Lyon 1

Mesh and Computational Geometry

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M2 ID3D
Image, Développement
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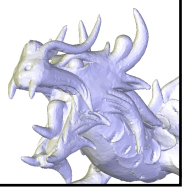

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Properties

- There are interesting interpretations of the Delaunay triangulation, lifting the points into a space of higher dimension
  - What is Delaunay in 1D?
    - Delaunay is like sorting and linking each point with the next one

    OR
    - Projecting the 1D points on the lower side of a 2D convex and computing the convex envelop of the 2D lifted points
  - Delaunay = lower envelope of points

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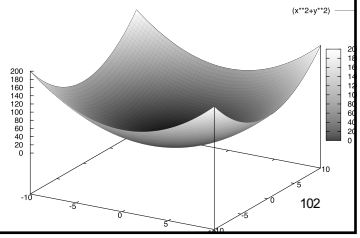
Properties

- Nice interpretation of Delaunay in the space (« space of spheres »)
  - Correspondence between the Delaunay triangulation and the lower convex envelope of the points lifted on the paraboloid

$$z = x^2 + y^2$$



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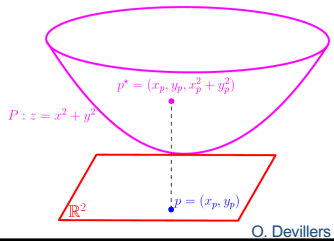
- Each 2D point x_P, y_P is lifted to $x_P, y_P, x_P^2+y_P^2$ on the paraboloid

$$z = x^2 + y^2$$



O. Devillers

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Demonstration

- Let us consider a circle of the plan

$$(M - C)^2 = R^2$$
$$(x - x_C)^2 + (y - y_C)^2 = R^2$$
$$x^2 + y^2 - 2xx_C - 2yy_C + x_C^2 + y_C^2 - R^2 = 0$$
$$x^2 + y^2 - 2ax - 2by + c = 0$$

  - Where are the points of the circle lifted on the paraboloid?

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Demonstration

- Circle $x^2 + y^2 - 2ax - 2by + c = 0$
  - all the points belonging to this circle are lifted to points belonging simultaneously to the paraboloid and the plane

$$z - 2ax - 2by + c = 0$$

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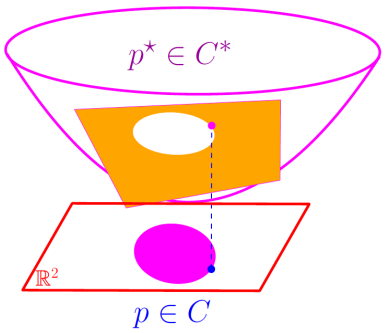
Demonstration



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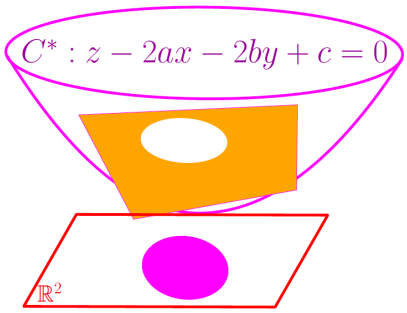
Demonstration



$C : x^2 + y^2 - 2ax - 2by + c = 0$

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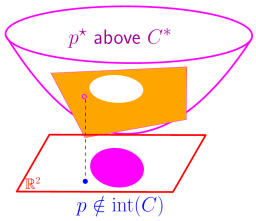
Demonstration

- What about the points outside the circle C when they are lifted?
  - They are lifted above the plane C*



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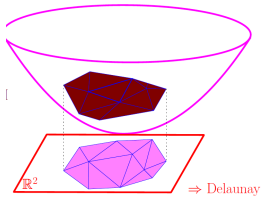
Demonstration

- Thus, if 3 points correspond to a Delaunay triangle, none of the other points can be lifted under the plane corresponding to the 3 points :
  - Correspondance between the Delaunay triangulation and the lower enveloppe of the points lifted to the paraboloid.



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Back to Lawson's algorithm

- Transformation of a 2D triangulation into a Delaunay triangulation
  - As long as there is a non-locally Delaunay edge (ie a non Delaunay subtriangulation of 4 points)
    - Replace the 4 points' subtriangulation by the alternative subtriangulation (flip)
- What complexity?

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Back to Lawson's algorithm

- By using Delaunay interpretation in the spheres space, let's show that **an edge cannot appear more than once in the queue** (even with other incident triangles)

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Back to Lawson's algorithm

- Interpretation of the *flip* in the "space of spheres"
  - Before the *flip*, each of the two triangles has the 4th point in its circumscribed circle
  - The lifting of each triangle on the paraboloid is located above the 4th point.

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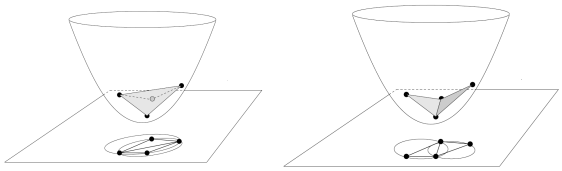
Back to Lawson's algorithm

- Interpretation of the *flip* in the "space of spheres"



- Each *flip*, allows the lifted surface to descend locally

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Back to Lawson's algorithm

- This means that a flipped edge cannot reappear a second time in the flow of the algorithm
- The number of edges that can be formed with n points is n(n-1)/2
- Lawson's algorithm is therefore in $O(n^2)$
- Much more efficient in practice!

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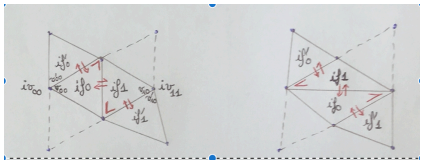
Operations for triangulating and flipping edges

- Updating the Mesh data structure
  - Division of a triangle into 3 triangles (Split)
  - Flip of an edge (Flip)



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Other use of the flip operation

- Insertion of a point P outside the convex-hull of a triangulation
  - New triangles joining P and the boundary edges that are visible from P
  - The infinite faces should be updated accordingly

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Other use of the flip operation

- Insertion of a 2D point P outside the convex-hull of a 2D triangulation
- A possible implementation using the infinite vertex and the flip operation
  - Let InfF be an infinite face incident to a visible edge of the convex-hull
  - Split InfF into 3
  - Iteratively flip the infinite edges bounding the modified area if they are incident to an other infinite face that should disappear (starting from InfF)

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Delaunay triangulation in any dimension n

- Paving the convex-hull of the points with n-simplices whose circumscribed sphere is empty
- Warning: Lawson's algorithm is only valid in 2D, because the notion of edge flip is more complicated in larger dimensions

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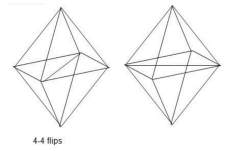
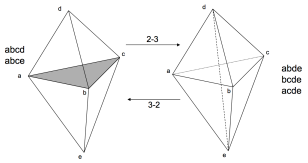
Flip in 3D

- 4-4 flip



- 2-3 and 3-2 flips



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Geometric algorithms implementation

- Distinction between exact, combinatorial vs. approximate objects
- Input data (considered as exacts) used to construct exact non combinatorial objects
  - points x,y or x,y,z
- Construction of combinatorial objects from the exact ones
- Approximate objets should be constructed for visualisation purpose only

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Algorithmic using predicates

- The progress of an algorithm should only depend on the sign of predicates evaluated accurately
  - Use of a controlled arithmetic
  - Is the arithmetic interval enough?
  - No use of inexact objects in the evaluation of predicates

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Algorithmic using predicates

- Without these precautions, there is a risk of aberrant behaviour of a geometric algorithm
- Example: How to express the simple insertion algorithm in a 2D triangulation according to these criteria?

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Algorithmic using predicates

- Algorithm of simple insertion in a 2D triangulation:
  - Using the three points orientation predicate
    - To perform inclusion tests in a triangle
    - To perform visibility tests on an edge of the convex envelope

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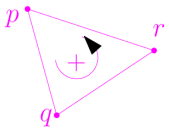
Algorithmic using predicates

- Three 2D points orientation predicate:

$$orientation(p,q,r) = sign(((q-p)\times(r-p)).Oz)$$

$$orientation(p,q,r) = sign((q_x-p_x)(r_y-p_y)-(q_y-p_y)(r_x-p_x))$$

$$= \begin{vmatrix} q_x-p_x & r_x-p_x \\ q_y-p_y & r_y-p_y \end{vmatrix}$$



$$orientation(p,q,r) = sign(\det \begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix})$$

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where the input coordinates belong to the regular grid of integers?
  - In the case where the input coordinates are rationals?

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case of the input coordinates belong tho the regular grid of integers?
  - In the case where the input coordinates are rationals?
  - In both cases the evaluation can be carried out accurately since we benefit from an exact multiplication, addition and subtraction for these types of numbers!

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where input coordinates are double?

$$\pm m2^e \qquad -1023 < e < 1024$$
$$m = 1.m_1m_2\ldots m_{52}\ (m_i \in \{0,1\})$$

  - The result of the arithmetic operations is rounded to the nearest double

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where input coordinates are double?
  - It is only the sign of the predicates that matters
  - The case where the three points are almost aligned can be error-prone

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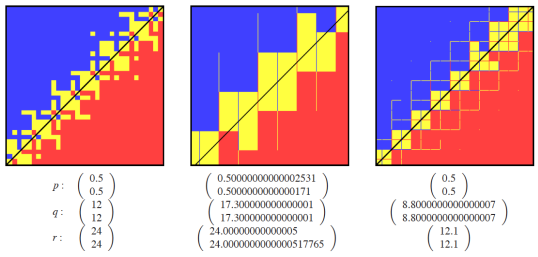
Algorithmic using predicates

- If the orientation predicate is evaluated using double arithmetic :
  - Result of orientation(p,q,r) with p(p_x+Xu_x, p_y+Yu_y)

$$0 \le X,Y \le 255 \qquad u_x = u_y = 2^{-53}$$



Images by S. Pion

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Algorithmic using predicates

- If the orientation predicate is evaluated using double arithmetic :
  - There are still values for which the sign is unambiguously certified (need to control the threshold)
  - Otherwise :
    - Consider the coordinates of p, q and r as rational (with a finer precision than that usually considered) and make the exact calculation

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Algorithmic unsing predicates

- Example : How to express Lawson algorithm using predicates?

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Algorithmic using predicates

- Lawson Algorithm :
  - An AB edge should be flipped if the circle circumscribed to one of its 2 incident triangles ABC contains point D located on the other side
  - **Do not** use an inaccurate temporary object (e. g. the centre of a circumscribed circle) when evaluating a predicate

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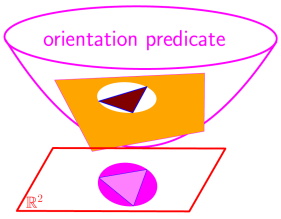
Algorithmic using predicates

- Reminder: the in-circle inclusion test can be expressed as an orientation test in a space of higher dimension (space of spheres)



Images par O. Devillers

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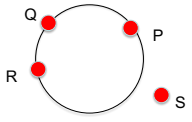
Algorithmic using predicates

- Predicate of inclusion of a point s in a circle circumscribed at p,q and r (orientation of the 4 points lifted on the paraboloid centered at p)

Φ = lifting operator



- In_cercle(p,q,r,s)

=-signe(((Φ(q)-Φ(p))x(Φ(r)-Φ(p))).(Φ(s)-Φ(p))

$$= - \; sign \begin{vmatrix} q_x - p_x & r_x - p_x & s_x - p_x \\ q_y - p_y & r_y - p_y & s_y - p_y \\ (q_x - p_x)^2 + (q_y - p_y)^2 & (r_x - p_x)^2 + (r_y - p_y)^2 & (s_x - p_x)^2 + (s_y - p_y)^2 \end{vmatrix}$$

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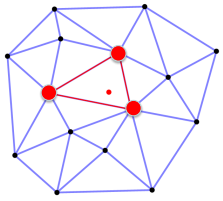
Back to Delaunay triangulation

- How to modify the incremental algorithm of insertion into a simple triangulation to obtain an incremental algorithm of insertion into a Delaunay triangulation?



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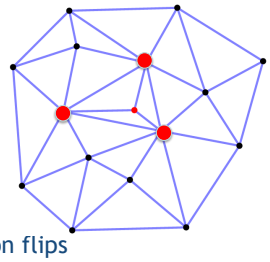
Incremental insertion into a Delaunay triangulation

- First perform a simple insertion :



  - Use Lawson flips
  - Is it necessary to test all the edges?

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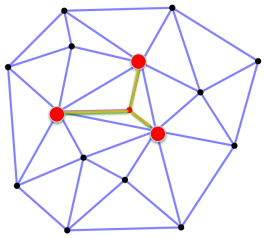
Delaunay incremental insertion

- Let Δ be the triangle in which P was inserted
  - After the simple insertion, the triangle is star-shaped with respect to P



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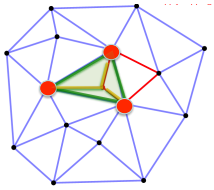
Incremental Delaunay insertion

- Let Δ be the triangle in which P was inserted
  - After the simple insertion, only the 3 edges of Δ are likely to be candidates for flipping
  - The 3 new edges incident to P could not be flipped
  - The others have not changed their pair of incident triangles



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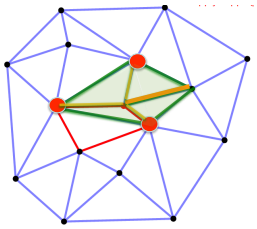
Incremental Delaunay insertion

- Each flip generates a new edge incident to P and two edges are added to the boundary of the modified area (in green)
- Green edges are likely to be flipped (since one of their incident triangle has been modified)



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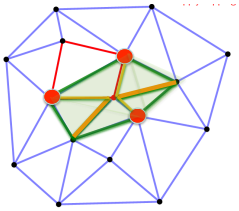

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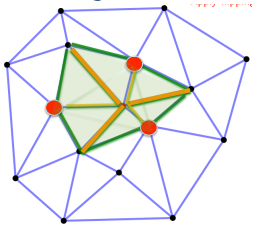

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Incremental Delaunay Algorithm

- The complexity of each insertion directly corresponds to the final number i of edges incident at the new vertex.
  - Every flipped edge gave birth to an edge incident to P ->there are i-3 flips
  - The flipping test was checked on each edge that was effectively flipped, and also on the green boundary of the resulting modified area (i edges)

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Incremental Delaunay Algorithm

- An insertion outside the convex envelope also starts as the simple insertion into a triangulation
  - Additional flips can be performed on the boundary of the modified area

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Delaunay incremental insertion (Alternative version)

- We just showed that the modified area of the triangulation is star-shaped around the inserted point P
- Alternative approach for incremental Delaunay :
  - Delete all the triangles whose circumscribed circle contains point P
  - Those triangles are said to be "in conflict" with P
  - Triangulate the conflict zone by star-shaping the conflict area around P

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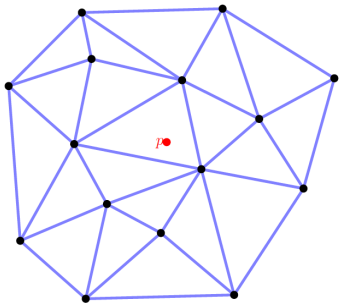
Incremental Delaunay Algorithm (Alternative Version)



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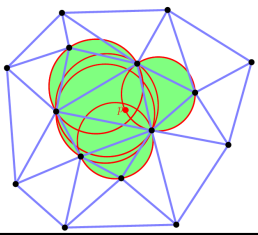
Delaunay incremental algorithm (Alternative Version)

- Determine the in-conflict triangles
  - Using breadth-first search (or deapth-first search) on the adjacency graph of triangles starting from Δ



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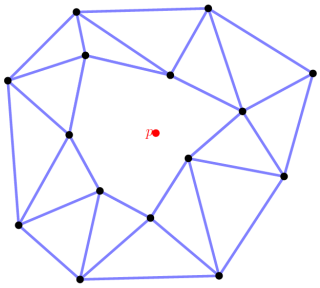
Delaunay incremental algorithm (Alternative Version)

- Conflict zone determination



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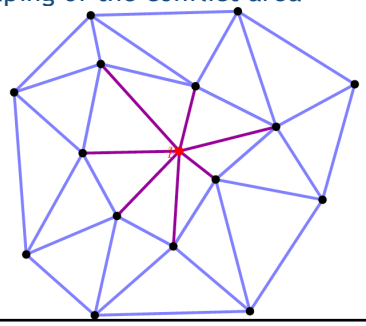
Delaunay incremental algorithm (Alternative Version)

- Star-shaping of the conflict area



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Delaunay incremental algorithm (Alternative Version)

- Star-shaping of the conflict area
- Validity of this algorithm in higher dimension
  - In 2D, the **edges of the boundary** of the conflict zone get connected to the inserted point by constructing new triangles
  - In 3D, the Delaunay triangulation is composed of tetrahedrons. The **triangles of the boundary** of the conflict zone get connected to the inserted point by constructing new tetrahedrons

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