Time Series Analysis AR, MA and ARIMA Models

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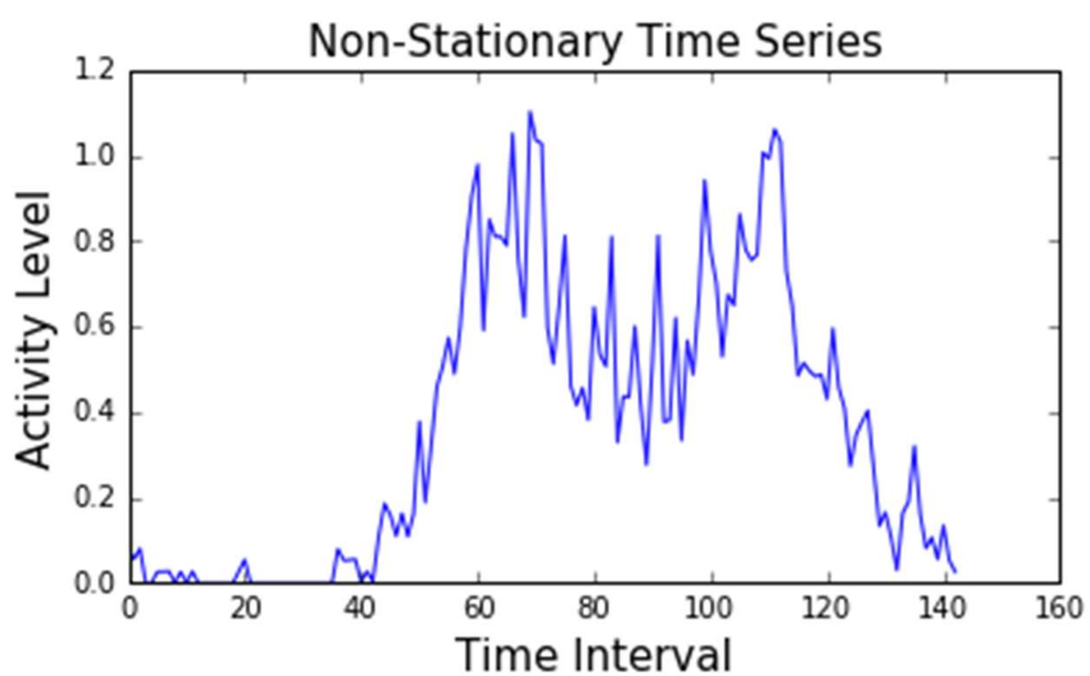
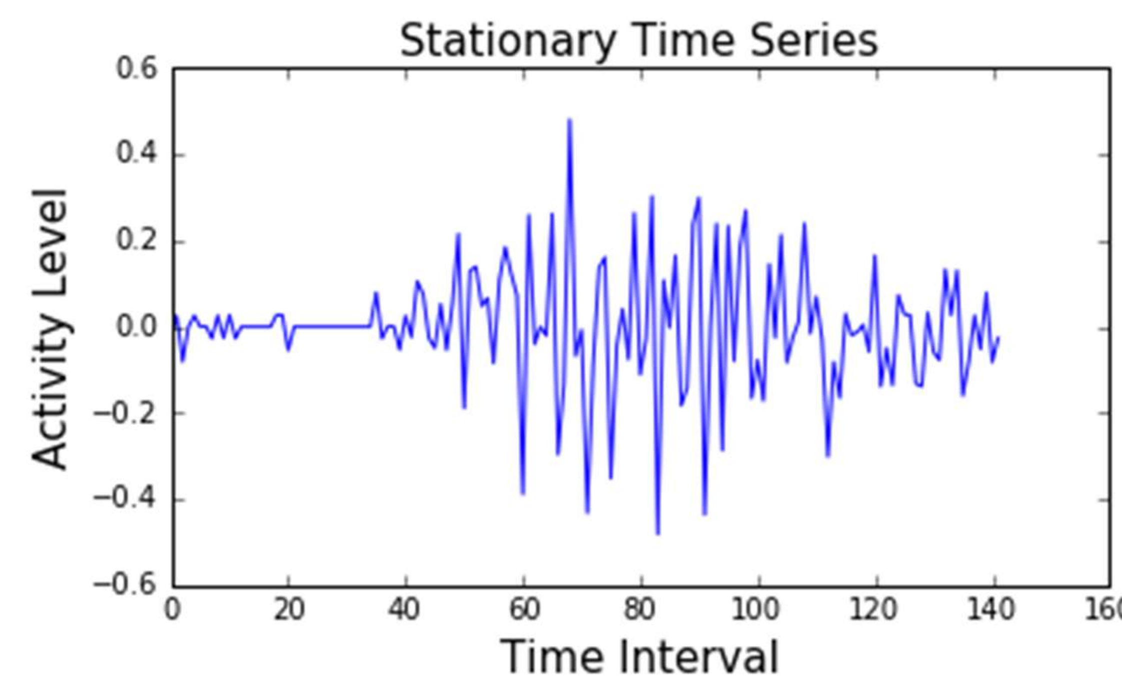
Stationarity in Time Series

- **What is Stationarity**
  - In the most intuitive sense, stationarity means that the **statistical property** (mainly mean and variance) of a process generating time series do not change over time.
  - It does not mean that the series does not change over time, just that the *way* it changes does not itself change over time.

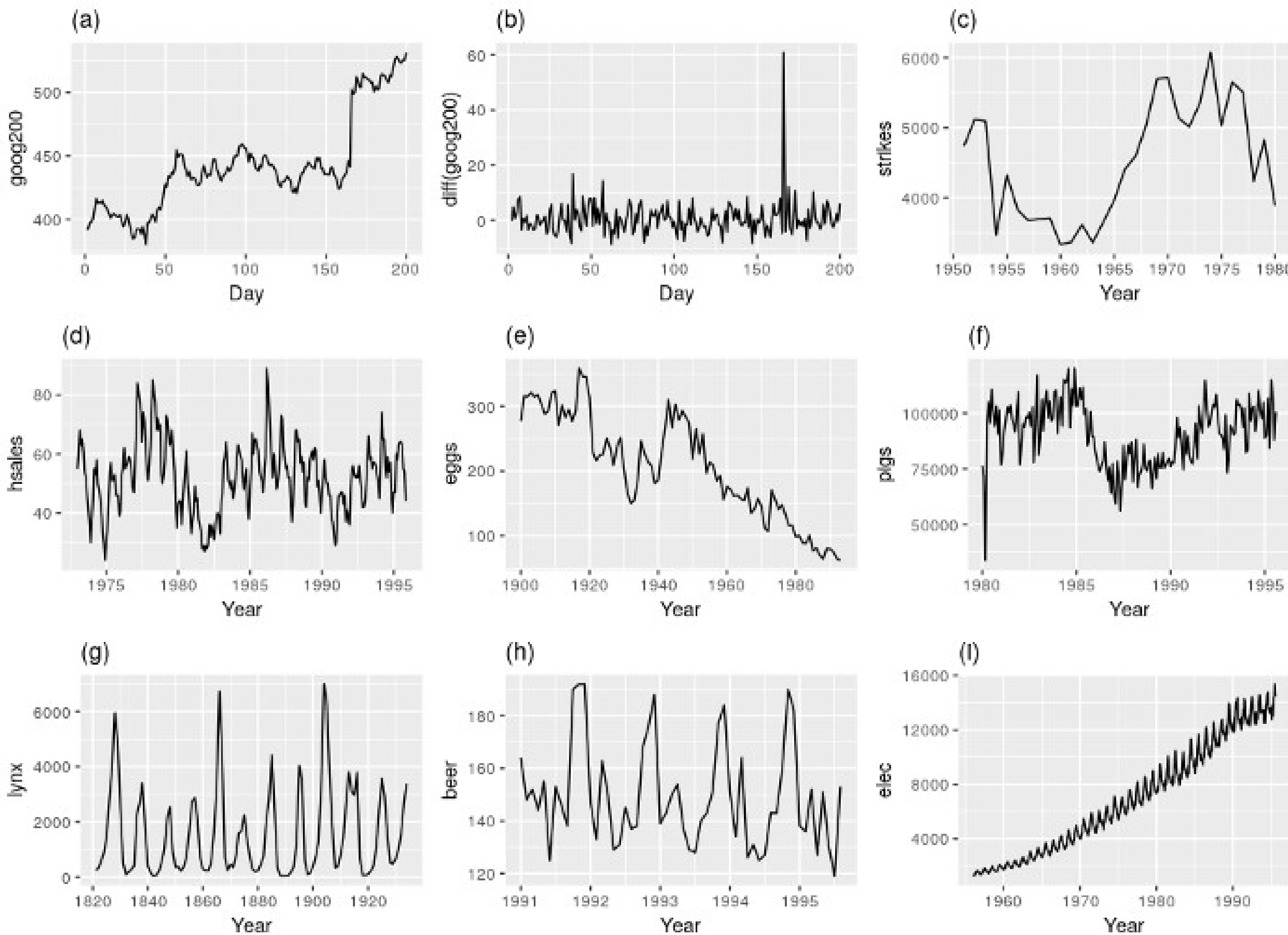




- **Why is it important**
  - Stationary data are much more simpler to analyse and model. Hence, for time series analysis and modelling if a series in non-stationary we make them stationary by differencing.

Detecting the stationarity in Time Series

- **Visualization**



$s(t) = \; 100 \quad 105 \quad 110 \quad 115 \quad 120 \ldots$

$s(t-1) = \ldots \quad 100 \quad 105 \quad 110 \quad 115 \ldots$

$5 \quad 5 \quad 5 \quad 5 \ldots$

(a) Google stock price for 200 consecutive days;
(b) Daily change in the Google stock price for 200 consecutive days;
(c) Annual number of strikes in the US;
(d) Monthly sales of new one-family houses sold in the US;
(e) Annual price of a dozen eggs in the US (constant dollars);
(f) Monthly total of pigs slaughtered in Victoria, Australia;
(g) Annual total of lynx trapped in the McKenzie River district of north-west Canada;
(h) Monthly Australian beer production;
(i) Monthly Australian electricity production.

$$\frac{s(t) - s(t-1)}{d(t) - d(t-1)}$$

Detecting the stationarity in Time Series

- **Statistical Tests**
  - Sometimes identifying the stationarity of a time series can't be done just by visualizing it.
  - Hence, we need to perform Statistical Tests in order to check if the time series is stationary.

- **The Dickey Fuller Tests**

– The Dickey-Fuller test was the first statistical test developed to test the null hypothesis that a unit root is present in an autoregressive model of a given time series, and that the process is thus not stationary. The original test treats the case of a simple lag-1 AR model.

- **The KPSS Tests**

– Another prominent test for the presence of a unit root is the KPSS test. [Kwiatkowski et al, 1992] Conversely to the Dickey-Fuller family of tests, the null hypothesis assumes stationarity around a mean or a linear trend, while the alternative is the presence of a unit root.

There are other tests to determine the stationarity of time series models.

Auto-Regressive Models

- **Intuition**
  - Auto-Regressive models are based on the idea that current value of the series, X_t, can be explained as a linear combination of p past values $X_{t-1}, X_{t-2}, \dots, X_{t-p}$, together with a random residual error (white noise) in the same series.

- **Definition**
  - An auto-regressive model of order $\boldsymbol{p}$, abbreviated as $AR(p)$ is of the form:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \dots + \phi_p X_{t-p} + \epsilon_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t$$

Where X_t is a stationary and ϵ_t is random residual error (usually modelled by normal distribution with zero mean) and $\phi_1, \phi_2, \dots, \phi_p$ ($\phi_p \neq 0$) are model parameters. The hyperparameter p represents the length of the "direct look back" in the series (i.e. how many past time stamps we want to look into). p is usually determined by model validation methods.

Auto-Regressive Models

- The simplest of AR process is $AR(0)$, which has no dependence between the terms. In fact, $AR(0)$ is just random residual error (also called white noise).

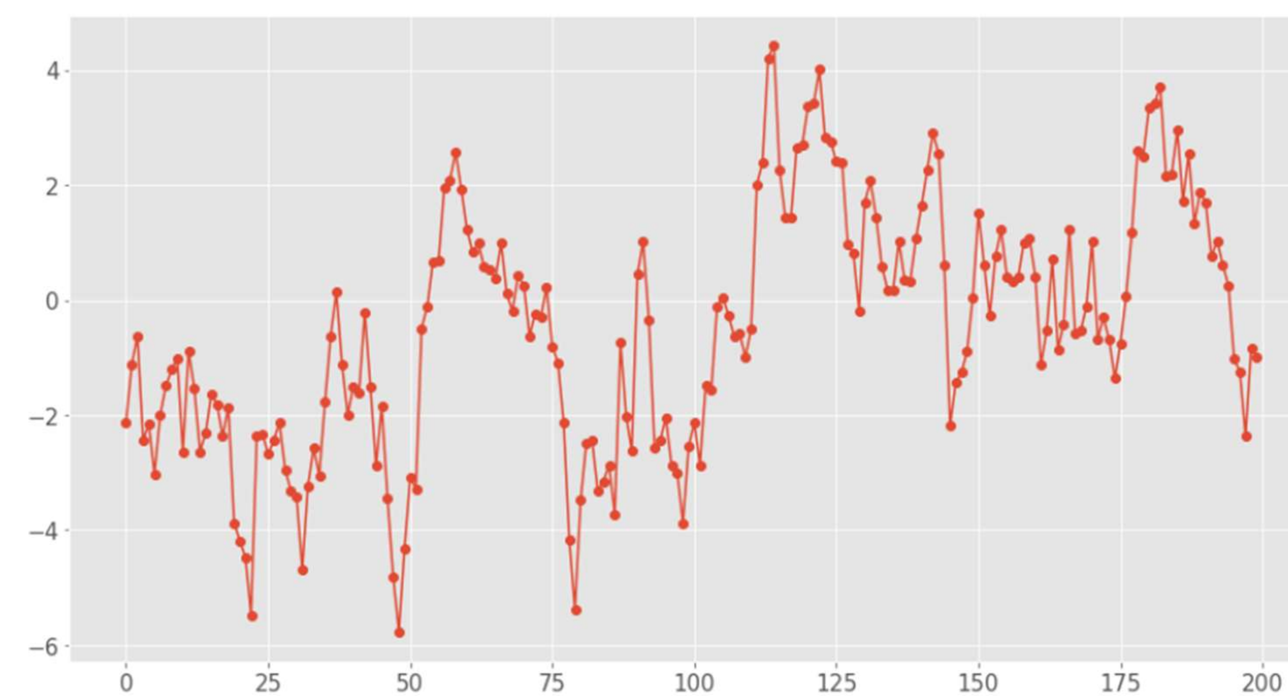
- $AR(1)$ can be given by $X_t = \phi_1 X_{t-1} + \epsilon_t$
  - Only the previous term in the process and the noise term contribute to the output.
  - If $|\phi_1|$ is close to 0, then the process still looks like a white noise.
  - If $\phi_1 < 0, X_t$ tends to oscillate between positive and negative values.
  - If $\phi_1 = 1$ then the process is equivalent to a random walk.

- Simulated $AR(1)$ process: $X_t = 0.9X_{t-1} + \epsilon_t$

  Mean: $E[X_t] = 0$

  Variance: $Var(X_t) = \frac{\sigma_\epsilon^2}{(1 - \phi_1^2)}$

  Where, σ_ϵ^2 is the variance of noise.



Moving Average (MA) Models

- The name might be misleading but moving average models should not be confused with the moving average smoothing.

- **Motivation**
  - Recall that in AR models, current observation X_t is regressed using the previous / past observations like: $X_{t-1}, X_{t-2}, \ldots, X_{t-p}$ plus an error term: ϵ_t (also called noise) at current time point.
  - One problem of AR model is the ignorance of the correlated noise structures (which is unobservable) in the time series.
  - In other words, the imperfectly predictable terms in the current time ϵ_t, and previous steps $\epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon_{t-q}$ are also informative for predicting observations.

- **Definition**
  - A moving average model of order $\boldsymbol{q}$, abbreviated as $MA(q)$ is of the form:

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3} + \ldots + \theta_q \epsilon_{t-q} = \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}$$

Where $\theta_q \neq 0$

Moving Average (MA) Models

- Although the equation of moving average (MA) model looks like a regression model, the difference is that the ϵ_t is not observable.

- Contrary to AR model, finite MA model is always stationary. (Yes, AR process can be non-stationary, for example: random walk)

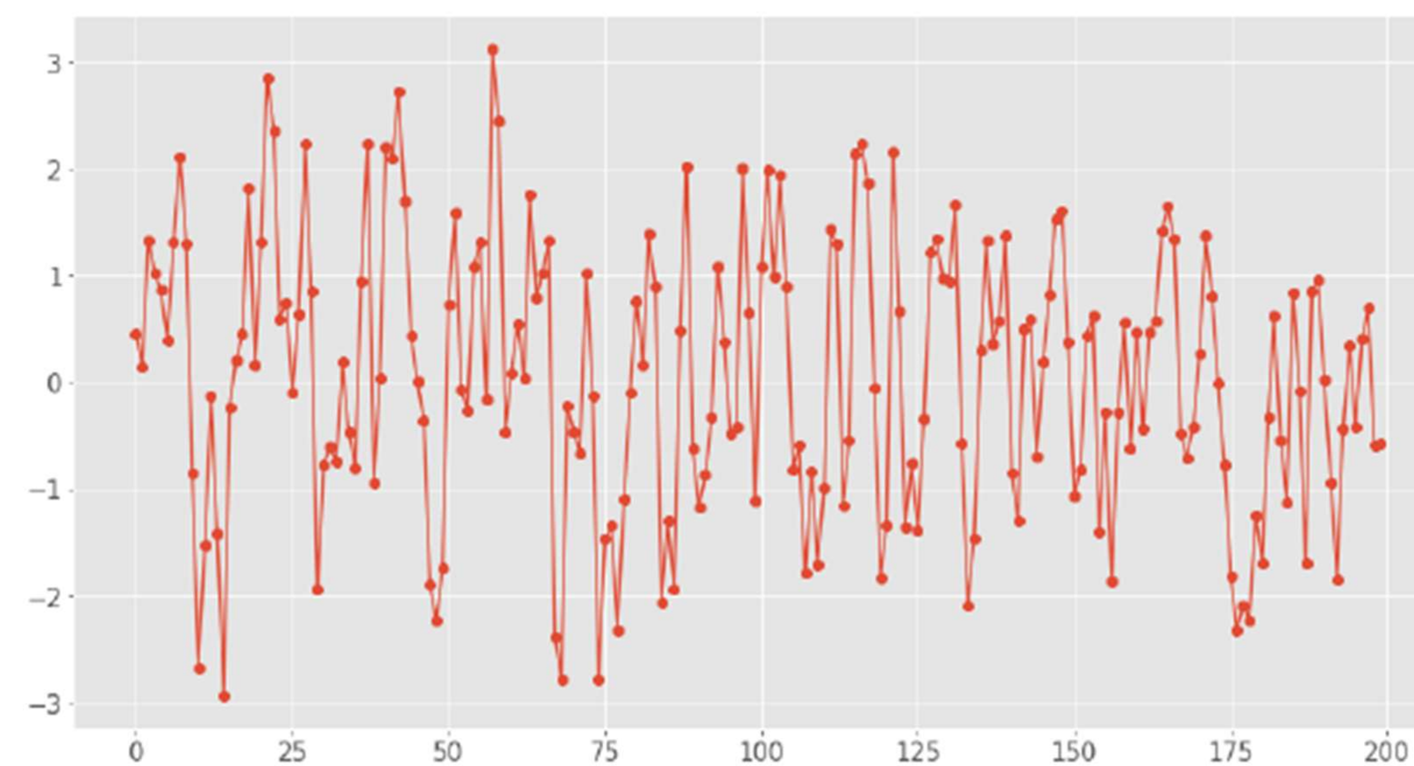
- Finite MA models are stationary because the observation is just a weighted moving average over past forecast errors.

- Simulated $MA(1)$ process: $X_t = \epsilon_t + 0.8\ \epsilon_{t-1}$

  Mean: $E[X_t] = 0$

  Variance: $Var(X_t) = \sigma_\epsilon^2(1 + \theta_1^2)$

  Where, σ_ϵ^2 is the variance of noise.



ARMA Models

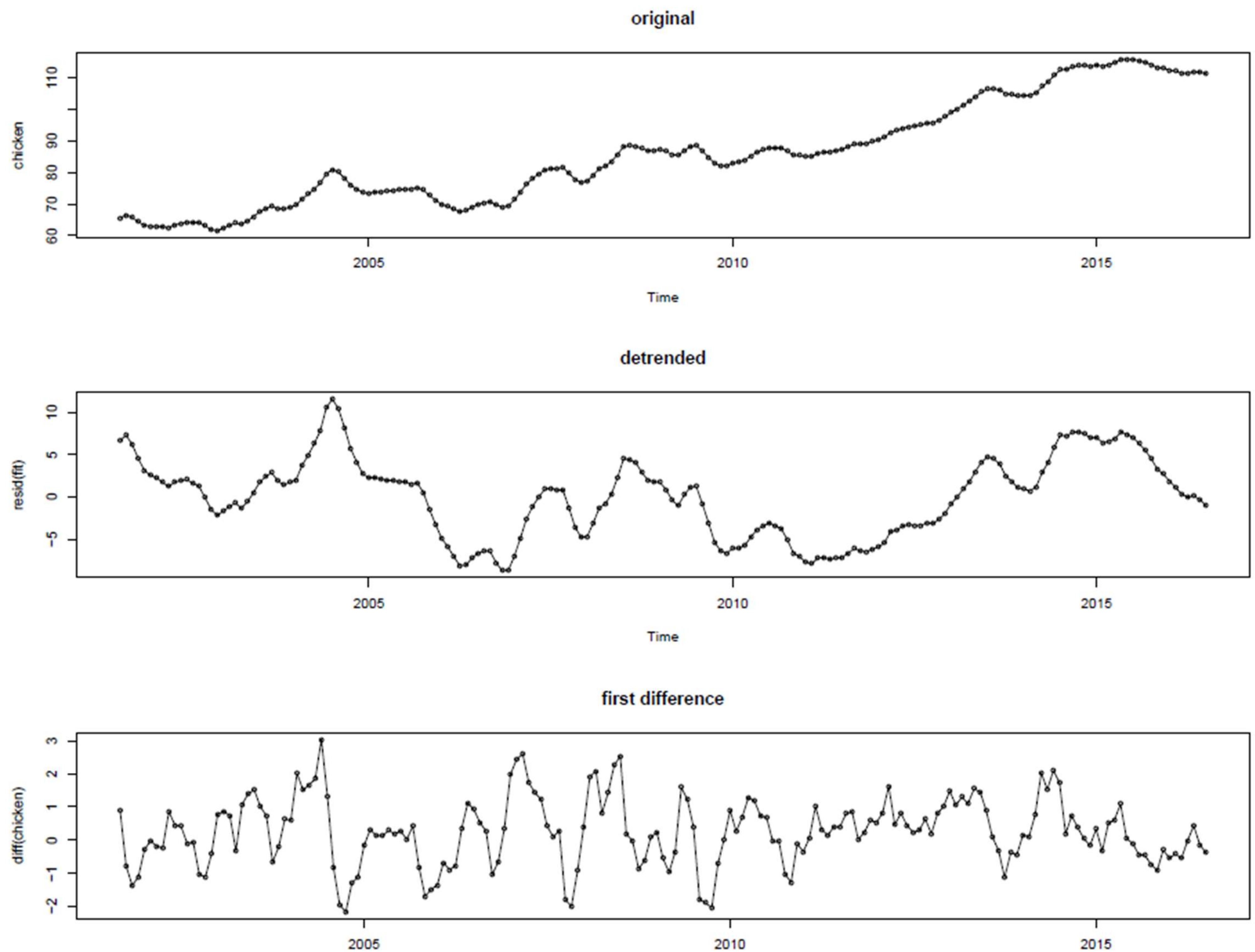
- **Intuition**
  - Auto-Regressive and Moving Average models can be combined together to form ARMA models.
- **Definition**
  - A time series $\{X_t \,; t = 0, \pm 1, \pm 2, \ldots\}$ is $ARMA(p, q)$ if it is stationary and

$$X_t = \epsilon_t + \sum_{i=1}^{p} \phi_i \, X_{t-i} + \sum_{j=1}^{q} \theta_j \, \epsilon_{t-j}$$

Where $\phi_p \neq 0 \; and \; \theta_q \neq 0$

- One limitation of ARMA models is the stationarity condition.
- In many situations, time series can be thought of as being composed of two components, a non-stationary trend series and a zero-mean stationary series. The strategies to **stationarize** time series models are:
  - **Detrending:** Subtract with an estimate for trend and deal with residual.
  - **Differencing:** Take the difference of the time series, often those are stationary.

Detrending and Differencing



ARIMA Model

ARIMA is an acronym that stands for **A**uto-**R**egressive **I**ntegrated **M**oving **A**verage.

Specifically,

- **AR** *Autoregression*. A model that uses the dependent relationship between an observation and some number of lagged observations.
- **I** *Integrated*. The use of differencing of raw observations in order to make the time series stationary.
- **MA** *Moving Average*. A model that uses the dependency between an observation and a residual error from a moving average model applied to lagged observations.

- Each of these components are explicitly specified in the model as a parameter.
- Note that **AR** and **MA** are two widely used linear models that work on stationary time series, and **I** is a preprocessing procedure to "stationarize" time series if needed.

ARIMA Model

A standard notation is used of ARIMA $(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q})$ where the parameters are substituted with integer values to quickly indicate the specific ARIMA model being used.

- $\boldsymbol{p}$ The number of lag observations included in the model, also called the lag order.
- $\boldsymbol{d}$ The number of times that the raw observations are differenced, also called the degree of differencing.
- $\boldsymbol{q}$ The size of the moving average window, also called the order of moving average.
- A value of $\mathbf{0}$ can be used for a parameter, which indicates to not use that element of the model.
- In other words, ARIMA model can be configured to perform the function of an ARMA model, and even a simple AR, I, or MA model.

Thank You