Problem Set 3 - Solutions

Intermediate Microeconomics - Summer 2020, NYU
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Rules:

- The problem set is due on **Thursday, June 18** before 4.05 PM Eastern time.
- The submission should be a PDF attachment (scanned or digital) via NYU classes. **Please don't upload multiple PDFs** - just put all your work in one PDF. An app such as CamScanner should help.
- Please **mention your NYU net ID** on your submission. e.g. my net ID is sg3992.
- Each of the five questions is for 3 points. The total score will be rescaled to 5. Even if you don't do everything right, you can still get a high mark - so try and attempt all questions, and if you hit a dead-end, try to explain how you got there and why you are stuck!
- If you feel there is an error or typo, please email me at sg3992@nyu.edu.
- You can work in groups, but must submit your own work. **If you work in groups, mention the name and net ID of students in your group.**
- **Please write legibly**. It helps the grading, *as well as your grades.* Also, when asked to plot something, please draw neat, well-labeled diagrams. Drawing larger figures increases legibility.
- Friendly reminder: Please do not circulate the problem set. **Please do not upload the problem set to the internet in any form**.

A. Profit maximization

This question is intended to illustrate the equivalence between the two views of profit maximization discussed in class.

Suppose a firm's production function is given by

$$f(K, L) = K^{\frac{1}{4}} L^{\frac{1}{3}}$$

Let $r$and w denote the input prices for K and L, and p denote the output price.

I. Choosing inputs to maximize profits (One person handling all operations for the firm):

(a) Set up the profit maximization problem with the two inputs as the choice variables. Find the optimal choices of K and L.

Answer:

$$\max_{K,L} pK^{\frac{1}{4}} L^{\frac{1}{3}} - rK - wL$$

FOCs are

$$\frac{1}{4} pK^{-\frac{3}{4}} L^{\frac{1}{3}} = r \tag{1}$$

$$\frac{1}{3} pK^{\frac{1}{4}} L^{-\frac{2}{3}} = w \tag{2}$$

which together give

$$\frac{L}{K} = \frac{4r}{3w}$$

Plugging this to eliminate L in the FOC wrt K

$$\frac{1}{4} pK^{-\frac{3}{4}} \left(\frac{4r}{3w} K\right)^{\frac{1}{3}} = r \implies K^* = \left(\frac{p}{4r}\right)^{\frac{12}{5}} \left(\frac{4r}{3w}\right)^{\frac{4}{5}}$$

and similarly,

$$L^* = \left(\frac{p}{3w}\right)^{\frac{12}{5}} \left(\frac{3w}{4r}\right)^{\frac{3}{5}}$$

(b) Suppose $w = 4$ and $r = 3$. What is the profit maximizing level of <u>output</u> as a function of p? What is this relationship called?

Answer:

With the values of input prices above

$$L^* = K^* = \left(\frac{p}{12}\right)^{\frac{12}{5}}$$

The optimal level of output is

$$y^* = f(K^*, L^*) = \left(\frac{p}{12}\right)^{\frac{7}{5}}$$

(c) What is the maximum profit? *[Note: This is also called the profit function. Just as the cost function is minimized cost, profit function is maximized profit. That's why I used the term "expression for profit" or just "profit" when talking about the difference between total revenue and total cost. "Profit function" is a more specific term. The terms 'difference between total revenue and total cost' and 'profit function' have the same relationship as 'utility' and 'indirect utility'.]*

Answer:

Profit function is given by

$$\begin{aligned} \pi^*(p) &= py^* - rK^* - wL^* \\ &= p\left(\frac{p}{4}\right)^{\frac{7}{5}} - 7\left(\frac{p}{4}\right)^{\frac{12}{5}} \end{aligned}$$

II. First choosing inputs and getting the lowest cost to produce each level of output (factory supervisor's problem - who reports this to the top brass of the firm), then choosing output to maximize profits (which is the top brass's job!)

(d) Find this firm's cost function as a function of w, r and y, where y denotes output level.

Answer:

The cost function is the solution to

$$C(y; r, w) = \min_{K,L} rK + wL \quad \text{s.t.} \quad y = K^{\frac{1}{4}}L^{\frac{1}{3}}$$

Set up the Lagrangean:

$$\mathcal{L}(K, L, \lambda) = rK + wL + \lambda\left[y - K^{\frac{1}{4}}L^{\frac{1}{3}}\right]$$

The first order conditions are

$$\frac{1}{4}\lambda K^{-\frac{3}{4}} L^{\frac{1}{3}} = r \tag{3}$$
$$\frac{1}{3}\lambda K^{\frac{1}{4}} L^{-\frac{2}{3}} = w \tag{4}$$

and the production function constraint. This gives

$$\frac{L}{K} = \frac{4r}{3w}$$

Plugging this in the production function constraint, we get

$$\begin{aligned} y &= K^{\frac{1}{4}} \left(\frac{4r}{3w} K\right)^{\frac{1}{3}} \\ &= \left(\frac{4r}{3w}\right)^{\frac{1}{3}} K^{\frac{7}{12}} \implies K^* = y^{\frac{12}{7}} \left(\frac{3w}{4r}\right)^{\frac{4}{7}} \end{aligned}$$

Similarly, we get

$$\begin{aligned} y &= \left(\frac{3w}{4r} L\right)^{\frac{1}{4}} L^{\frac{1}{3}} \\ &= \left(\frac{3w}{4r}\right)^{\frac{1}{4}} L^{\frac{7}{12}} \implies L^* = y^{\frac{12}{7}} \left(\frac{4r}{3w}\right)^{\frac{3}{7}} \end{aligned}$$

The cost function is given by

$$\begin{aligned} C(y; r, w) &= r y^{\frac{12}{7}} \left(\frac{3w}{4r}\right)^{\frac{4}{7}} + w y^{\frac{12}{7}} \left(\frac{4r}{3w}\right)^{\frac{3}{7}} \\ &= y^{\frac{12}{7}} \left[r \left(\frac{3w}{4r}\right)^{\frac{4}{7}} + w \left(\frac{4r}{3w}\right)^{\frac{3}{7}} \right] := \gamma y^{\frac{12}{7}} \end{aligned}$$

(e) How does the average cost change vary with y? What does this imply about the shape of the cost function?

Answer: The average cost is given by

$$AC(y) = \frac{\gamma y^{\frac{12}{7}}}{y} = \gamma y^{\frac{5}{7}}$$

and

$$AC'(y) = \frac{5}{7} \gamma y^{-\frac{2}{7}} > 0,$$

which implies AC is increasing in y. This also means that the cost function is convex, as can be checked by taking the second derivative of $C(y)$.

(f) Now, use the cost function from part (d) to set up the profit maximization problem with y as the choice variable. What is the firm's supply function in terms of w, r and p?

Answer:

$$\max_y py - \gamma(w,r)y^{\frac{12}{7}}$$

where

$$\gamma(w,r) = \left[r\left(\frac{3w}{4r}\right)^{\frac{4}{7}} + w\left(\frac{4r}{3w}\right)^{\frac{3}{7}}\right]$$

which gives FOC

$$p = \frac{12}{7}\gamma(w,r)y^{\frac{5}{7}} \implies y^* = \left(\frac{7p}{12\gamma(w,r)}\right)^{\frac{7}{5}}$$

We know SOC hold because $C(y)$ is convex.

(g) Suppose $w = 4$ and $r = 3$. What is the profit maximizing level of output? What is the maximum profit? [Verify that these answers match the answers to part (b) and part (c).]

Answer:

$$\gamma(4,3) = 7$$

which implies

$$y^* = \left(\frac{p}{12}\right)^{\frac{7}{5}}$$

and maximized profit is

$$p\left(\frac{p}{4}\right)^{\frac{7}{5}} - 7\left(\frac{p}{4}\right)^{\frac{12}{5}}.$$

(f) Compare the first order conditions from (a) and (d). Do you see the similarity between p in the former and λ in the latter? What is the interpretation of λ? What does this mean for optimality condition for profit maximization?

Answer:

Comparing (1) with (3), and (2) with (4), we see that p gets replaced by λ. This implies that FOCs from problem (d) can be optimal iff $\lambda^* = p$. λ^* tells us how much the optimized function changes by if we are to loosen the constraint a bit (this is the envelope theorem result). In this case, the constraint is y and the optimized function is total cost, which implies that λ^* is the marginal cost at output level y i.e. $C'(y)$. This is exactly the optimality condition of part (f). Once again, this is a duality result.

B. Housekeeping

This problem is meant to show the math behind the AVC and MC curves on the slide. It is also meant to give you practice in doing (slightly) abstract derivations. You should be able to do this question with some algebra.

(a) For a cost function $C(y) = F + V(y)$ (assuming input prices are fixed), show that

$$AVC(y) = \frac{V(y)}{y}$$

is minimized at $\bar{y}$ where

$$AVC(\bar{y}) = C'(\bar{y})$$

<u>if C is convex.</u>

[Hint: You just need to take the FOC of $AVC(y)$ and rearrange terms. Remember to check SOC.]

Answer:

$$AVC(y) = \frac{V(y)}{y}$$

so the FOC to minimize AVC is

$$AVC'(y) = \frac{V'(y)y - V(y)}{y^2} = 0 \implies V'(y) = \frac{V(y)}{y} \tag{5}$$

and we can see from $C(y) = F + V(y)$ that

$$V'(y) = C'(y).$$

Therefore, the value of y solving the FOC is $\bar{y}$ such that

$$AC(\bar{y}) = C'(\bar{y})$$

This is the result we want to show, so long as we are sure that this FOC indeed picks a minima, and not a maxima. To see that

$$\begin{aligned} AVC''(y) &= \frac{[V''(y)y + V'(y) - V'(y)]\, y^2 - 2y\,[V'(y)y - V(y)]}{y^4} \\ &= \frac{V''(y)y^3 - 2V'(y)y^2 + 2yV(y)}{y^4} \end{aligned}$$

But we know from FOC that $V'(y) = \frac{V(y)}{y}$, so we can substitute out $V'(y)$ from the above expression. We get

$$\begin{aligned} AVC''(y) &= \frac{V''(y)y^3 - 2\left(\frac{V(y)}{y}\right)y^2 + 2yV(y)}{y^4} \\ &= \frac{V''(y)y^3 - 2yV(y) + 2yV(y)}{y^4} \\ &= \frac{V''(y)y^3}{y^4} > 0, \end{aligned}$$

which is the second order condition for minima. Note that we were given C is convex, which can only be true if V is convex i.e. $V''(y) > 0$

(b) Show that

$$AC(y) = \frac{C(y)}{y}$$

is also minimized where the C' function intersects AC, but at a point different from $\bar{y}$. Is this point above or below $\bar{y}$?

Answer:

We can make same arguments for AC as well.

$$AC(y) = \frac{V(y) + F}{y}$$

so we have

$$AC'(y) = \frac{V'(y)y - [V(y) + F]}{y^2} = 0 \implies V'(y) = \frac{C(y)}{y}. \tag{6}$$

Comparing equations (5) and (6), we can see that while

$$\bar{y} = V'^{-1}\left(\frac{V(y)}{y}\right),$$

the solution to (6) is

$$\tilde{y} = V'^{-1}\left(\frac{V(y)+F}{y}\right) \neq \bar{y}.$$

In fact, because marginal cost is convex, $V'' > 0$, and, therefore, $(V'^{-1})' > 0$ - the inverse of an upward sloping function is upward sloping (draw a graph to check) - implying that $V'^{-1}$is an increasing function. Since

$$\frac{V(y)+F}{y} > \frac{V(y)}{y}$$

for $F > 0$,

$$\tilde{y} > \bar{y}.$$

(c) Suppose the *marginal cost* (as a function of y) is U-shaped (initially decreasing, then increasing). What does the total cost (as a function of y) look like? (You can just draw a diagram).

Answer:

Initially concave, then convex.

(d) Show that for a U-shaped marginal cost function, the firm will not produce on the downward sloping part of the marginal cost function.

[Hint: Look at the figure in (c). Profits are just the difference between the curve plotted there and total revenue (given by py). Try and translate this intuition into math by solving the profit maximization problem $\max_y py - C(y)$ and checking the second order condition.]

Answer:

On the downward sloping part

$$C''(y) < 0$$

which means that if the y that solves

$$p = C'(y)$$

is such that it minimizes profits. Check SOC.

C. Shut down costs

This is just going over the solved example done in class. It might be useful to review the "confusing" math in the slides on this topic after doing this (and the previous) question.

A firm has the following cost function

$$C(y) = y^4 - 5y^3 + 2y + 10$$

Suppose p denote the output price.

Follow the steps in the lecture to find the short-run supply curve of the firm (under shutdown option II).

[Hint: You will have to solve a quadratic equation as a part of this problem and, therefore, you will have two candidate solutions. You should be able to choose the right candidate based on your answer to question B(d). Also, you don't need to invert the marginal cost function.]

Answer: First, we need to find the supply curve for the operational firm.

$$p = 3y^3 - 15y^2 + 2 = C'(y)$$

Now, we need to find the shutdown point. Given all that we did in class, we first need to find y where AVC is minimized. For this, note

$$V(y) = y^4 - 5y^3 + 2y$$

so

$$AVC(y) = y^3 - 5y^2 + 2.$$

The FOC to minimize this is

$$AVC'(y) = 3y^2 - 10y = 0$$

Solving this we get

$$y = 0 \text{ or } \frac{10}{3}.$$

Which y is the minimizer? For that, we need to evaluate the second derivative of AVC at each of the y values and check which one is positive.

$$AVC''(y) = 6y - 10.$$

It's easy to see that

$$AVC''(0) = -10 < 0$$
$$AVC''\left(\frac{10}{3}\right) = 10 > 00$$

so we pick

$$\bar{y} = \frac{10}{3}.$$

The corresponding price level on the marginal cost curve is

$$C'(\bar{y}) = \bar{p} = 3\left(\frac{10}{3}\right)^3 - 15\left(\frac{10}{3}\right)^2 + 2 < 0$$

Therefore, the firm never finds it optimal shut down as long as the prices are positive. The short run supply curve is given by

$$y : p = 3y^3 - 15y^2 + 2.$$

For your practice: An example with positive shut-down price:
Suppose the cost function was

$$C(y) = y^4 - 5y^3 + 2y^2 + 10$$

First, we need to find the supply curve for the operational firm.

$$p = 3y^3 - 15y^2 + 4y = C'(y)$$

Now, we need to find the shutdown point. Given all that we did in class, we first need to find y where AVC is minimized. For this, note

$$V(y) = y^4 - 5y^3 + 2y^2$$

so

$$AVC(y) = y^3 - 5y^2 + 2y.$$

The FOC to minimize this is

$$AVC'(y) = 3y^2 - 10y + 2 = 0$$

Solving this we get

$$y = \frac{10 \pm \sqrt{100 - 12}}{6} = \frac{10 \pm \sqrt{88}}{6}.$$

Which y is the minimizer? For that, we need to evaluate the second derivative of AVC at each of the y values and check which one is positive.

$$AVC''(y) = 6y - 10.$$

It's easy to see that

$$AVC''\left(\frac{10 + \sqrt{88}}{6}\right) = +\sqrt{88} > 0$$
$$AVC''\left(\frac{10 - \sqrt{88}}{6}\right) = -\sqrt{88} < 0$$

so we pick

$$\bar{y} = \frac{10 + \sqrt{88}}{6}.$$

The corresponding price level on the marginal cost curve is

$$C'(\bar{y}) = \bar{p}.$$

The short run supply curve is given by

$$y^*(p) = \begin{cases} 0 & \text{if } p < \bar{p} \\ y : p = 3y^3 - 15y^2 + 4y & \text{if } p \geq \bar{p} \end{cases}$$

D. Leontief production function

This question is meant to give you an understanding of what happens when we try to maximize profits with CRS production functions.

A farmer produces rice R using seeds S and fertilizer F. The production function is given by

$$R = \min\{S, F\}$$

Suppose seeds cost p_S and fertilizer costs p_F.

(a) Is the production function homogenous? Is it homothetic?

Answer: The function is homogenous:

$$\min\{S, F\} = \begin{cases} S & \text{if } S \leq F \\ F & \text{if } S \geq F \end{cases}$$

Proportionately changing inputs:

$$\min\{cS, cF\} = \begin{cases} cS & \text{if } cS \leq cF \\ cF & \text{if } cS \geq cF \end{cases}$$

which is the same thing as

$$\min\{cS, cF\} = \begin{cases} cS & \text{if } S \leq F \\ cF & \text{if } S \geq F \end{cases}$$

which is the same as

$$c \min\{S, F\}.$$

The degree of homogeneity is 1.

Its homothetic because its homogeneous. You can draw a graph and check that the slope of contour lines along any line passing through the origin is the same (except for the 45 degree line, where the slope is not defined)

(b) What is the optimal bundle of seeds and fertilizer needed to produce R units of delicious rice. What is the cost function in terms of R, p_S and p_F?

Answer:

The cost minimizing bundle to produce R units of rice is

$$S^* = F^* = R.$$

Therefore, the cost function is just

$$C(R, p_S, p_F) = (p_S + p_F)R$$

(c) Suppose this farmer gets a price of p for the rice she sells. Write the expression for farmer's profit, in terms of output price, output level and input prices. What is her marginal revenue and marginal cost?

Answer:

The expression for farmer's profit is

$$\pi(R) = pR - (p_S + p_F)R$$

where marginal revenue is p and marginal cost is $(p_S + p_F)$.

(d) Suppose marginal revenue is higher than marginal cost, and she is producing output level R_1. How much does her profit increase by if she increases production to $R_2 > R_1$? What does this imply for her optimal choice of output? [Hint: R_1 was an arbitrary output level.]

Answer:

We can see that

$$\pi'(R) = p - (p_S + p_F),$$

so if $p - (p_S + p_F) > 0$, then any increase in R will increase her profits. Her optimal choice would to be unbounded production i.e. $R^* \to \infty$.

(e) Following up on (d), what is her optimal choice of output if marginal revenue is lower than marginal cost? What if marginal revenue is equal to marginal cost?

Answer:

By the same logic, if $p - (p_S + p_F) < 0$, she would choose $R^* = 0$.

If $p - (p_S + p_F) = 0$, she would be indifferent between producing any level of output. The optimal choice is not unique, $R^* \in [0, \infty)$. Note that 0 is included in the set because $p - (p_S + p_F) = 0$ implies 0 profit.

E. Many markets, many plants

This question is meant to give you a better understanding of the optimality condition for profit maximization. It will also give you practice in doing unconstrained optimization with two variables. It is slightly different from what you've seen in class, but the translation of words into math is easy. You will have to set up a different profit maximization problem for each subpart. Remember profit is total revenue minus total cost. Once you have done the translation, its very easy optimization math.

(a) Consider a firm that has two plants (or factories), each with a different cost function. Factory A can produce output at cost

$$C_A(y_A) = y_A^2$$

and factory B can produce output at cost

$$C_B(y_B) = \exp(y_B)$$

This firm can sell the total output from both plants (in only one market) at the price p.

How should the firm optimally allocate production between the two plants? Interpret your answer in terms of marginal costs and marginal revenue.

Answer:

The profit maximization problem is

$$\max_{y_A, y_B} p(y_A + y_B) - y_A^2 - \exp(y_B)$$

This is an unconstrained maximization problem is two variables y_A and y_B. The FOCs (respectively) are:

$$\begin{aligned} p - 2y_A = 0 &\implies y_A^* = 0.5p \\ p - \exp(y_B) = 0 &\implies y_B^* = \ln p \end{aligned}$$

Intuitively, the firm would like to equalize marginal revenue and marginal cost for each of the plants. If marginal revenue is below marginal cost, and because the marginal cost is an increasing function (convex costs), the firm can increase profits by reducing production. (In this case, marginal revenue is fixed and reducing output reduces marginal cost.)

The FOCs also jointly imply that marginal cost in both plants is the same. If the marginal cost is higher in one plant, the firm can transfer some of the production from that plant to the other plant, thereby reducing overall costs. This is again due to the convexity of the cost function - an (increasing) convex function rises really fast so its better to stay at lower levels of output as much as possible. The firms feasibility of doing that is limited as it has only two plants, but within those two plants its not optimal to burden one plant too much. The firm would like to *smooth production* *across plants.*

(b) Consider a firm that can sell its output the US as well as Canada. The output price in US is p_U and in Canada is p_C. Suppose $p_U > p_C$. This firm only has one plant that produces at cost

$$C(y) = y^2.$$

How should this firm allocate production from its only plant between the two markets? Think intuitively if you are stuck - what would you do if you managed this firm but had never attended intermediate micro? [Math note: $(a+b)^2 \neq a^2 + b^2$]

Answer:

In this case, the profit maximization problem is

$$\max_{y_U, y_C} p_U y_U + p_C y_C - (y_U + y_C)^2$$

The FOCs are

$$\begin{aligned} p_U &= 2(y_U + y_C) \\ p_C &= 2(y_U + y_C) \end{aligned}$$

Now, obviously these two cannot simultaneously hold because the LHS of both equations are different but RHS are the same. In this case, the firm would optimally supply only to US because p_U is higher for each additional level of output.

The key thing to realize here is that marginal profit is marginal revenue (MR) minus marginal cost (MR). As marginal cost in increasing, the firm will want to increase production upto the point where MC equals MR. It will not go beyond that level because it will get negative profits. Lets call MR(y)-MC(y) marginal <u>net</u> profit at output level y. If there are multiple opportunities that give different marginal net profits, then the firm will chase the one with the highest marginal net profit. This will provide the <u>greatest increase</u> in total profits. Because of convex costs, as the firm mines this opportunity, the marginal <u>net</u> profit from this opportunity will keep falling. Once its low enough the firm will move to the next best opportunity.

In this problem, the marginal <u>net</u> profit in the US is higher that the marginal net profit in Canada <u>for all levels of output</u> (shown below). The reason why this has bite, unlike part (a), is because the firm effectively has one choice variable - total output. This is exactly what FOCs suggest (going back to linear algebra, you can see that they are linearly dependent equations). This is another case where its important to be careful about mechanically applying the FOC apparatus.

Suppose the firm is supplying to both markets and its producing a total of $y = y_U + y_C$. Then, marginal net profit in the US is

$$p_U - 2y$$

and in Canada is

$$p_C - 2y$$

and because $p_U > p_C$, we have, for all y,

$$p_U - 2y > p_C - 2y.$$

So if the firm is at all supplying to Canada, it can just transfer some of that supply to US and get higher total profits. Ideally, as in the two plants example above, the firm will stop doing so when $p_U - 2y = p_C - 2y$, but that never happens because of the structure of the problem.