Problem Set 1

Intermediate Microeconomics - Summer 2020, NYU
Instructor: Skand Goel

Rules:

- The problem set is due on Monday, June 1 before 4.05 PM Eastern time.
- The submission should be made as a PDF attachment (scanned or digital) via NYU classes.
- Each of the seven questions is for 3 points. The total score will be rescaled to 5.
- If you feel there is an error or typo, please email me at sg3992@nyu.edu.
- You can work in groups, but must submit your own work. If you work in groups, mention the name of students in your group.

A. Budget constraints

1 Intertemporal budgets

Dwight works for a paper company. He has a two-year contract with his firm. This year the firm will pay him m_1 dollars. Next year, he will get m_2 dollars.

Dwight loves beets - in fact, beets are all he consumes. He would like to consume beets now, as well as in the future. Denote his current consumption of beets by c_1 and his next period consumption by c_2. One tonne of beets cost p dollars, whether he buys them this year or the next.

Dwight also has access to a savings bank account, where he can save money at the end of the first year. Denote his savings by s. He gets an interest of $(100 \times r)\,\%$ i.e. for every dollar saved today, he gets $(1 + r)$ dollars next year.

(a) Given that Dwight saved s at the end of the first period, what is his budget line in the second period?

Answer:

The budget line is given by

$$pc_2 = (1+r)s + m_2$$

(b) With the understanding that Dwight might be borrowing from his future income, what is his budget line for the first period?

Answer:

The budget line is given by

$$pc_1 = b + m_1$$

Dwight is an upstanding citizen - he will not cheat the bank. Therefore, savings is just negative borrowing.

(c) Use this idea to combine the two budget lines into one budget line. [Hint: The combined "intertemporal" budget line should not have any terms containing b or s.]

Answer:

Rewriting the equation in (a)

$$s = \frac{pc_2 - m_2}{1+r}$$

Rewriting equation (b)

$$b = pc_1 - m_1$$

Using $s = -b$,

$$\frac{pc_2 - m_2}{1+r} = -pc_1 + m_1$$
$$\Longrightarrow pc_1 + \frac{pc_2}{1+r} = m_1 + \frac{m_2}{1+r}$$

(d) What is the price consumption today, relative to consumption tomorrow?

Answer:

$m_1 + \frac{m_2}{1+r}$ *is the net present value of total income over the two periods. With this understanding, price of c_1 is p, and price of c_2 is $\frac{p}{1+r}$. The relative price is given by* *the negative of the slope of the line, which is* $(1 + r)$*.*

(e) (Harder question) Suppose Dwight's contract lasted T periods, $T \geq 3$. Denote consumption in period t by c_t, and the income in period t as m_t. Write down the budget line. [Hint: Use the same idea as above - net present value of consumption must equal net present value of income.]

Answer:

$$\Sigma_{t=1}^{T} \frac{pc_t}{(1+r)^{t-1}} = \Sigma_{t=1}^{T} \frac{m_t}{(1+r)^{t-1}}$$

2 Labor budgets

Angela is Dwight's co-worker. She likes to go to work - it gives her money for her everyday needs - but she also likes to spent time at home with her cats. Lets denote "everyday needs" by c (for consumption) and days spent at home by l (for leisure). Unlike Dwight, Angela gets a wage of w every hour she shows up to work.

(a) Consider Angela's budget for a 12-hour workday. What is her total income if she spends l hours of this time at home?

Answer:

Her income is $(12 - l)\, w$*.*

(b) Suppose price of c is \$1. Write down her budget constraint. What is the price of leisure?

Answer:

Her budget constraint is

$$c \leq (12 - l)w$$

which can be written as

$$c + wl \leq 12w$$

(c) How does the budget line move if the government impose a $t\%$ income tax on every dollar earned? Plot the budget set in the (l, c)-plane.

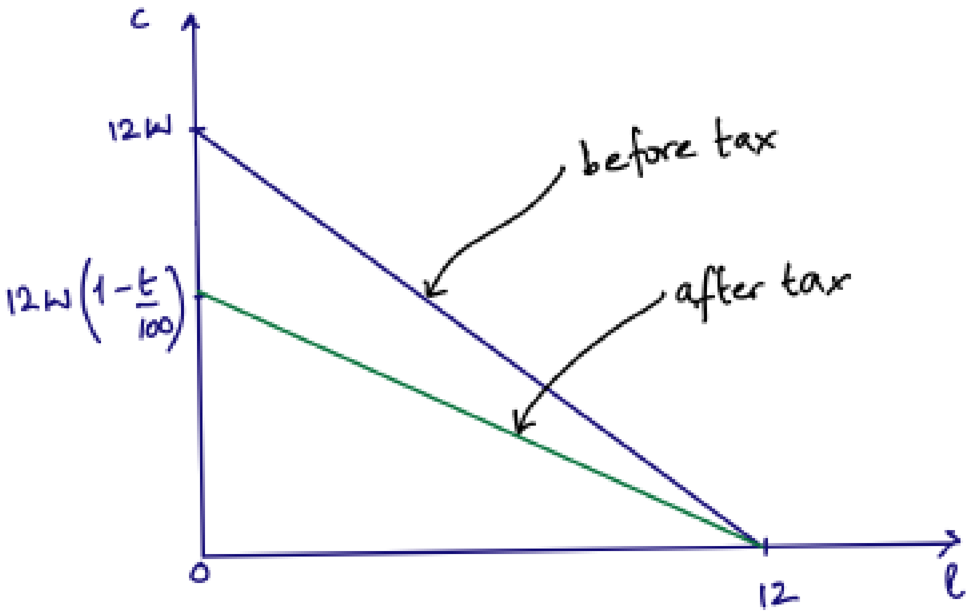
Answer:

Budget line before tax

$$c + wl = 12w$$

Budget line after tax

$$c + w\left(1 - \frac{t}{100}\right)l = 12w\left(1 - \frac{t}{100}\right)$$



(d) Lets go back to the case without taxes. Suppose she is allowed to work 3 hours over time (over and above the 12 hour work day). For each overtime hour worked, she get $2w$. Write down her budget constraint with the overtime option. Plot the budget set in the (l, c)-plane.

Answer:

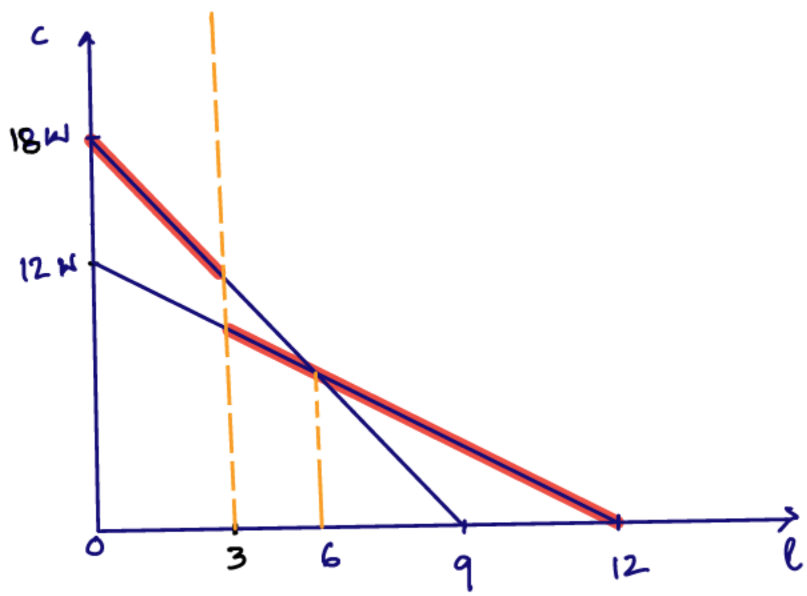
She gets overtime pay only if she spends less than 3 hours at home, out of her 15-hour workday. When she get overtime, she gets paid w for 12 hours worked, and $2w$ *for* $3 - l$ *overtime hours worked.*

$$c \leq (12 - l)w \text{ } if \text{ } l \geq 3$$
$$c \leq 12w + [3 - l]\, 2w \text{ } if \text{ } l < 3$$

which can be rearranged to give

$$c + wl \leq 12w \;\; \textit{if} \; l \geq 3$$
$$c + 2wl \leq 18w \;\; \textit{if} \; l < 3$$

This is plotted as the red line in the figure below.



B. Preferences

3 Preference sets

Show that for any $x \in \mathbb{X}$,

$$\mathbb{I}_x \cap \mathbb{SP}_x = \phi,$$

where ϕ refers to the empty set. In other words, show that there is no bundle common between the indifference set and strictly preferred set (c.f. lecture slides).

Answer:

We'll prove using contradiction. Suppose it is true that there exists some bundle y that lies in the intersection of the indifference set and strictly preferred set. In notation, *suppose*

$$\exists y \in \mathbb{I}_x \cap \mathbb{SP}_x.$$

This implies that

$$y \in \mathbb{I}_x \textit{ and } y \in \mathbb{SP}_x.$$

$$y \in \mathbb{I}_x \implies y \succsim x \textit{ and } x \succsim y$$
$$y \in \mathbb{SP}_x \implies y \succsim x \textit{ and } \textbf{\textit{not}} \ x \succsim y$$

*$x \succsim y$ and **not** $x \succsim y$ cannot be true together. This is a contradiction, hence proved.*

4 Utility representation

You want to model Jim's consumption behavior. You ask him to give you his preference ordering. He only gives you a list of his indifference sets. This list has at least two indifferent sets - there are at least some bundles that Jim prefers more than others. Can you use this information to derive his utility function? [Hint: Think about the properties a bilateral relation must satisfy in order to admit a functional representation.]

Answer:

You cannot derive a utility function. The indifference relation is not complete. Consider two bundles, each from a different indifference set. Jim doesn't tell you how these are ranked relative to each other.

5 Convex preferences

(a) Show that the quasi-linear utility function given by

$$u(x, y) = \log x + y$$

is a concave function. [Hint: Use the convex combination definition from the slides. You can use calculus to show that log is a concave function.]

Answer:

Take two arbitrary points in the commodity space $a = (x_1, y_1)$ *and* $b = (x_2, y_2)$. *Fix an arbitrary* $\alpha \in (0,1)$[1].

The convex combination of a *and* b *is given by*

$$\begin{aligned} c &= \alpha a + (1-\alpha)b \\ &= (\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \end{aligned}$$

The image of the convex combination is given by

$$u(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) = \log\left(\alpha x_1 + (1-\alpha)x_2\right) + \left(\alpha y_1 + (1-\alpha)y_2\right). \tag{1}$$

The convex combination of the *images of* a*and* b *is given by*

$$\begin{aligned} &\alpha\left[\log x_1 + y_1\right] + (1-\alpha)\left[\log x_2 + y_2\right] \\ =&\alpha \log x_1 + (1-\alpha)\log x_2 + (\alpha y_1 + (1-\alpha)y_2) \end{aligned} \tag{2}$$

u *is concave iff the convex combination of the images (given by (2)) is greater than or equal to the image of convex combination (given by (1)).*

Option 1: Proof by contradiction.

Suppose u *is not concave. Then*

$$\log\left(\alpha x_1 + (1-\alpha)x_2\right) + \left(\alpha y_1 + (1-\alpha)y_2\right) < \alpha \log x_1 + (1-\alpha)\log x_2 + \left(\alpha y_1 + (1-\alpha)y_2\right)$$

which implies

$$\log\left(\alpha x_1 + (1-\alpha)x_2\right) < \alpha \log x_1 + (1-\alpha)\log x_2 \tag{3}$$

because the $y-$*terms cancel out.*

However, (3) cannot be true, as log is a concave function (shown below). Therefore, we have a contradiction. Hence, proved.

Option 2: Constructive proof.

[1] *Notation:* $\alpha \in (0,1)$ *is the same as* $0 \leq \alpha \leq 1$

$$
\begin{aligned}
u(c) &= \log\left(\alpha x_1 + (1-\alpha)x_2\right) + (\alpha y_1 + (1-\alpha)y_2) \\
&> \alpha \log x_1 + (1-\alpha)\log x_2 + (\alpha y_1 + (1-\alpha)y_2) \\
&= \alpha u(a) + (1-\alpha)u(b)
\end{aligned}
$$

where the inequality is true because log is a concave function. This shows that u is a concave function.

Concavity of log.

It still remains to be shown that log is a concave function. As log is differentiable (at least twice) we can just check the sign on the second derivative.

$$
\begin{aligned}
\frac{d}{dx}\log x &= \frac{1}{x} \\
\frac{d^2}{dx^2}\log x &= -\frac{1}{x^2} < 0
\end{aligned}
$$

which is what is needed for a concave function.

(b) i. Show that $u : \mathbb{R}_+ \to \mathbb{R}$

$$u(x) = x^2$$

is a convex function but it is quasi-concave.

Answer:

$$
\begin{aligned}
u'(x) &= 2x \\
u''(x) &= 2 > 0
\end{aligned}
$$

which implies u is a convex function.

To show u is quasi-concave, we need to show that every upper level set is a convex set. Let's pick an arbitrary $c \in \mathbb{R}$, the co-domain (or range) of u. The upper level set relative to c is given by

$$
\begin{aligned}
\mathbb{WP}_c &= \{x \in \mathbb{R}_+ | u(x) \geq c\} \\
&= \left\{x \in \mathbb{R}_+ | x^2 \geq c\right\} \\
&= \left\{x \in \mathbb{R}_+ | x \geq \sqrt{c}\right\}
\end{aligned}
$$

where the third inequality is true because $x \geq 0$. Otherwise, we would have had not only all $x \geq \sqrt{c}$ but also all $x \leq -\sqrt{c}$.

We need to show that $\mathbb{WP}_c$ is a convex set. Take any two points in this set $x_1, x_2 \in \mathbb{WP}_c$ and fix any $\alpha \in (0,1)$.

We need to show that

$$\alpha x_1 + (1-\alpha)x_2 \in \mathbb{WP}_c$$

Using the information given to us

$$\begin{aligned} x_1 \in \mathbb{WP}_c &\implies x_1 \geq c \implies \alpha x_1 \geq \alpha c \\ x_2 \in \mathbb{WP}_c &\implies x_2 \geq c \implies (1-\alpha)x_2 \geq (1-\alpha)c \end{aligned}$$

Adding the two final inequalities, we get

$$\alpha x_1 + (1-\alpha)x_2 \geq \alpha c + (1-\alpha)c = c$$

which implies

$$\alpha x_1 + (1-\alpha)x_2 \in \mathbb{WP}_c.$$

This is true for every x_1, x_2, α and c. Hence, proved.

ii. Is this true if we include negative numbers in the domain of u i.e. $u : \mathbb{R} \to \mathbb{R}$ and

$$u(x) = x^2.$$

You can use a diagram to argue your case.

Answer:

If the domain is the entire real line, then u is not quasi-concave. As argued above, in this case the upper level set is

$$\begin{aligned} \mathbb{WP}_c &= \{x \in \mathbb{R} | u(x) \geq c\} \\ &= \left\{x \in \mathbb{R}_+ | x \geq \sqrt{c} \text{ or } x \leq -\sqrt{c}\right\} \end{aligned}$$

We just need to show that there is one upper level set which is not convex. To show that, we just need to find two points for which just one convex combination does not *belong to the upper level set. Let* $c = 100$.

$$11 \in \mathbb{WP}_{100} \text{ and -11} \in \mathbb{WP}_{100}$$

Their convex combination with $\alpha = 0.5$ *is* 0. *Clearly* $0 \notin \mathbb{WP}_{100}$. *Hence, proved.*

C. Choice

6 Intertemporal choices

Consider the two period problem in question 1. Suppose Dwight's utiltiy function is given by

$$u(c_1, c_2) = \log c_1 + \beta \log c_2$$

where β is known as the discount factor.

(a) Find Dwight's optimal choice of beet consumption today vs tomorrow. (Second order conditions are satisfied)

Answer:

The utility function is monotonic, so we can work with an equality constraint.

$$\mathcal{L}(c_1, c_2, \lambda) = \log c_1 + \beta \log c_2 + \lambda \left[m_1 + \frac{m_2}{1+r} - pc_1 - \frac{pc_2}{1+r} \right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda p = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{\beta}{c_2} - \lambda \frac{p}{1+r} = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m_1 + \frac{m_2}{1+r} - pc_1 - \frac{pc_2}{1+r} = 0 \tag{6}$$

Combining (4) and (6), we get

$$\frac{c_2}{c_1} = \beta (1+r)$$

This intertemporal first order condition is also called the Euler equation.

Plugging c_2 from this equation into the budget line, we get

$$m_1 + \frac{m_2}{1+r} - pc_1 - p\beta c_1 = 0 \implies c_1^* = \frac{1}{p}\left(\frac{1}{1+\beta}\right)\left(m_1 + \frac{m_2}{1+r}\right)$$
$$c_2^* = \frac{(1+r)}{p}\left(\frac{\beta}{1+\beta}\right)\left(m_1 + \frac{m_2}{1+r}\right)$$

(b) How does his choice of c_1 change with today's income? With tomorrow's income?

Answer:

$$\frac{\partial c_1^*}{\partial m_1} = \frac{1}{p}\left(\frac{1}{1+\beta}\right)$$
$$\frac{\partial c_1^*}{\partial m_2} = \frac{1}{p\,(1+r)}\left(\frac{1}{1+\beta}\right)$$
$$\frac{\partial c_1^*}{\partial m_1} > \frac{\partial c_1^*}{\partial m_2}$$

Both income shocks increase consumption today. Today's income shock affects today's consumption more than tomorrow's income shock does.

(c) Is Dwight borrowing or saving?

Answer:

Saving is the excess of today's income over today's expenditure. This is equal to

$$m_1 - pc_1^* = m_1 - \left(\frac{1}{1+\beta}\right)\left(m_1 + \frac{m_2}{1+r}\right)$$
$$= \left(\frac{\beta}{1+\beta}\right)m_1 - \frac{1}{(1+r)\,(1+\beta)}m_2$$

Negative saving implies Dwight is borrowing. Whether this is true depends on the value of the parameters of the model. Borrowing is positive

$$
\begin{aligned}
&\left(\frac{\beta}{1+\beta}\right) m_1 - \frac{1}{(1+r)(1+\beta)} m_2 \geq 0 \\
\Longrightarrow\ & \beta m_1 \geq \frac{1}{(1+r)} m_2 \\
\Longrightarrow\ & \frac{m_1}{m_2} \geq \frac{1}{\beta(1+r)}.
\end{aligned}
$$

(d) If Dwight was not allowed to save or borrow, what would his optimal choice be?

Answer:

If Dwight couldn't save or borrow, then consumption opportunities would be limited to those offered by contemporaneous income. He would optimally spend all of m_1 *in period 1 and all of* m_2 *in period 2. This is because preferences are monotonic for both* c_1 *and* c_2*. Therefore,*

$$
\begin{aligned}
c_1^* &= \frac{m_1}{p} \\
c_2^* &= \frac{m_2}{p}
\end{aligned}
$$

Suppose Dwight's contract lasted 3 periods and his utility was given by

$$u(c_1, c_2, c_3) = \log c_1 + \beta \log c_2 + \beta^2 \log c_3$$

(e) Use your answer from 1(e). Write down the lagrangean and the associated first order conditions.

Answer:

$$\mathcal{L}(c_1, c_2, \lambda) = \log c_1 + \beta \log c_2 + \beta^2 \log c_3 + \lambda \left[m_1 + \frac{m_2}{1+r} + \frac{m_3}{(1+r)^2} - pc_1 - \frac{pc_2}{1+r} - \frac{pc_3}{(1+r)^2} \right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda p = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{\beta}{c_2} - \lambda \frac{p}{1+r} = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial c_3} = \frac{\beta^2}{c_3} - \lambda \frac{p}{(1+r)^2} = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m_1 + \frac{m_2}{1+r} + \frac{m_3}{(1+r)^2} - pc_1 - \frac{pc_2}{1+r} - \frac{pc_3}{(1+r)^2} = 0 \tag{10}$$

(f) Interpret these conditions using marginal arguments.

Answer:

We can combine first order conditions for two consecutive periods. We have two such combinations

$$\frac{c_2}{c_1} = \beta(1+r)$$
$$\frac{c_3}{c_2} = \beta(1+r)$$

β *tells us how much Dwight values the present relative to the future.* $(1+r)$ *dictates how much sacrificing one dollar today gives tomorrow. The left hand side of each of these conditions is the ratio of marginal utilities between two periods. The first order condition says that the marginal dollar spent on consumption today should give the same marginal utility as that dollar spent on consumption tomorrow.*

7 Labor supply

We'll consider the case from question 2 *without taxes or overtime*. Suppose Angela's utility function is given by

$$u(c, l) = \log c + l$$

Use the budget constraint from question 2.

(a) Find her optimal choices. (Second order conditions are satisfied).

Answer:

Utility is increasing in c *as well as* l*, so we can work with the equality constraint.*

Writing the Lagrangean

$$\mathcal{L}(c, l, \lambda) = \log c + l + \lambda [12w - c - wl]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial l} = 1 - \lambda w = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 12w - c - wl = 0 \tag{13}$$

From equations (11) and (12), we have

$$c^* = w$$

Plugging this into the budget constraint, we get

$$l^* = 11$$

(b) How does Angela's choice of l change as her wage w changes? What about her choice of c?

Answer:

Angela always chooses $l^* = 11$, *regardless of her wage.* c^* *increases one-to-one with wage.*

Suppose Angela had a fixed income of m dollars per day (not depending on hours worked). Also, but spending time with cats cost her money - p dollars per hour.

(c) What would her budget constraint look like? [Hint: This is your standard budget constraint]

Answer:

The budget constraint is given by

$$c + pl \leq m$$

(d) What would be her optimal choices now?

Answer:

$$\mathcal{L}(c, l, \lambda) = \log c + l + \lambda \left[m - c - pl\right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial l} = 1 - \lambda p = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - c - pl = 0 \tag{16}$$

Solving these equations gives us

$$c^* = p$$

and

$$l^* = \frac{m}{p} - 1.$$

We need to be careful about a constraint that we were not explicit about, which is that $l \geq 0$. *(This is not the budget constraint).* l *cannot be negative. So if* $\frac{m}{p} < 1$, *the optimal choice of* l *will not satisfy the first order condition. Of course, it would be ideal to* consume negative l if there was no constraint on l, but we can't do that. Therefore, she'll consume $l^* = 0$.

(e) How does Angela's choice of l change as her wage m changes? What about her choice of c?

Answer:

Now, we get the opposite result. Increase in income increases l^* *but does not change consumption.*