Rigidity for the discrete Bonnet-Myers diameter bound.

Which graphs look like a sphere?

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Introduction

The Bonnet-Myers theorem [6] gives an estimate of the diameter in terms of a positive Ricci curvature bound of a manifold M:

$$\mathrm{diam}(M) \le \pi\sqrt{\frac{n-1}{K}} \tag{1}$$

with $K = \inf \mathrm{Ric}_M(v) > 0$, and the infimum is taken over all unit tangent vectors v of M. Moreover, Cheng's Rigidity theorem [1] states that this diameter estimate (1) is sharp if and only if M is the n-dimensional round sphere.

In the discrete setting of graphs, Ollivier's notion of Ricci curvature provides a discrete analogue of the Bonnet-Myers theorem. In view of Cheng's rigidity result, it is natural to ask which graphs are *Bonnet-Myers sharp*?

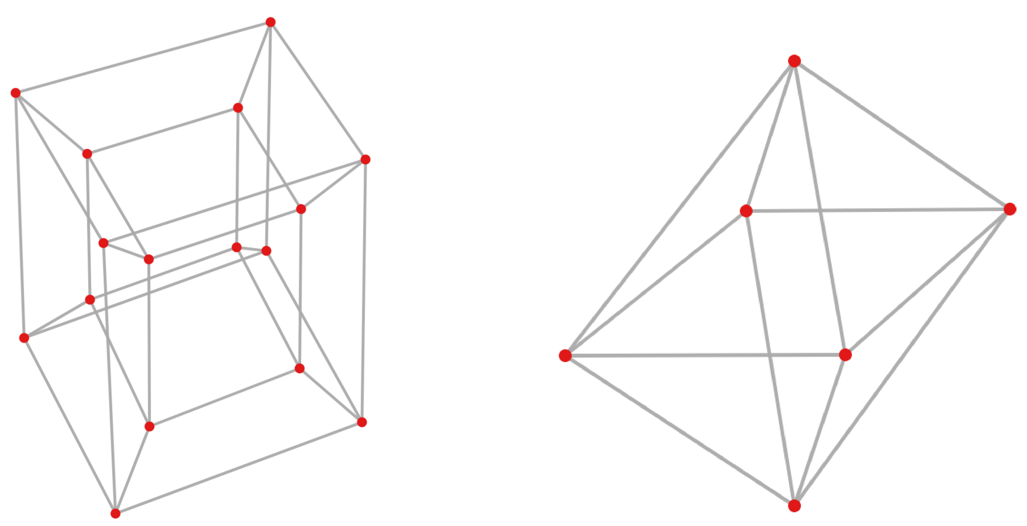

Figure 1: Hypercube Q^4 and Octahedron $CP(3)$

Ollivier Ricci curvature on graphs

A graph $G = (V, E)$ with vertex set V and edge set E is always assumed to be undirected, simple (i.e. no loops and multiple edges), and equipped with a combinatorial distance function d (i.e. the length of the shortest path between given two vertices). To facilitate a modification of Ollivier's definition introduced by Lin-Lu-Yau in [5], it is further assumed that G is regular, i.e. every vertex has the same degree.

Definition 1. Let $G = (V, E)$ be a connected D-regular graph. The (modified) Ollivier Ricci curvature is defined on an edge $\{x, y\} \in E$ to be

$$\kappa(x,y) = \frac{D+1}{D}\left(1 - W_1(\mu_x, \mu_y)\right),$$

where W_1 is 1-Wasserstein distance, and μ_x is the probability measure on V after performing a one-step simple random walk from a vertex x, i.e.

$$\mu_x(v) = \begin{cases} \frac{1}{D+1} & \text{if } v = x \text{ or } v \sim x, \\ 0 & \text{otherwise} \end{cases}$$

Let us now state the discrete Bonnet-Myers Theorem for Ollivier Ricci curvature and introduce the associated notion of Bonnet-Myers sharpness for this curvature notion:

Theorem 2 (Discrete Bonnet-Myers, see [5, 7])**.** *Let $G = (V, E)$ be a connected D-regular graph satisfying $K := \inf_{x\sim y} \kappa(x, y) > 0$. Then G has finite diameter and*

$$\mathrm{diam}(G) \le \frac{2}{K}. \tag{2}$$

Such a graph G is then called ***Bonnet-Myers sharp*** *(with respect to Ollivier Ricci curvature) if* (2) *holds with equality.*

Strongly spherical graphs

To formulate our main results, we require the following concepts of being an interval, self-centered, antipodal, and strongly spherical.

Definition 3. Let $G = (V, E)$ be a connected graph. For any two vertices $x, y \in V$, an *interval* $[x, y]$ is the set of all vertices lying on geodesics from x to y, that is

$$[x,y] = \{z \in V \mid d(x,z) + d(z,y) = d(x,y)\}.$$

• G is *self-centered* if for every vertex $x \in V$, there exists a vertex $\overline{x} \in V$ such that $d(x, \overline{x}) = \mathrm{diam}(G)$.
• G is *antipodal* if for every vertex $x \in V$, there exists a vertex $\overline{x} \in V$ such that $[x, \overline{x}] = V$.
• G is *strongly spherical* if G is antipodal, and the induced subgraphs of all its intervals are antipodal.

Theorem 4 (Classification of strongly spherical graphs, see [4])**.** *Strongly spherical graphs are precisely the Cartesian products $G_1 \times G_2 \times \cdots \times G_k$, where each factor G_i is either* ***a hypercube****,* ***a cocktail party graph $CP(n)$****,* ***a Johnson graph $J(2n, n)$****,* ***an even dimensional demicube****, or* ***the Gosset graph****.*

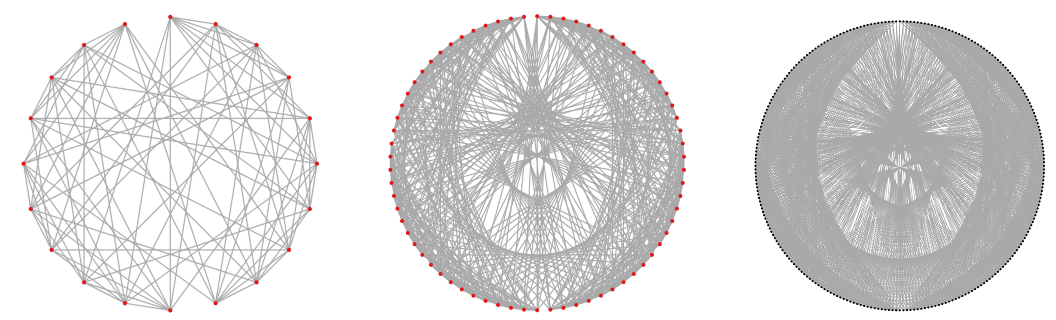

Figure 2: Johnson graphs $J(6, 3)$, $J(8, 4)$ and $J(10, 5)$

Graphs in the above classification are illustrated in Figures 1,2 and 3. Figures 1 and 2 were drawn with the aid of *Graph curvature calculator* software, see [3]. Figure 3 was excerpted from Wikimedia commons [8] and [9].

Main Results

The ultimate goal is to classify all Bonnet-Myers sharp graphs. We proved in [2] the following results towards this goal:

(i) **Cartesian products:** $G_1 \times G_2 \times \cdots \times G_k$ is Bonnet-Myers sharp if and only if all factors G_i are Bonnet-Myers sharp and satisfy

$$\frac{D_1}{L_1} = \frac{D_2}{L_2} = \cdots = \frac{D_k}{L_k}, \tag{3}$$

where D_i and L_i are the vertex degrees and the diameters of the graphs G_i, respectively.

(ii) Self-centered Bonnet-Myers sharp graphs are strongly spherical.

(iii) **Classification of self-centered Bonnet-Myers sharp graphs:** Self-centered Bonnet-Myers sharp graphs are precisely the following ones: hypercubes, cocktail party graphs $CP(n)$, Johnson graphs $J(2n, n)$, even-dimensional demi-cubes, the Gosset graph and all Cartesian products of them satisfying the condition (3).

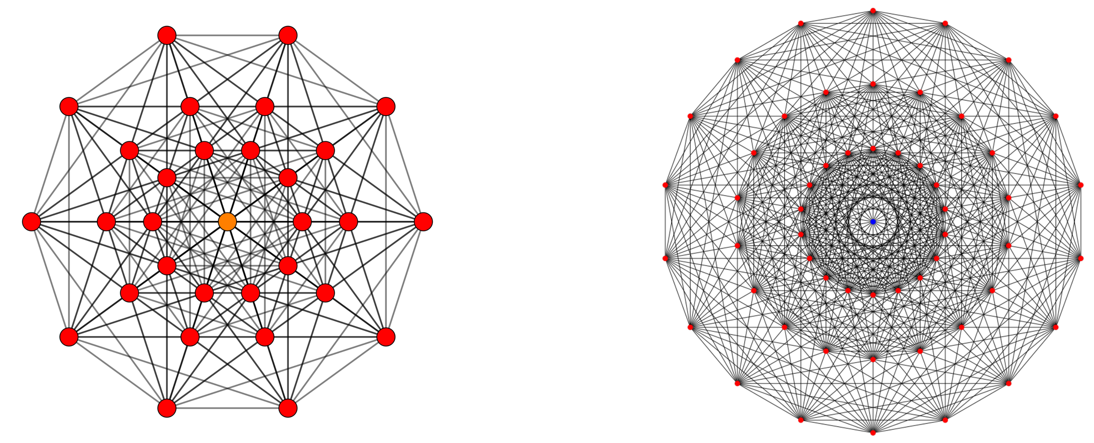

Figure 3: 6-demicubeand the Gosset graph

In order to prove (ii), we developed a useful concept of **transport geodesics**, based on Optimal Transport Theory.

Remark. *It is an interesting question whether the condition of* ***self-centeredness can be removed*** *in the classification (iii). If this were possible, we could view this result as another example where* local *properties have a strong* global *implication.*

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