# ISyE 3103 Supply Chain Modeling: Transportation and Logistics

**Summer 2011**

**Winters' Method Forecasting Example**

A scatterplot of the following actual demand data suggests the existence of seasonal (quarterly) variation coupled with an overall increasing trend; consequently, this data is a strong candidate for utilizing Winters' Method (aka Triple Exponential Smoothing).

| | **2008** | **2009** | **2010** |
|---|---|---|---|
| Quarter 1 | 146 | 192 | 272 |
| Quarter 2 | 96 | 127 | 155 |
| Quarter 3 | 59 | 79 | 98 |
| Quarter 4 | 133 | 186 | 219 |

In this note we will fit a Winters model to this data set, compute the *a posteriori* forecast errors for 2008-2010, and provide a demand forecast for 2011. The smoothing constants are $\alpha = 0.2$, $\beta = 0.1$, and $\gamma = 0.05$.
Recall that the underlying demand process assumed for the Winters model is

$$D_t = (a + bt)\, I_t + \epsilon_t$$

In order to initialize the model, we need estimates, $\hat{a}_0$, $\hat{b}_0$, and $\hat{I}_{-3}, \hat{I}_{-2}, \hat{I}_{-1}, \hat{I}_0$. There are various methods to initialize the model. Here we consider one such method.
Now we are ready to initialize the model by computing the initial estimates for $\hat{a}_0$ and $\hat{b}_0$ using the data from 2008 and 2009. The estimate of the slope ($\hat{b}_0$) is found by taking the difference in the average demand for the two years and dividing by the number of seasonal periods:

$$\hat{b}_0 = \frac{\Delta D}{4} = \frac{\bar{D}_{2009} - \bar{D}_{2008}}{4} = \frac{146 - 108.5}{4} = 9.38.$$

We can solve for the level estimate ($\hat{a}_0$) using the following equation

$$\bar{D}_{2008} = \hat{a}_0 + \hat{b}_0(2.5),$$

where the trend value equals 2.5 because that was the average period number for the year 2008 ($\frac{1+2+3+4}{4}$). Solving the equation, we obtain $\hat{a}_0 = 85.05$.

The next step in the initialization process is to compute trend line estimates for the two years of data using the trend function $\hat{D}_t = 85.05 + 9.38t$, $t = 1, \ldots, 8$. Some representative calculations are $\hat{D}_1 = 85.05 + 9.38 * 1 = 94.43$ and $\hat{D}_7 = 85.05 + 9.38 * 7 = 150.71$.

| | **2008** | **2009** |
|---|---|---|
| Quarter 1 | 94.43 | 131.95 |
| Quarter 2 | 103.81 | 141.33 |
| Quarter 3 | 113.19 | 150.71 |
| Quarter 4 | 122.57 | 160.09 |

We can use these trend line estimates to develop initial seasonal indices by taking the actual demand for each period and dividing by the trend line estimate for that period. For example, $\frac{146}{94.43} = 1.55$ and $\frac{127}{141.33} = 0.90$. We average the two indices for each quarter to obtain the average index for that quarter, and then we normalize the average indices by dividing each by 4.06 and multiplying by 4.00 to ensure that they all add up to 4.

| | **2008** | **2009** | **Average Index** | **Initial Index** |
|---|---|---|---|---|
| Quarter 1 | 1.55 | 1.46 | 1.51 | 1.4828 |
| Quarter 2 | 0.92 | 0.90 | 0.91 | 0.8965 |
| Quarter 3 | 0.52 | 0.52 | 0.52 | 0.5123 |
| Quarter 4 | 1.09 | 1.16 | 1.12 | 1.1084 |
| | | | $\Sigma = 4.06$ | $\Sigma = 4.00$ |

The values in the "Initial Index" column above are our initial estimates of the seasonal factors at time $t = 1$ (which denotes the first quarter of 2008). Specifically, we have $\hat{I}_{-3} = 1.4828$, $\hat{I}_{-2} = 0.8965$, $\hat{I}_{-1} = 0.5123$, and $\hat{I}_0 = 1.1084$. We are now in position to make the first forecast, which is for the first quarter of 2008, according to the following formula.

$$f_{1;0} = \left(\hat{a}_0 + \hat{b}_0(1)\right)\hat{I}_{-3} = (85.05 + 9.38)\,1.4828 = 140.0169$$

Since we have a new data realization ($D_1 = 146$), we can incorporate it into the model via the Winters model updating equations to obtain new estimates for the parameters. The general Winters' method updating equations to be used at the end of period $t$ are

$$\begin{aligned}
\hat{a}_t &= \alpha\frac{D_t}{\hat{I}_{t-m}} + (1-\alpha)\left(\hat{a}_{t-1} + \hat{b}_{t-1}\right) \\
\hat{b}_t &= \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) + (1-\beta)\hat{b}_{t-1} \\
\hat{I}_t &= \gamma\frac{D_t}{\hat{a}_t} + (1-\gamma)\hat{I}_{t-m}.
\end{aligned}$$

Using these equations, we obtain the following updated parameter estimates at the end of time period 1.

$$\begin{aligned}
\hat{a}_1 &= \alpha\frac{D_1}{\hat{I}_{-3}} + (1-\alpha)\left(\hat{a}_0 + \hat{b}_0\right) = .2\frac{146}{1.4828} + 0.8(85.05 + 9.38) = 95.2370 \\
\hat{b}_1 &= \beta\left(\hat{a}_1 - \hat{a}_0\right) + (1-\beta)\hat{b}_0 = .1(95.2370 - 85.05) + 0.9 * 9.38 = 9.4607 \\
\hat{I}_1 &= \gamma\frac{D_1}{\hat{a}_1} + (1-\gamma)\hat{I}_{-3} = 0.05\frac{146}{95.2370} + 0.95 * 1.4828 = 1.4853
\end{aligned}$$

We can also compute the forecast for period 2 according to

$$f_{2;1} = \left(\hat{a}_1 + \hat{b}_1(1)\right)\hat{I}_{-2} = (95.2370 + 9.4607)0.8965 = 93.8669.$$

Continuing in this fashion, we obtain the following results for the three years of data.

| Year | Quarter | $t$ | $D_t$ | $f_{t;t-1}$ | $e_t$ | $\lvert e_t\rvert$ | $\frac{\lvert e_t\rvert}{D_t}$ |
|---|---|---|---|---|---|---|---|
| 2008 | 1 | 1 | 146 | 140.0169 | 5.9831 | 5.9831 | 0.0427 |
| | 2 | 2 | 96 | 93.8669 | 2.1331 | 2.1331 | 0.0227 |
| | 3 | 3 | 59 | 58.7533 | 0.2467 | 0.2467 | 0.0042 |
| | 4 | 4 | 133 | 137.7666 | −4.7666 | 4.7666 | 0.0346 |
| 2009 | 1 | 5 | 192 | 197.3449 | −5.3449 | 5.3449 | 0.0271 |
| | 2 | 6 | 127 | 126.9841 | 0.0159 | 0.0159 | 0.0001 |
| | 3 | 7 | 79 | 77.3070 | 1.6930 | 1.6930 | 0.0219 |
| | 4 | 8 | 186 | 178.1543 | 7.8457 | 7.8457 | 0.0440 |
| 2010 | 1 | 9 | 272 | 255.1068 | 16.8932 | 16.8932 | 0.0662 |
| | 2 | 10 | 155 | 165.1318 | −10.1318 | 10.1318 | 0.0614 |
| | 3 | 11 | 98 | 98.1232 | −0.1232 | 0.1232 | 0.0013 |
| | 4 | 12 | 219 | 222.6918 | −3.6918 | 3.6918 | 0.0166 |

The RMSE for this forecasting model is 6.8179, the MAD is 4.9058, and the MAPE is 0.0286. The last measure means that our forecast differs from the actual demand by an average of 2.86%, which represents a very good model specification. The forecasts for 2011 can be found using the following equations and our latest estimates (calculations are not shown) of the model parameters ($\hat{a}_{12} = 200.1812$, $\hat{b}_{12} = 9.4987$, $\hat{I}_9 = 1.4875$, $\hat{I}_{10} = 0.8951$, $\hat{I}_{11} = 0.5128$, and $\hat{I}_{12} = 1.1080$).

2011Q1. $f_{13;12} = (200.1812 + 9.4987(1))1.4875 = 311.9056$

2011Q2. $f_{14;12} = (200.1812 + 9.4987(2))0.8951 = 196.1950$

2011Q3. $f_{15;12} = (200.1812 + 9.4987(3))0.5128 = 117.2708$

2011Q4. $f_{16;12} = (200.1812 + 9.4987(4))1.1080 = 263.9047$