Vector matrix multiplication

$\vec{v}^T M \longrightarrow$ post multiplication of vector $\vec{v}$ with matrix M.

$$\vec{v}_{m\times 1} \quad M_{m\times n} \longrightarrow (\vec{v}^T M)_{1\times n}$$

$M\vec{v} \longrightarrow$ Pre-multiplication of vector $\vec{v}$ with matrix M

$$M_{m\times n} \quad \vec{v}_{n\times 1} \longrightarrow (M\vec{v})_{m\times 1}$$

$$\begin{bmatrix} 2 & 3 & 0 & -1 \\ 6 & 7 & -2 & 1 \end{bmatrix}_{2\times 4} \begin{bmatrix} 0.5 \\ 1 \\ -1 \\ 0 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}_{2\times 1}$$

$2\times 0.5 + 3\times 1 + \cancel{0\times -1} + \cancel{(-1)\times 0}$
$= 1 + 3 = 4$

$6\times 0.5 + 7\times 1 + (-2)(-1) + \cancel{(1)(0)}$
$= 3 + 7 + 2 = 12$

Matrix - Matrix multiplication

$$\vec{v}^{T} = [1, 2, 3, 4]_{\underline{1} \times \underline{4}}$$

$$\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

$$M = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 1 & -1 & -2 & 7 \\ -3 & 5 & 6 & 0 \end{bmatrix}_{3\times 4}$$

The rules of matrix multiplication

$$\underline{A}\,\underline{B} = \underline{C}$$

$A_{r_a \times c_a}$

$r_a \rightarrow$ number of rows in A

$c_a \rightarrow$ number of columns in A

$B_{r_b \times c_b}$

$r_b \rightarrow$ number of rows in B

$c_b \rightarrow$ number of columns in B

$A_{r_a \times c_a} \times B_{r_b \times c_b}$ is possible only when $\boxed{c_a = r_b}$ ✓

number of columns in first matrix = number of rows in second matrix.

$$A_{r_a \times c_a} \times B_{r_b \times c_b} = C_{r_a \times c_b}$$

→ number of rows in the resultant matrix = number of rows in first matrix

$$A_{3\times 4} \times B_{4\times 5} = C_{3\times 5}$$

→ number of columns in the resultant matrix = number of columns in the second matrix.

$A_{2\times 3} \times B_{4\times 3}$ → multiplication of these two matrices is not possible

$$A = \begin{bmatrix} \underset{a_{11}}{2} & \underset{a_{12}}{3} & \underset{a_{13}}{4} & \underset{a_{14}}{5} \\ \underset{a_{21}}{6} & \underset{a_{22}}{7} & \underset{a_{23}}{8} & \underset{a_{24}}{9} \\ \underset{a_{31}}{1} & \underset{a_{32}}{-1} & \underset{a_{33}}{2} & \underset{a_{34}}{3} \end{bmatrix} \quad \underset{r_a \times c_a}{3 \times 4}$$

$$B = \begin{bmatrix} \underset{b_{11}}{1} & \underset{b_{12}}{3} \\ \underset{b_{21}}{-1} & \underset{b_{22}}{0} \\ \underset{b_{31}}{0} & \underset{b_{32}}{-2} \\ \underset{b_{41}}{2} & \underset{b_{42}}{1} \end{bmatrix} \quad \underset{r_b \times c_b}{4 \times 2}$$

$C = A \times B$

$(C)_{3 \times 2}$

$i \rightarrow$ row no. $\quad j \rightarrow$ col. no.

$18 + 0 + (-16) + 9 = 11$

$$C = \begin{bmatrix} c_{11} & \overset{3}{c_{12}} \\ c_{21} & c_{22} \\ \underset{8}{c_{31}} & c_{32} \end{bmatrix} \; 11 \quad \underset{r_a \times c_b}{3 \times 2}$$

$A = [a_{ij}]$ $\quad i = \{1,2,3\}$, $j = \{1,2,3,4\}$

$B = [b_{ij}]$ $\quad i = \{1,2,3,4\}$, $j = \{1,2\}$

$C = [c_{ij}]$ $\quad i = \{1,2,3\}$, $j = \{1,2\}$

$$c_{ij} = \sum_{k=1}^{\cancel{c_a}\,4} \underline{a_{ik}} \times \underline{b_{kj}}$$

$i = 1$
$j = 2$

$$c_{12} = \underbrace{a_{11} b_{12}}_{k=1} + \underbrace{a_{12} b_{22}}_{k=2} + \underbrace{a_{13} b_{32}}_{k=3} + \underbrace{a_{14} b_{42}}_{k=4}$$

$= 2 \times 3 + 3 \times 0 + 4 \times (-2) + 5 \times 1$

$= 6 + 0 + (-8) + 5$

$= 3$

$$A = \begin{bmatrix} \boxed{2 \quad 3 \quad 4 \quad 5} \\ 6 \quad 7 \quad 8 \quad 9 \\ 1 \quad -1 \quad 2 \quad 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & -2 \\ 2 & 1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 0 & -2 & 1 \end{bmatrix}$$

$A = [[2, 3, 4, 5], [6, 7, 8, 9], [1, -1, 2, 3]]$

$B = [[1, 3], [-1, 0], [0, -2], [2, 1]]$

$B^T = [[1, -1, 0, 2], [3, 0, -2, 1]]$

$$A_{n\times n} \times B_{n \times n} = C_{n \times n}$$

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} = ? \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-35 + 14 + 21 = 0$

$$A_{n\times n} \times B_{n\times n} = I_n$$

$$A = B^{-1} \quad \text{or} \quad B = A^{-1}$$

$$A_{n\times n} (A)^{-1}_{n\times n} = I_n$$

$$(A)^{-1} A = I$$

Inverse of a square matrix :—

System of linear equations : $\begin{cases} 3x + 5y = 1 \\ x + 2y = -2 \end{cases}$ find x & y

$$\Rightarrow \underbrace{\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{b}$$

$$AX = b$$

$$\boxed{X = A^{-1} b}$$

Solution of the system of linear equations.

$3x + 5y = 1$ — ①

$3x + 6y = -6$ --- ②

② − ①

$y = -6 - 1 = -7$

$x = -2 - 2y$

$= -2 + 14 = 12$

$$\boxed{\begin{aligned} x &= 12 \\ y &= -7 \end{aligned}}$$

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

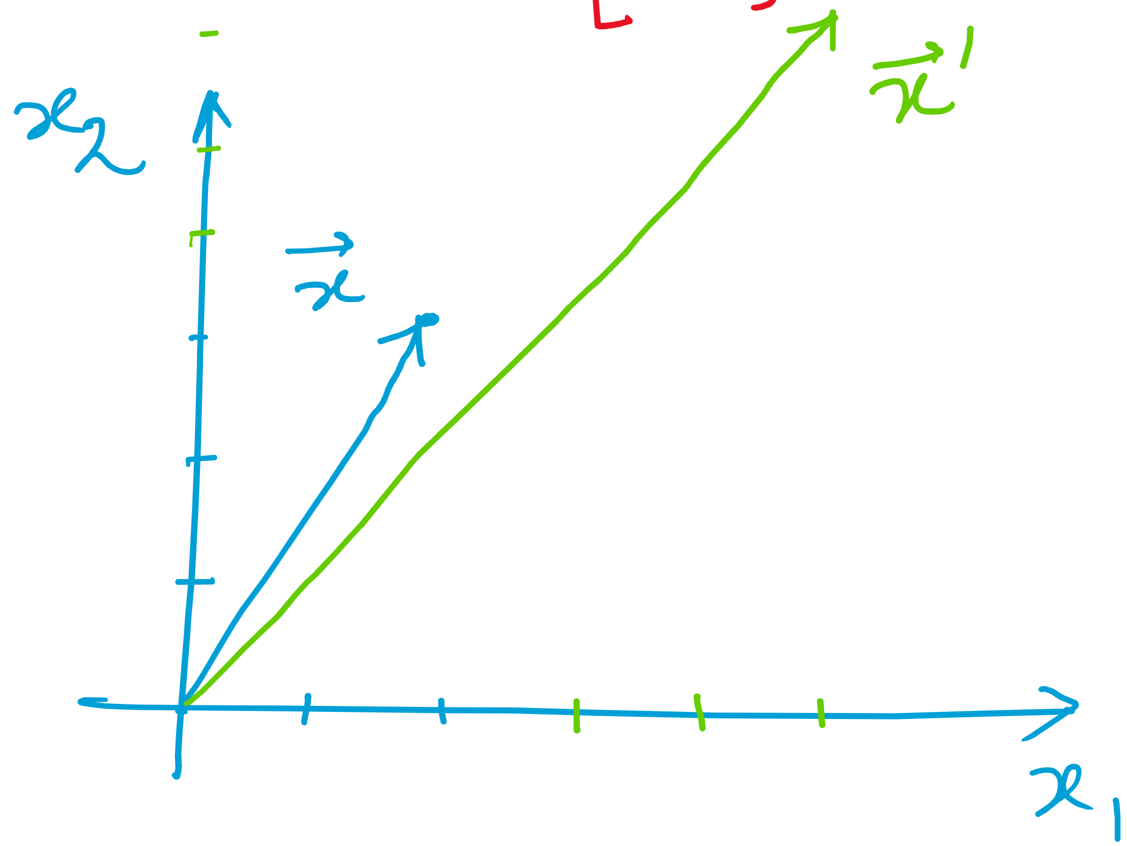
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10 \\ -1-6 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \end{bmatrix}$$

<u>Vector</u>

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T \qquad \|\vec{x}\| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$$



$$\vec{x}' = A\vec{x} \qquad A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

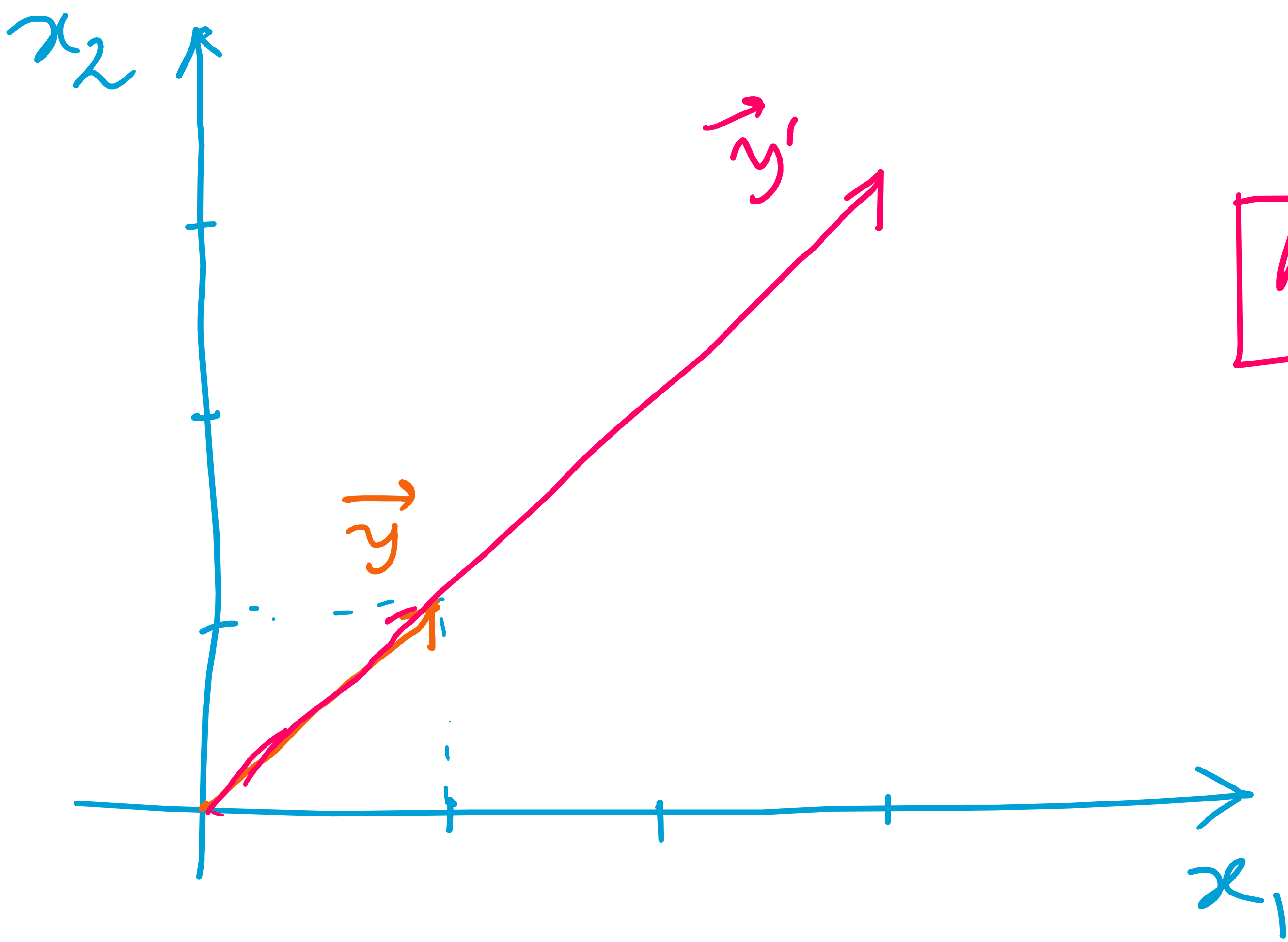
$$\|\vec{x}'\| = \sqrt{5^2+7^2} = \sqrt{25+49} = \sqrt{74}$$

When we multiply a vector $\vec{x}$ with a matrix A, both its direction & magnitude changed.

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2} \qquad \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{y}\,' = A\vec{y}$$

$$\vec{y}\,' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \underline{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\vec{y}$$



$\vec{y}\,' = 3\vec{y}$

$$\boxed{A\vec{y} = 3\vec{y}}$$

$\vec{y}\,'$ & $\vec{y}$ are having same direction but different magnitude.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen-vector of matrix A.

$$\boxed{\underline{A}\,\underline{\vec{x}} = \underline{\lambda}\,\underline{\vec{x}}}$$

then $\vec{x}$ is an eigenvector of A & λ is called an eigenvalue of A.

$A_{n\times n} \rightarrow$ <u>n eigenvalues</u>. (they may not be all real / distinct)

Corresponding to each distinct eigenvalue (λ) there exist an unit vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. $\vec{v}$ is called the eigenvector of A corresponding to eigenvalue λ.

How to find eigenvalue for a 2×2 matrix:—

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x} = \lambda \vec{x}$$

$$A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Rightarrow (A - \lambda I)\vec{x} = 0$$

$$\Rightarrow |A - \lambda I| = 0$$

$$\begin{vmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{vmatrix} = 0$$

$$\Rightarrow \boxed{(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} \cdot a_{21} = 0}$$

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (4-\lambda)\cdot(1-\lambda) - 2(-1) = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 2\lambda + 6 = 0$$

$$\Rightarrow \lambda(\lambda - 3) - 2(\lambda - 3) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0 \rightarrow \boxed{\lambda = 2} \quad \boxed{\lambda = 3}$$