Identity matrix of order-n

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why is it called identity matrix?

For any kind of operation (*) there exist one identity element. $a * b = b$ → identity element

ex: for addition of two numbers the identity element is 0.

$$2 + 0 = 2$$

for multiplication of two numbers the identity element is 1.

$$5 \times 1 = 5$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$AI = IA = A \qquad I \rightarrow \text{identity matrix}$$

Eigen-Value problem

$A\vec{x} = \lambda\vec{x}$ $\Rightarrow$ λ is the eigenvalue &
$\vec{x}$ is the eigenvector

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$ab = 0$

$a = 0$ or $b = 0$

$\Rightarrow$ $a_{11}x_1 + a_{12}x_2 = \lambda x_1$ & $a_{21}x_1 + a_{22}x_2 = \lambda x_2$

$\Rightarrow$ $\underline{(a_{11}-\lambda)x_1 + a_{12}x_2 = 0}$ & $\underline{a_{21}x_1 + (a_{22}-\lambda)x_2} = 0$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$A\vec{x} = \lambda\vec{x}$

$\Rightarrow$ $A\vec{x} - \lambda\vec{x} = 0$

$\Rightarrow$ $\boxed{(A - \lambda I)\vec{x} = 0}$

$$\lambda I = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

if $\vec{x} = \vec{0}$ $\rightarrow$ that satisfies the above equation. (Trivial solution) $\rightarrow$ useless.

$(A-\lambda I)\vec{x} = 0 \Rightarrow |A - \lambda I| = 0$ the determinant value of $(A-\lambda I)$ is 0.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0 \longrightarrow$ quadratic polynomial equation of $\lambda \longrightarrow \lambda$ will have two solutions.

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow \text{find its eigen values.}$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)(1-\lambda) - (2)(-1) = 0$

$\Rightarrow 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\Rightarrow \boxed{\lambda^2 - 5\lambda + 6 = 0}$

$\Rightarrow \lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$\Rightarrow \lambda(\lambda - 3) - 2(\lambda - 3) = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0 \Rightarrow \boxed{\lambda = 2 \text{ or } \lambda = 3}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = (a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22}-\lambda \\ a_{31} & a_{32} \end{vmatrix}$$

$$= (a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}\,a_{32}\right] - \cdots\cdots$$

$$= \underline{(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)} + \cdots\cdots\cdots$$

$\rightarrow$ polynomial of degree 3. $\Big\}$ 3 solutions.

$A_{n \times n}$ matrix $\quad \underbrace{|A - \lambda I|}_{O(\lambda^n)} = 0$

$O(\lambda^n) \longrightarrow$ polynomial of degree - n. $\Rightarrow$ n - solutions.

$|A - \lambda I| = 0 \longrightarrow$ characteristic equation of square matrix A.

A
$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow$ the characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$

$A^2 = A \cdot A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 10 & -1 \end{bmatrix}$

$A^2 - 5A + 6I = 0$

$= \begin{bmatrix} 14 & -5 \\ 10 & -1 \end{bmatrix} - \begin{bmatrix} 20 & -5 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 14 - 20 + 6 & -5 + 5 + 0 \\ 10 - 10 + 0 & -1 - 5 + 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Cayley - Hamilton's Theorem

Every square matrix satisfies its own characteristic equation.

We are not interested in trivial solution.

$(A - \lambda I)\vec{x} = \vec{0} \Rightarrow$ $A - \lambda I$ which when multiplied to $\vec{x}$ gives $\vec{0}$

$|A - \lambda I| = 0$ Think determinant as value of the matrix.

$ab = 0 \Rightarrow a = 0$ or $b = 0$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) = 0$$

$2, 3$

✓ For a diagonal matrix the eigenvalues are the diagonal elements.

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Upper triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

Lower triangular matrix

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \Rightarrow \begin{vmatrix} a-\lambda & b \\ 0 & c-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(c-\lambda) = 0$$

$$\Rightarrow \lambda = a \text{ or } c.$$

For triangular matrix also the eigen values are its diagonal elements.

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad \det(A) = 6 \quad , \quad \underbrace{\text{trace}}_{\text{sum of diagonal values}}(A) = 5$$

$$\lambda_1 = 2, \quad \lambda_2 = 3$$

$$\lambda_1 + \lambda_2 = 5$$

$$\lambda_1 \lambda_2 = 6$$

$$\sum_{i=1}^{n} \lambda_i = \text{trace}(A) \quad \& \quad \prod_{i=1}^{n} \lambda_i = \det(A)$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow \left.\begin{matrix} \lambda_1 = 2 \\ \lambda_1 = 3 \end{matrix}\right\}$$ We need to find eigenvectors.

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\vec{x} = \lambda \vec{x}$$

for $\lambda = 2$. $\Rightarrow A\vec{x} = 2\vec{x}$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

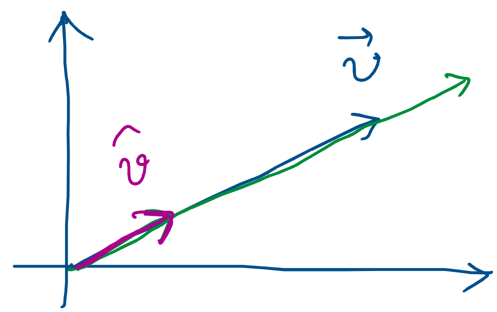
$\Rightarrow 4x_1 - x_2 = 2x_1$ | $2x_1 + x_2 = 2x_2$

$\Rightarrow 2x_1 = x_2$ ✓ | $\Rightarrow 2x_1 = x_2$ ✓

Let $x_2 = 2 \rightarrow x_1 = 1$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Any multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ will be an eigenvector with corresponding eigenvalue $= 2$



$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{[1 \;\; 2]^T}{\sqrt{1^2 + 2^2}} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$\underline{\lambda = 3}$

$A\vec{x} = 3\vec{x}$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$\Rightarrow \quad 4x_1 - x_2 = 3x_1 \quad \Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad 2x_1 + x_2 = 3x_2 \quad \Rightarrow \quad 2x_1 = 2x_2$

$\Rightarrow \quad x_1 = x_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \rightarrow \text{eigenvector.}$$

$$\underbrace{\underset{\lambda = 2}{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}} \qquad \underset{\lambda = 3}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$AB = I \Rightarrow B = A^{-1}$

$A = \begin{bmatrix} \boxed{\begin{matrix} 4 \\ 2 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 1 \end{matrix}} \end{bmatrix}$

$$\boxed{A = V \underline{\Lambda} \underline{V}^{-1}}$$

Eigen-value decomposition

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\therefore$ $\text{Cofactor}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(A) = ad - bc \Rightarrow A^{-1} = \dfrac{\text{adj}(A)}{\det(A)} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \underline{2}A = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(A) = 5$

$\det(2A) = 20$
$= 2^2 \times 5$

$\det(A_{n \times n})$

$\det(KA) = k^n \cdot \det(A)$

$(AB)^{-1} = B^{-1} A^{-1}$ $(AB)^T = B^T \cdot A^T$