Introduction to Linear Algebra

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Roadmap

1. Chapter 1: Vectors
2. Chapter 2: Matrices

1.1 What is a Vector?

Definition

A vector is an ordered list of numbers. In computer science, we often think of this as a 1D array.

Notation: We denote a vector with a bold lowercase letter, e.g., $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$$

Example (Numerical Example)

A house has an area of 1200 sqft and 3 bedrooms. We can represent this as a feature vector:

$$\mathbf{h} = \begin{bmatrix} 1200 \\ 3 \end{bmatrix}$$

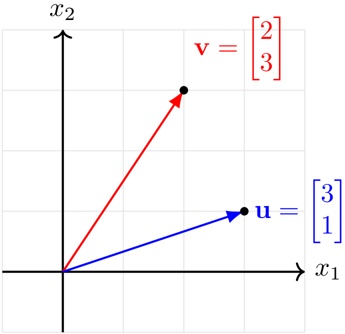
Machine Learning Connection

In ML, vectors represent a single data point (e.g., a customer, an image pixel row, a house).

1.2 Dimensions and Plotting

Dimensions: The number of elements in the vector.

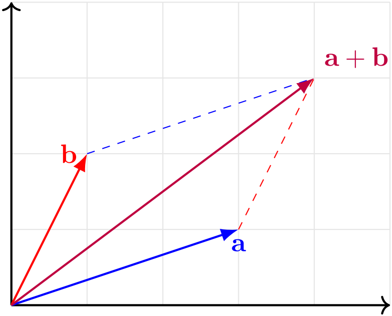
- $\mathbf{v} = [2, 3]^T$ is 2-Dimensional (2D).
- $\mathbf{w} = [1, 5, -2]^T$ is 3-Dimensional (3D).



1.3 Vector Addition (Parallelogram Law)

[Image of Parallelogram law of vector addition]
Adding vectors is done element-wise.

$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \mathbf{a} + \mathbf{b} = \begin{bmatrix} 3+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



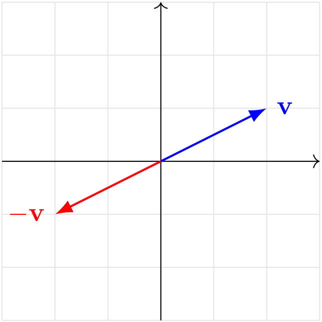
1.4 Vector Subtraction Negation

Negative of a vector: Flips the direction.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies -\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Subtraction: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

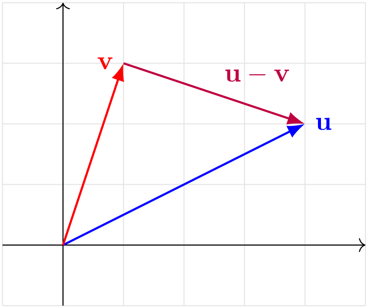
$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



1.5 Vector Subtraction: Geometric View

To find $\mathbf{u} - \mathbf{v}$, draw a vector from the **head of $\mathbf{v}$** to the **head of $\mathbf{u}$**.

$$\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \implies \mathbf{u} - \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$



1.6 Magnitude (Norms)

The "size" or "length" of a vector.

L_2 Norm (Euclidean Distance)

Most common. Straight line distance.

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Example: $\mathbf{x} = [3, 4]^T \implies \|\mathbf{x}\|_2 = \sqrt{9 + 16} = 5$.

- L_1 **Norm (Manhattan):** Sum of absolute values. $|x_1| + |x_2|$. Example: $|3| + |4| = 7$.
- L_p **Norm:** General case $(\sum |x_i|^p)^{1/p}$.

Machine Learning Connection

Used in **Regularization** (Lasso L1, Ridge L2) to prevent overfitting by keeping model weights small.

1.7 L1 vs L2 Norm: Geometry

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

L2 Norm (Euclidean): "Straight line"

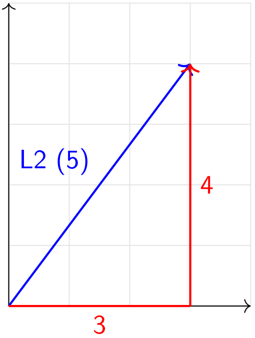
$$\|\mathbf{x}\|_2 = \sqrt{3^2 + 4^2} = 5$$

Blue Line

L1 Norm (Manhattan): "City blocks"

$$\|\mathbf{x}\|_1 = |3| + |4| = 7$$

Red Path

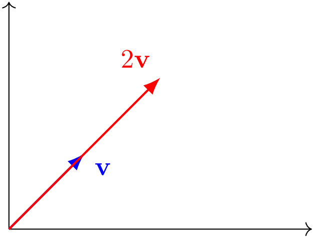


Unit Vector: A vector with length 1.

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Scalar Multiplication: Stretching or shrinking. Let $c = 2$ and $\mathbf{v} = [1, 1]^T$.

$$c \cdot \mathbf{v} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



1.9 Inner Product (Dot Product)

Crucial for calculating similarity and projections.
Algebraic: Sum of element-wise products.

$$\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i = a_1 b_1 + a_2 b_2$$

Example: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (1)(3) + (2)(4) = 3 + 8 = 11.$
Geometric:

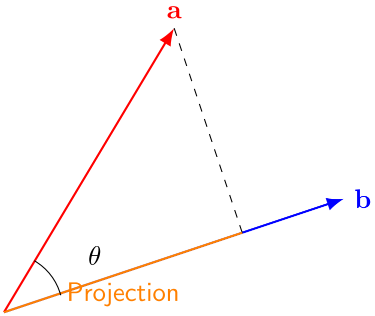
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$$

Note

If $\mathbf{a} \cdot \mathbf{a}$, result is $\|\mathbf{a}\|^2$ (Magnitude squared).

1.9 Inner Product: The Projection

Geometry: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$.

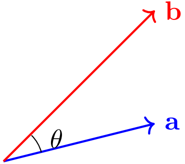


1.10 Cosine Similarity

Measures the angle between two vectors, irrespective of magnitude.

$$\text{similarity} = \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$$

- If $\cos(\theta) = 1$: Vectors point in same direction (Similar).
- If $\cos(\theta) = 0$: Vectors are orthogonal (90°) (Unrelated).
- If $\cos(\theta) = -1$: Vectors are opposite.



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Used in "Natural Language Processing" and "Recommender Systems" to find similar documents or users.

1.10 Cosine Similarity

Similarity is about **orientation**, not magnitude.

$$\text{Sim}(\mathbf{a}, \mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$$

Numerical Example

Documents as vectors: Doc A (Sports): $\mathbf{a} = [1, 5]$. Doc B (Politics): $\mathbf{b} = [5, 1]$.

$$\mathbf{a} \cdot \mathbf{b} = 5 + 5 = 10$$

$$\|\mathbf{a}\| = \sqrt{26} \approx 5.1, \quad \|\mathbf{b}\| = \sqrt{26} \approx 5.1$$

$$\text{Sim} = \frac{10}{5.1 \times 5.1} \approx 0.38 \quad \text{(Low Similarity)}$$

Chapter 1 Quiz: Test Your Skills

1. Let $\mathbf{u} = [2, -2]$. What is the unit vector $\hat{\mathbf{u}}$?
2. If $\|\mathbf{a}\|_2 = 3$ and $\|\mathbf{b}\|_2 = 4$ and the angle between them is $90°$, what is $\mathbf{a} \cdot \mathbf{b}$?
3. Calculate the L1 norm of $\mathbf{v} = [-5, 10, -2]$.
4. True or False: $\mathbf{a} \cdot \mathbf{a}$ can be negative.
5. Given $\mathbf{x} = [1, 2]$ and $\mathbf{y} = [2, 4]$. Calculate Cosine Similarity without a calculator.
6. If $\mathbf{a} \cdot \mathbf{b} = 0$, what does this geometrically imply?

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Answers: 1. $[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$ 2. 0 3. $5 + 10 + 2 = 17$ 4. False (sum of squares) 5. 1 (They are collinear) 6. Vectors are orthogonal (perpendicular).

2.1 Introduction to Matrices

A matrix is a rectangular grid of numbers.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

This is a 2×3 matrix (2 rows, 3 columns).
Data Science View:

- Rows $=$ Samples (e.g., Users).
- Columns $=$ Features (e.g., Age, Income, Height).

In Data Science, a matrix X usually represents our entire dataset.

Square Feet (x_1)	**Bedrooms (x_2)**	**Age (x_3)**
2100	3	20
1400	2	15
2500	4	5

This becomes the matrix $X \in \mathbb{R}^{3\times 3}$:

$$X = \begin{bmatrix} 2100 & 3 & 20 \\ 1400 & 2 & 15 \\ 2500 & 4 & 5 \end{bmatrix}$$

Row: A single house (sample). **Column:** A feature across all houses.

2.3 Types of Matrices

- **Square Matrix:** Rows = Columns ($n \times n$).

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- **Diagonal Matrix:** Non-zero values only on the main diagonal.

$$\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- **Identity Matrix (I):** Diagonal matrix with 1s. Acts like "1" in multiplication.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.3 Types of Matrices

- **Symmetric Matrix:** A square matrix is **Symmetric** if $A = A^T$. *Mirror image across the main diagonal.*

$$A = \begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Machine Learning Connection

Covariance Matrix: In Machine Learning, covariance matrices (describing how features vary together) are always symmetric.

2.4 Matrix Operations (Basic)

Transpose (A^T): Swap rows and columns.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Addition: Element-wise. We can only add matrices of the **same dimensions**.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

Frobenius Norm: The "magnitude" of a matrix. Square root of sum of squared elements.

$$\|A\|_F = \sqrt{\sum \sum |a_{ij}|^2}$$

Example:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

$$\|A\|_F = \sqrt{3^2 + (-1)^2 + 2^2 + 5^2}$$

$$= \sqrt{9 + 1 + 4 + 25} = \sqrt{39} \approx 6.24$$

2.5 Matrix Multiplication

1. **Hadamard Product (Element-wise):** $A \odot B$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$$

2. **Dot Product (Matrix Multiplication):** Row of A $\cdot$ Col of B.

$$C = A \times B$$

Important: Order matters! $A \times B \neq B \times A$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(0) + (2)(1) & (1)(1) + (2)(0) \\ (3)(0) + (4)(1) & (3)(1) + (4)(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

2.5 Matrix Multiplication

Conformability: To do $A \times B$:

$$(m \times \mathbf{n}) \times (\mathbf{n} \times p) = (m \times p)$$

Inner dimensions must match!
Example (2×3 times 3×2):

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 1 \\ 2 & 3 \end{bmatrix}$$

Row 1 $\cdot$ Col 1: $(1)(7) + (2)(9) + (3)(2) = 7 + 18 + 6 = 31$

$$= \begin{bmatrix} 31 & 19 \\ 85 & 55 \end{bmatrix}$$

Non-Commutative: If we swap order, we get a 3×3 matrix! Result is totally different.

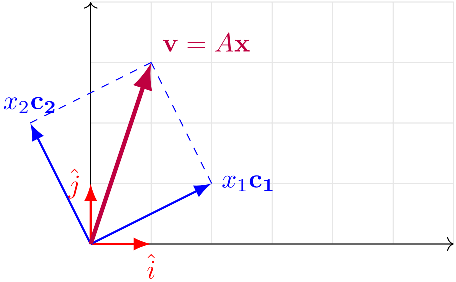
2.5 Matrix-Vector Multiplication

A matrix (A) can be thought of as a collection of column vectors.

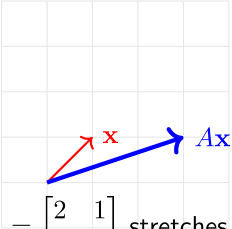
$$A = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} \end{bmatrix}$$

Multiplication $A\mathbf{x}$ transforms vector $\mathbf{x}$. It is a linear combination of the columns of A.

$$A\mathbf{x} = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\mathbf{c_1} + x_2\mathbf{c_2}$$



Multiplication $A\mathbf{x}$ transforms vector $\mathbf{x}$ into a new space.



Example: $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ stretches and shears.

Rotation Example: To rotate a vector by 90° counter-clockwise:

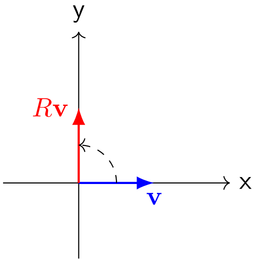
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Matrices act as **functions** that transform vectors (rotate, scale, shear).

Let vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (Unit x-axis vector). Rotation Matrix (90°): $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$R\mathbf{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The vector rotated from X-axis to Y-axis.



2.6 Eigenvalues and Eigenvectors

For a square matrix A, an eigenvector $\mathbf{v}$ is a vector that does not change direction when A is applied, only its length scales.

$$A\mathbf{v} = \lambda\mathbf{v}$$

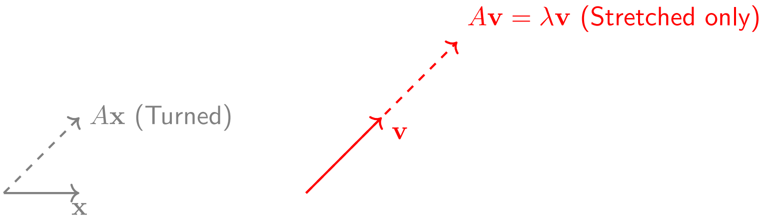
- $\mathbf{v}$: Eigenvector
- λ: Eigenvalue (scaling factor)

Machine Learning Connection

Principal Component Analysis (PCA): We use eigenvectors of the data covariance matrix to reduce dimensions (e.g., compressing 100 features into 2).

2.6 Eigenvectors: The Invariant Directions

Most vectors change direction when multiplied by matrix A. **Eigenvectors** stay on the same span (line), they just stretch/shrink.



Insight

λ tells you the magnitude of stretch. If $\lambda = 1$, the vector is unchanged. If $\lambda < 1$, it shrinks.

2.7 Determinant of a square matrix

A scalar value indicating how a matrix "scales" area (2D) or volume (3D).

2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \det(A) = ad - bc$$

Example: $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \implies \det(A) = (4)(3) - (2)(1) = 10.$

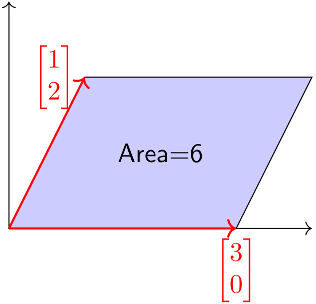
Why it matters:

- If $\det(A) = 0$, the matrix A is called a **Singular Matrix**. The singular matrix squishes space into a line or point.
- If $\det(A) = 0$, the inverse A^{-1} does **not** exist.

2.7 Determinant: Geometric Interpretation (2x2)

The determinant of a 2×2 matrix is the **Area of the Parallelogram** formed by the column vectors.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \implies \det(A) = (3)(2) - (1)(0) = 6$$



For higher order matrices, we need Minors (M_{ij}) and Cofactors (C_{ij}).

Matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

Step 1: Minor M_{ij}**:** Ignore Row i, Col j.

$$M_{11} = \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = 24 - 0 = 24$$

Step 2: Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$**.** Since $1 + 1 = 2$ (even), $C_{11} = 24$.

2.7 Determinant of 3x3 Matrix

Formula: Expansion along the first row.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$:

1. $a_{11} = 1, M_{11} = 24 \implies 1(24)$ 2. $a_{12} = 2, M_{12} = \det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} = -5$. Sign is neg: $-2(-5) = 10$. 3. $a_{13} = 3, M_{13} = \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = -4$. Sign is pos: $3(-4) = -12$.

$$\det(A) = 24 + 10 - 12 = 22$$

2.8 Solving Eigenvalues and Eigenvectors

Consider the triangular matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Step 1: Find Eigenvalues
Solve $\det(A - \lambda I) = 0$. Since it is triangular, λ are just the diagonal entries!

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

Step 2: Find Eigenvector for $\lambda = 3$
Solve $(A - 3I)\mathbf{v} = 0$:

$$\begin{bmatrix} 2-3 & 1 & 0 \\ 0 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

From row 2: $-y = 0 \implies y = 0$.
From row 1: $-x + y = 0 \implies -x + 0 = 0 \implies x = 0$.
z is free. Let $z = 1$.
Eigenvector: $\mathbf{v}_3 = [0, 0, 1]^T$.

2.8 Solving Eigenvalues and Eigenvectors

Matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$.

1. Solve $\det(A - \lambda I) = 0 \implies (2-\lambda)[(3-\lambda)(9-\lambda) - 16] = 0$.
2. Roots: $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 11$.

Solve for $\mathbf{v}$ when $\lambda = 2$:

$$(A - 2I)\mathbf{v} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rows 2 and 3 imply $y = 0, z = 0$. x can be anything.
Eigenvector: $\mathbf{v_1} = [1, 0, 0]^T$.

Note

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of a square matrix.

2.9 Significance and Properties of Eigenvalues

Eigenvalues are not just scaling factors; they reveal the internal structure of a matrix $A_{n \times n}$.

The Sum and Product Properties

- **The Trace Property:** The sum of the eigenvalues is equal to the **Trace** of the matrix (the sum of the diagonal elements).

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A)$$

- **The Determinant Property:** The product of the eigenvalues is equal to the **Determinant** of the matrix.

$$\prod_{i=1}^{n} \lambda_i = \det(A)$$

Numerical Example

For $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$:

- $\text{tr}(A) = 4 + 3 = 7$. Indeed, eigenvalues are $\lambda_1 = 5, \lambda_2 = 2 \implies 5 + 2 = 7$.
- $\det(A) = (4)(3) - (2)(1) = 10$. Indeed, $5 \times 2 = 10$.

Data Science Intuition: In PCA, the sum of eigenvalues represents the **total variance** of the dataset. If one eigenvalue is 0, the product (determinant) is 0, meaning the data is redundant and exists in a lower dimension.

2.10 Inverse of a Matrix

A^{-1} is the matrix such that $AA^{-1} = I$.

For a 2x2 Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Numerical Example

Let $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$.

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Warning

If determinant is 0, the matrix is "Singular" and has no inverse.

2.10 Inverse of 3x3: The Algorithm

To invert higher order matrices, we use Minors and Cofactors.

$$A^{-1} = \frac{1}{\det(A)}\mathsf{adj}(A)$$

Where $\mathsf{adj}(A)$ is the **Transpose of the Cofactor Matrix**.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Step 1: Determinant (expand Row 1)

$$\det(A) = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1(0) + 1(-1) = -1$$

Since $\det \neq 0$, inverse exists.

Step 2: Cofactor Matrix

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find Cofactors C_{ij}.

$$C_{11} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{12} = - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(-1) = 1$$

$$C_{13} = + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{21} = - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(-1) = 1$$

$$C_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{23} = - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

(Skipping Row 3 for brevity, assume similar steps).

$$\text{Cofactor Matrix } C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

2.10 Inverse of 3x3: Final Calculation

Step 3: Adjugate (Transpose of Cofactor Matrix)

$$\text{adj}(A) = C^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(It happened to be symmetric).

Step 4: Divide by Determinant (-1)

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Check: $A \cdot A^{-1} = I$.

Note

Numerical computation packages like NumPy implements matrix inverse using Gaussian Elimination Process (not the above algorithm).

Chapter 2 Quiz: Test Your Skills

1. Can you multiply a 4×2 matrix with a 4×2 matrix?
2. If A is symmetric, what is $A - A^T$?
3. If $\det(A) = 5$, what is $\det(2A)$ for a 2×2 matrix?
4. If a matrix has a column of all zeros, what is its determinant?
5. If $\lambda = 0$ is an eigenvalue, is the matrix invertible?
6. Given $A = \begin{bmatrix} 1 & k \\ 2 & 4 \end{bmatrix}$, for what k is it singular?
7. True or False: $(AB)^{-1} = A^{-1}B^{-1}$.
8. True or False: $(AB)^T = B^T A^T$

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4. If a matrix has a column of all zeros, what is its determinant?
5. If $\lambda = 0$ is an eigenvalue, is the matrix invertible?
6. Given $A = \begin{bmatrix} 1 & k \\ 2 & 4 \end{bmatrix}$, for what k is it singular?
7. True or False: $(AB)^{-1} = A^{-1}B^{-1}$.
8. True or False: $(AB)^T = B^T A^T$

Answers: 1. No 2. Zero matrix 3. 20 ($2^n \times \det$) 4. 0 5. No 6. $k = 2$ 7. False (Order swaps: $B^{-1}A^{-1}$) 8. True.