Problem Set 2 - Solutions

Intermediate Microeconomics - Summer 2020, NYU
Instructor: Skand Goel

Rules:

- The problem set is due on Tuesday, June 9 before 4.05 PM Eastern time.
- The submission should be a PDF attachment (scanned or digital) via NYU classes. **Please don't upload multiple PDFs** - just put all your work in one PDF. An app such as CamScanner should help.
- Please **mention your NYU net ID** on your submission. e.g. my net ID is sg3992.
- Each of the four questions is for 3 points. The total score will be rescaled to 5. Even if you don't do everything right, you can still get a high mark - so try and attempt all questions, and if you hit a dead-end, try to explain how you got there and why you are stuck!
- If you feel there is an error or typo, please email me at sg3992@nyu.edu.
- You can work in groups, but must submit your own work. If you work in groups, mention the name of students in your group.
- **Please write legibly**. It helps the grading, *as well as your grades.* Also, when asked to plot something, please draw neat, well-labeled diagrams. Drawing larger figures increases legibility.
- Friendly reminder: Please do not circulate the problem set. **Please do not upload the problem set to the internet in any form**.

A. Perfect substitutes

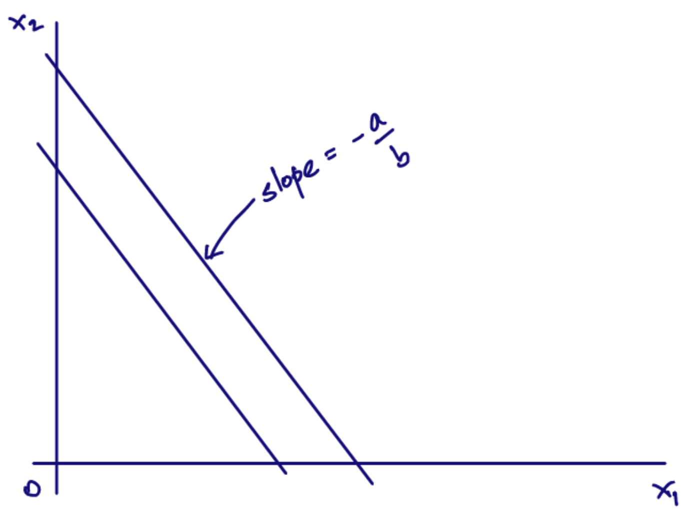
A consumer has the following preferences

$$u(x_1, x_2) = ax_1 + bx_2$$

Suppose the price of good 1 is p_1 and the price of good 2 is p_2. The consumer has income m.

(a) Plot this consumer's indifference curves. What is the marginal rate of substitution for this consumer?

Answer:



The MRS is $\frac{a}{b}$. Its alright to define it as $-\frac{a}{b}$ too.

(b) Solve the utility maximization problem and write down the Marshallian demand function for good 1 in terms of p_1 (taking p_2 and m as fixed). Also, Plot the demand curve. [Hint: The demand function will display unusual behavior when the price ratio is in the neighborhood of 1.]

Answer:

The solution to the utility maximization problem is

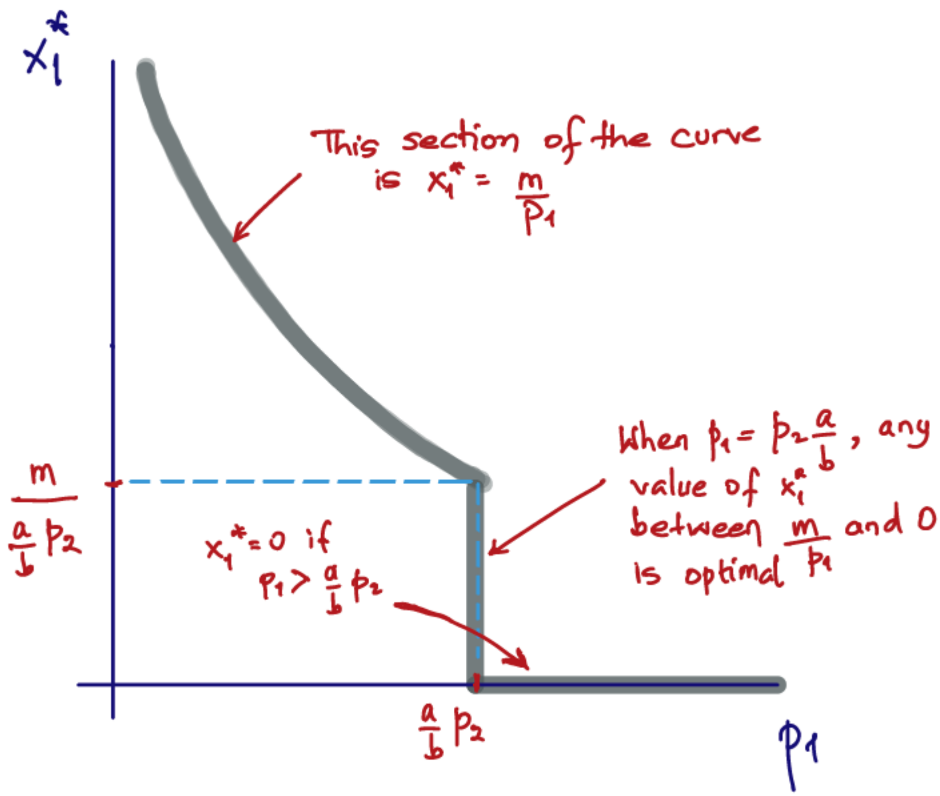
$$(x_1^*, x_2^*) = \begin{cases} \left(\frac{m}{p_1}, 0\right) & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ \left(t, \frac{m - p_1 t}{p_2}\right) & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ \left(0, \frac{m}{p_2}\right) & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

for $t \in \left[0, \frac{m}{p_1}\right]$.

The Marshallian demand is given by

$$x_1^*(p_1; p_2, m) = \begin{cases} \frac{m}{p_1} & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ t \in \left[0, \frac{m}{p_1}\right] & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ 0 & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

where p_2 and m are treated as fixed.



(c) Suppose $\frac{p_1}{p_2} > \frac{a}{b}$. What is the cheapest way to obtain a utility level $\bar{u}$. What if

$\frac{p_1}{p_2} < \frac{a}{b}$? What if $\frac{p_1}{p_2} = \frac{a}{b}$? Follow the clues and write down the consumer's Hicksian demand function for good 1.

Answer:

If $\frac{p_1}{p_2} > \frac{a}{b}$, it is best to spend only on good 2. To obtain utility level $\bar{u}$, we need to consume

$$x_2 = \frac{\bar{u}}{b}.$$

Analogous reasoning gives us the optimal choices

$$(x_1^h, x_2^h) = \begin{cases} \left(\frac{\bar{u}}{a}, 0\right) & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ \left(t, \frac{\bar{u}-at}{b}\right) & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ \left(0, \frac{\bar{u}}{b}\right) & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

for $t \in \left[0, \frac{\bar{u}}{a}\right]$.

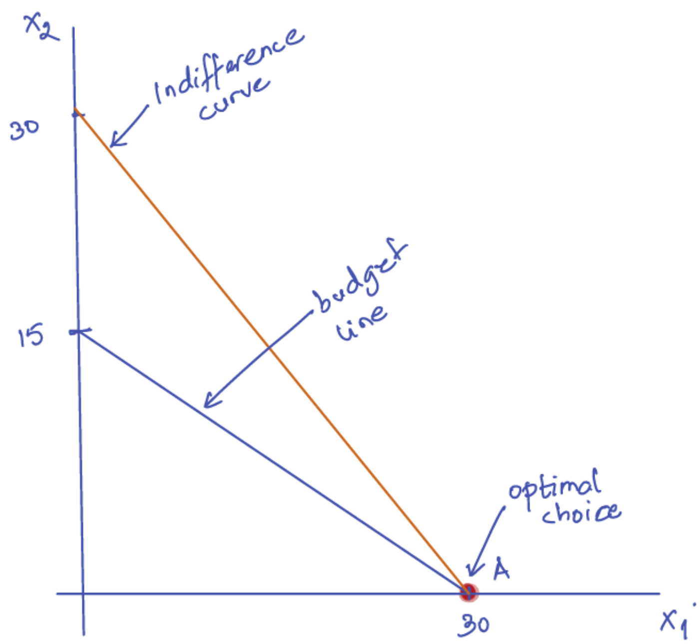
The Hicksian demand for good 1 is given by

$$x_1^h(p_1; p_2, \bar{u}) = \begin{cases} \frac{\bar{u}}{a} & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ t \in \left[0, \frac{\bar{u}}{a}\right] & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ 0 & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

where p_2 and $\bar{u}$ are treated as fixed.

(d) For all subparts below, set $a = b = 1$. The consumer has $m = 30$ and faces $p_1 = 1$ and $p_2 = 2$. Plot the budget line, the optimal choice and the indifference curve passing through the optimal choice.

Answer:

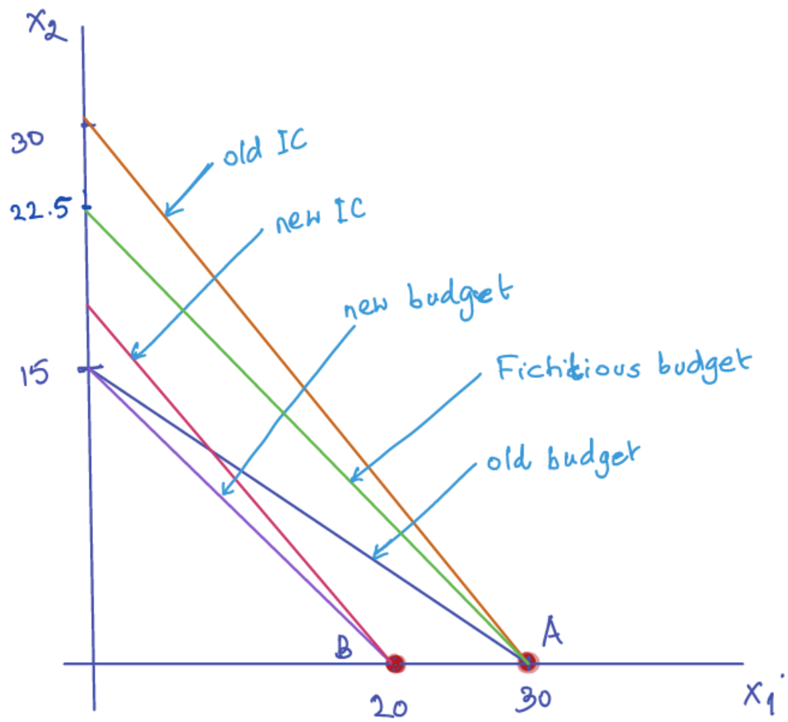


(e) In part (d), suppose p_1 increases to $p_1' = 1.5$. On the same graph as above, plot the new budget line, the new choice and the indifference curve passing through the new choice. State (in numbers) and depict (on the figure) the income effect and the substitution effect for both goods. (In order to depict substitution and income effects, it might be clearer to label the relevant bundles on the figure and state the income and substitution effects in terms of these label, as opposed to directly labeling them on the diagram.)

Answer:

In the figure below, the green line labeled "fictitious budget" line passes through the old bundle (A) and has the same slope as the new budget line. We can see that the utility maximizing bundle at this budget line (or the expenditure minimizing bundle at the old utility level and new prices) is the same as the old bundle A. Therefore, the substitution effect is 0 for both goods.

The move from A to B captures the change due income effect. This change for good 1 is 10, and for good 2 is 0.

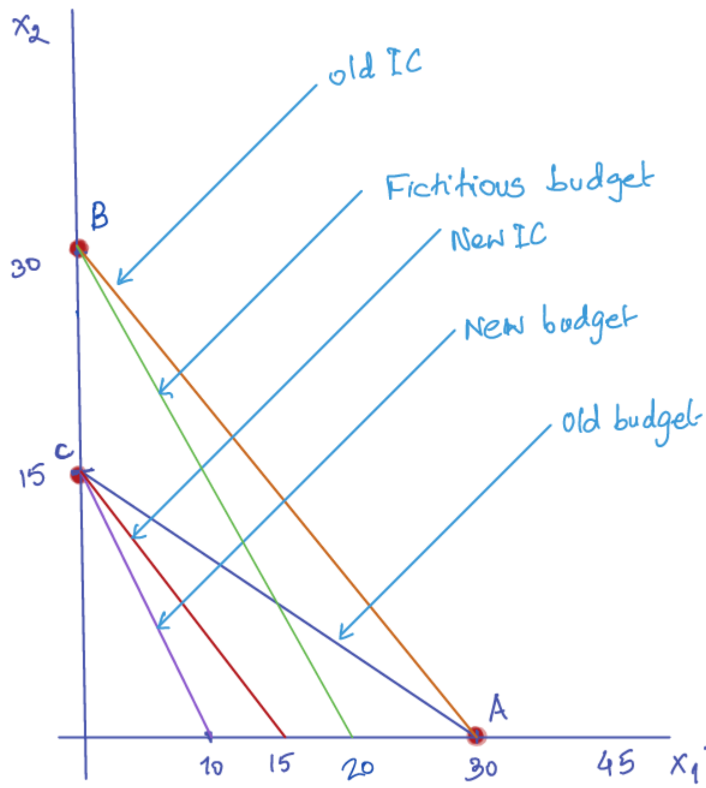


(f) Going back to part (d), suppose p_1 had increased to increases to $p_1' = 3$. On a new graph, once again plot the choice sitaution in part (d). Also, plot the choice situation corresponding to $p_1' = 3$. State and depict income and substitution effects.

Answer:

In the figure below, the green "fictitious budget" line (i.e the budget line that passes through the old bundle (A) and has the same slope as the new budget line). The utility maximizing bundle at this budget line (or the expenditure minimizing bundle at the old utility level and new prices) is given by point B ($x_1 = 0$ and $x_2 = 30$). The move from A to B is *the change due to* substitution effect (SE). For good 1, this change is -30 and for good 2, this change is +30. We can see that SE is negative i.e. the demand goes down for the good that saw an increase in relative price and vice-versa.

The move from B to C ($x_1 = 0$ and $x_2 = 15$), which is the utility maximizing bundle at new prices, is *the change due to* income effect (IE). For good 1, this change is 0 and for good 2, this change is -15. We can see that IE is negative i.e. the demand goes down as real income (or purchasing power) goes down (to be clear, its weakly negative because the change due to IE is 0 for good 1).



(g) Use the answer in part (a) to derive the indirect utilty function. What is the value of the indirect utility function at $m = 30$, $p_1 = 4$ and $p_2 = 2$?

Answer:

$$V(p_1, p_2, m) = \begin{cases} a\frac{m}{p_1} & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ m & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ b\frac{m}{p_2} & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

For the case $\frac{p_1}{p_2} = \frac{a}{b}$, note that indifference curve is the same as the budget line, and we know that the budget line has level m.

This value of the indirect utility function is given by

$$V(4, 2, 30) = \frac{30}{2} = 15$$

(h) Use the answer in part (b) to derive the expenditure function. What is the value of the expenditure function at $p_1 = 4$, $p_2 = 2$ and the utility level found in (g)? What does this tell us about duality?

Answer:

$$E(p_1, p_2, \bar{u}) = \begin{cases} p_1 \frac{\bar{u}}{a} & \text{if } \frac{p_1}{p_2} < \frac{a}{b} \\ \bar{u} & \text{if } \frac{p_1}{p_2} = \frac{a}{b} \\ p_2 \frac{\bar{u}}{b} & \text{if } \frac{p_1}{p_2} > \frac{a}{b} \end{cases}$$

At $\bar{u} = V(4, 2, 30) = 15$, and $p_1 = 4$, $p_2 = 2$,

$$E(4, 2, 30) = 2 \times 15 = 30,$$

which is the income at which $V(4, 2, 30)$ was calculated. This is in accordance with duality.

B. Quasilinear preferences and welfare

A consumer has the following preferences

$$u(x_1, x_2) = \log(x_1) + x_2$$

Suppose the price of good 1 is p_1 and the price of good 2 is p_2. The consumer has income m.

(a) Find the optimal choices for the utility maximization problem in terms of p_1, p_2 and m. Denote the lagrange multiplier by λ.

Answer:

$$\mathcal{L}(x_1, x_2, \lambda) = \log(x_1) + x_2 + \lambda [m - p_1 x_1 - p_2 x_2]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 = 0 \implies x_1 = \frac{1}{\lambda p_1}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_2 = 0 \implies \lambda = \frac{1}{p_2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0$$

Putting together the implications from the first two FOCs,

$$x_1^* = \frac{p_2}{p_1}$$

Plugging this in the budget line,

$$x_2^* = \frac{m}{p_2} - 1.$$

(b) How do the optimal choices change as m increases? What does the income offer curve (also called the income expansion path) look like for this consumer? (You can show it on a diagram.)

Answer:

x_1^* does not change w.r.t. income.

$$\frac{\partial x_2^*}{\partial m} = \frac{1}{p_2}.$$

The income offer curve looks like a vertical line at $x_1^* = \frac{p_2}{p_1}$.

(c) What is the slope of the Marshallian demand curve for good 1? Use the slutsky equation to find the slope of the Hicksian demand curve for good 1, without actually solving the expenditure minimization problem.

Answer:

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{p_2}{p_1^2}$$

The slutsky equation tells us

$$\frac{\partial x_1^*}{\partial p_1} = \frac{\partial x_1^h}{\partial p_1} - x_1^* \frac{\partial x_1^*}{\partial m}.$$

We just showed that

$$\frac{\partial x_1^*}{\partial m} = 0,$$

which implies $\frac{\partial x_1^*}{\partial p_1} = \frac{\partial x_1^h}{\partial p_1}$. Therefore,

$$\frac{\partial x_1^h}{\partial p_1} = -\frac{p_2}{p_1^2}$$

(d) For a utility level $\bar{u}$, solve the expenditure minimization problem and find the optimal choices in terms of p_1, p_2 and $\bar{u}$. Denote the lagrange multiplier by μ.

Answer:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \mu \left[\bar{u} - \log(x_1) - x_2\right]$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \mu \frac{1}{x_1} = 0 \implies x_1 = \frac{\mu}{p_1} \\
\frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \mu = 0 \implies p_2 = \mu \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \bar{u} - \log(x_1) - x_2 = 0
\end{aligned}$$

Putting together the implications from the first two FOCs,

$$x_1^h = \frac{p_2}{p_1}$$

Plugging this into the utility constraint gives

$$x_2^h = \bar{u} - \log\left(\frac{p_2}{p_1}\right)$$

(e) Find the Hicksian demand curve for good 1. What is the slope of this curve? Does it match your answer in (c)?

Answer: The Hicksian demand is given by

$$x_1^h(p_1; p_2, \bar{u}) = \frac{p_2}{p_1}$$

We can see that this is the same as the Marshallian demand found above. Slope is the same as above.

(f) Find the expenditure function. Find its partial derivative with respect to p_2. Provide an interpretation of this derivative in terms of choice behavior.

Answer:

The expenditure function is given by

$$\begin{aligned}
E(p_1, p_2, \bar{u}) &= p_1 x_1^h + p_2 x_2^h \\
&= p_2 + p_2 \left(\bar{u} - \log\left(\frac{p_2}{p_1}\right)\right)
\end{aligned}$$

$$\frac{\partial E(p_1, p_2, \bar{u})}{\partial p_2} = 1 + \left(\bar{u} - \log\left(\frac{p_2}{p_1}\right)\right) - 1 = \bar{u} - \log\left(\frac{p_2}{p_1}\right)$$

As we can see, this is just the Hicksian demand for good 2 from part (d). We can also see how the terms containing indirect effects are the ones to cancel out while finding the partial derivative.

(g) Use the answer in (e) to find compensating variation for a change in p_1 from a to b, $a < b$. [Hint:

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

and the fundamental theorem of calculus tell us

$$\int_a^b cf'(t)\,dt = c\,(f(b) - f(a))$$

where c is a constant.]

Answer:

$$\begin{aligned} CV &= \int_a^b \frac{p_2}{p_1}\,dp_1 \\ &= p_2 \int_a^b \frac{1}{p_1}\,dp_1 \\ &= p_2 \int_a^b \frac{d}{dx}\log(p_1)\,dp_1 \\ &= p_2\,(\log(b) - \log(a)) \end{aligned}$$

(h) Use the answer in (f) to find compensating variation for a change in p_1 from a to b. Does this match your answer in (g)?

Answer:

$$\begin{aligned} CV &= E(b, p_2, \bar{u}) - E(a, p_2, \bar{u}) \\ &= p_2 + p_2\left(\bar{u} - \log\left(\frac{p_2}{b}\right)\right) - \left[p_2 + p_2\left(\bar{u} - \log\left(\frac{p_2}{a}\right)\right)\right] \\ &= p_2\,(\log(b) - \log(a)) \end{aligned}$$

which matches the above answer.

(i) What is the consumer surplus lost as a result of the change in p_1 from a to b,

$a < b$?

Answer:

We showed in this question that Marshallian and Hicksian demand for good 1 are the **same everywhere** (in general, duality tells us that they same only at some points). In particular, neither depend on the constraint values (m and $\bar{u}$ respectively). Therefore, the loss in consumer surplus is the same as CV, which is equal to $p_2 (\log(b) - \log(a))$.

C. Cobb-Douglas preferences

A consumer has the following preferences

$$u(x_1, x_2) = \log(x_1) + \log(x_2)$$

Suppose the price of good 1 is p_1 and the price of good 2 is p_2. The consumer has income m.

(a) Write down optimal choices for the utility maximization problem.

Answer:

You should set up the Lagrangean and solve this. The result will be

$$x_1^* = \frac{m}{2p_1}$$
$$x_2^* = \frac{m}{2p_2}$$

(b) What is the own-price elasticity of demand for good 1? What is the cross-price elasticity of demand for good 1?

Answer:

Own price elasticity of demand for good 1 is given by

$$\frac{\partial x_1^*}{\partial p_1} \cdot \frac{p_1}{x_1^*} = -\frac{m}{2p_1^2} \cdot \frac{p_1}{\frac{m}{2p_1}}$$
$$= -1$$

The cross price elasticity of demand for good 1 is

$$\frac{\partial x_1^*}{\partial p_2} \cdot \frac{p_2}{x_1^*} = 0 \cdot \frac{p_2}{\frac{m}{2p_1}} = 0$$

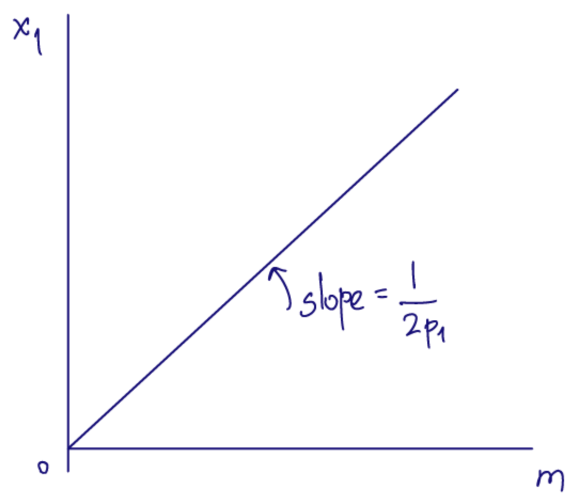
(c) What is the income elasticity of demand for good 1? Plot the Engel curve for good 1.

Answer:

The income elasticity of good 1 is

$$\frac{\partial x_1^*}{\partial m} \cdot \frac{m}{x_1^*} = \frac{1}{2p_1} \cdot \frac{m}{\frac{m}{2p_1}} = 1$$

See figure below for Engel curve. It is typically drawn with m on the x-axis. I think I miswrote during the lecture and put m on the y-axis. That's an acceptable answer, though its good to know the convention.



(d) Solve the expenditure minimization problem and find the optimal solution.

Answer:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \mu \left[\bar{u} - \log(x_1) - \log(x_2)\right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \mu \frac{1}{x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \mu \frac{1}{x_2} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \mu} = \bar{u} - \log(x_1) - \log(x_2) = 0$$

First two FOCs imply

$$p_1 x_1 = p_2 x_2$$

Solving this together with the budget line implies

$$x_1^h = \sqrt{\frac{p_2}{p_1}\exp(\bar{u})}$$
$$x_2^h = \sqrt{\frac{p_1}{p_2}\exp(\bar{u})}$$

(e) Find compensating variation from change in p_2 from a to b using a method of your choosing ($a < b$). Is this the same as consumer surplus lost due to this change? [Hint: You don't need to calculate CS.]

Answer:

I will integrate Hicksian demand for good 2.

$$\begin{aligned} CV &= \int_a^b \sqrt{\frac{p_1}{p_2}\log(\bar{u})}\, dp_2 \\ &= \sqrt{p_1 \log(\bar{u})} \int_a^b p_2^{-\frac{1}{2}}\, dp_2 \\ &= \sqrt{p_1 \log(\bar{u})} \left[\frac{b^{\frac{1}{2}}}{\frac{1}{2}} - \frac{a^{\frac{1}{2}}}{\frac{1}{2}} \right] \end{aligned}$$

This is not equal to CS because x_2^h and x_2^* are different functions.

D. Risky investment

An individual with wealth w is deciding how much to invest in the stock market. Denote the investment by z. With probability π the price of this stock will go up by $r \times 100\%$, and with probability $(1-\pi)$ the price of this stock will go down by $r \times 100\%$. The current price of the stock is \$1.

(a) What is her total wealth when she makes a profit on her investment? What is her total wealth when she makes a loss on her investment?

Answer:

Her total wealth in the event of profit is

$$(1 + r)z + w - z = w + rz$$

and her total wealth in the event of loss is

$$w - rz$$

Her Bernoulli utility function is given by

$$u(x) = x^{\alpha}, \qquad \alpha \in (0, 1)$$

(b) What is her expected utility from the investment level z? [Hint: Each lottery in this case is denoted by a different level of z because nothing else varies across lotteries. So we can think of the expected utility as a function of z.]

Answer:

Her expected utility from the investment level is (abusing the expected utility notation)

$$U(z) = \pi(w + rz)^{\alpha} + (1 - \pi)(w - rz)^{\alpha}$$

(c) What is her expected utility maximizing level of investment in terms of π, w and r?

Answer:

Take the first order conditon with respect to z:

$$U'(z) = \pi\alpha r(w + rz)^{\alpha-1} - (1 - \pi)\alpha r(w - rz)^{\alpha-1} = 0$$

which implies

$$z^* = \left(\frac{1-\gamma}{1+\gamma}\right)\frac{w}{r}$$

where $\gamma = \left(\frac{\pi}{1-\pi}\right)^{\frac{1}{\alpha-1}}$.

(d) If $\pi = 0.5$, how much does she invest? What if $\pi = 1$?

Answer:

At $\pi = 0.5$, $\gamma = 1$, and therefore $z^* = 0$.

At $\pi = 1$, her expected utility is just

$$U(z) = (w + rz)^{\alpha}$$

which is strictly increasing in z. Therefore, she should invest all her budget. $z^* = w$. Her marginal utility from investment is increasing at the constraint - if there was no constraint, she would have increased her investment. Therefore, her constraint binds.

The above was the easier way to think about it. Alternatively, some of plugged in $\pi = 1$ in the expression for z^* from part (c) and got $z^* = \frac{w}{r}$. Let's think about this solution. Suppose you plug this into $U(z)$, you get expected utility level equal to

$$U\left(z^* = \frac{w}{r}\right) = (2wr)^{\alpha}$$

Instead, if you plugged $z^* = w$, you get

$$U\left(z^* = w\right) = (w(1 + r))^{\alpha}$$

If $z^* = \frac{w}{r}$ is indeed the correct answer, then

$$U\left(z^* = \frac{w}{r}\right) > U\left(z^* = w\right)$$

which implies

$$(2wr)^{\alpha} > (w(1 + r))^{\alpha}$$

which implies

$$2 > (1 + r) \iff r < 1.$$

However, if $r < 1$, then $z^* = \frac{w}{r} > w$, which is not feasible!

It is also straightforward to check that if $r \leq 1$, then

$$U\left(z^* = \frac{w}{r}\right) \leq U\left(z^* = w\right).$$

Logic and math never disagree. If they do, one (or both!) of the arguments are incorrect or incomplete.

(e) Suppose the parameter values are $\alpha = 0.5$, $\pi = \frac{4}{5}$ and $r = 1$. How much does she invest?

Answer:

At these parameter values, $z^* = \frac{15}{17}w < w$. This is the unconstrained maximizer and is also feasible; therefore, it is optimal. If from the FOC solution $z^* > w$, then it would be optimal to invest all w. If from the FOC $z^* < 0$, then it would be optimal to not invest anything (you'd need to paid to make this investment!).

(f) What is the certainty equivalent and risk premia for the lottery corresponding to z? What is the risk premium?

Answer:

The certainty equivalent c must satisfy

$$U(z) = u(c)$$

which implies

$$\pi(w + rz)^\alpha + (1 - \pi)(w - rz)^\alpha = c^\alpha$$

which gives

$$c = [\pi(w + rz)^\alpha + (1 - \pi)(w - rz)^\alpha]^{\frac{1}{\alpha}}$$

The risk premium is given by

$$\begin{aligned} RP &= \{\pi(w + rz) + (1 - \pi)(w - rz)\} - [\pi(w + rz)^\alpha + (1 - \pi)(w - rz)^\alpha]^{\frac{1}{\alpha}} \\ &= w + (2\pi - 1)rz - [\pi(w + rz)^\alpha + (1 - \pi)(w - rz)^\alpha]^{\frac{1}{\alpha}} \end{aligned}$$

because for a given z, the term in braces above is the expectation of the stock market lottery.

(g) What happens to the risk premium as $\alpha \to 1$? Provide an intuition for this result?

Answer:

The Bernoulli utility function becomes the straight straight line at $\alpha = 1$. Therefore, the expected utility is just the expected value (this is the world before Bernoulli proposed the expected utility hypothesis!), and certainty equivalent tends to the expected value. Therefore, the risk premium, which is the difference between expected value and certainty equivalent tends to zero.

We can see this in the expression for RP above: At $\alpha = 1$, $RP = 0$.