ECE 603
Probability and Random Processes

Lessons 7-9

Chapter 4

Continuous and Mixed Random Variables

Objectives

- Explore tools used and methods used to work with continuous random variables.
- Examine mixed random variables that are mixtures of discrete and continuous random variables.

Rationale

- Consider that a continuous random variable *X* has a range in the form of an interval or a union of non-overlapping interview on the real line. This leads to the need for new tools to help you focus on continuous random variables.
- The theory of continuous random variables is analogous to the theory of discrete random variables. So, you may take any formula about discrete random variables and replace *sums* with *integrals* to come up with the corresponding formula for continuous random variables.
- Chapter 4 focuses on these relationships.

Prior Learning

- Basic Concepts
- Counting Methods
- Random Variables
- Access to the online textbook: https://www.probabilitycourse.com/

Continuous Random Variables

So far, we have discussed discrete random variables:

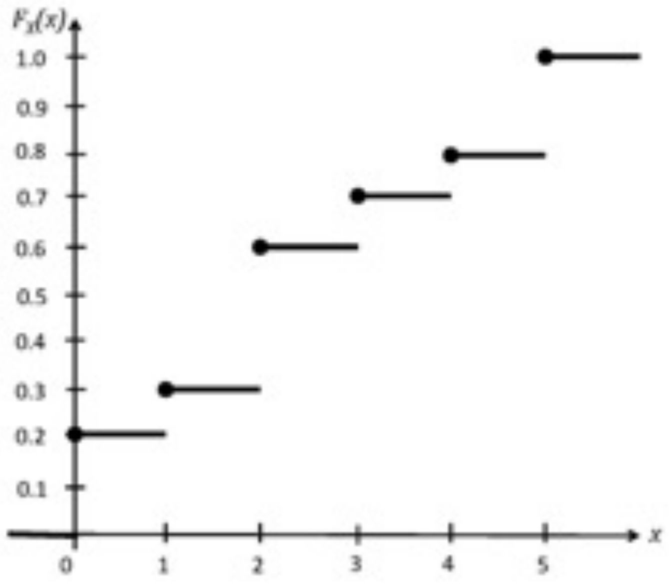
For a discrete random variable X, $R_X = \{x_1, x_2, \ldots\}$, is a countable set, e.g.,

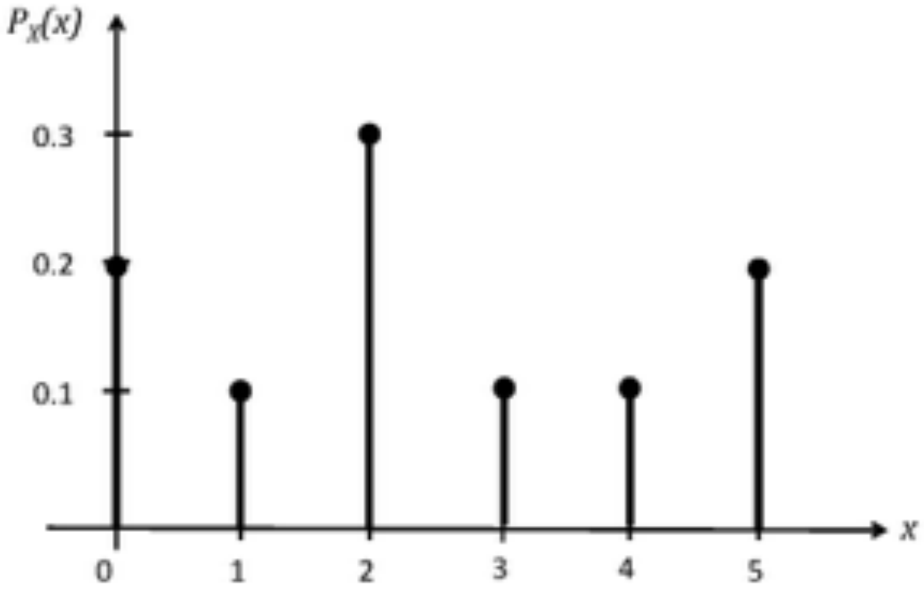
$\text{Range}(X) = \{1, 2, 3, 4, 5, 6\}$ or $\text{Range}(X) = \{1, 2, 3, \cdots\}$, etc.

the CDF $F_X(x) = P(X \leq x)$, looks like a series of steps with jumps at $x_1, x_2, x_3 \cdots$.

Discrete Random Variables: PMF & CDF

The jump at $x = x_k$ is given by the **PMF** $P_X(x_k)$





Cumulative Distributive Function (CDF)

Recall the general properties of a CDF:

1) $F_X(-\infty) = 0, \quad F_X(+\infty) = 1$

2) $y \geq x \;\Rightarrow\; F_X(y) \geq F_X(x)$

Continuous Random Variables

If $\mathrm{Range}(X)$ is not countable then X is not a discrete random variable.

Example. $\mathrm{Range}(X) = [a, b]$,

Remember: $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}, (a, b] = \{x \in \mathbb{R}, a < x \leq b\}$, **etc.**

Example.

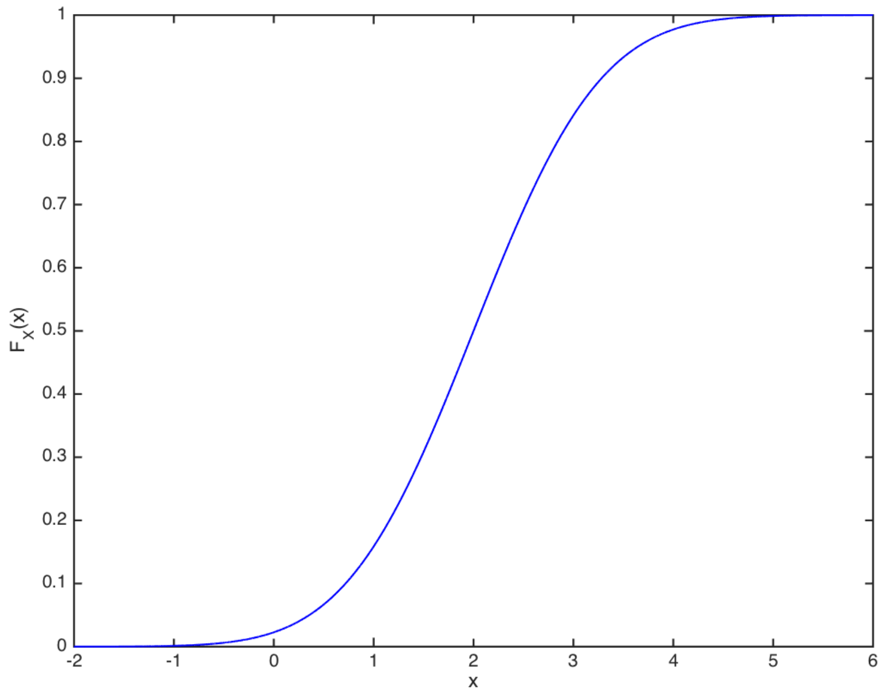
- T: **Lifetime of a light bulb,** $\mathrm{Range}(T) = [0, \infty)$.
- V: **Voltage across a resistor,** $\mathrm{Range}(V) = [0, V_{max}]$.

Continuous Random Variables

Now suppose we have a continuous function having those properties – for example:

- **A function like this is also a valid CDF $F_X(x)$ – that is, it can also represent $P(X \leq x)$ for some random variable X.**

$$F_X(x) = Prob\{X \leq x\}.$$



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Continuous Random Variables

Definition. A random variable X having a CDF $F_X(x)$ that is a continuous function for all x in $\mathbb{R}$ is said to be a **continuous random variable**.

Example. Let $[a, b]$ be an interval in the real line (where a and b are real numbers with $a < b$). Let X be a number chosen at random from that interval.

"Chosen at random" means: If $a \leq x_1 \leq x_2 \leq b$, then

$$P(X \in [x_1, x_2]) = \frac{x_2 - x_1}{b - a},$$

Continuous Random Variables

Let's find the CDF $F_X(x) = P(X \leq x)$:

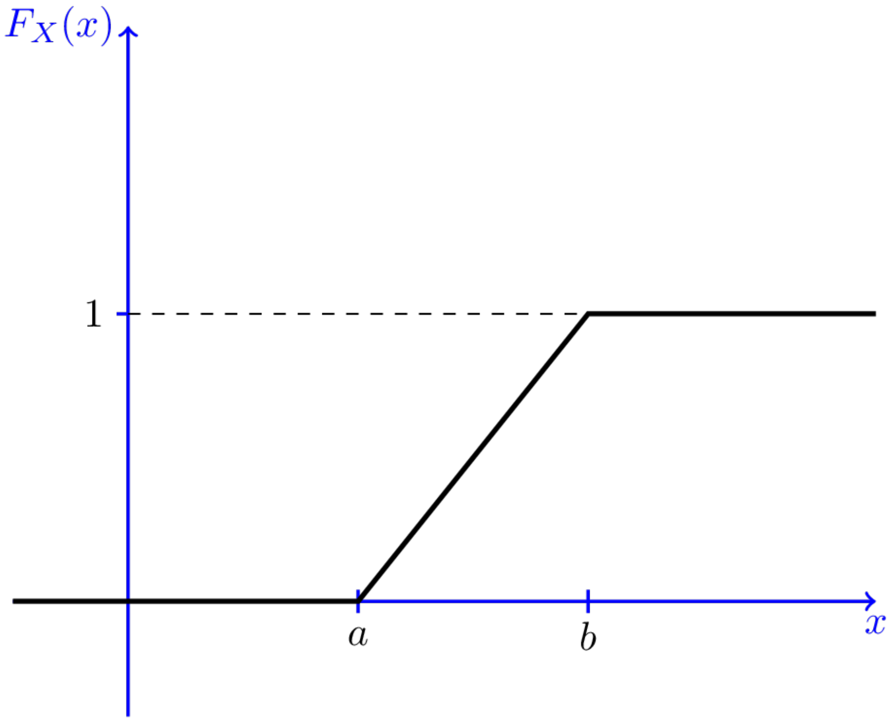
a) $F_X(x) = 0, \qquad \text{for } x < a,$

b) $F_X(x) = \dfrac{x-a}{b-a}, \qquad \text{for } a \leq x \leq b,$

c) $F_X(x) = 1, \qquad \text{for } x \geq b.$

Continuous Random Variables

In this case, X is called a $Uniform(a, b)$ random variable.



Continuous Random Variables

Example. Consider a line of length 1, i.e., $[0, 1]$. Place a dot at random on the line.

X = Location of the dot, $\mathrm{Range}(X) = [0, 1]$.

a) What is $\mathrm{Prob}\{0 \leq X \leq 0.5\}$? 0.5

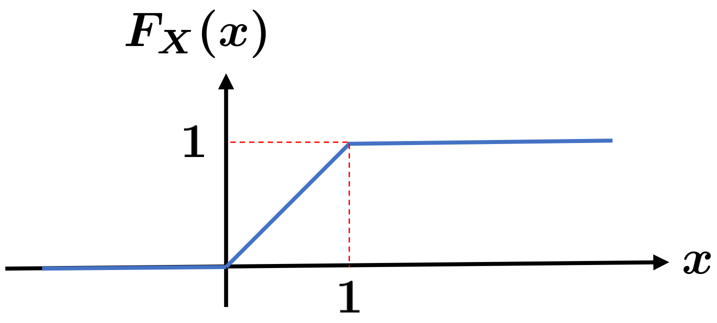
b) What is $\mathrm{Prob}\{0.2 \leq X \leq 0.8\}$? 0.6

c) For $0 \leq a \leq b \leq 1$, what is $\mathrm{Prob}\{a \leq X \leq b\}$? $a - b$

d) Obtain the CDF of X.

Continuous Random Variables

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 0 \end{cases}$$



e) What is $\mathrm{Prob}\{X = 0.5\}$? why?

$$P(X = 0.5) = 0,$$

$$P(X = 0.5) \le P(0.5 - \epsilon \le X \le 0.5 + \epsilon) = 2\epsilon,$$

$$\text{For all } \epsilon > 0, \Rightarrow \ P(X = 0.5) = 0.$$

Continuous Random Variables

- For a discrete random variable: the rate of increase in the CDF $F_X(x)$ is characterized by the PMF $P_X(x)$ - that is, by the locations and sizes of the jumps in the CDF.

- What characterizes the rate of increase for a continuous function?

The derivative of the function

Continuous Random Variables

- If the CDF $F_X(x)$ is a continuous function, then X is said to be a continuous random variable.
- PMF is Not well-defined for continuous random variables. Instead, we define probability density function (pdf).

Probability Density Function (PDF)

Definition. **pdf:**

$$f_X(x_1) = \lim_{\Delta \to 0} \frac{P(x_1 < X \leq x_1 + \Delta)}{\Delta}.$$

Remember that,

$$P(x_1 < X \leq x_1 + \Delta) = F_X(x_1 + \Delta) - F_X(x_1).$$

$$\Rightarrow \; f_X(x_1) = \lim_{\Delta \to 0} \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} = \frac{dF_X(x_1)}{dx_1} = F'_X(x_1).$$

Probability Density Function (PDF)

Definition. Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. Then we have

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

is called the probability density function (PDF) of X.

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)d\alpha.$$

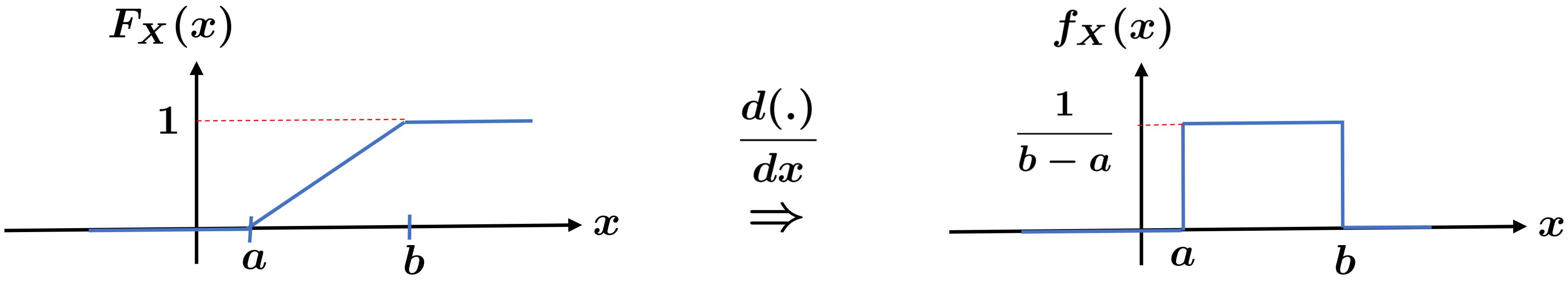
Probability Density Function (PDF)

Theorem.

$$
\begin{aligned}
P(a < X < b) &= F_X(b) - F_X(a) \\
&= \int_{-\infty}^{b} f_X(\alpha)d\alpha - \int_{-\infty}^{a} f_X(\alpha)d\alpha \\
&= \int_{a}^{b} f_X(\alpha)d\alpha.
\end{aligned}
$$

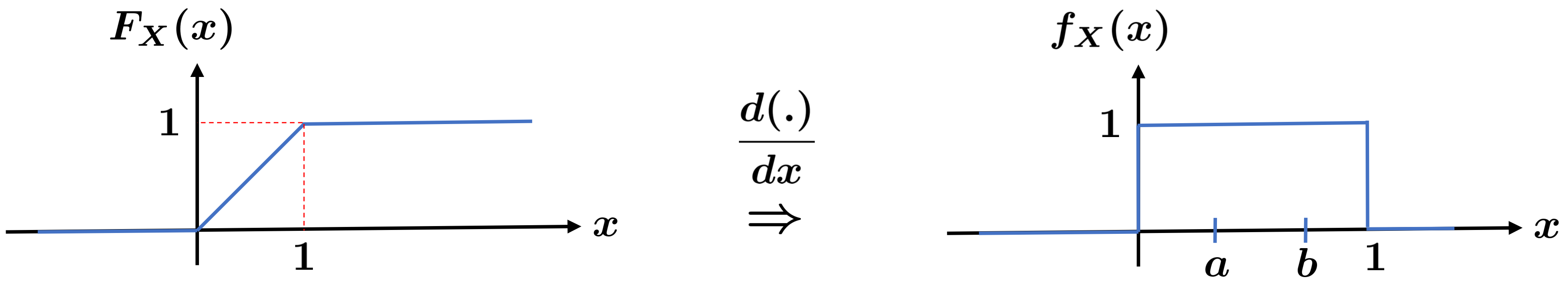
Probability Density Function (PDF)

Example. Say that $X \sim Uniform(a, b)$.



Probability Density Function (PDF)

Back to our dot and line example:



$$P(a \leq X \leq b) = \int_a^b f_X(x)dx = \int_a^b 1\ dx = b - a, \quad 0 \leq a \leq b \leq 1$$

Probability Density Function (PDF)

Comparison with discrete random variable:

$$\text{Prob}\{a \le X < b\} = \sum_{a \le x_k < b} P_X(x_k) \longrightarrow \int_a^b f_X(x)dx$$

discrete

continuous

If we integrate over the entire real line, we must get 1, i.e.,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Probability Density Function (PDF)

In math: **A density function for a quantity is a non-negative function that is integrated to give that quantity.**

Let's look at the properties of a PDF:

1) **Since $F_X(x)$ is monotone non-decreasing, its derivative must satisfy**

$$f_X(x) \geq 0 \text{ for all } x \in \mathbb{R}.$$

2) $$\int_{-\infty}^{\infty} f_X(u)du = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1.$$

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Probability Density Function (PDF)

3) In general:

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du.$$

4) We can extend the property 3 to:

$$P(X \in A) = \int_A f_X(u)du, \quad \text{for any set } A.$$

Probability Density Function (PDF)

- So, for a discrete random variable having range $R_X = \{x_1, x_2, x_3, \ldots\}$, to find $P(X \in A)$, we sum the PMF over the points $x_k \in A$.

- For a continuous random variable, to find $P(X \in A)$, we integrate the PDF over the set A .

Probability Density Function (PDF)

Example. Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} Ae^{-x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) Find A .

b) Find $F_X(x)$.

c) Find $P(1 < X < 3)$.

Probability Density Function (PDF)

Important note: $f_X(x)$ is not equal to $P(X = x)$. In fact, for a continuous random variable X we have $P(X = x) = 0$ for every point x. Also, it is possible in general to have $f_X(x) > 1$ for some values of x.

Expected Value and Variance

$$\sum_{k=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}; \quad P_X(x_k) \longrightarrow f_X(x),$$

So, we get:

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Expected Value and Variance

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Expected Value and Variance

Remember that the variance of any random variable is defined as

$$\mathrm{Var}(X) = E\left[(X - \mu_X)^2\right] = EX^2 - (EX)^2.$$

So for a continuous random variable, we can write

$$\begin{aligned}\mathrm{Var}(X) = E\left[(X - \mu_X)^2\right] &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx \\ = EX^2 - (EX)^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X^2.\end{aligned}$$

Expected Value and Variance

Example. In a medical ultrasound system, the energy in a pulse scattered by diffuse reflectors is a random variable X having a pdf of the form:

$$f_X(x) = \begin{cases} Ae^{-x/10} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) Find the value of A.

b) Find $E[X]$.

Expected Value and Variance

c) Find $\mathrm{Var}(X)$.

d) Find $E(5X + 3)$.

e) Let $Y = e^{-X/10}$, find $E[Y]$.

Functions of Continuous Random Variables

Suppose that X is a continuous random variable having CDF $F_X(x)$ and PDF $f_X(x)$. Let $g: R \rightarrow R$ be some function, and let $Y = g(X)$.

Since X is a random variable, so is Y. We already know that we can find $E[Y]$ by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Functions of Continuous Random Variables

Question: **How do we find the CDF and PDF for Y ?**

Usually: **It is easier to first find the CDF for Y , and then take the derivative to find the PDF.**

Functions of Continuous Random Variables

Example. Suppose that $X \sim Uniform(0,1)$ and $Y = X^3$.

Find the CDF and PDF of Y.

Note that:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Functions of Continuous Random Variables

Example. Suppose that $X \sim Uniform(-1,3)$ and $Y = X^2$.

Find the CDF and PDF of Y.

Note that:

$$f_X(x) = \begin{cases} \frac{1}{4} & -1 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

Functions of Continuous Random Variables

There's another approach called the Method of Transformations that sometimes gives a quicker way of finding $f_Y(y)$ directly from $f_X(x)$.

Functions of Continuous Random Variables

If X is a continuous random variable and $Y = g(X)$ is a function of X, then Y itself is a random variable.

Steps:

1) Find R_Y.

2) Find CDF of Y : $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$.

3) Find $f_Y(y) = \frac{d}{dy} F_Y(y)$.

Functions of Continuous Random Variables

Example. Now suppose that $X \sim Uniform(-1, 3)$, and $Y = X^2$.

Find CDF and PDF of Y.

Summary of Continuous Random Variable

- **PDF:** $f_X(x) = \dfrac{dF_X(x)}{dx}$
- **Expected Value:** $EX = \int_{-\infty}^{\infty} x f_X(x) dx$
- **LOTUS:** $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
- $P(a < X < b) = \int_a^b f_X(x) dx = F_X(b) - F_X(a)$

Summary of Continuous Random Variable

Definition. Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha) d\alpha.$$

Summary of Continuous Random Variable

Consider a continuous random variable X with PDF $f_X(x)$. We have

1) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.
2) $\int_{-\infty}^{\infty} f_X(u)du = 1.$
3) $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du.$
4) More generally, for a set A, $P(X \in A) = \int_A f_X(u)du.$

Summary of Continuous Random Variable

Discrete RVs		Continuous RVs
PMF	$\longrightarrow$	**PDF**
$\sum$	$\longrightarrow$	$\int$
$EX = \sum_{x_k \in R_X} x_k P_X(x_k)$	$\longrightarrow$	$EX = \int_{-\infty}^{\infty} x f_X(x) dx$
$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k)$	$\longrightarrow$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

$$\text{Var}(X) = EX^2 - (EX)^2$$

Summary of Continuous Random Variable

Functions of Continuous RVs:

$$X \longrightarrow f_X(x); \quad Y = g(X)$$

Steps:

1) $R_Y = \{g(x);\; x \in R_X\}.$

2) $F_Y(y) = P(Y \leq y) = P(g(X) \leq y).$

Read method of Transformations (4.1.3).

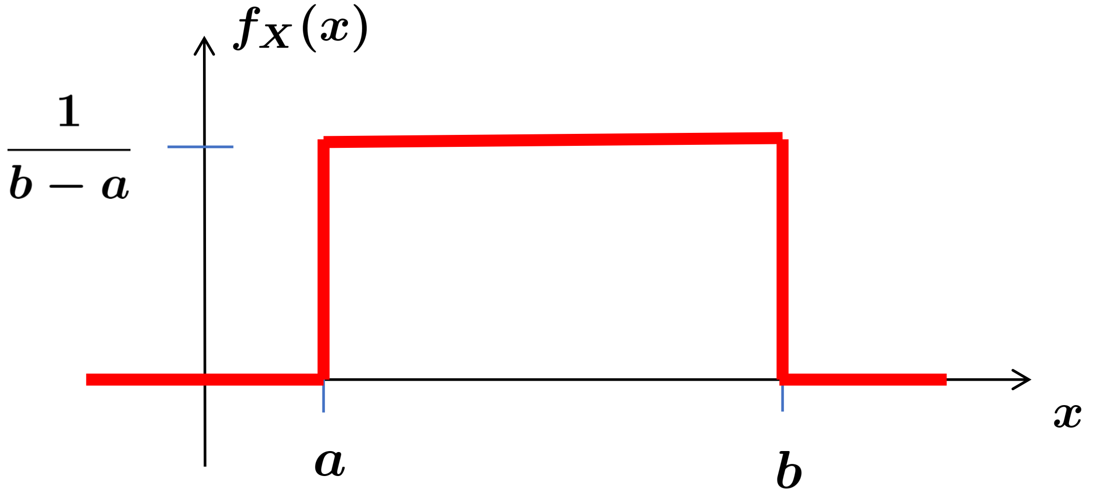
Special Distributions

- **Uniform Distribution:** $X \sim Uniform(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$$EX = \frac{a+b}{2},$$

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{(b-a)^2}{12}. \end{aligned}$$



Special Distributions

Example. **Let** $X \sim Uniform(-1, 3),$

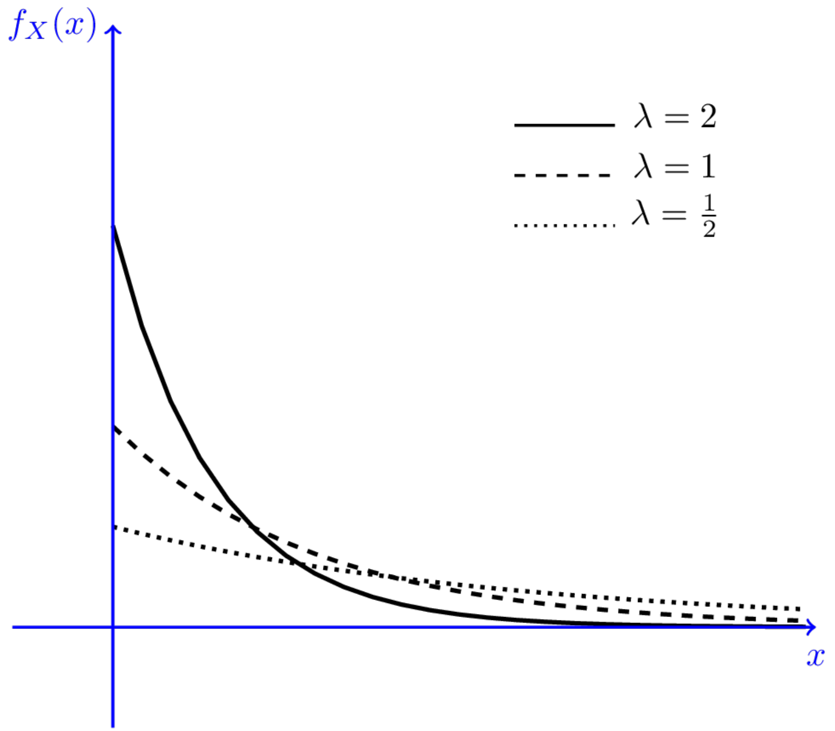
$$E[X] = \frac{3 + (-1)}{2} = 1,$$

$$\mathrm{Var}(X) = \frac{(3 - (-1))^2}{12} = \frac{16}{12} = \frac{4}{3}.$$

Special Distributions

- **Exponential Distribution:** $X \sim Exponential(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$$
$$= \lambda e^{-\lambda x}\, u(x)$$



Where $u(x)$ denote the unit step function.

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Special Distributions

We find the **CDF** using the equation

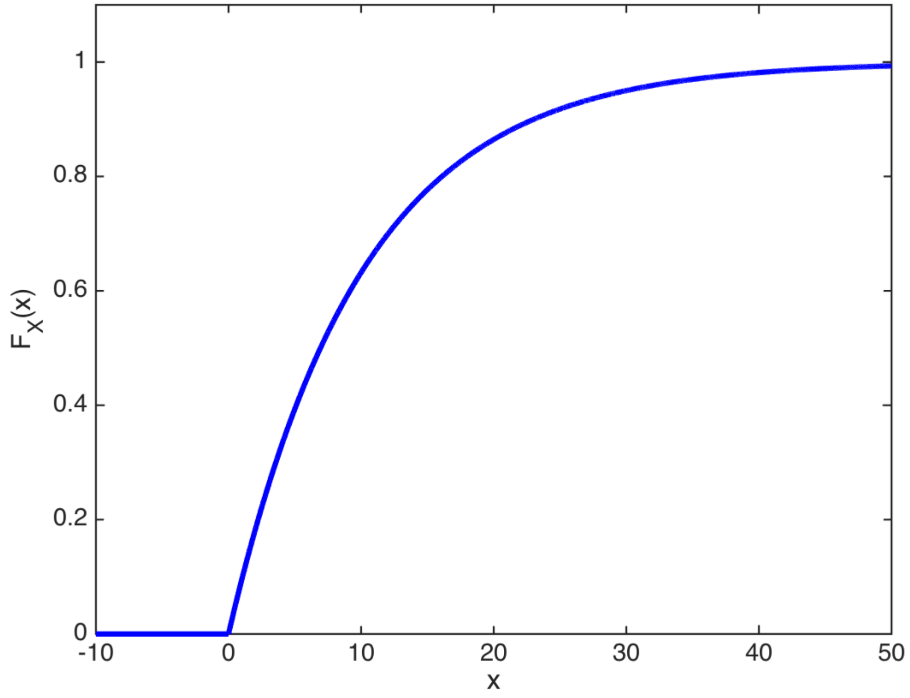
$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x \lambda e^{-\lambda t}dt = 1 - e^{-\lambda x}.$$

So

$$F_X(x) = (1 - e^{-\lambda x})u(x).$$

Special Distributions

Example. **CDF of** $Exponential(\frac{1}{10})$



Special Distributions

Expected value

$$
\begin{aligned}
EX &= \int_0^\infty x\lambda e^{-\lambda x} dx \\
&= \frac{1}{\lambda}\int_0^\infty y e^{-y} dy \qquad\qquad \text{choosing } y = \lambda x \\
&= \frac{1}{\lambda}\Big[-e^{-y} - y e^{-y}\Big]_0^\infty \\
&= \frac{1}{\lambda}.
\end{aligned}
$$

Special Distributions

$$\mathbf{Var}(X): \quad E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}.$$

Thus, we obtain

$$\mathbf{Var}(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Special Distributions

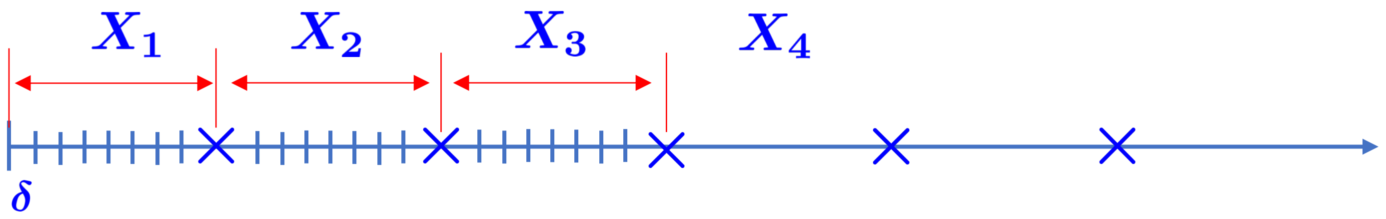
If $X \sim Exponential(\lambda)$, then

$$EX = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

Special Distributions

If X is exponential with parameter $\lambda > 0$, then X is a **memoryless** random variable, that is

$$P(X > x + a \mid X > a) = P(X > x), \qquad \text{for } a, x \geq 0.$$



Special Distributions

Proof:

$$
\begin{aligned}
P(X > x + a | X > a) &= \frac{P\big(X > x + a, X > a\big)}{P(X > a)} = \frac{P(X > x + a)}{P(X > a)} \\
&= \frac{1 - F_X(x + a)}{1 - F_X(a)} = \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} \\
&= P(X > x).
\end{aligned}
$$

Special Distributions

Exponential random variables are often used to model times between arrivals at service centers (these are called interarrival times) – the memoryless property says that no matter how long you have waited for an arrival ($X > a$), the probability that you will have to wait for more than x more time instants is always $P(X > x)$ (that is, it doesn't depend on a).

Special Distributions

- **Normal (Gaussian) Distribution:**

  The normal distribution is by far the most important probability distribution. One of the main reasons for that is the Central Limit Theorem (CLT).

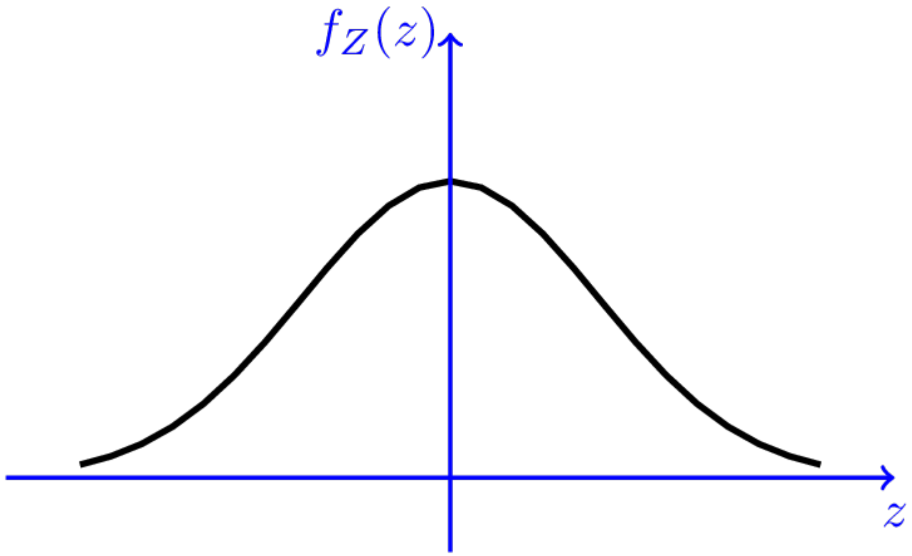
Special Distributions

As we'll see later: The Central Limit Theorem says that random variables that are formed as sums of many independent random variables have distributions that are approximately Gaussian (e.g., the random voltage caused by electronic noise that results from the summed contributions of a very large number of randomly-located charge carriers).

Special Distributions

- **Standard normal random variable:** $Z \sim N(0, 1)$

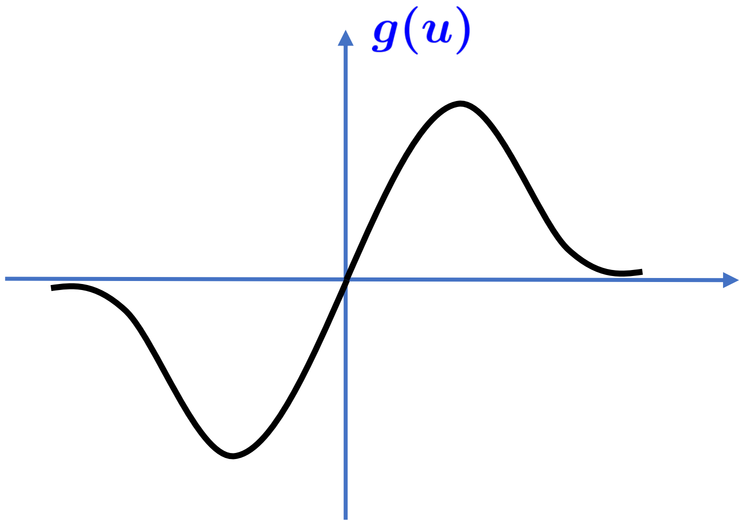
$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad \text{for all } z \in \mathbb{R}.$$



Special Distributions

Expected value of the standard normal:

$$E[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u e^{-\frac{u^2}{2}}}_{g(u)} du = 0,$$



Since $g(u)$ is and odd function.

Special Distributions

Variance of the standard normal:

$$\text{Var}(Z) = E[Z^2] - \underset{\substack{\| \\ 0}}{(EZ)^2} = E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} du = 1.$$

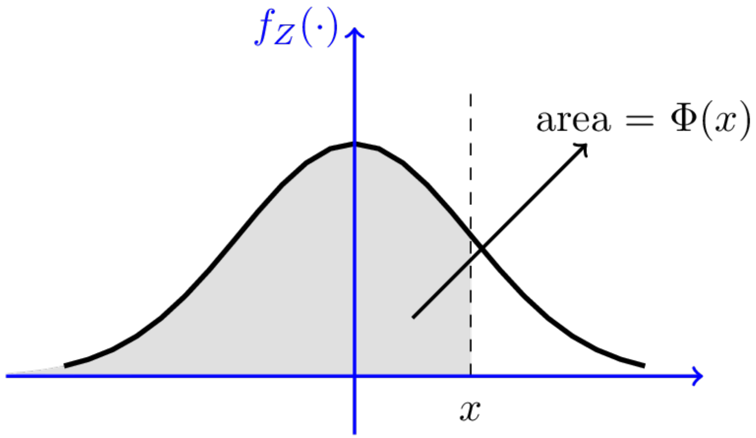
Integration by parts.

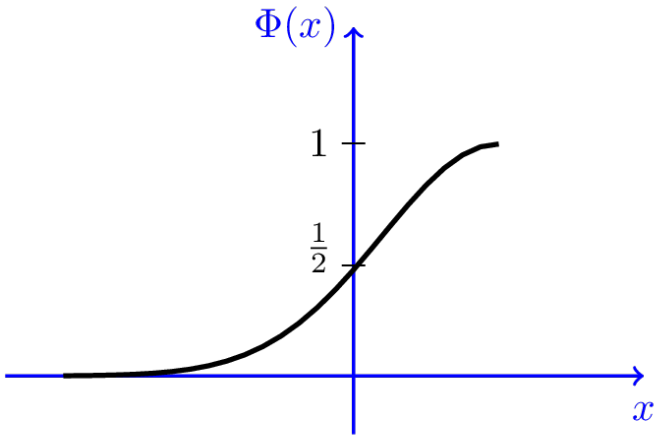
$$Z \sim N(0,1) \Rightarrow \begin{cases} E[Z] = 0 \\ \text{Var}(Z) = 1 \end{cases}$$

Special Distributions

CDF of the standard normal: **The CDF of the standard normal distribution is denoted by the Φ function.**

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$





Special Distributions

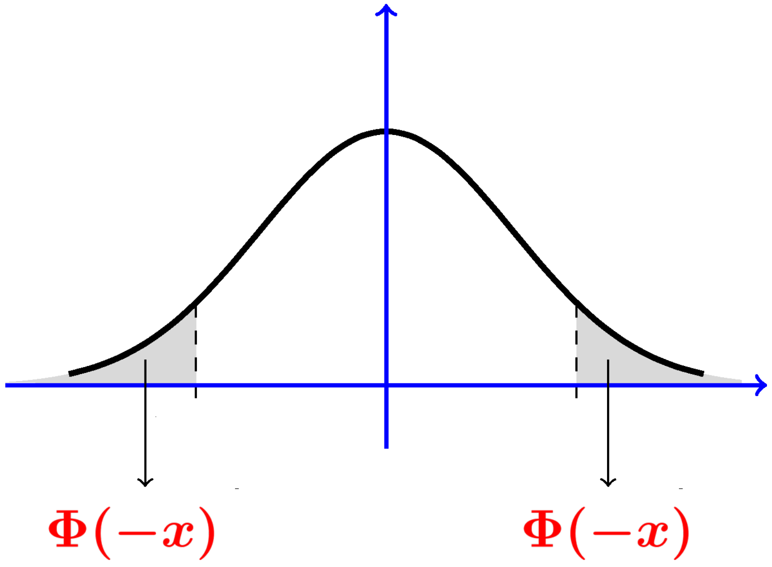
Here are some properties of the Φ function:

1) $\lim_{x \to \infty} \Phi(x) = 1, \quad \lim_{x \to -\infty} \Phi(x) = 0;$

2) $\Phi(0) = \frac{1}{2};$

3) $\Phi(-x) = 1 - \Phi(x), \text{ for all } x \in \mathbb{R};$

4) $\Phi(-x) + \Phi(x) = 1.$



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Special Distributions

The standard normal CDF $\Phi(z)$ can't be evaluated in closed form (except at a few specific values of z), but numerical approximations are widely available (e.g., the normcdf command in MATLAB).

Special Distributions

- **General Normal random variables:**

$$X = \sigma Z + \mu, \qquad \text{where } \sigma > 0,\ \mu \in \mathbb{R}.$$

$$Z \sim N(0,1)$$

$$\Rightarrow \ X \sim N(\mu, \sigma^2).$$

$$EX = E[\sigma Z + \mu] = \sigma \underbrace{EZ}_{0} + \mu = \mu, \quad \text{(linearity of expectation)}$$

Special Distributions

- **General Normal random variables:**

$$\mathrm{Var}(X) = \mathrm{Var}(\sigma Z + \mu) = \sigma^2 \underbrace{\mathrm{Var}(Z)}_{1} = \sigma^2.$$

$$X \sim N(\mu, \sigma^2) \;\Rightarrow\; \begin{cases} E[X] = \mu \\ \mathrm{Var}(X) = \sigma^2 \end{cases}$$

Special Distributions

CDF and PDF of Normal random variables ($X \sim N(\mu, \sigma^2)$):

$$F_X(x) = P(X \leq x) = P(\sigma Z + \mu \leq x)$$
$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Special Distributions

If $X \sim N(\mu, \sigma^2)$, $E[X] = \mu$; $\mathrm{Var}(X) = \sigma^2$,

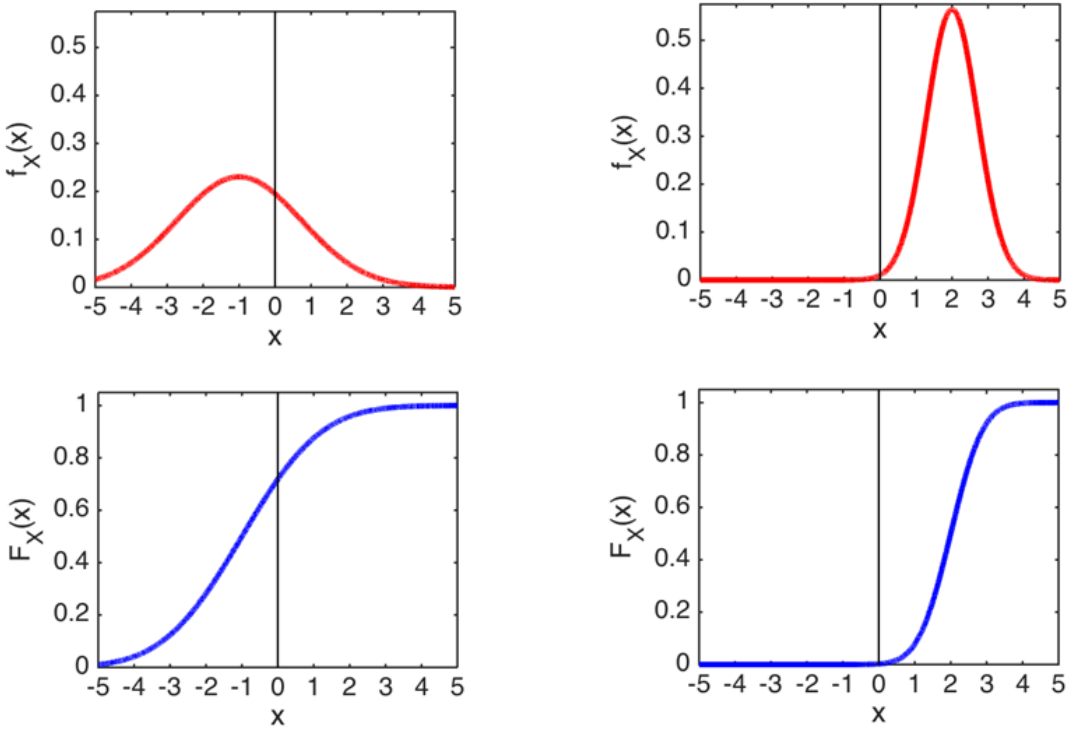
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$P(a < X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Special Distributions

Example. **PDF (top) and CDF (bottom) of $N(-1,3)$ random variable (left) and $N(2,0.5)$ random variable (right):**



Special Distributions

Example. **Say that** $X \sim N(2, 0.5)$**, find** $P(1 < X \leq 3)$.

Special Distributions

Example. The IQ of randomly selected individuals is often assumed to follow a Normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$. Find the probability that a randomly selected individual has an IQ:

a) Above 140

b) Between 120 and 130

c) Find the value of X such that 99% of the population has IQ at least X.

Special Distributions

Note: In MATLAB, the command $\text{normcdf}(x, m, s)$ calculates

$$\Phi\left(\frac{x-m}{s}\right).$$

- So, for example: If $X \sim N(2, 0.5)$ and we want to calculate $F_X(3) - F_X(1)$, we can use the command:

$$\text{normcdf}(3, 2, sqrt(0.5)) - \text{normcdf}(1, 2, sqrt(0.5))$$

Special Distributions

Example. Suppose that $X \sim N(\mu, \sigma^2)$. Then:

$$\begin{aligned} P(|X-\mu| \leq \sigma) &= P(\mu-\sigma \leq X \leq \mu+\sigma) \\ &= \Phi\left(\frac{\mu+\sigma-\mu}{\sigma}\right) - \Phi\left(\frac{\mu-\sigma-\mu}{\sigma}\right) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827 \end{aligned}$$

Special Distributions

$$\begin{aligned}
P(|X-\mu| \le 3\sigma) &= P(\mu - 3\sigma \le X \le \mu + 3\sigma) \\
&= \Phi\left(\frac{\mu + 3\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3\sigma - \mu}{\sigma}\right) \\
&= \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.9973
\end{aligned}$$

Special Distributions

- So, the probability that a Gaussian random variable takes a value within 1 standard deviation from its mean is 0.6827.

- The probability that a Gaussian random variable takes a value within 3 standard deviations from its mean is 0.9973.

Special Distributions

Theorem. If $X \sim N(\mu_X, \sigma_X^2)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

$$E[Y] = E[aX + b] = aEX + b = a\mu + b,$$

$$\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) = a^2\sigma^2.$$

Special Distributions

Example. Suppose that $X \sim N(2, 0.5)$, and let $Y = 2X - 1$.

Find $P(1 < Y \leq 3)$.

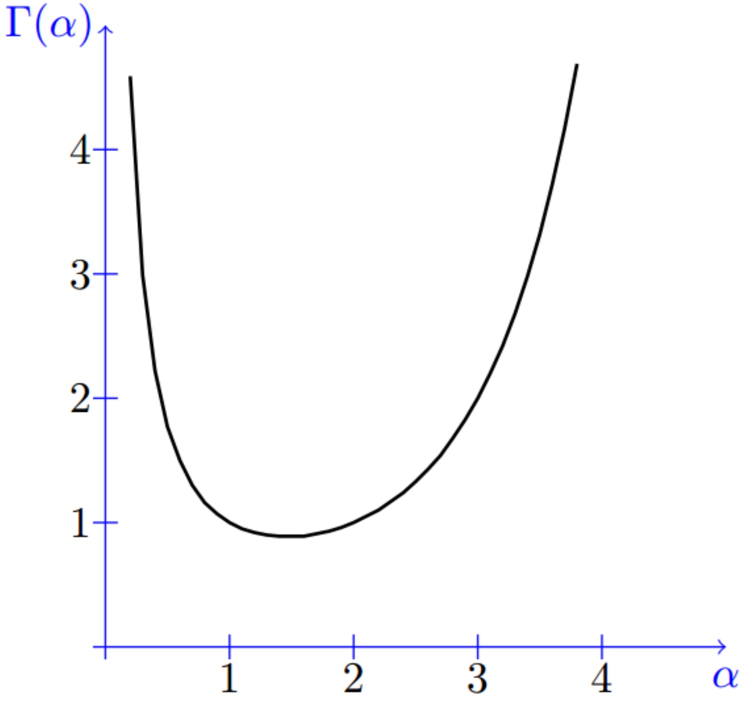
Gamma Distribution

Gamma function: $X \sim Gamma(\alpha, \lambda)$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x, \quad \text{for } \alpha > 0.$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \text{for } \alpha > 0.$$

$$\Gamma(n) = (n-1)!, \ \text{for } n \in \mathbb{N};$$



Gamma Distribution

Gamma Distribution:

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \qquad x > 0$$

If $\alpha = 1$ **:**

- $f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$
- $Gamma(1, \lambda) = Exponential(\lambda)$

Gamma Distribution

If $X_1, X_2, \cdots, X_n$ are independent, then

$$X_1, X_2, \cdots, X_n \sim Exponential(\lambda)$$

$$X_1 + X_2 + \cdots + X_n \sim Gamma(n, \lambda)$$

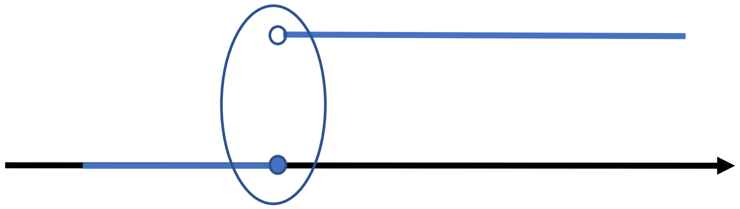
Mixed Random Variables

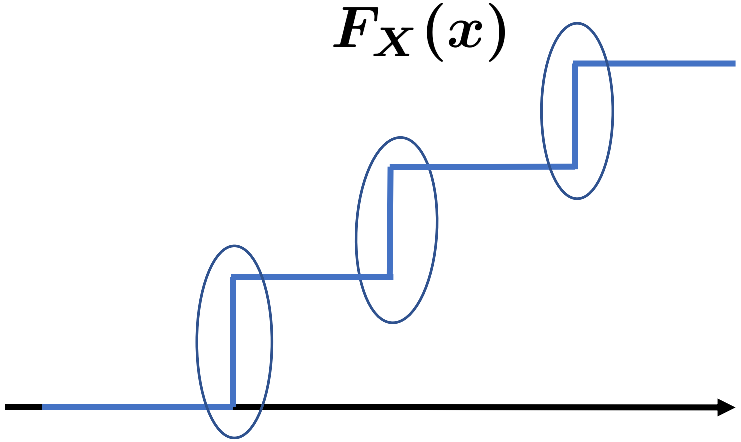
We will:

1) **Define PDF (generalized PDF) for discrete random variables by using the delta function.**
2) **Introduce mixed random variables.**

Mixed Random Variables

$$f_X(x) = \frac{d}{dx} F_X(x),$$





CDF for discrete RVs

Mixed Random Variables

Idea: Define the "derivative" of $u(x)$, $\frac{d}{dx}u(x)$ and that we can extend the PDF to the discrete RVs.

$\Rightarrow$ **delta function** $\delta(x)$

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{else} \end{cases}$$

Mixed Random Variables

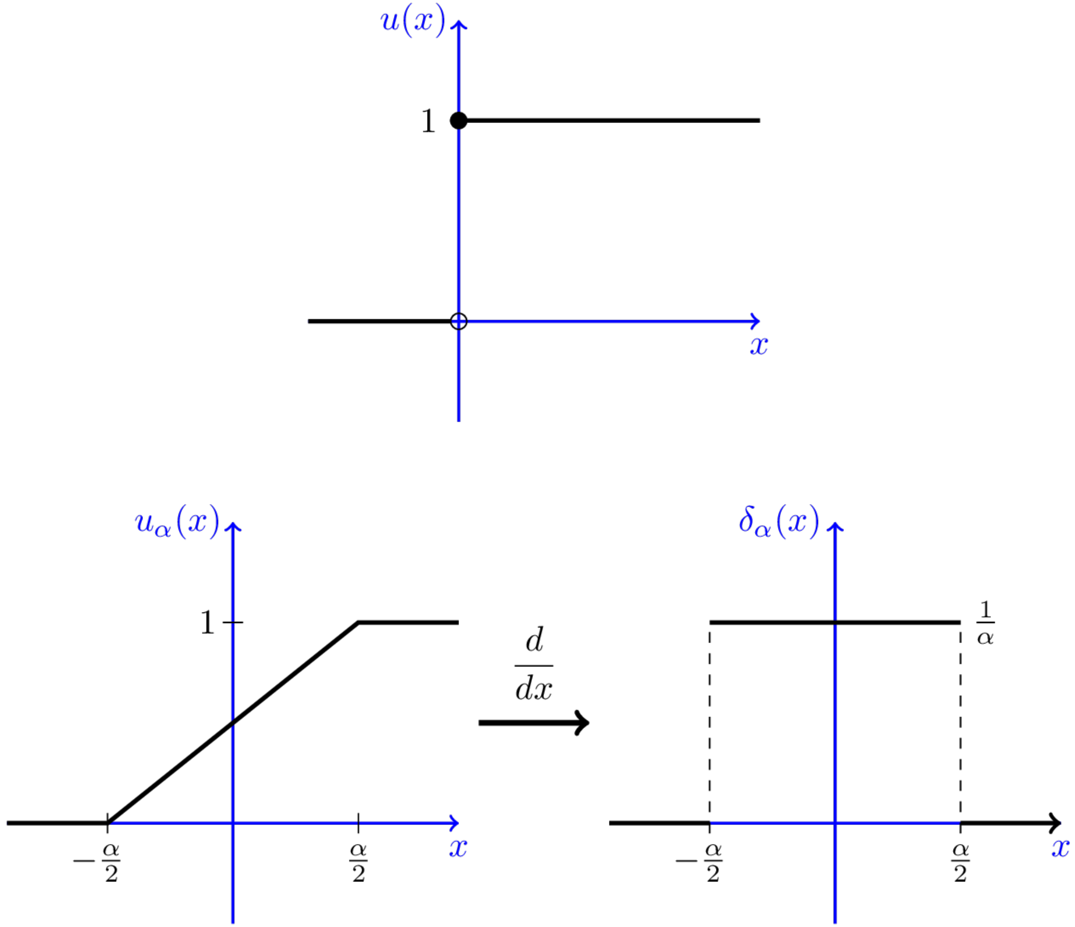
Delta function: $\delta(x)$

Where α is small.

$$\int \delta(x)dx = 1$$

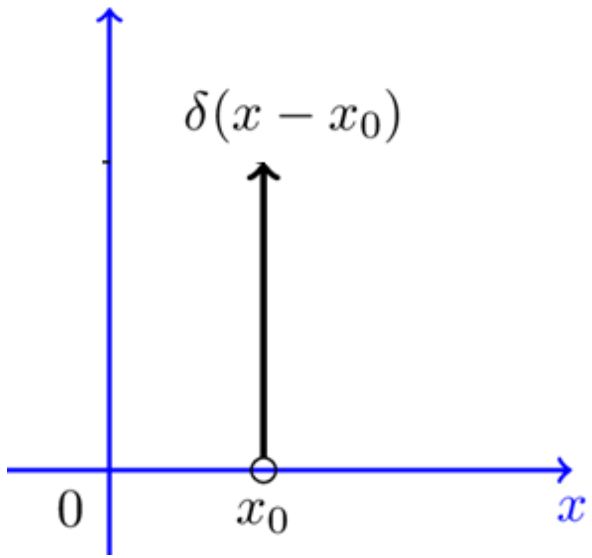
$$\alpha = 0 \Rightarrow \delta(0) = \frac{1}{\alpha} = \infty$$

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{else} \end{cases}$$



Mixed Random Variables

$$\delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & \text{else} \end{cases}$$



Mixed Random Variables

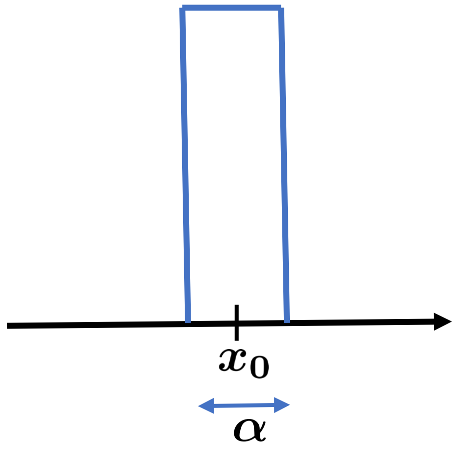
Lemma. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. We have

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0).$$

$$g(x).\frac{1}{\alpha} \approx \frac{1}{\alpha}g(x_0)$$

Proof:

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx \approx \alpha\left(g(x_0).\frac{1}{\alpha}\right) = g(x_0).$$



Mixed Random Variables

Example. **Let** $g(x) = 2(x^2 + x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x-1)dx = g(1) = 2(1^2+1) = 4.$$

Mixed Random Variables

Definition: **Properties of the delta function**

We define the delta function $\delta(x)$ as an object with the following properties:

1) $$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{else} \end{cases}$$

2) $\delta(x) = \dfrac{d}{dx}u(x)$, **where $u(x)$ is the unit step function.**

3) $\displaystyle\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$ **for any** $\epsilon > 0.$

Mixed Random Variables

4) For any $\epsilon > 0$ and any function $g(x)$ that is continuous over $(x_0 - \epsilon, x_0 + \epsilon)$, we have

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = \int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)dx = g(x_0).$$

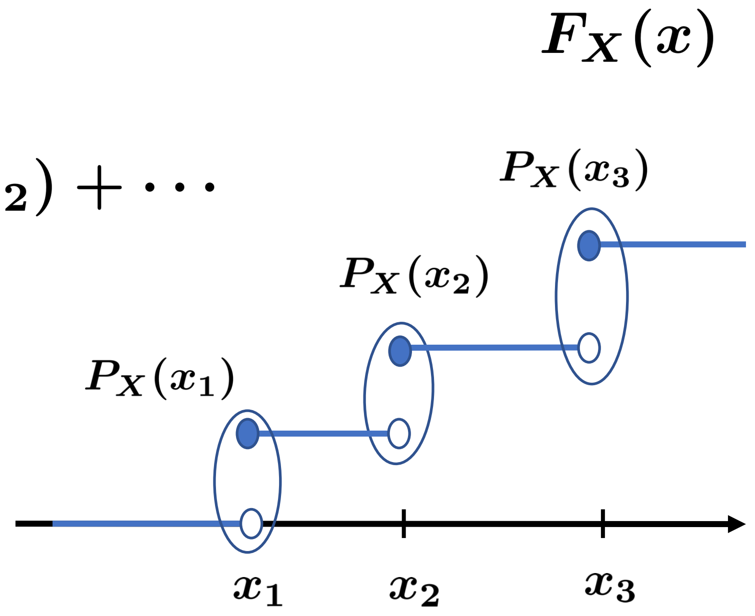
Mixed Random Variables

Discrete Random Variable X

$$R_X = \{x_1, x_2, x_3, \cdots\}$$

$$F_X(x) = \sum_{x_k \in R_X} P_X(x_k) u(x - x_k)$$

$$= P_X(x_1) u(x - x_1) + P_X(x_2) u(x - x_2) + \cdots$$



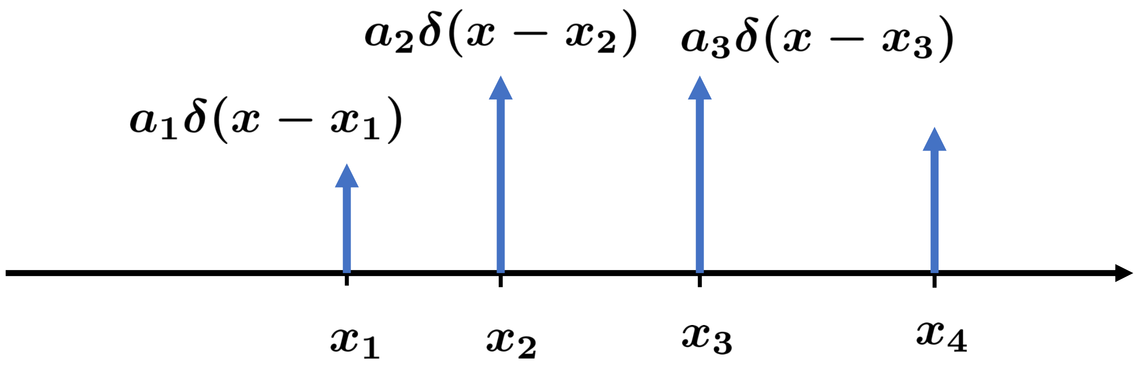
Mixed Random Variables

generalized PDF

$$a_k = P(X = x_k) = P_X(x_k),$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \sum_{x_k \in R_X} a_k u(x - x_k)$$

$$= \sum_{x_k \in R_X} a_k \delta(x - x_k).$$



Mixed Random Variables

For a discrete random variable X with range $R_X = \{x_1, x_2, x_3, ...\}$ and PMF $P_X(x_k)$, we define the (generalized) probability density function (PDF) as

$$f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\delta(x - x_k).$$

Orchestrated Conversation: Mixed Random Variables

Is it true $\displaystyle\int_{-\infty}^{+\infty} f_X(x)dx = 1$ **?**

$$\int_{-\infty}^{+\infty} f_X(x)dx = \int_{-\infty}^{+\infty} \sum_{x_k \in R_X} a_k \delta(x - x_k).dx$$

$$= \sum_{x_k \in R_X} a_k \int_{-\infty}^{+\infty} \delta(x - x_k).dx \qquad \left(\int_{-\infty}^{+\infty} \delta(x - x_k).dx = 1\right)$$

$$= \sum_{x_k \in R_X} a_k = \sum_{x_k \in R_X} P_X(x_k) = 1.$$

Mixed Random Variables

Example. $EX = \sum_k x_k P_X(x_k).$

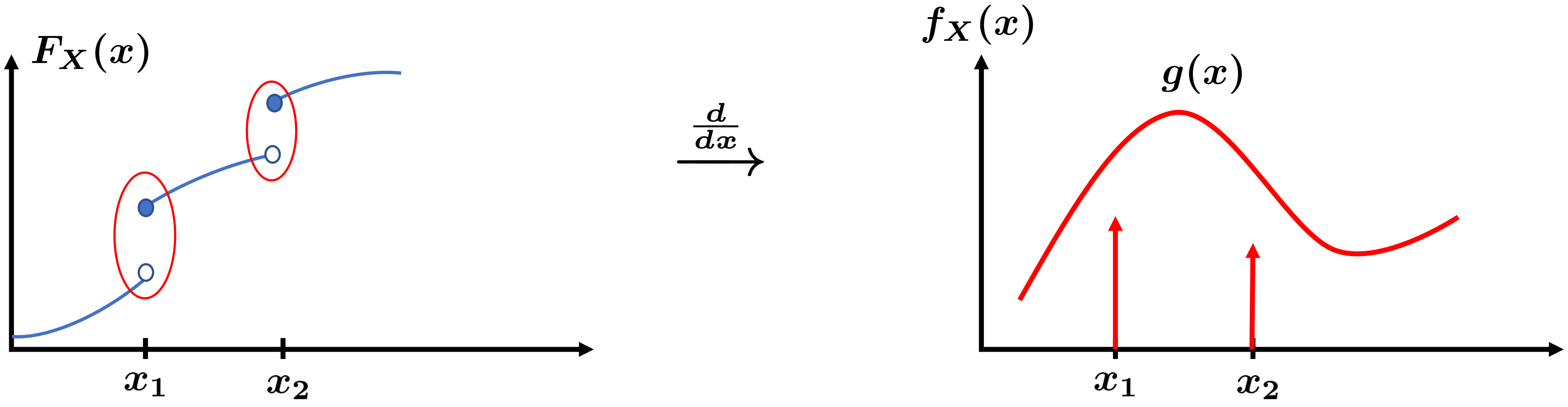
$$EX = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \sum_{x_k \in R_X} a_k \delta(x - x_k)dx$$

$$= \sum_{x_k \in R_X} a_k \underbrace{\int_{-\infty}^{\infty} x\delta(x - x_k)dx}_{x_k} = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Mixed Random Variables

Random variables:

- Discrete
- Continuous
- Mixed random variable



Mixed Random Variables

The (generalized) PDF of a mixed random variable can be written in the form

$$f_X(x) = \underbrace{\sum_k a_k \delta(x - x_k)}_{\text{Discrete}} + \underbrace{g(x)}_{\text{Continuous}},$$

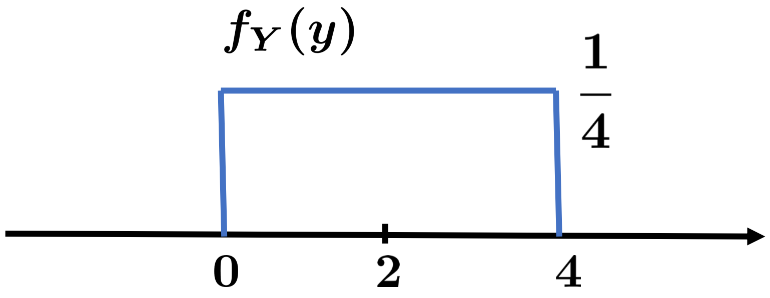
where $a_k = P(X = x_k)$, and $g(x) \geq 0$ does not contain any delta functions. Furthermore, we have

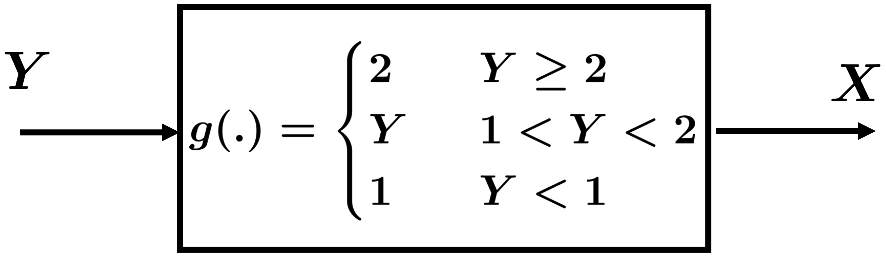
$$\int_{-\infty}^{\infty} f_X(x)dx = \sum_k a_k + \int_{-\infty}^{\infty} g(x)dx = 1.$$

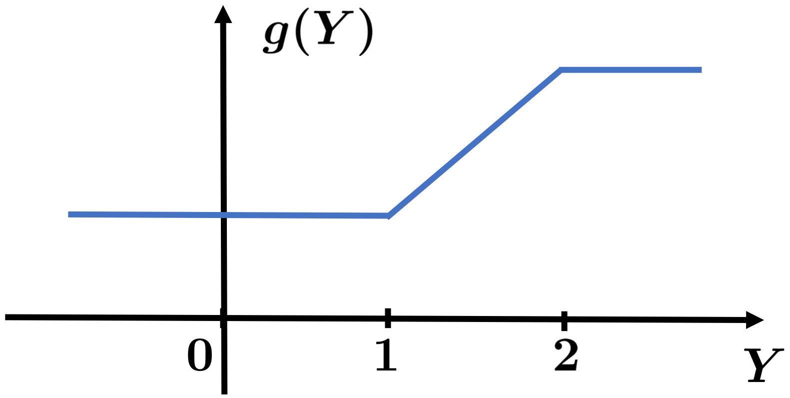
Mixed Random Variables

Example. Let $Y \sim Uniform(0,4)$,

$$X = \begin{cases} 2 & Y \geq 2 \\ Y & 1 < Y < 2 \\ 1 & Y < 1 \end{cases}$$







Mixed Random Variables

a) Find the CDF of X.

b) Find the generalized PDF of X.

c) Find EX and $\mathrm{Var}(X)$.

Mixed Random Variables

So mixed random variable:

$$f_X(x) = \underbrace{\sum_k P(X = x_k)\delta(x - x_k)}_{\text{Discrete part}} + \underbrace{g(x)}_{\text{Continuous part}}.$$

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx = \sum_k x_k P(X = x_k) + \int_{-\infty}^{\infty} x g(x) dx.$$

Orchestrated Conversation: Mixed Random Variables

Example. Let $f_X(x) = \frac{1}{3}\delta(x+1) + \frac{1}{6}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$,

a) Find $P(X=-1), P(X=0)$ and $P(X=+1)$.

b) Find $P(X \geq 0)$.

c) Find $P(X=+1|X \geq 0)$.

d) Find EX and $\mathrm{Var}(X)$.

Post-work for Lesson

- Complete homework assignment for Lessons 7-9: HW#4 and HW#5

Go to the online classroom for details.

To Prepare for the Next Lesson

- Read Chapter 5 in your online textbook:

  https://www.probabilitycourse.com/chapter5/5_1_0_joint_distributions.php

- Complete the Pre-work for Lessons 10-13.

Visit the online classroom for details.