Lyon 1

Mesh and Computational Geometry

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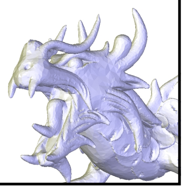
M2 ID3D
Image, Développement
et Technologie 3D
et 3A Centrale



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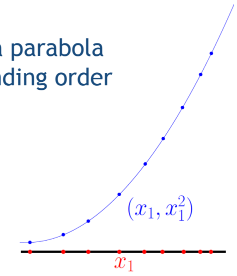
Incremental Delaunay Algorithm

- Complexity analysis
- Worst case :
  - Points distributed on a parabola and inserted in descending order of abscissa.



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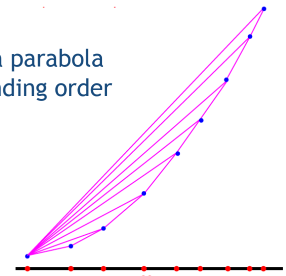
Incremental Delaunay Algorithm

- Complexity analysis
- Worst case :
  - Points distributed on a parabola and inserted in descending order of abscissa.



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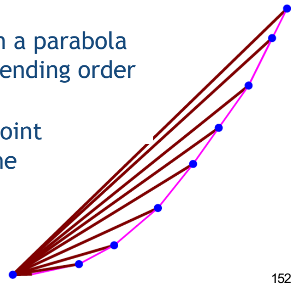
Incremental Delaunay Algorithm

- Complexity analysis
- Worst case :
  - Points distributed on a parabola and inserted in descending order of abscissa.
  - Each new inserted point conflicts with ALL the triangles

$\Omega(n^2)$



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Incremental Delaunay Algorithm

- Complexity analysis
- In average :
  - Complexity dependent on the strategy used to locate the triangle containing the point to be inserted

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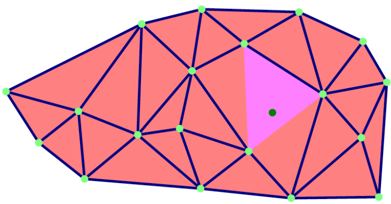
Location strategies

- Exhaustive search among all the triangles



$O(n)$

O. Devillers

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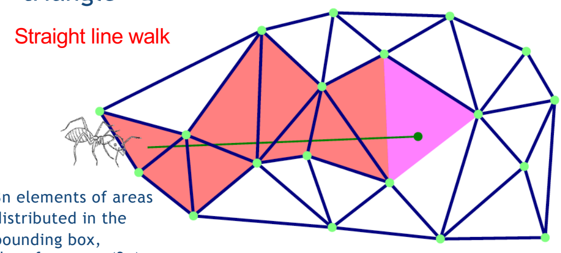
Location strategies

- Search from a first randomly selected triangle



3n elements of areas distributed in the bounding box, therefore squrt(3n) elements encountered on average on a straight line

$O(\sqrt{n})$ on average

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Location strategies

- Search from a first randomly selected triangle
  - Straight walk
  - Requires a predicate of segments intersection

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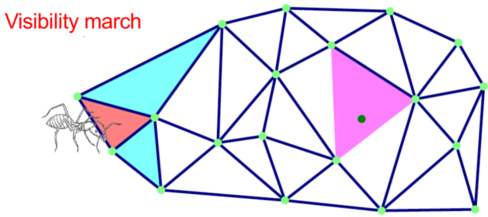
Location strategies

- Search from a first randomly selected triangle
  - Some minor deviations from the straight line walk



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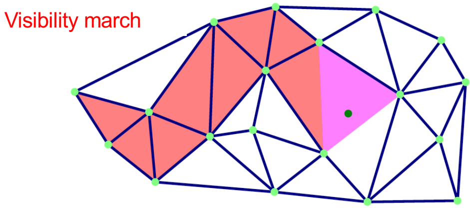
Location strategies

- Search from a first randomly selected triangle
  - Some minor deviations from the straight line walk



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Location strategies

- Search from a first randomly selected triangle
  - Visibility walking
  - Only requires an orientation predicate to find the next triangle to walk in

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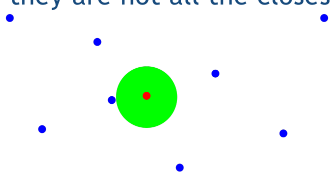
Delaunay and proximity in space

- Delaunay triangulation allows to model the notion of proximity between points
- Each point is thus connected to nearby points around it
- Be careful, they are not all the closest!



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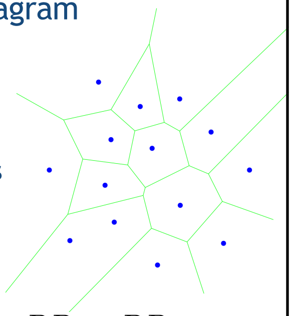
Voronoi diagram

- Voronoi's cell of a site P_i is the set of points closer to this site than to other sites



$$V_i = \{P \in \mathbb{R}^k \ t.\ que\ PP_i < PP_j \ pour\ tout\ j \neq i\}$$

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Voronoi diagram

- Given a set E of points in $\mathbb{R}^k$, the partitioning of $\mathbb{R}^k$ into cells composed of points having the same nearest neighbour in E is called a Voronoi diagram of E

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Voronoi diagram

- Possible construction:
  - V_i : intersection of half-spaces h_{ij}^i where h_{ij} is the mediator of segment P_iP_j and h_{ij}^i is the half-space delimited by h_{ij} containing P_i

In practice we will proceed differently!

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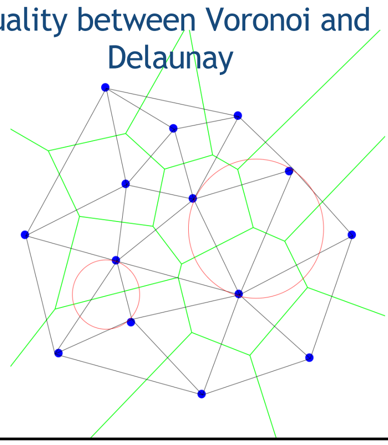
Duality between Voronoi and Delaunay



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Duality between Voronoi and Delaunay

- Each Voronoi vertex is located at the center of the circumscribed circle of a Delaunay triangle
- Two Voronoi vertices are connected if they are associated with adjacent triangles

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Coordinates of the centre of the circle circumscribed to a triangle ABC

- Useful for displaying the Voronoi diagram
- 1st possibility:
  - Write the equation of the mediator for each edge
    ex: For the edge AB, set of points M such that $MA^2=MB^2$
  - Solving a system of 2 equations with 2 unknowns (it is enough to take 2 mediators)
  - Numerically unstable

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Coordinates of the centre of the circle circumscribed to a triangle ABC

- ^{2nd} possibility :
  - Let's consider the angles
    $$\hat{A} = \widehat{CAB} \; \hat{B} = \widehat{ABC} \; \hat{C} = \widehat{BCA}$$
  - Then the barycentric coordinates of the center H of the circumscribed circle with respect to A, B and C are elegantly expressed :

$$H(tan\hat{B}+tan\hat{C}, tan\hat{C}+tan\hat{A}, tan\hat{A}+tan\hat{B})$$

  - Reminder :

$$\tan(\widehat{ABC}) = \frac{\sin(\widehat{ABC})}{\cos(\widehat{ABC})} = sign((\overrightarrow{BC} \times \overrightarrow{BA}) \cdot \overrightarrow{k}) \frac{\left\|\overrightarrow{BC} \times \overrightarrow{BA}\right\|}{\overrightarrow{BC} \cdot \overrightarrow{BA}}$$

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Coordinates of the centre of the circle circumscribed to a triangle ABC

$$H = Barycenter((A, \tan\hat{B} + \tan\hat{C}), (B, \tan\hat{C} + \tan\hat{A}), (C, \tan\hat{A} + \tan\hat{B}))$$

- Reminder :

$$\tan(\widehat{ABC}) = \frac{\sin(\widehat{ABC})}{\cos(\widehat{ABC})} = sign((\overrightarrow{BC} \times \overrightarrow{BA}) \cdot \overrightarrow{k}) \frac{\left\|\overrightarrow{BC} \times \overrightarrow{BA}\right\|}{\overrightarrow{BC} \cdot \overrightarrow{BA}}$$

$$Barycenter((A, \alpha a), (B, \alpha b), (C, \alpha c))$$
$$= Barycenter((A, a), (B, b), (C, c))$$

- Ensure to have no more denominators in the expression of your barycentric coordinates (normalization performed afterwards)

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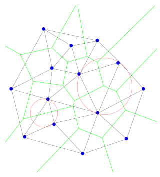
Duality between Voronoi and Delaunay

- Each Delaunay vertex is dual to one Voronoi cell
- Each Delaunay edge is dual to a Voronoi edge
- Each Voronoi vertex is dual to a Delaunay triangle



- What about Delaunay, Voronoi and their duality in 3D?

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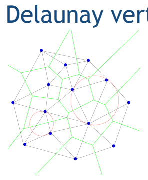
Duality between Voronoi and Delaunay

- Which data structure for Voronoi?
  - Walking around a Voronoi face is performed by walking through the faces/edges incident at a Delaunay vertex.
  - To move from one Voronoi cell to an adjacent cell is like moving from a Delaunay vertex to an adjacent vertex.



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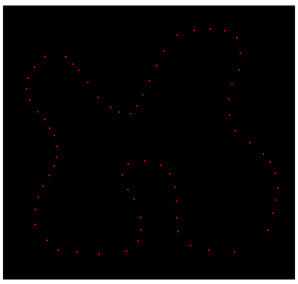
Use of Voronoi Center for line reconstruction from a 2D point set (Crust)

- Let consider a set of points sampled on a line
- The points are not ordered
- How to approximate the input line with a polygonal line?



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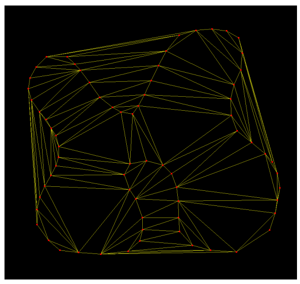
Use of Voronoi Center for line reconstruction from a point set (Crust)

- If the sampling is dense enough, Delaunay encloses a good candidate
- How to remove the edges crossing the shape?



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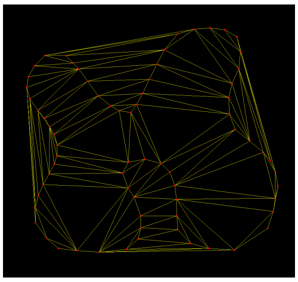
Use of Voronoi Center for line reconstruction from a point set (Crust)

- Add points as far as possible from the input line
  - Points located on the internal and external skeleton of the shape
  - Centers of maximal empty circles



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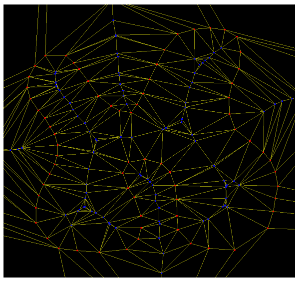
Use of Voronoi Center for line reconstruction from a point set (Crust)

- The Voronoi centers are close to the shape skeleton
  - Let's add them to break edges that cross the shape while preserving the boundaries



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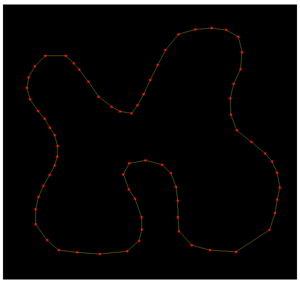
Use of Voronoi Center for line reconstruction from a point set (Crust)

- Keep the edges that join initial input points
- Correctness of the algorithm if the input point set is locally denser than a given proportion **ε** of the distance to the skeleton (**ε-sampling**)



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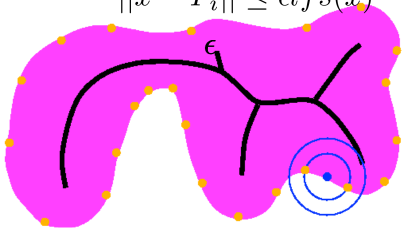
Local Feature Size (lfs) and **ε**- sampling

- Given ε, an **ε-sampling** of a shape is a set of samples P_i such that for each x there is a i such that $||x - P_i|| \leq \epsilon lfs(x)$



Sampling density locally proportional to 1/lfs
Measure of thickness and curvature

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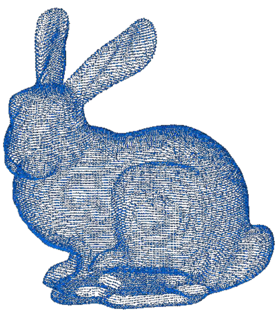
Crust in 3D?

In 3D the Delaunay triangulation provides a tetrahedrisation of the shape



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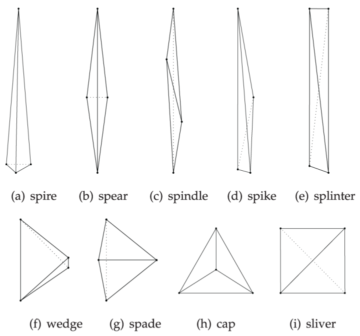
Crust in 3D?

- Zoology of Delaunay tetrahedra based on point samples from an object's surface



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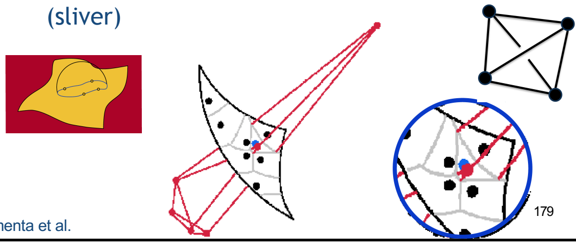
Crust in 3D

- Some Voronoi centers can be far from the skeleton and close to the surface...
  - No control over the position of the centers of flat tetrahedrons: 4 neighboring points almost cocyclic may have their center near the surface (sliver)



Amenta et al.

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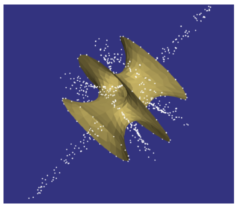
3D Crust

- Need to filter Voronoi centers



- Consider only the poles!
  - ie. Voronoi vertices certified to be far from the surface by one of the point samples

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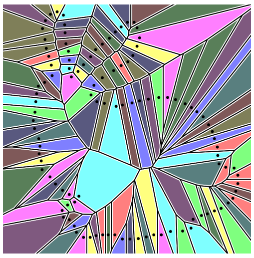
Poles

- In presence of a dense and non noisy sampling
  - long and thin Voronoi cells,
  - direction similar to the normal to the surface.



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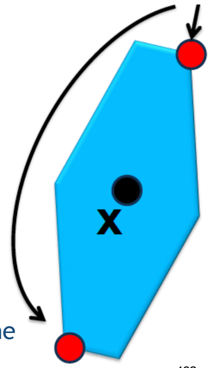
Poles

- Let V_x be the Voronoi's cell of a point x
- Positive pole p^+ : Voronoi Vertex of V_x further away from x
- Vector xp^+ : approximation of the normal direction at x
- The negative pole p^- : farthest vertex of V_x in the opposite direction to the vector xp^+



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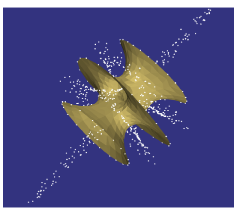
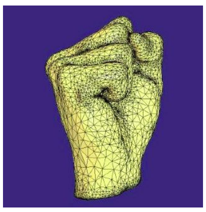
Crust in 3D

- Adding poles in the 3D triangulation





Amenta et al 98

- Reconstructed surface composed of Delaunay faces relying on 3 input point samples

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