Note 3a The Electric Potential

As you can see from the results of the electric fields due to the various source charge configurations, the electrostatic force is almost never constant. In order for us to calculate what happens to target charges from these sources, we have to integrate the force function over displacements or elapsed times.

Integration of the force over same displacement is the **work** and integration of the force over the elapsed time is the **impulse**.

Now, it turns out that the electrostatic force is a conservative force. This is not surprising since Coulomb's law has the same form as the law of universal gravitation. Therefore, it is much simple to deal with it as a potential energy, the electrostatic potential energy. The two primary characteristics of a potential energy are that the work done by (or the change the potential energy of) the force is path independent and that the change in the potential energy over a close path is zero.

The Electric Work

The work done by the electrostatic force is this

$$W_{by\ electrostatic\ force} \equiv W_e = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_e \cdot d\vec{r} = q_{target} \int_{\vec{r}_i}^{\vec{r}_f} \vec{E}_{source} \cdot d\vec{r}$$

The work done by the electrostatic force can be expressed as a change in the potential energy of a charge.

$$W_e = -\Delta U_e$$

In terms of the electric field, the change in the potential energy is this.

$$\boxed{\Delta U_e = -q_{target} \int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r}}$$

Just like a mass moving against the gravitational force increases the gravitational potential energy, a charge moving against the electrostatic force increases the electrostatic potential energy. But now, there are two types of source charges that two types of target charges can move against making the situation a bit more complex.

Finally, if we remove the target charge from consideration, we have the **electric potential difference**.

$$-\int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r} = -\frac{1}{q_{target}} \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_e \cdot d\vec{r} = \frac{1}{q_{target}} \Delta U_e = \Delta V$$

$$\boxed{\Delta V = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r}}$$

Unit

The unit for the electric potential is the volt, V. Thus, the unit of energy is the joule which you can see is also coulombs times volt. Or the volt is joules per coulombs.

$$[U] = J = [q][V] = CV \quad \Rightarrow \quad J = CV \quad \Rightarrow \quad V = \frac{J}{C}$$

Superposition

The principle of superposition applies to the electric potential energy and the electric potential as well since they are just integrals of the electric field and electric force and integration is equivalent to addition. The total electric potential due to multiple source charges is just the sum of the electric potential due to each source charge.

$$V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) + V_3(\vec{r}) + \ldots$$

The total energy of a system of charges is just the sum of the potential energies between all pairs of charges.

$$U(\vec{r}) = U_1(\vec{r}) + U_2(\vec{r}) + U_3(\vec{r}) + \ldots$$

Relationship between the Electric Field Vectors and Electric Potential Lines

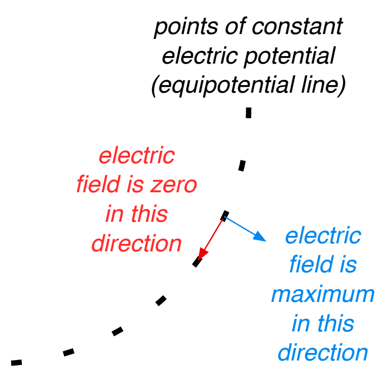
Here is the electric potential.

$$\Delta V = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r}$$

The reverse is this.

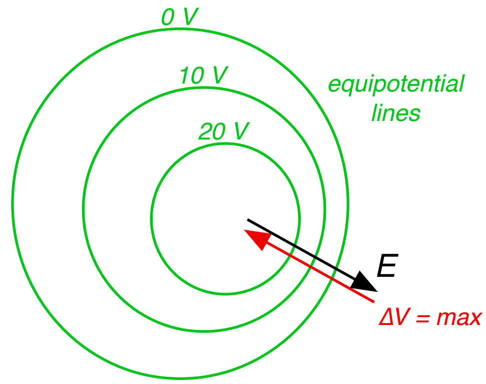
$$\vec{E} = -\vec{\nabla} V = -\left[\left(\frac{\partial V}{\partial x}\right) \cdot \hat{i} + \left(\frac{\partial V}{\partial y}\right) \cdot \hat{j} + \left(\frac{\partial V}{\partial z}\right) \cdot \hat{k}\right]$$

The operator here is the gradient operator. It points in the direction of greatest change (increase). With the negative sign, it points in the direction of greatest decrease. Thus, a positive charge (at rest) slides in this direction like a mass (at rest) slides down hill in the direction of steepest slope. In the direction of no electric potential change, the electric field is zero.

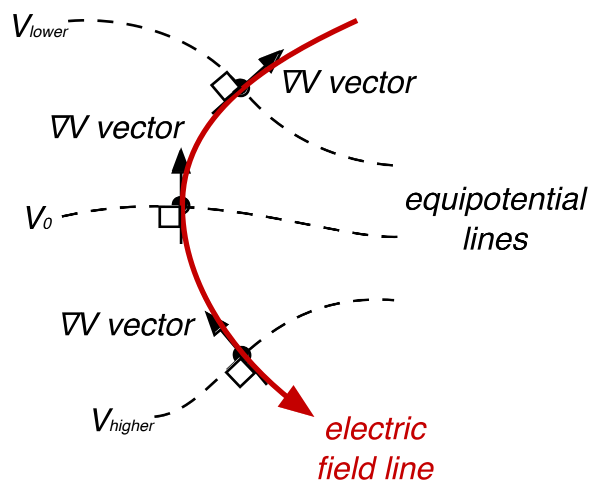


This means the electric field is perpendicular to the equipotential line.

Here is a fuller picture of this relationship.



Connecting electric fields along a line forms what are called **electric field lines** (red). Since there is a negative sign, the electric potential increases in the direction shown.



The Electric Potentials of a Point Charge

For a point charge, the electric field due to a source charge at the origin is

$$\vec{E} = kq\frac{\hat{r}}{r^2}$$

The electric potential difference due to a point charge when it moves from point r_i to point r_f is this.

$$\Delta V = -\int_{r_i}^{r_f} kq\frac{\hat{r}}{r^2}\cdot d\vec{r} = -\int_{r_i}^{r_f} kq\frac{\hat{r}}{r^2}\cdot dr\cdot\hat{r} = -kq\int_{r_i}^{r_f}\frac{dr}{r^2}\cdot\hat{r}\cdot\hat{r} = -kq\int_{r_i}^{r_f}\frac{dr}{r^2}$$

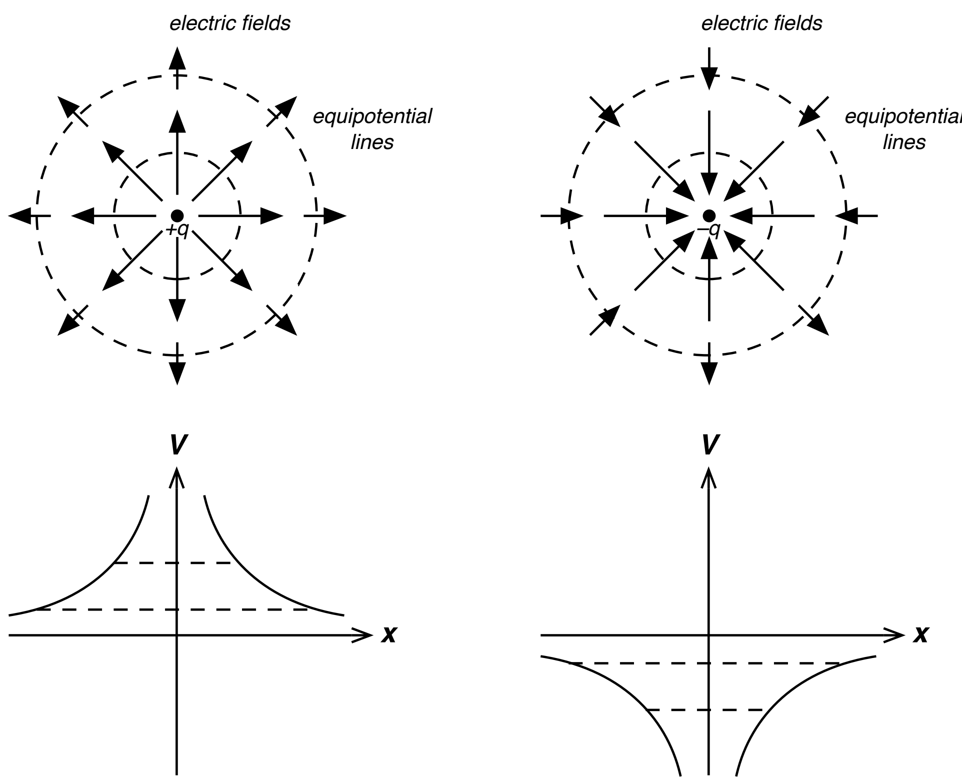
$$\Delta V = -kq\int_{r_i}^{r_f} r^{-2}\,dr = -\,kq\left[-r^{-1}\right]_{r_i}^{r_f} = kq\left[r^{-1}\right]_{r_i}^{r_f} = kq\left[\frac{1}{r_f}-\frac{1}{r_i}\right]$$

$$\boxed{\Delta V = kq\left[\frac{1}{r_f}-\frac{1}{r_i}\right]}$$

Where is the electric potential zero? For this function, the zero value happens at r = infinity. Thus, if we start with a charge at r = infinity, then the initial potential is zero. This is the standard way that the potential is written. It **assumes** that the position starts at infinity.

$$\boxed{V(r) = kq\frac{1}{r}}$$

The graph of this function is simpler than the electric field. It looks like this for a positive source charge and a negative source charge at the origin.



A positive charge released from rest will "slide down" this function.

Here is the electric field for a point charge calculated from the electric potential.

$$\vec{E}(x,y,z) = -\vec{\nabla}V(x,y,z) = -\vec{\nabla}\left[kq\frac{1}{r}\right] = -kq\left[\vec{\nabla}\frac{1}{r}\right]$$

Let's do this in cartesian coordinates.

$$\vec{E}(x,y,z) = -kq\left[\hat{i}\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} + \hat{j}\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}} + \hat{k}\frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\vec{E}(x,y,z) = -kq\left[\hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2} + \hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-1/2} + \hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-1/2}\right]$$

$$\vec{E}(x,y,z) = -kq\left[\hat{i}\left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2x) + \hat{j}\left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2y) + \hat{k}\left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2z)\right]$$

$$\vec{E}(x,y,z) = kq\left[\hat{i}\frac{x}{\sqrt{x^2+y^2+z^2}^{\,3}} + \hat{j}\frac{y}{\sqrt{x^2+y^2+z^2}^{\,3}} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}^{\,3}}\right]$$

$$\vec{E}(x,y,z) = kq\left[\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}^{\,3}}\right] = kq\frac{\vec{r}}{r^3} = kq\frac{1}{r^2}\hat{r}$$

In polar coordinates (the natural coordinates here) and in 2 dimension,

$$\vec{E}(r,\theta) = -kq\left[\hat{r}\cdot\frac{\partial}{\partial r}\frac{1}{r} + \hat{\theta}\cdot r\frac{\partial}{\partial\theta}\frac{1}{r}\right]$$

$$\vec{E}(r,\theta) = -kq\left[\hat{r}\cdot-\frac{1}{r^2}\right] = kq\frac{1}{r^2}\hat{r}$$