

Lyon 1

Mesh and Computational Geometry

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Local surface variations

- How to study and characterize surface variations?
 - When defined as a height function over a plane?
 $z=f(x,y)$
Multivariate scalar function
 - When parameterized as a 3D vector function?

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, (u,v) \in \mathbb{R}^2$$

$$\Omega \subset \mathbb{R}^2 \xrightarrow{P} S \subset \mathbb{R}^3$$

$$U$$
 - When no parameterization is provided?

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Interest

- Local surface variations
 - Used to compute geometric information such as normal vectors, curvature, and higher order information ...
- Useful for :
 - Surface analysis
 - Surface rendering
 - Surface texturing
 - Constructing a better parameterization with less distortion

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Differential operators

- Reminder :
 - Given a univariate function $f : \mathbb{R} \rightarrow \mathbb{R}$
- The derivative of f is another function $\frac{\partial f}{\partial x}$ that describes the growing speed of f

$f' : \mathbb{R} \rightarrow \mathbb{R}$

$$f'(x) = \lim_{a \rightarrow 0} \frac{f(x+a) - f(x)}{a}$$

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Differential operators

- Reminder :
 - The Laplacian (second derivative) of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measure of the difference between the value of f at any point P and the average value of f in the vicinity of P
- It is linked to the curvature of the curve (inverse of the osculating circle radius)

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

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Differential operators

- Laplace equation

$$f : U \in \mathbb{R}^n \rightarrow \mathbb{R}$$

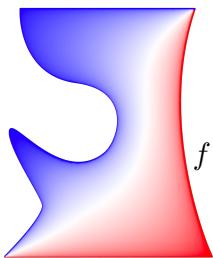
$$\Delta f = 0$$
- Solutions of Laplace equation :
 - Harmonic functions = kernel of the Laplacian operator
 - No local maxima or minima in U
 - Minimizing the Dirichlet energy that measures the smoothness of a function

$$E(f) = \int_U \langle \nabla f, \nabla f \rangle dA$$

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Differential operators

- Solutions of Laplace equation :
 - Used for interpolation

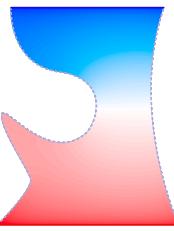


$$\begin{aligned}\Delta f(x) &= 0 & x \in U \\ f(x) &= f_0(x) & x \in \delta U\end{aligned}$$

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Differential operators

- Solutions of Laplace equation :
 - Used for interpolation



$$\begin{aligned}\Delta f(x) &= 0 & x \in U \\ f(x) &= f_0(x) & x \in \delta U_D \text{ (Dirichlet boundary)} \\ \nabla f \cdot n &= g_0(x) & x \in \delta U_N \text{ (Neumann boundary)}\end{aligned}$$

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Differential operators

- Harmonic functions not to be misunderstood with the eigenvectors and eigen values of the Laplacian operator

$$\Delta f = \lambda f$$

When $U = \mathbb{R}$ eigenvectors are the sines and cosines functions that are used within Fourier framework

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Differential operators

- In a multivariate context, differential operators are often used in differential equations

- Generally expressed in \mathbb{R}^k

- Spatial notation "nabla" :

Think of it as a vector
of partial derivative operators!

- Gradient operator of a scalar function f : ∇f
- Divergence operator of a vector function \mathbf{V} : $\nabla \cdot \mathbf{V}$
- Laplacian operator of a scalar function f : $\Delta f = \nabla \cdot \nabla f$
- Curl (rotational) operator of a vector function \mathbf{V} : $\nabla \times \mathbf{V}$

- Expression of nabla depending on the coordinate system of the input domain

Cartesian coordinates u, v

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Differential operator

- A few reminders :
 - Gradient : direction (u_{\max}, v_{\max}) of maximal slope
 - Divergence : flow traversing a unit element around (u, v)
 - Laplacian : measures the difference between the function and its mean value in a small neighborhood
 - Useful for several geometry processing tasks (interpolation by heat diffusion, spectral analysis, mean-curvature, smoothing)
 - Curl (Rotational) : does the vector field locally turn around one vector?
- Questions :
 - How could you define the "gradient" of a vector field? Jacobian matrix
 - How could you define the "laplacian" of a vector field? Vector with the Laplacian of each component

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Other use of Nabla

- Nabla transpose

${}_x \delta$: Derivative of the scalar field on the left

$$\nabla^t = [{}_x \delta, {}_y \delta, {}_z \delta]$$

- Jacobian matrix of a vector field \mathbf{V}

$$\mathbb{J}_{\mathbf{V}} = \mathbf{V} \nabla^t = \begin{bmatrix} \delta_x v_x, \delta_y v_x, \delta_z v_x \\ \delta_x v_y, \delta_y v_y, \delta_z v_y \\ \delta_x v_z, \delta_y v_z, \delta_z v_z \end{bmatrix}$$

- Remark : circulation of \mathbf{V} around small parallelogram \mathbf{A}, \mathbf{B} : $\text{curl}(\mathbf{V}) \cdot (\mathbf{A} \times \mathbf{B}) = \mathbb{J}_{\mathbf{V}} \mathbf{A} \cdot \mathbf{B} - \mathbb{J}_{\mathbf{V}} \mathbf{B} \cdot \mathbf{A}$

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Differential operator

- Functions defining a curve

$\mathbf{x} : [0, L] \rightarrow \mathbb{R}^3$ de classe C^3

Tangent, normal and binormal vectors?

$$\text{Vecteur tangent : } \mathbf{t}(s) = \frac{\mathbf{x}'(s)}{\|\mathbf{x}'(s)\|}$$

$$\text{Vecteur normal : } \mathbf{m}(s) = \frac{\mathbf{x}''(s)}{\|\mathbf{x}''(s)\|}$$

$$\text{Vecteur binormal : } \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{m}(s)$$

$$\text{Repère de Serret-Frénet : } (\mathbf{x}(s); \mathbf{t}(s), \mathbf{m}(s), \mathbf{b}(s))$$

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Differential operator

- Let s be the curvilinear abscissa

– s : length of curve between 0 and t

$$s(t) = \int_0^t \|\mathbf{X}'(t)\| dt$$

– parameterize the curve wrt s

– $\mathbf{x}'(s)$ is a unit vector

$$\text{Courbure : } \kappa(s) = \mathbf{x}''(s)$$

► Mesure la déviation par rapport à une droite

$$\text{Torsion : } \tau(s) = \frac{\det[\mathbf{x}'(s), \mathbf{x}''(s), \mathbf{x}'''(s)]}{\kappa^2(s)}$$

► Mesure le défaut de planarité

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Differential operator

- Functions defining a surface

$\mathbf{X} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ de classe C^r

Analogues vecteurs tangent/normal/binormal ?

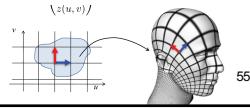
On note $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ et $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$

Plan tangent en p : $T_p \mathbf{X}$ = plan passant par p et engendré par les vecteurs

$\mathbf{X}_u(p)$ et $\mathbf{X}_v(p)$

Vecteur normal : $\mathbf{n}(p) = \frac{\mathbf{X}_u(p) \times \mathbf{X}_v(p)}{\|\mathbf{X}_u(p) \times \mathbf{X}_v(p)\|}$

$(\mathbf{X}(p); \mathbf{X}_u(p), \mathbf{X}_v(p), \mathbf{n}(p))$ forme aussi un repère local



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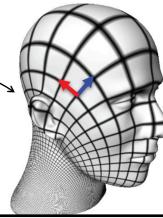
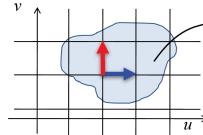
Differential operator

- Functions defining a surface

- How do unitary vectors are transformed by the parameterization?

– Length and angles distortion?

$\mathbf{X} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ de classe C^r



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Differential operator

- Functions defining a surface?

On note de manière analogue $\mathbf{X}_{uu} = \frac{\partial \mathbf{X}_u}{\partial u} = \frac{\partial^2 \mathbf{X}}{\partial u^2}$, etc.

Première forme fondamentale :

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

Angle change

Length change

Seconde forme fondamentale :

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{vu}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

Opérateur de forme/Application de Weingarten :

$$\mathbf{W} := \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - gF \end{bmatrix} = (D_u \mathbf{n} \ D_v \mathbf{n})$$

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\mathbf{I} = outil géométrique (**tenseur métrique**)

► Permet de mesurer aires locales, longueurs de courbes sur la surface, angles, ... $dA = \sqrt{EG - F^2} du dv$

► Exemple : anisotropie locale de la surface : décomposition spectrale de \mathbf{I}

Propriétés différentielles ne dépendant que de \mathbf{I} sont dites **intrinsèques**

- Ne dépendent pas de la paramétrisation
- Ne dépendent pas de l'espace 3D

Eigenvalues of \mathbf{I} : maximal/minimal stretching of a tangent vector

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II = propriétés **extrinsèques** de la surface

- Dépendant du plongement dans l'espace ambiant \mathbb{R}^3

W détermine les directions de courbure locale de la surface

- Valeurs propres = courbures principales
- Vecteurs propres = directions principales de courbure

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Courbures principales et directions principales de courbure :

$$\mathbf{W} = \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix}^{-1}$$

Courbure moyenne : $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} \text{trace}(\mathbf{W})$

Courbure de Gauss : $K = \kappa_1 \cdot \kappa_2 = \det(\mathbf{W})$

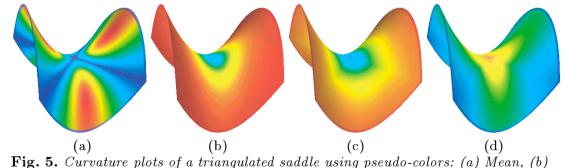


Fig. 5. Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum.

[Meyer et al. 2003]

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Differential operators

- Fait au tableau
 - Gradient de la fonction qui donne l'aire d'un triangle quand son sommet s0 varie

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Differential operators

- Functions defined on a surface
 - Let u be a scalar function defined on a surface
 - We would like to express local variations of u
- Let consider the discrete case of a surface being approximated by a simplicial mesh
 - Function u discretized on vertices
 - Gradient of u discretized on triangles
 - Divergence of a vector (defined on triangles) discretized on vertices
- Bibliography : Keenan Crane

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Differential operators

- Gradient d'une fonction définie sur les sommets d'un triangle
 - Gradient de l'interpolation linéaire de la fonction sur le triangle
- Laplacien en un sommet d'une fonction définie sur les sommets
 - Différence entre la moyenne des valeurs sur les voisins et la valeur au sommet
 - Quels coefficients choisir pour faire la moyenne?
 - Choisir des coefficients cohérents avec le fait que le laplacien de la fonction de position des points sur une surface doit être lié à la courbure moyenne et à la normale à la surface (et correspond au gradient de l'aire locale de la surface quand la position du point varie).

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Differential operators

- Simplicial meshes
 - Laplacian Δ of u at vertex i = sum over neighbor vertices j

$$(Lu)_i = \frac{1}{2A_i} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$
 - Gradient ∇ of u inside a triangle = sum over the 3 vertices I

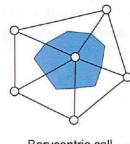
$$\nabla u = \frac{1}{2A_f} \sum_i u_i (N \times e_i)$$
 - Divergence at vertex i of a vector X defined on faces = sum over incident faces j

$$2A_f \nabla \cdot X = \frac{1}{2} \sum_j \cot \theta_1 (e_1 \cdot X_j) + \cot \theta_2 (e_2 \cdot X_j)$$

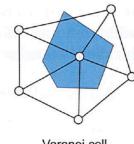
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Area A_i

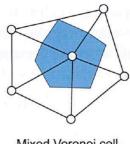
- Computed by duality to a vertex



Barycentric cell



Voronoi cell



Mixed Voronoi cell

Courtesy M. Botsch et al.

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Differential geometry

- How to generalize normal, curvatures?
- Consider u corresponding to each coordinate function in turn
 - $u = t(x,y,z)$

$$\Delta_x u = -2Hn$$

H : mean curvature

n : normal vector

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Signals being studied in Computer Graphics

- Signals defined on \mathbb{R}^2
 - Scalar signals :
 - height value (terrain)
 - density of some fluid flowing in the plane
 - Vector fields
 - Surface parameterization $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$
 - Displacement field of some fluid flowing in the plane
- Signals defined in \mathbb{R}^3
 - Density of a volume material (scalar)
 - Displacement field of some fluid (vector)

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Signals being studied in Computer Graphics

- Signals defined on a surface S
 - Scalar values :
 - Temperature, grey color ...
 - Position coordinates (x, y or z)
 - Vector fields
 - Normal vector
 - Maximal/minimal curvature direction
 - Displacement field
 - ...

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