ECE 603
Probability and Random Processes

Lessons 10-13

Chapter 5

Joint Distributions

Objectives

- **Explore tools for studying joint distributions of random variables.**
- **Compare joint distributions of discrete random variables with continuous random variables.**

Rationale

- If you were to examine the population of your community, you might notice that each household has a different number of people. Each of those household members has a different age, a different income, a different number of hobbies, etc.
- Each of these results is a random variable. In this Lesson, you explore the concept of comparing two or more random variables, because you grasp comparing two, the extension to *n* random variables is straightforward.

Prior Learning

- Basic Concepts
- Counting Methods
- Random Variables
- Access to the online textbook: https://www.probabilitycourse.com/

Joint Probability Mass Function (PMF)

PMF:

$$P_X(x_k) = P(X = x_k), \quad R_X = \text{Range}(X).$$

Joint Probability Mass Function (PMF) for X and Y:

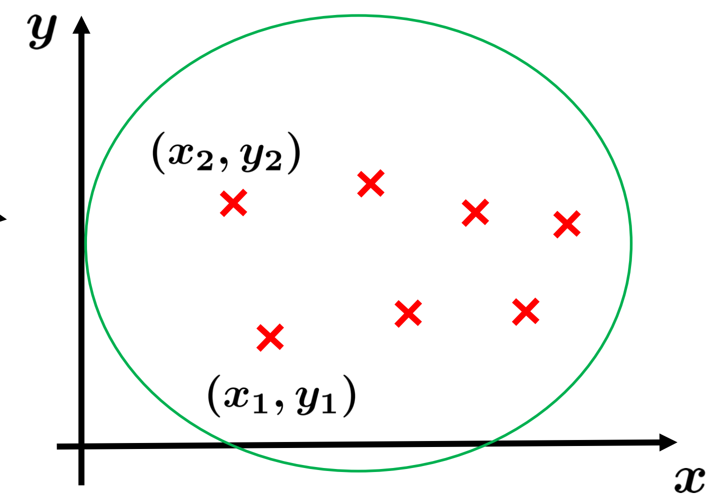
$$P_{XY}(x_j, y_j) = P(X = x_k, Y = y_j).$$

Joint Probability Mass Function (PMF)

R_{XY} = all possible value for (X, Y).

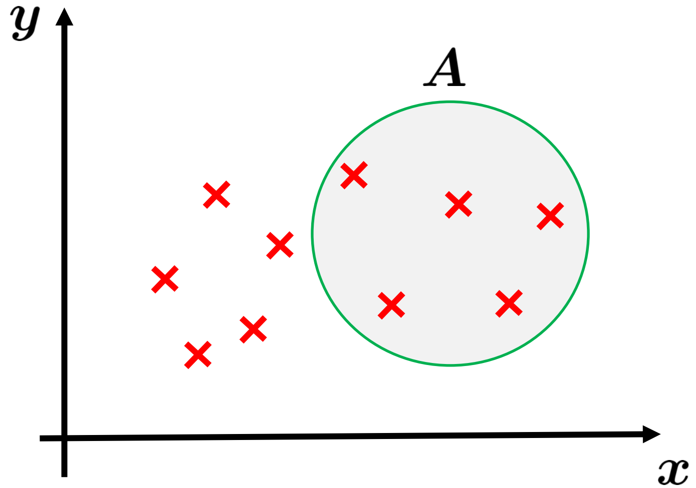
$$= \{(x_i, y_j) | x_i \in R_X, y_j \in R_Y\}.$$

$$\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x_i, y_j) = 1$$



Joint Probability Mass Function (PMF)

$$P\big((X,Y)\in A\big) = \sum_{(x_i,y_j)\in A} P_{XY}(x_i,y_j)$$



Joint Probability Mass Function (PMF)

Marginal PMFs

I have $P_{XY}(x_i y_j)$, how do I find PMF of X, $P_X(x_i)$?

$$
\begin{aligned}
P_X(x_i) &= P(X = x_i) \\
&= \sum_{y_j \in R_Y} P(X = x_i, Y = y_j) \qquad \text{law of total probablity} \\
&= \sum_{y_j \in R_Y} P_{XY}(x_i, y_j).
\end{aligned}
$$

Joint Probability Mass Function (PMF)

Marginal PMFs

$$P_X(x_i) = \sum_{y_j \in R_Y} P_{XY}(x_i, y_j), \qquad \text{for any } x_i \in R_X$$

$$P_Y(y_j) = \sum_{x_i \in R_X} P_{XY}(x_i, y_j), \qquad \text{for any } y_j \in R_Y$$

Joint Probability Mass Function (PMF)

Example. **Consider two random variables X and Y with joint PMF given in Table.**

a) Find the marginal PMFs of X and Y.

b) Find $P(Y=0|X=0)$.

c) Are X and Y independent?

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{2}$	$\frac{1}{3}$
$X=1$	$\frac{1}{6}$	0

Joint Cumulative Distributive Function (CDF)

Remember that, for a random variable X , we define the CDF as

$$F_X(x) = P(X \leq x).$$

The joint cumulative distribution function of two random variables

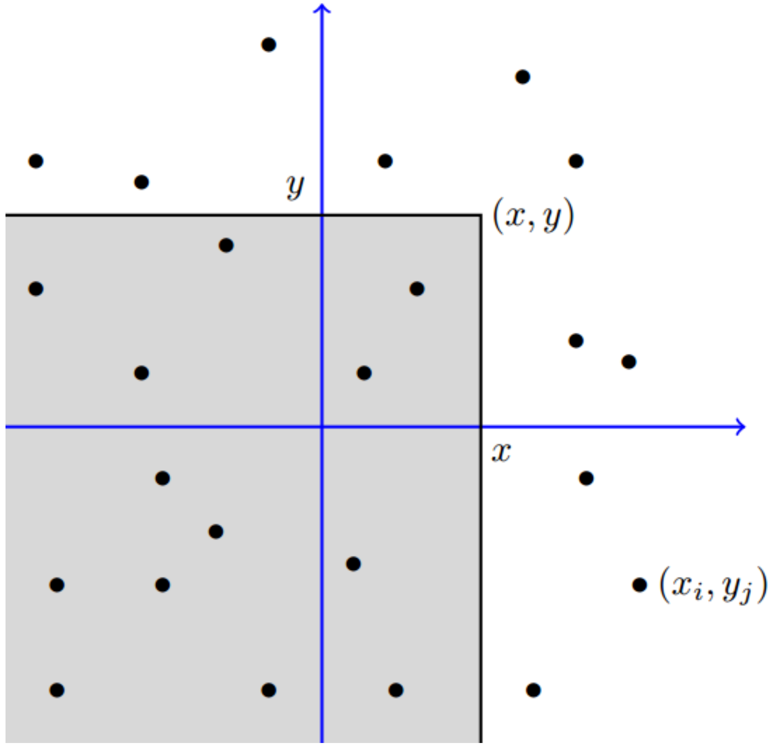
$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

and

Joint Cumulative Distributive Function (CDF)

$$F_{XY}(x, y) = P(\boxed{}).$$

$$F_{XY}(1, 2) = P(X \leq 1, Y \leq 2).$$



Joint Cumulative Distributive Function (CDF)

Marginal CDFs of X and Y:

$$F_{XY}(x,\infty) = P(X \le x, Y < \infty) = P(X \le x) = F_X(x),$$

$$F_{XY}(\infty,y) = P(X < \infty, Y < y) = P(Y \le y) = F_Y(y),$$

$$F_{XY}(\infty,\infty) = 1,$$

$$F_{XY}(-\infty,y) = 0, \qquad \text{for any } y,$$

$$F_{XY}(x,-\infty) = 0, \qquad \text{for any } x.$$

$$0 \le F_{XY}(x,y) \le 1$$

Joint Cumulative Distributive Function (CDF)

Example. Toss a fair coin twice,

$$\text{First: } \begin{cases} X = 1 & H \\ X = 0 & T \end{cases} \qquad \text{Second: } \begin{cases} Y = 1 & H \\ Y = 0 & T \end{cases}$$

X and Y are independent. Find $F_{XY}(3, 1)$.

Joint Cumulative Distributive Function (CDF)

Two discrete random variables X and Y are independent if

$$P_{XY}(x, y) = P_X(x)P_Y(y), \qquad \text{for all } x, y.$$

Equivalently, X and Y are independent if

$$F_{XY}(x, y) = F_X(x)F_Y(y), \qquad \text{for all } x, y.$$

Joint Cumulative Distributive Function (CDF)

So far:

Joint PMF

- $P_{XY}(x_i, y_j) = P(X = x_i, Y = y_j)$.
- $R_{XY} =$ **all possible value for** (X, Y).
- **Marginal PMFs**

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \qquad \text{LOTP}$$

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \qquad \text{LOTP}$$

Joint Cumulative Distributive Function (CDF)

Joint CDF:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y).$$

$$F_{XY}(3,2) = P(X \leq 3, Y \leq 2).$$

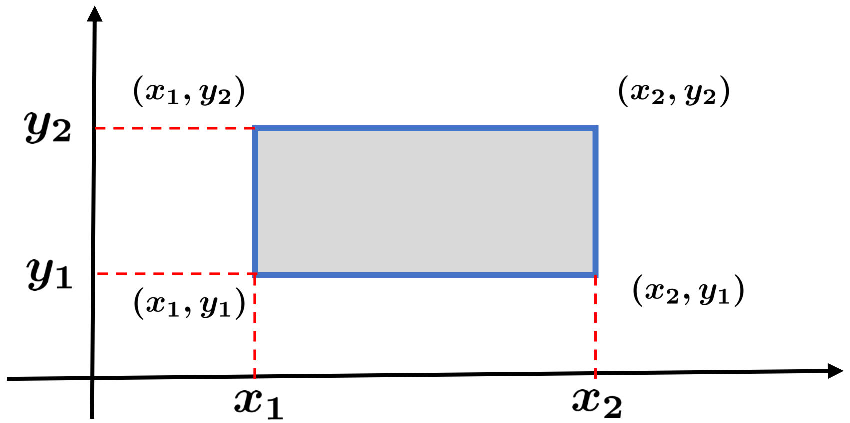
Remember:

$$P(a < X \leq b) = F_X(b) - F_X(a),$$

Joint Cumulative Distributive Function (CDF)

Lemma. For two random variables X and Y, and real numbers $x_1 \le x_2,\ y_1 \le y_2$, we have

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1).$$



Conditioning and Independence

Conditioning:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ \textbf{when} \ P(B) > 0.$$

$$P(X = x_i | A) = \frac{P(X = x_i \ \textbf{and} \ A)}{P(A)},$$

For example $A: \quad Y = y_j.$

Conditioning and Independence

PMF: $P_X(x_i) = P(X \leq x_i)$

Conditional PMF: $P_{X|A}(x_i) = P(X = x_i|A),$

Conditional CDF: $F_{X|A}(x) = P(X \leq x|A),$

Conditioning and Independence

Conditional PMF and CDF:

For a discrete random variable X and event A, the conditional PMF of X given A is defined as

$$\begin{aligned} P_{X|A}(x_i) &= P(X = x_i | A) \\ &= \frac{P(X = x_i \text{ and } A)}{P(A)}, \quad \text{for any } x_i \in R_X. \end{aligned}$$

Similarly, we define the conditional CDF of X given A as

$$F_{X|A}(x) = P(X \leq x | A).$$

Conditioning and Independence

Let $A: \quad Y = y_j.$

Conditional PMF of X given $Y = y_j$:

$$\begin{aligned} P_{X|Y}(x_i|y_j) &= P(X = x_i|Y = y_j) \\ &= \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} \\ &= \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}. \end{aligned}$$

Conditioning and Independence

Similarly, we can define the conditional probability of Y given X :

$$\begin{aligned} P_{Y|X}(y_j|x_i) &= P(Y = y_j|X = x_i) \\ &= \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}. \end{aligned}$$

Conditioning and Independence

For discrete random variables X and Y, the conditional PMFs of X given Y and vice versa are defined as

$$P_{X|Y}(x_i|y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)},$$

$$P_{Y|X}(y_j|x_i) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}$$

for any $x_i \in R_X$ and $y_j \in R_Y$.

Conditioning and Independence

Example. **Consider two random variables X and Y with joint PMF given in the following Table.**

Find $P_{X|Y}(x|2)$, conditional PMF of X given $Y=2$.

	$Y=1$	$Y=2$
$X=1$	$\frac{1}{3}$	$\frac{1}{12}$
$X=2$	$\frac{1}{6}$	0
$X=4$	$\frac{1}{12}$	$\frac{1}{3}$

Conditioning and Independence

Independent Random Variables:

Two discrete random variables X and Y are independent if

$$P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j), \quad \text{for all } x_i, y_j.$$

Equivalently

$$P_{X|Y}(x_i|y_j) = P_X(x_i), \quad P_{Y|X}(y_j|x_i) = P_Y(y_j).$$

Equivalently

$$F_{XY}(x, y) = F_X(x)F_Y(y), \quad \text{for all } x, y.$$

Conditioning and Independence

Conditional Expectation:

$$E[X] = \sum_{x_i \in R_X} x_i P_X(x_i),$$

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i),$$

$$E[X|Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y_j)$$

Conditioning and Independence

Example. Consider two random variables X and Y with joint PMF given in Table.

Find $E[X|Y=2]$ and $\mathrm{Var}(X|Y=2)$.

	$Y=1$	$Y=2$
$X=1$	$\frac{1}{3}$	$\frac{1}{12}$
$X=2$	$\frac{1}{6}$	0
$X=4$	$\frac{1}{12}$	$\frac{1}{3}$

Conditioning and Independence

Law of Total Probability:

$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A | Y = y_j) P_Y(y_j), \quad \text{for any set } A.$$

Conditioning and Independence

Law of Total Probability:

If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S, then we have

$$P(A) = \sum_j P(A \cap B_j) = \sum_j P(A|B_j)P(B_i).$$

$B_j : \; Y = y_j,$

$$P(A) = \sum_j P(A|Y = y_j)P(Y = y_j).$$

Conditioning and Independence

Law of Total Expectation:

If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S, then we have

$$EX = \sum_j E[X|B_j]P(B_j).$$

$$B_j : \; Y = y_j,$$

$$EX = \sum_{y_j \in R_Y} E[X|Y = y_j]P_Y(y_j).$$

Orchestrated Conversation: Conditioning and Independence

Example. Suppose that the number of customers visiting a fast food restaurant in a given day is $N \sim Poisson(\lambda)$. Assume that each customer purchases a drink with probability p, independently from other customers and independently from the value of N. Let X be the number of customers who purchase drinks. Find EX.

Functions of Two Random Variables

Let X and Y are two random variables, and suppose that $Z = g(X, Y)$,

$\Rightarrow$ Z is random variable.

Law of the unconscious statistician (LOTUS) for two discrete random variables:

$$E[g(X)] = \sum_{x_i \in R_X} g(x_i) P_X(x_i)$$

$$E[g(X,Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) P_{XY}(x_i, y_j)$$

Functions of Two Random Variables

Example.

Linearity of Expectation: For two discrete random variables X and Y, show that $E[X + Y] = EX + EY$.

Functions of Two Random Variables

General scenario:

$Z = g(X, Y),$

Find PMF of Z.

1) $R_Z = \{g(x_i, y_j);\ (x_i, y_j) \in R_{XY}\},$

2) For $n \in R_Z;\ P_Z(n) = P(Z = n) = P(g(X, Y) = n)$

$$= \sum_{\substack{(x_i, y_j) \in R_{XY} \\ g(x_i, y_j) = n}} P_{XY}(x_i, y_j).$$

Functions of Two Random Variables

Example.

Let $X \sim Geometric(\frac{1}{2})$ **and** $Y \sim Geometric(\frac{3}{4})$ **be two independent random variables and** $Z = X + Y$ **. Find PMF of** Z**.**

Conditional Expectation and Variance

Conditional Expectation as a Function of a Random Variable:

$$E[X|Y=y] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y).$$

Note that $E[X|Y=y]$ depends on the value of y, so we can write

$$g(y) = E[X|Y=y].$$

Conditional Expectation and Variance

Thus, we can think of $E[X|Y=y]$ as a function of the value of the random variable Y. We then write

$$g(Y) = E[X|Y].$$

If Y is a random variable with range $R_Y = \{y_1, y_2, \cdots\}$, then

$$E[X|Y] = \begin{cases} E[X|Y=y_1] & \text{with probability } P(Y=y_1) \\ E[X|Y=y_2] & \text{with probability } P(Y=y_2) \\ \quad\quad\quad . & \quad\quad . \\ \quad\quad\quad . & \quad\quad . \\ \quad\quad\quad . & \quad\quad . \end{cases}$$

Conditional Expectation and Variance

Example. **Consider two random variables X and Y with joint PMF given in the following Table $Z = E[X|Y]$.**

a) **Find the Marginal PMFs of X and Y.**

b) **Find the conditional PMF of X given $Y = 0$ and $Y = 1$.**

c) **Find the PMF of Z.**

d) **Find EZ, and check that $EZ = EX$.**

e) **Find $\mathrm{Var}(Z)$.**

Conditional Expectation and Variance

Table. Joint PMF of X and Y in example.

	$Y = 0$	$Y = 1$
$X = 0$	$\frac{1}{5}$	$\frac{2}{5}$
$X = 1$	$\frac{2}{5}$	0

Conditional Expectation and Variance

Rule: Taking out what is known.

$$E[g(X)h(Y)|X] = g(X)E[h(Y)|X].$$

Proof: Note that $E[g(X)h(Y)|X]$ is a random variable that is a function of X.
If $X = x$ then

$$\begin{aligned} E[g(X)h(Y)|X = x] &= E[g(x)h(Y)|X = x] \\ &= g(x)E[h(Y)|X = x] \qquad (g(x) \text{ is a constant}). \end{aligned}$$

$$\Rightarrow \; E[g(X)h(Y)|X] = g(X)E[h(Y)|X].$$

Conditional Expectation and Variance

Iterated Expectations:

Let $g(Y) = E[X|Y]$,

Then,

$$\begin{aligned} E[X] &= \sum_{y_j \in R_Y} E[X|Y = y_j] P_Y(y_j) \\ &= \sum_{y_j \in R_Y} g(y_j) P_Y(y_j) \\ &= E[g(Y)] \qquad\qquad \text{by LOTUS} \\ &= E[E[X|Y]]. \end{aligned}$$

Conditional Expectation and Variance

Iterated Expectations:

$$\text{Law of Iterated Expectations: } E[X] = E[E[X|Y]]$$

This is equal to the low of total Expectation.

Conditioning and Independence

Example. Suppose that the number of customers visiting a fast food restaurant in a given day is $N \sim Poisson(\lambda)$. Assume that each customer purchases a drink with probability p, independently from other customers and independently from the value of N. Let X be the number of customers who purchase drinks. Find EX.

Conditional Expectation and Variance

If X and Y are **independent** random variables, then

1. $E[X|Y] = EX;$
2. $E[g(X)|Y] = E[g(X)];$
3. $E[XY] = EXEY;$
4. $E[g(X)h(Y)] = E[g(X)]E[h(Y)].$

Conditional Expectation and Variance

Conditional Variance:

Let $\mu_{X|Y}(y) = E[X|Y=y]$, **then**

$$\begin{aligned}\mathrm{Var}(X|Y=y) &= E\left[(X-\mu_{X|Y}(y))^2|Y=y\right]\\ &= E\left[X^2|Y=y\right]-\mu_{X|Y}(y)^2.\end{aligned}$$

Note that $\mathrm{Var}(X|Y=y)$ **is a function of** y**. We define** $\mathrm{Var}(X|Y)$ **is a function of** Y**. That is,** $\mathrm{Var}(X|Y)$ **is a random variable whose value equals** $\mathrm{Var}(X|Y=y)$.

Conditional Expectation and Variance

Example. Consider two random variables X and Y with joint PMF given in the following Table. Let $V = \text{Var}(X|Y)$.

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
$X=1$	$\frac{2}{5}$	0

a) Find the PMF of V.

b) Find EV.

c) Check $\text{Var}(X) = E(V) + \text{Var}(Z)$.

Conditional Expectation and Variance

Law of Total Variance:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

Conditional Expectation and Variance

Example. Let N be the number of customers that visit a certain store in a given day. Suppose that we know $E[N]$ and $\mathrm{Var}(N)$. Let X_i be the amount that the $i\mathrm{th}$ customer spends on average. We assume X_i's are independent of each other and also independent of N. We further assume they have the same mean and variance

$$\begin{aligned} EX_i &= EX, \\ \mathrm{Var}(X_i) &= \mathrm{Var}(X). \end{aligned}$$

Conditional Expectation and Variance

Let Y be the store's total sales, i.e.,

$$Y = \sum_{i=1}^{N} X_i.$$

Find EY and $\mathrm{Var}(Y)$.

Two Continuous Random Variables

PDF:

$$P(X \in A) = \int_A f_X(x)dx,$$

Joint PDF:

$$P\big((X, Y) \in A\big) = \iint\limits_A f_{XY}(x, y)dxdy.$$

Two Continuous Random Variables

If we choose $A = \mathbb{R}^2$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Two Continuous Random Variables

Definition. Two random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{XY}: \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \iint\limits_A f_{XY}(x,y)dxdy,$$

The function $f_{XY}(x,y)$ is called **the joint probability density function (PDF)** of X and Y.

Two Continuous Random Variables

CDF:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du,$$

Joint CDF:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)dudv,$$

and

Two Continuous Random Variables

$$f_X(x) = \frac{d}{dx} F_X(x),$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Two Continuous Random Variables

Example. **Let X and Y be two random variables with joint PDF given by**

$$f_{XY}(x,y) = \begin{cases} c(x+2y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

a) **Find c.**

b) **Find $F_{XY}(x,y)$.**

Two Continuous Random Variables

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)\,du\,dv,$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Two Continuous Random Variables

The joint cumulative function of two random variables X and Y is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

The joint CDF satisfies the following properties:

1) $F_X(x) = F_{XY}(x, \infty)$, for any x (marginal CDF of X);

2) $F_Y(y) = F_{XY}(\infty, y)$, for any y (marginal CDF of Y);

3) $F_{XY}(\infty, \infty) = 1$;

Two Continuous Random Variables

4) $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0;$

5) $P(x_1 < X \le x_2,\ \ y_1 < Y \le y_2) =$

$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

6) if X and Y are independent, then $F_{XY}(x, y) = F_X(x)F_Y(y)$.

Two Continuous Random Variables

Marginal PDFs:

For discrete random variables:

$$P_X(x) = \sum_{y \in R_Y} P_{XY}(x, y).$$

For continuous random variables:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy, \qquad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx, \qquad \text{for all } y.$$

Two Continuous Random Variables

Example. Let X and Y be two random variables with joint PDF given by

$$f_{XY}(x,y) = \begin{cases} \frac{2}{3}(x+2y) & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Conditioning and Independence

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

conditional CDF:

$$F_{X|A}(x) = P(X \leq x|A).$$

Conditioning and Independence

Example. Let $A : a \le X \le b,\ X$: continuous

$$F_{X|A}(x) = P(X \le x|A) = P(X \le x|a \le X \le b) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > b \end{cases}$$

If $a \le x \le b$:

$$F_{X|A}(x) = \frac{P(X \le x, a \le X \le b)}{P(a \le X \le b)} = \frac{P(a \le X \le x)}{P(a \le X \le b)}$$

$$= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}.$$

Conditioning and Independence

Finally, if $x > b$, then $F_{X|A}(x) = 1$. Thus, we obtain

$$F_{X|A}(x) = \begin{cases} 1 & x > b \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

$$A : \{a \leq X \leq b\}$$

Then the conditional PDF of X given A is given by

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{F_X(b) - F_X(a)} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Conditioning and Independence

In general, for a random variable X and an event A, we have the following:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx,$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx,$$

$$\mathrm{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Conditioning and Independence

Conditioning by Another Random Variable:

For discrete random variables: the conditional PMF of X given $Y = y$ is given by

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}.$$

Conditioning and Independence

For continuous random variables:

The conditional PMF of X given $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

The conditional PMF of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Conditioning and Independence

Example. **Choose a point (X, Y) Uniformly at random in the unit disc ,**
$\{(x, y) : \ x^2 + y^2 \leq 1\}$.

a) **Find** $f_{XY}(x, y), \ f_X(x)$ **and** $f_Y(y)$.

b) **Find** $f_{Y|X}(y|x)$.

c) **Find** $E[Y|X = \frac{\sqrt{3}}{2}], \ \mathrm{Var}(Y|X = \frac{\sqrt{3}}{2})$.

Conditioning and Independence

For two jointly continuous random variables X and Y, we have

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx,$$

$$E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)dx,$$

$$\mathrm{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

Summary of Conditioning

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

conditional CDF and PDF given A : (e.g., $A = \{a \leq X \leq b\}$)

$$F_{X|A}(x) = P(X \leq x|A),$$

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x).$$

Summary of Conditioning

In general, for a random variable X and an event A, we have the following:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx,$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx,$$

$$\mathrm{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Summary of Conditioning

Conditioning by Another Random Variable:

For discrete random variables: the conditional PMF of X given $Y = y$ is given by

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}.$$

Summary of Conditioning

For continuous random variables:

The conditional PMF of X given $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$

The conditional PMF of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

Summary of Conditioning

For two jointly continuous random variables X and Y, we have

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx,$$

$$E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)dx,$$

$$\mathrm{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)dxdy.$$

Conditioning and Independence

Law of Total Probability:

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) f_X(x)\, dx,$$

Law of Total Expectation:

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x)\, dx = E[E[Y|X]].$$

Conditioning and Independence

Law of Total Variance:

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X]).$$

Conditioning and Independence

Example. Suppose $X \sim Uniform(0,1)$ and given $X = x$, then

$$Y|X = x \quad \sim \quad Uniform(0, x).$$

Find $P(Y \leq X^2)$.

Conditioning and Independence

Independent Random Variables:

Discrete:

$$P_{XY}(x_i, y_j) = P_X(x_i) P_Y(y_j) \quad \text{for all } x_i, y_j.$$

Continuous:

$$f_{XY}(x, y) = f_X(x) f_Y(y), \quad \text{for all } x, y.$$

#:

$$F_{XY}(x, y) = F_X(x) F_Y(y), \quad \text{for all } x, y.$$

Functions of Two Continuous Random Variables

LOTUS for two continuous random variables:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y).$$

Functions of Two Continuous Random Variables

Example. Let X and Y be two independent $Uniform(0, 1)$ random variables, and $Z = XY$. Find the CDF and PDF of Z.

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Functions of Two Continuous Random Variables

Theorem. Let X and Y be two jointly continuous random variables. Let $(Z, W) = g(X, Y) = (g_1(X, Y), g_2(X, Y))$, where $g: \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a continuous one-to-one (invertible) function with continuous partial derivatives. Let $h = g^{-1}$, i.e., $(X, Y) = h(Z, W) = (h_1(Z, W), h_2(Z, W))$. Then Z and W are jointly continuous and their joint PDF, $f_{ZW}(z, w)$, for $(z, w) \in R_{ZW}$ is given by

$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w))|J|,$$

Functions of Two Continuous Random Variables

Where J is the Jacobian of h defined by

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z} . \frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial w} .$$

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Functions of Two Continuous Random Variables

Example.

Let X and Y be two independent random variables with PDF $f_{XY}(x, y)$ defined

$$\begin{cases} Z = X + Y \\ W = X \end{cases}$$

Find $f_{ZW}(z, w)$ and $f_Z(z)$.

Functions of Two Continuous Random Variables

If X and Y are two jointly continuous random variables and $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z-w)dw = \int_{-\infty}^{\infty} f_{XY}(z-w, w)dw.$$

If X and Y are also independent, then

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w)dw. \end{aligned}$$

Summary of Independence

Two continuous random variables X and Y are independent if

$$f_{XY}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x, y.$$

Conditioning and Independence

Example. **Determine whether X and Y are independent:**

$$f_{XY}(x, y) = e^{-(x+y)} \;, \quad x, y \geq 0$$

Conditioning and Independence

Two continuous random variables X and Y are independent, then we have

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,\overbrace{f_{XY}(x,y)}^{f_X(x)f_Y(y)}\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} yf_Y(y)\left[\int_{-\infty}^{\infty} xf_X(x)dx\right]dy = \underbrace{\int_{-\infty}^{\infty} xf_X(x)dx}_{EX}\underbrace{\int_{-\infty}^{\infty} yf_Y(y)dy}_{EY}
\end{aligned}
$$

Conditioning and Independence

More generally: X and Y independent

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$

Covariance and Correlation

The covariance between X and Y is defined as

$$\mathrm{Cov}(X,Y) = E\big[(X - EX)(Y - EY)\big] = E[XY] - (EX)(EY).$$

Proof:

$$\begin{aligned} E\big[(X - EX)(Y - EY)\big] &= E\big[XY - X(EY) - (EX)Y + (EX)(EY)\big] \\ &= E[XY] - (EX)(EY) - (EX)(EY) + (EX)(EY) \\ &= E[XY] - (EX)(EY). \end{aligned}$$

Covariance and Correlation

Example. **Suppose** $X \sim Uniform(1,2)$ **and given** $X = x$**,** Y **is**

$$Y|X = x \sim Exponential(\lambda = x).$$

Find $\mathrm{Cov}(X, Y)$.

$$f_X(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{else} \end{cases}, \quad f_{Y|X}(y|x) = \lambda e^{-\lambda y} u(y).$$

Covariance and Correlation

Lemma. **The covariance has the following properties:**

1) $\mathrm{Cov}(X,X) = E[XX] - EXEX = E[X^2] - (EX)^2 = \mathrm{Var}(X).$

2) $X \& Y$ **independent:**

$$\mathrm{Cov}(X,X) = E[XY] - EXEY = E[X]E[Y] - EXEY = 0.$$

3) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

4) $\mathrm{Cov}(aX,Y) = a\mathrm{Cov}(X,Y) \quad a \in \mathbb{R}$

Covariance and Correlation

5) $\text{Cov}(X + c, Y) = \text{Cov}(X, Y)$

6) $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

7) $\text{Cov}(X+Y, Z+W) = \text{Cov}(X, Z)+\text{Cov}(X, W)+\text{Cov}(Y, Z)+\text{Cov}(Y, W)$

$$\text{Cov}(2X + Y, 3Z + W) = 6\text{Cov}(X, Z) + 2\text{Cov}(X, W) + 3\text{Cov}(Y, Z) + \text{Cov}(Y, W)$$

Covariance and Correlation

More generally

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \mathrm{Cov}(X_i, Y_j).$$

Covariance and Correlation

Variance of a sum:

If $Z = X + Y$, **then**

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Cov}(Z, Z) \\
&= \mathrm{Cov}(X + Y, X + Y) \\
&= \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) + \mathrm{Cov}(Y, Y) \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y).
\end{aligned}$$

Covariance and Correlation

More generally, for $a, b \in \mathbb{R}$, we conclude:

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y).$$

Covariance and Correlation

Correlation Coefficient:

$$\rho_{XY} = \rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var(X)\ Var(Y)}}} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

Covariance and Correlation

Properties of the correlation coefficient:

1) $-1 \leq \rho(X,Y) \leq 1;$

2) $\rho(aX+b, cY+d) = \rho(X,Y) \text{ for } a,c > 0;$

3) $\rho(X,Y) = 1 \quad \text{if } Y = aX + b \;\; a > 0;$
$\rho(X,Y) = -1 \;\; \text{if } Y = aX + b \;\; a < 0.$

Covariance and Correlation

Definition. Consider two random variables X and Y :

1) If $\rho(X,Y) = 0$, we say that X and Y are uncorrelated.

2) If $\rho(X,Y) > 0$, we say that X and Y are positively correlated.

3) If $\rho(X,Y) < 0$, we say that X and Y are negatively correlated.

Bivariate Normal Distribution

Definition. Two random variables X and Y are said to be **bivariate normal**, or **jointly normal**, if $aX + bY$ has a normal distribution for all $a, b \in \mathbb{R}$.

$$\boldsymbol{b = 0,\ a = 1 \ \longrightarrow X : \text{Normal}}$$

$$\boldsymbol{a = 0,\ b = 1 \ \longrightarrow Y : \text{Normal}}$$

Bivariate Normal Distribution

Example. **Let X and Y be two bivariate normal random variables, i.e.,**

$$X \sim N(1,4),\ Y \sim N(0,1),\ \text{and } \mathrm{Cov}(X,Y) = 1.$$

If $Z = 3X + 2Y$, find the PDF of Z.

Bivariate Normal Distribution

Definition. Two random variables X and Y are said to have a **bivariate normal distribution** with parameters $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ and ρ, if their joint PDF is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

Where $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $\rho \in (-1,1)$ are all constants.

Bivariate Normal Distribution

If $\rho = 0$ ($X\&Y$ are uncorrelated) :

$$f_{XY}(x,y) = c \cdot \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}$$

$$= c' \underbrace{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}}_{\text{Function of } X} . \, d \underbrace{\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}}_{\text{Function of } Y}$$

Bivariate Normal Distribution

Theorem. If X and Y are bivariate normal and uncorrelated, then they are independent.

$$f_{XY}(x,y) = f_X(x) f_Y(y).$$

Bivariate Normal Distribution

Example. Let X and Y be two bivariate normal random variables. Find the conditional PDF of $Y|X = x$.

Bivariate Normal Distribution

Theorem. Suppose X and Y are jointly normal random variables with parameters $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ and ρ. Then, given $X = x$, Y is normally distributed with

$$E[Y|X = x] = \mu_Y + \rho\sigma_Y \frac{x - \mu_X}{\sigma_X},$$

$$\mathrm{Var}(Y|X = x) = (1 - \rho^2)\sigma_Y^2.$$

Bivariate Normal Distribution

Example. Let X and Y be two bivariate normal random variables and uncorrelated, $X \sim N(0,1),\ Y \sim N(1,4)$.

We define $Z = X - 2Y,\ W = 3X + 4Y,$

a) Find $EZ, EW, \mathrm{Var}(Z)$ and $\mathrm{Var}(W)$.

b) Find $\mathrm{Cov}(Z, W)$.

c) Find $f_{ZW}(z, w)$.

d) Find $f_{Z|W}(z|w)$.

Post-work for Lesson

- **Complete homework assignment for Lessons 10-13:**

HW#6 and HW#7

Go to the online classroom for details.

To Prepare for the Next Lesson

- Read Chapter 6 in your online textbook:

  https://www.probabilitycourse.com/chapter6/6_0_0_intro.php

- Complete the Pre-work for Lessons 14-16.

Visit the online classroom for details.