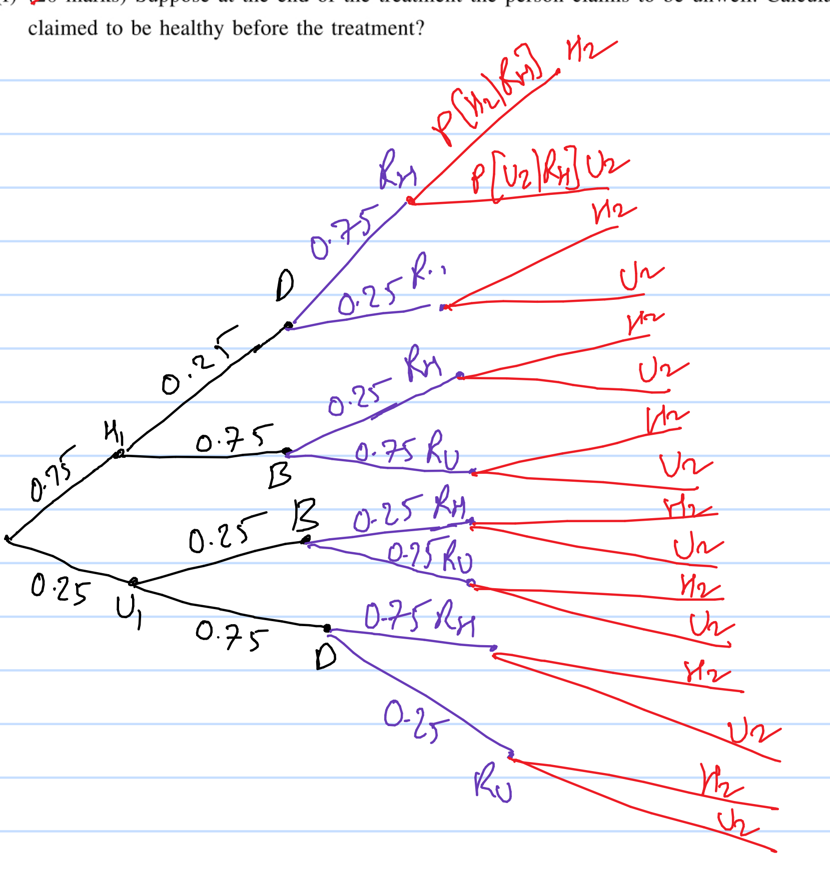
(b) (5 marks) Calculate the probability that the test reports a person to be healthy.

(c) (~~15~~ 20 marks) Calculate the probability that at the end of the treatment, a person whose test report says unwell, claims to be healthy.

(d) (~~15~~ 20 marks) Calculate the probability that at the end of the treatment, a person whose test report says healthy, claims to be healthy. (1 mark for valid assumption)

(e) ~~(20 marks)~~ Calculate the probability that at the end of the treatment a person claims to be healthy.

(f) (~~20~~ 25 marks) Suppose at the end of the treatment the person claims to be unwell. Calculate the probability that the person claimed to be healthy before the treatment?



A tree diagram (OR) correctly expressed given probabilities 4/80

We have been given:

$$P[H_1] = 0.75 \qquad P[D|H_1] = 0.25 \qquad P[D|U_1] = 0.75$$
$$P[U_1] = 0.25 \qquad P[B|H_1] = 0.75 \qquad P[B|U_1] = 0.25$$

$$P[R_H|D] = 0.75 \qquad P[R_H|B] = 0.25$$
$$P[R_U|D] = 0.25 \qquad P[R_U|B] = 0.75$$
$$P[R_U|H_2] = 0.1 \qquad P[R_U|U_2] = 0.9$$

(a) Probability that the person is given a placebo is

$$P[B] = P[H_1 \cap B] + P[U_1 \cap B]$$ 1.5 + 1.5 = 3

$$= \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{10}{16} = \frac{5}{8}.$$ 1 + 1 = 2.

(b) Probability that the test reports a person to be healthy is

$$P[R_H] = P[R_H, D, H_1] + P[R_H, B, H_1] + P[R_H, B, U_1] + P[R_H, D, U_1]$$ 2 marks for the probabilities

$$= (0.75)(0.25)(0.75) + (0.75)(0.75)(0.25) + (0.25)(0.25)(0.25) + (0.25)(0.75)(0.75)$$ A total of 3 for the correct calculation of probabilities

$$= 3(0.75)^2(0.25) + \left(\frac{1}{4}\right)^3$$
$$= \left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{28}{64} = \frac{7}{16}.$$

(c) We want

$$P[\text{Person at end of treatment Claims to be Healthy} \mid \text{Test Report Says Unwell}]$$

$$= P[H_2|R_U]$$
$$= \frac{P[R_U|H_2]\,P[H_2]}{P[R_U]}$$ 5 + 5

$$= \frac{P(R_U|H_2)\,P[H_2]}{P[R_U|H_2]\,P[H_2] + P[R_U|U_2]\,P[U_2]}$$ 5

$$= \frac{(0.1)\,P[H_2]}{(0.1)\,P[H_2] + (0.9)(1 - P[H_2])} = 0.1 \quad \left[\text{Assuming } P[H_2] = 0.5\right]$$ Substitutions → answer 5

(d) $$P[H_2|R_H] = \frac{P[R_H|H_2]\,P[H_2]}{P[R_H]}$$ 5 + 5

$$= \frac{P[R_H|H_2)\,P[H_2]}{P[R_H|H_2]\,P[H_2] + P[R_H|U_2]\,P[U_2]}$$ 5

$$= \frac{(1 - P[R_U|H_2])\,P[H_2]}{P[R_H|H_2]\,P[H_2] + P[R_H|U_2]\,P[U_2]}$$

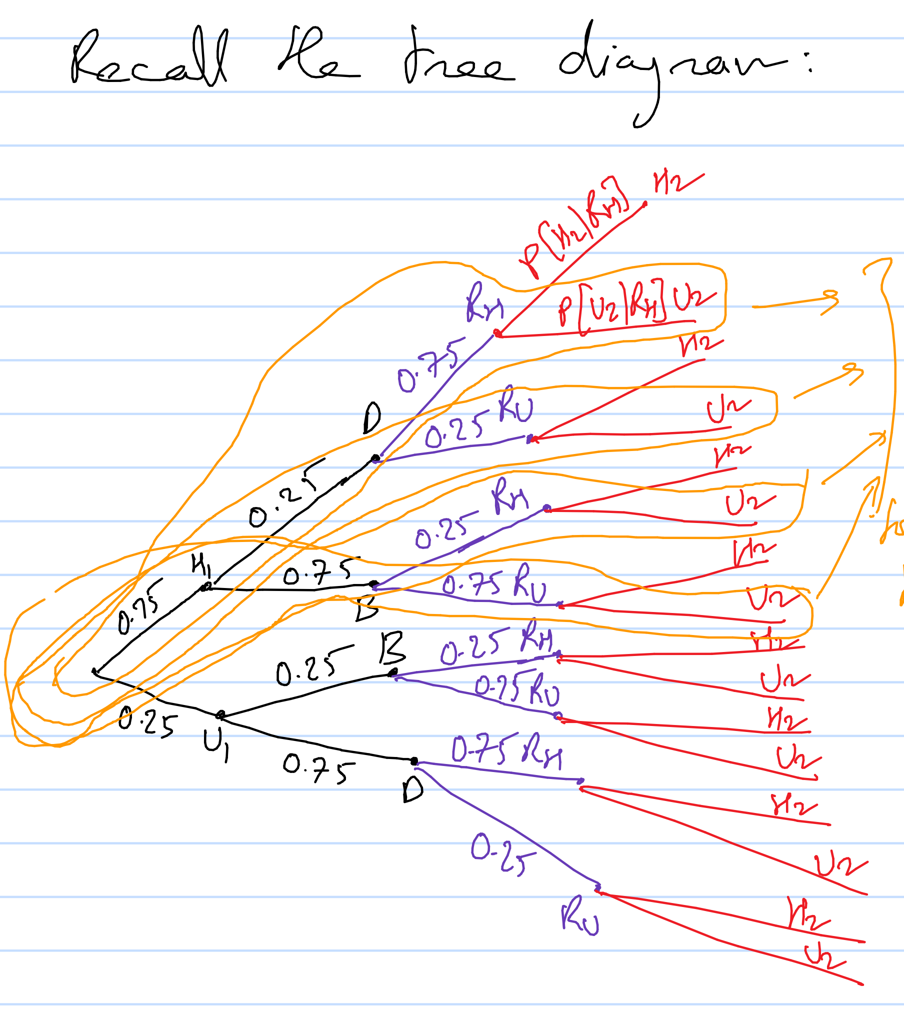
$$= \frac{(0.9)\,P[H_2]}{(0.9)\,P[H_2] + (0.1)(1 - P[H_2])} = 0.9 \quad \left[\text{Assuming } P[H_2] = 0.5\right]$$ 5

(e) We will assume $0 < P[H_2] < 1$. 1

(f) We want

$$P(H_1|U_2] = \frac{P[H_1, U_2]}{P[U_2]}$$ 5

Recall the tree diagram:



Mutually exclusive paths that correspond to the event $H_1 \cap U_2$.

$$P[H_1 \cap U_2] = P[H_1 \cap D \cap R_H \cap U_2] + P[H_1 \cap D \cap R_U \cap U_2] + P[H_1 \cap B \cap R_H \cap U_2] + P[H_1 \cap B \cap R_U \cap U_2]$$ The events! 10

$$= (0.75)(0.25)(0.75)(1 - P[H_2|R_H]) + (0.75)(0.25)(0.25)(1 - P[H_2|R_U]) + (0.75)(0.75)(0.25)(1 - P[H_2|R_H]) + (0.75)(0.75)(0.75)(1 - P[H_2|R_U])$$

$P[H_2|R_H] = 0.9$
$P[H_2|R_U] = 0.1$

Assignment of probabilities 10

$$= \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)(0.1) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2(0.9) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)(0.1) + \left(\frac{3}{4}\right)^3(0.9)$$

$$= \left(\frac{3}{4}\right)(0.9)\left(\frac{5}{8}\right) + \left(\frac{9}{16}\right)\left(\frac{1}{2}\right)(0.1)$$

$$= \frac{14.4}{32} = \frac{0.9}{2} = 0.45.$$