Generalized Fibonacci Sequence

In this project, I will explore some aspects of a new Fibonacci-like sequence and try to form conjectures with computational evidence. We saw that we can get new Fibonacci-like sequences by changing either a1, a2, b1, or b2 in the following:

$S_0 = \text{a1}$

$S_1 = \text{a2}$

$S_n = \text{b1}\, S_{n-1} + \text{b2}\, S_{n-2}$

In this project, I will be considering the sequence you get when you have

a1 = 1

a2 = 5

b1 = 1

b2 = 1

The first view terms in this sequence are:

1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309, 500, 809

Here is some preliminary code to get us started.

```
In[ ]:= gibonacciList[a_, b_, n_] := Module[{gibList = Table[0, n + 1]},
          (* set the initial values F0 and F1 *)
          gibList[[1]] = a;
          gibList[[2]] = b;

          (* compute F2 through Fn and store them in the list *)
          Do[
           gibList[[i]] = gibList[[i - 1]] + gibList[[i - 2]]
           , {i, 3, n + 1}];

          (* return the list *)
          gibList
         ]
```

```
In[ ]:= siblist = gibonacciList[1, 5, 10000];
        (*Generalized fibonacci sequence where intial terms are S0 = 1 and S1=5*)
```

```
In[ ]:= s[n_] := siblist[[n + 1]]
```

```
In[]:= Table[s[n], {n, 0, 20}]
Out[]= {1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309, 500,
 809, 1309, 2118, 3427, 5545, 8972, 14517, 23489, 38006}
```

Lets examine if $\frac{S_{n+1}}{S_n}$ converges to anything

```
In[]:= Table[s[n + 1]/s[n], {n, 1, 10}] // N
Out[]= {1.2, 1.83333, 1.54545, 1.64706, 1.60714, 1.62222, 1.61644, 1.61864, 1.6178, 1.61812}
```

It looks like $\frac{S_{n+1}}{S_n}$converges to the golden ratio. This makes sense, because I haven’t changed the basic recursive formula of the Fibonacci Sequence, only the initial values. We saw the Lucas sequence had a similar convergence, and it also had the same recursive formula but with different initial values.

CONJECTURE:

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n}=\left(\frac{1+\sqrt{5}}{2}\right)$$

Lets see if there is any pattern with the convergence of $\frac{S_{n+m}}{S_n}$

There were also identities in the form $\frac{S_{n+m}}{S_n}$ for the Fibonnacci and Lucas sequence. Let’s see if there is one for S_n sequence

The code below makes a table where each column has a different value of m and each row increases n.

In[∘]:=
```
TableForm[
  Table[s[n + m]/s[n], {n, 1, 10}, {m, 1, 10}],
  TableHeadings → {{"n = 1", "n = 2", "..."}, {"m = 1", "m = 2", "..."}}] // N
```

Out[∘]//TableForm=

	m = 1	m = 2	...					
n = 1	1.2	2.2	3.4	5.6	9.	14.6	23.6	38.2
n = 2	1.83333	2.83333	4.66667	7.5	12.1667	19.6667	31.8333	51.5
...	1.54545	2.54545	4.09091	6.63636	10.7273	17.3636	28.0909	45.45
	1.64706	2.64706	4.29412	6.94118	11.2353	18.1765	29.4118	47.58
	1.60714	2.60714	4.21429	6.82143	11.0357	17.8571	28.8929	46.75
	1.62222	2.62222	4.24444	6.86667	11.1111	17.9778	29.0889	47.06
	1.61644	2.61644	4.23288	6.84932	11.0822	17.9315	29.0137	46.94
	1.61864	2.61864	4.23729	6.85593	11.0932	17.9492	29.0424	46.99
	1.6178	2.6178	4.2356	6.8534	11.089	17.9424	29.0314	46.97
	1.61812	2.61812	4.23625	6.85437	11.0906	17.945	29.0356	46.98

Take a look at the bottom row of this table, where we start to see convergence in each column. The number that each column is converging to is (golden ratio)^m

In[∘]:=
```
1.61812^2
```

Out[∘]= 2.61831

In[∘]:=
```
1.61812^3
```

Out[∘]= 4.23674

The Lucas sequence had the same pattern, that is $\lim_{n \to \infty}, \frac{L_{n+m}}{L_n} = \left(\frac{\sqrt{5}+1}{2}\right)^m$

CONJECTURE:

$$\lim_{n \to \infty}, \frac{S_{n+m}}{S_n} = \left(\frac{\sqrt{5}+1}{2}\right)^m$$

Is there an analog of Cassini’s identity for the Sibonacci Sequence?

$$S_{n+1} S_{n-1} - S_n^2 = ?$$

Cassini' s identity for the Fibonacci Sequence was the following :
$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

In[∘]:=
```
TableForm[
 Table[
  {n, s[n + 1] s[n - 1] - s[n]^2},
  {n, 1, 10}],
 TableHeadings → {{}, {"n", "S_{n+1} S_{n-1}-S_n^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+1}\ S_{n-1} - S_n^2$
1	-19
2	19
3	-19
4	19
5	-19
6	19
7	-19
8	19
9	-19
10	19

For different values of n, $S_{n+1}\ S_{n-1} - S_n^2$ oscillates between -19 and 19. This is similar to the analogous identity with the fibonacci sequence which oscillated between -1 and 1.

CONJECTURE:

$S_{n+1} S_{n-1} - S_n^2 = (-1)^n\ 19$

Is there a more general form to this conjecture? Let’s try

$S_{n+m} S_{n-m} - S_n^2 = ?$

For the fibonacci seqence, we saw that:

$F_{n+k}\,F_{n-k} - F_n{}^2 = (-1)^{n+k-1}\,F_k{}^2$

Is there a similar identity for the S_n?

I will check this by checking this identity for different values of m and see if there is a pattern.

In[∘]:=
```
(*m = 2 case*)TableForm[
 Table[
  {n, s[n + 2] s[n - 2] - s[n]^2},
  {n, 2, 10}],
 TableHeadings → {{}, {"n", "S_{n+2} S_{n-2}-S_n^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+2}\ S_{n-2}-S_n{}^2$
2	-19
3	19
4	-19
5	19
6	-19
7	19
8	-19
9	19
10	-19

In[∘]:=
```
(*m = 3 case*)TableForm[
 Table[
  {n, s[n + 3] s[n - 3] - s[n]^2},
  {n, 3, 10}],
 TableHeadings → {{}, {"n", "S_{n+3} S_{n-3}-S_n^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+3}\ S_{n-3}-S_n{}^2$
3	-76
4	76
5	-76
6	76
7	-76
8	76
9	-76
10	76

All ready, we notice this will be different than the Fibonacci case, because the oscillating values are not always the same depending on m.

In[∘]:=
```
(*m = 4 case*)TableForm[
 Table[
  {n, s[n + 4] s[n - 4] - s[n]^2},
  {n, 4, 10}],
 TableHeadings → {{}, {"n", "S_{n+4} S_{n-4}-S_n^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+4}\ S_{n-4}-S_n{}^2$
4	-171
5	171
6	-171
7	171
8	-171
9	171
10	-171

In[∘]:= `171 / 19`

Out[∘]= 9

One thing I am noticing is that the oscillating value is always divisible by 19.

In[∘]:=
```
(*m = 5 case*)TableForm[
 Table[
  {n, s[n + 5] s[n - 5] - s[n]^2},
  {n, 5, 10}],
 TableHeadings → {{}, {"n", "S_{n+5} S_{n-5}-S_n^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+5}\ S_{n-5}-S_n{}^2$
5	-475
6	475
7	-475
8	475
9	-475
10	475

In[∘]:=
```
(*m = 6 case*)TableForm[
 Table[
  {n, s[n + 6] s[n - 6] - s[n]^2},
  {n, 6, 10}],
 TableHeadings → {{}, {"n", "Sn+6 Sn-6-Sn^2"}}
]
```

Out[∘]//TableForm=

n	$S_{n+6}\ S_{n-6}-S_n{}^2$
6	-1216
7	1216
8	-1216
9	1216
10	-1216

In[∘]:= 475 / 19

Out[∘]= 25

In[∘]:= 1216 / 19

Out[∘]= 64

The magnitude of the number which oscillates is $19\,(1)^2$, $19\,(1)^2$, $19\,(2)^2$, $19\,(3)^2$, $19\,(5)^2$, $19\,(8)^2$, as m increases. The numbers in the parenthesis follow the Fibonacci sequence, which is mth term of the Fibonacci sequence. We can add this additional factor into our previous conjecture to get a more general conjecture:

CONJECTURE:

$$S_{n+m}\, S_{n-m} - S_n{}^2 = (-1)^{m+n+1}\,(19)\,(F_m)^2,$$

for n > m.

Polynomial Identities with S_n

Lets see if there is an identity in the form $S_{3\,n} = a\,S_n^3 + b\,S_n^2 + c\,S_n + d$

To investigate this, I will need 4 equations to solve for the 4 unknowns. I can use Solve[ ] to solve for the unknowns.

In[◦]:=
```
poly3[n_] := Module[{lhs, rhs},
  s[3 n] == a s[n]^3 + b s[n]^2 + c s[n] + d
 ]
```

In[◦]:=
```
Solve[
 {poly3[0], poly3[1], poly3[2], poly3[3]}, {a, b, c, d}
]
```

Out[◦]= $\left\{\left\{a \to -\frac{71}{100},\ b \to \frac{741}{50},\ c \to -\frac{6441}{100},\ d \to \frac{513}{10}\right\}\right\}$

Does this solve for any other value of n?

The code below will answer if $S_{3n} = a\,S_n^3 + b\,S_n^2 + c\,S_n + d$ is true or false with n as the argument of the poly3[ ] function. I can put the values of the a, b ,c d constants with /. {a->__ , b->__ , c->__ , d->__ }

In[◦]:= poly3[4] /. $\left\{a \to -\frac{71}{100},\ b \to \frac{741}{50},\ c \to -\frac{6441}{100},\ d \to \frac{513}{10}\right\}$

Out[◦]= False

In[◦]:= poly3[5] /. $\left\{a \to -\frac{71}{100},\ b \to \frac{741}{50},\ c \to -\frac{6441}{100},\ d \to \frac{513}{10}\right\}$

Out[◦]= False

In[◦]:= poly3[6] /. $\left\{a \to -\frac{71}{100},\ b \to \frac{741}{50},\ c \to -\frac{6441}{100},\ d \to \frac{513}{10}\right\}$

Out[◦]= False

Doesn’t look like there are any other solutions

Lets just just even n for $S_{3n} = a\,S_n^3 + b\,S_n^2 + c\,S_n + d$ and see if there is a common solution

In[◦]:=
```
Solve[
 {poly3[2], poly3[4], poly3[6], poly3[8]}, {a, b, c, d}
]
```

Out[◦]= $\left\{\left\{a \to \frac{4\,676\,731}{29\,521\,492},\ b \to -\frac{1368}{7\,380\,373},\ c \to \frac{53\,563\,527}{29\,521\,492},\ d \to -\frac{1\,445\,463}{14\,760\,746}\right\}\right\}$

In[◦]:= poly3[10] /. $\left\{a \to \frac{4\,676\,731}{29\,521\,492},\ b \to -\frac{1368}{7\,380\,373},\ c \to \frac{53\,563\,527}{29\,521\,492},\ d \to -\frac{1\,445\,463}{14\,760\,746}\right\}$

Out[◦]= False

Did not work for even n, lets try odd n

In[ ]:=
```
Solve[
 {poly3[1], poly3[3], poly3[5], poly3[7]}, {a, b, c, d}
]
```

Out[ ]= $\left\{\left\{a \to \frac{19199}{121210}, b \to \frac{152}{60605}, c \to -\frac{227563}{121210}, d \to \frac{6365}{12121}\right\}\right\}$

In[ ]:= `poly3[9] /.` $\left\{a \to \frac{19199}{121210}, b \to \frac{152}{60605}, c \to -\frac{227563}{121210}, d \to \frac{6365}{12121}\right\}$

Out[ ]= False

Did not work.

No identity with $S_{3n} = a\, S_n^3 + b\, S_n^2 + c\, S_n + d$

Lets try a polynomial identity in the form: $S_{4n} = a\, S_n^4 + b\, S_n^3 + c\, S_n{}^2 + d\, S_n + f$

In[ ]:=
```
poly4[n_] := Module[{},
  s[4 n] == a s[n]^4 + b s[n]^3 + c s[n]^2 + d s[n] + f
 ]
```

Lets try evens -->

In[ ]:=
```
Solve[
 {poly4[2], poly4[4], poly4[6], poly4[8], poly4[10] }, {a, b, c, d, f}
]
```

Out[ ]= $\left\{\left\{a \to \frac{355527805}{5638604972}, b \to \frac{513}{402757498}, c \to \frac{5403085385}{5638604972}, d \to \frac{105165}{15490673}, f \to \frac{2462261471}{1409651243}\right\}\right\}$

In[ ]:= `poly4[12] /.` $\left\{a \to \frac{355527805}{5638604972}, b \to \frac{513}{402757498}, c \to \frac{5403085385}{5638604972}, d \to \frac{105165}{15490673}, f \to \frac{2462261471}{1409651243}\right\}$

Out[ ]= False

Even n do not share a common solution, how about odd?

In[ ]:=
```
Solve[
 {poly4[1], poly4[3], poly4[5], poly4[7], poly4[9] }, {a, b, c, d, f}
]
```

Out[ ]= $\left\{\left\{a \to \frac{69733}{1105955}, b \to \frac{26334}{582838285}, c \to -\frac{560727544}{582838285}, d \to \frac{3215142}{34284605}, f \to \frac{136310750}{116567657}\right\}\right\}$

In[ ]:= `poly4[11] /.` $\left\{a \to \frac{69733}{1105955}, b \to \frac{26334}{582838285}, c \to -\frac{560727544}{582838285}, d \to \frac{3215142}{34284605}, f \to \frac{136310750}{116567657}\right\}$

Out[ ]= False

Odds do not share a common solution

Lets try a polynomial identity in the form :

$$S_{5\,n} = a\,S_n^5 + b\,S_n^4 + c\,S_n{}^3 + d\,S_n{}^2 + f\,S_n + g$$

In[∘]:=
```
poly5[n_] := Module[{},
  s[5 n] == a s[n]^5 + b s[n]^4 + c s[n]^3 + d s[n]^2 + f s[n] + g
 ]
```

Odds ->

In[∘]:=
```
Solve[
 {poly5[1], poly5[3], poly5[5], poly5[7], poly5[9], poly5[11]}, {a, b, c, d, f, g}
]
```

Out[∘]=
$$\left\{\left\{a \to \frac{5\,789\,729\,021\,737}{230\,704\,295\,513\,265},\ b \to \frac{2\,866\,853}{26\,366\,205\,201\,516},\ c \to -\frac{22\,002\,070\,580\,782}{46\,140\,859\,102\,653},\ d \to \frac{46\,091\,186\,537}{26\,366\,205\,201\,516},\ f \to \frac{816\,190\,792\,440\,911}{461\,408\,591\,026\,530},\ g \to \frac{1\,929\,602\,519\,213}{6\,591\,551\,300\,379}\right\}\right\}$$

In[∘]:=
$$\text{poly5[13] /. }\left\{a \to \frac{5\,789\,729\,021\,737}{230\,704\,295\,513\,265},\ b \to \frac{2\,866\,853}{26\,366\,205\,201\,516},\ c \to -\frac{22\,002\,070\,580\,782}{46\,140\,859\,102\,653},\ d \to \frac{46\,091\,186\,537}{26\,366\,205\,201\,516},\ f \to \frac{816\,190\,792\,440\,911}{461\,408\,591\,026\,530},\ g \to \frac{1\,929\,602\,519\,213}{6\,591\,551\,300\,379}\right\}$$

Out[∘]= False

Evens ->

In[∘]:=
```
Solve[
 {poly5[2], poly5[4], poly5[6], poly5[8], poly5[10], poly5[12]}, {a, b, c, d, f, g}
]
```

Out[∘]=
$$\left\{\left\{a \to \frac{195\,561\,010\,362\,989}{7\,792\,552\,071\,304\,000},\ b \to -\frac{23\,693}{20\,240\,394\,990\,400},\ c \to \frac{743\,132\,485\,098\,903}{1\,558\,510\,414\,260\,800},\ d \to -\frac{988\,954\,807}{20\,240\,394\,990\,400},\ f \to \frac{220\,850\,886\,651\,524}{121\,758\,626\,114\,125},\ g \to -\frac{1\,142\,372\,822\,733}{55\,661\,086\,223\,600}\right\}\right\}$$

In[∘]:=
$$\text{poly5[14] /. }\left\{a \to \frac{195\,561\,010\,362\,989}{7\,792\,552\,071\,304\,000},\ b \to -\frac{23\,693}{20\,240\,394\,990\,400},\ c \to \frac{743\,132\,485\,098\,903}{1\,558\,510\,414\,260\,800},\ d \to -\frac{988\,954\,807}{20\,240\,394\,990\,400},\ f \to \frac{220\,850\,886\,651\,524}{121\,758\,626\,114\,125},\ g \to -\frac{1\,142\,372\,822\,733}{55\,661\,086\,223\,600}\right\}$$

Out[∘]= False

Lets try a polynomial identity in the form :

$$S_{6n} = a\, S_n^6 + b\, S_n^5 + c\, S_n{}^4 + d\, S_n{}^3 + f\, S_n{}^2 + g\, S_n + h$$

In[∘]:=
```
poly6[n_] := Module[{},
  s[6 n] == a s[n]^6 + b s[n]^5 + c s[n]^4 + d s[n]^3 + f s[n]^2 + g s[n] + h
 ]
```

In[∘]:=
```
Solve[{poly6[1], poly6[3], poly6[5], poly6[7], poly6[9], poly6[11], poly6[13]},
 {a, b, c, d, f, g, h}]
```

Out[∘]=
$$\left\{\left\{a \to \frac{17\,083\,992\,593\,301\,652}{1\,710\,354\,660\,069\,825\,315},\ b \to \frac{921\,272}{5\,429\,697\,333\,555\,001},\ c \to -\frac{25\,967\,679\,983\,805\,920}{114\,023\,644\,004\,655\,021},\ d \to \frac{106\,162\,530\,419}{5\,429\,697\,333\,555\,001},\ f \to \frac{739\,290\,421\,413\,209\,476}{570\,118\,220\,023\,275\,105},\ g \to \frac{182\,034\,250\,381\,577}{5\,429\,697\,333\,555\,001},\ h \to -\frac{3\,801\,834\,161\,125\,615}{2\,874\,545\,647\,176\,177}\right\}\right\}$$

In[∘]:=
```
poly6[3]
```

Out[∘]=
$$14\,517 == 1\,771\,561\, a + 161\,051\, b + 14\,641\, c + 1331\, d + 121\, f + 11\, g + h$$

In[∘]:=
```
Solve[{poly6[2], poly6[4], poly6[6], poly6[8], poly6[10], poly6[12], poly6[14]},
 {a, b, c, d, f, g, h}]
```

Out[∘]=
$$\left\{\left\{a \to \frac{2\,925\,637\,918\,505\,616\,163}{292\,898\,654\,704\,103\,448\,000},\ b \to \frac{638\,609}{915\,308\,295\,950\,323\,275},\ c \to \frac{6\,670\,454\,434\,998\,159\,683}{29\,289\,865\,470\,410\,344\,800},\ d \to \frac{3\,072\,149\,358\,823}{14\,644\,932\,735\,205\,172\,400},\ f \to \frac{7\,455\,076\,792\,625\,879\,337}{5\,743\,110\,876\,551\,048\,000},\ g \to \frac{4\,507\,061\,457\,997\,617}{4\,881\,644\,245\,068\,390\,800},\ h \to \frac{62\,392\,675\,604\,946\,073}{57\,431\,108\,765\,510\,480}\right\}\right\}$$

I am struggling to find a polynomial identity for the S_n's

I will try a different method, to quickly find solutions

This is a modified version of the matrix solver from class (I added 1 more column in the matrix for the $S[n]^0$ term). The Module below checks even n's and finds the solution to for a given q for:

$S_{q\,n} = \sum^{q}{}_{i=0}\ a_i\, S_n^i$, for n = 2 to n = 2q + 2

```
In[]:= findsibCoefficients[q_] := Module[{nvals, matrix, vector, sol},
  (* set up the nvals *)
  nvals = Range[2, 2 q + 2, 2];
  (*Print["nvals:",nvals];*)

  (* set up the matrix *)
  matrix = Table[s[n]^i, {n, nvals}, {i, q, 0, -1}];
  (*Print["matrix:",MatrixForm[matrix]];*)

  (* set up the vector *)
  vector = Table[s[q n], {n, nvals}];
  (*Print["vector:",vector];*)

  (* solve the system *)
  sol = LinearSolve[matrix, vector ];
  (*Print["solution:",sol];*)

  (* return the solution *)
  sol
 ]

In[]:= findsibCoefficients[2]
```

Out[]= $\left\{\frac{1591}{4004}, \frac{171}{4004}, \frac{4883}{2002}\right\}$

```
In[]:= {a1, a2, a3}.{b1, b2, b3}
```

Out[]= a1 b1 + a2 b2 + a3 b3

The following module will use the findsibCoefficients[ ] module (which is the same as the one we made in class the find the coefficients of the Fibonacci polynomial identities) and then solve for the next even n equation. For example, if we did solvenextterm[q = 2], it would find the a, b , and c coefficients for $S_{2n} = a\, S_n^2 + b\, S_n + c$ using this equation when n = 2 , n = 4, and n = 6 and then see if those coefficients satisfy the next equation which is the case when n = 8. If it does satisfy the next equation, it will return true, if it does not, it will return false. This module only uses even n’s because we saw that there were common solutions to even n’s for the fibonacci and lucas sequences to their polynomial identities, and I am seeing if there is a similar pattern here for the S_n sequence.

```
In[]:= solvenextterm[q_] := Module[{lhs, rhs},
  lhs = s[q (2 q + 4)] (*q times n in the argument, where n is the new n*);
  rhs =
   findsibCoefficients[q].
    Table[s[2 q + 4]^i, {i, q, 0, -1}];
  (* this dot product just writes the Equation for the next n case,
  connecting the coefficients to the s[n]^i's*)
  (*Print[rhs];*)
  lhs == rhs]
```

In[•]:=
```
findsibCoefficients[2]
solvenextterm[2]
```

Out[•]= $\left\{\frac{1591}{4004}, \frac{171}{4004}, \frac{4883}{2002}\right\}$

$\frac{792251}{143}$

Out[•]= False

In[•]:=
```
findsibCoefficients[2].
 Table[s[8]^i, {i, 2, 0, -1}]
```

Out[•]= $\frac{792251}{143}$

The code above shows that my module works as indended

solvenextterm[q_] solves the matrix: $S_{q\,n} = \sum^{q}{}_{i=0}\; a_i\, S_n^i$, for n = 2 to n = 2q + 2.

Then it takes those a_i coefficients and tests them for the next even case, where n = 2q + 4.

I will test this for many q-->

In[•]:=
```
Table[solvenextterm[q], {q, 2, 50}]
```

Out[•]=
```
{False, False, False, False, False, False, False, False, False, False, False, False, False,
 False, False, False, False, False, False, False, False, False, False, False, False,
 False, False, False, False, False, False, False, False, False, False, False, False,
 False, False, False, False, False, False, False, False, False, False, False, False}
```

The code above just checked if there are common solutions for even n ' s for each of the $S_{q\,n}$ set of equaitons. Because we only have false outputs, there is no polynomial identity $S_{q\,n} = \sum_{i=0}^{q} a_i\, S_n^i$ which satisfy for even n ' s for any q between 2 and 50.

Is there an identity that involves $S_n\,F_n$?

Preliminary code

In[∘]:=
```
fibonacciList[n_] := Module[{fibList = Table[0, n + 1]},
  (* set the initial values F0 and F1 *)
  fibList[[2]] = 1;

  (* compute F2 through Fn and store them in the list *)
  Do[
   fibList[[i]] = fibList[[i - 1]] + fibList[[i - 2]]
   , {i, 3, n + 1}];

  (* return the list *)
  fibList
 ]
```

In[∘]:=
```
fibList = fibonacciList[1000];
```

In[∘]:=
```
f[n_] := fibList[[n + 1]]
```

In[∘]:=
```
Table[f[n] s[n], {n, 0, 10}]
```

Out[∘]=
```
{0, 5, 6, 22, 51, 140, 360, 949, 2478, 6494, 16995}
```

In[∘]:=
```
TableForm[
 Table[{n, f[n] s[n], f[n], s[n]}, {n, 0, 10}],
 TableHeadings → {{}, {"n", "f[n]s[n]", "f[n]", "s[n]"}}
]
```

Out[∘]//TableForm=

n	f[n]s[n]	f[n]	s[n]
0	0	0	1
1	5	1	5
2	6	1	6
3	22	2	11
4	51	3	17
5	140	5	28
6	360	8	45
7	949	13	73
8	2478	21	118
9	6494	34	191
10	16995	55	309

Above I have a list of F_n S_n values. After taking some time to look at them, I cannot discern any identity with them.