Principles of Counting and Introduction to Probability Theory

Sourav Karmakar

Outline

Counting Principles

The Mathematics of "How Many?"

1. The Multiplication Rule of Counting

Fundamental Counting Principle

If a task can be broken down into a sequence of k steps, where:

- Step 1 can be done in n_1 ways,
- Step 2 can be done in n_2 ways, ...,
- Step k can be done in n_k ways,

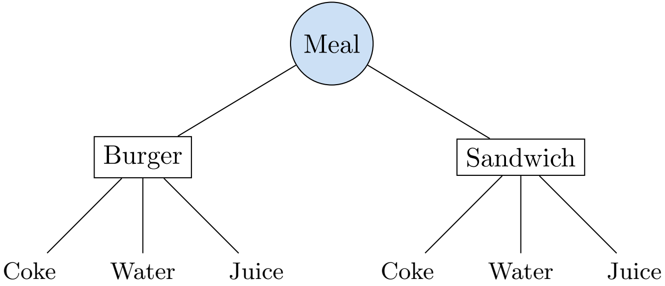
Then the total number of ways to perform the task is:

$$N = n_1 \times n_2 \times \cdots \times n_k$$

Multiplication Rule: Visual Example

Scenario: Ordering a "Combo Meal".

- **Main Dish:** Burger or Sandwich (2 options).
- **Drink:** Coke, Water, or Juice (3 options).



Total Combinations $= 2 \times 3 = 6$ distinct meals.

Multiplication Rule: Numerical Examples

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- Pos 3: 10 options
- Pos 4: 10 options
- **Total:** $10 \times 10 \times 10 \times 10 = 10,000$ PINs.

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Example 2: Buying a Car

You are configuring a new car online. The options are: 1. Model: (Sedan, SUV, Hatchback) $\rightarrow$ 3 options. 2. Color: (Red, Black, White, Silver) $\rightarrow$ 4 options. 3. Transmission: (Manual, Automatic) $\rightarrow$ 2 options.

Total Configurations: $3 \times 4 \times 2 = 24$ unique cars.

2. The Addition Rule

Rule of Sum

If a task can be done in n_A ways OR in n_B ways, and these sets of ways are mutually exclusive (you cannot do both at once), then:

$$\text{Total ways} = n_A + n_B$$

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Example: Commute Options

You want to go to the office.

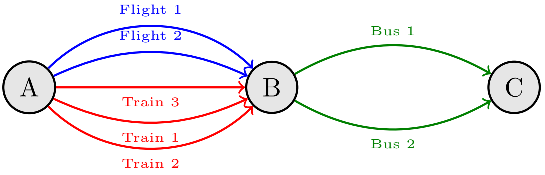
- You can take a **Bus** (3 different routes).
- OR you can take a **Train** (2 different lines).

You cannot take both simultaneously.
Total Choices: $3 + 2 = 5$ ways to get to work.

Combining Addition and Multiplication

Problem: Travel from City A to City C via City B.

- **A to B:** 2 Flights OR 3 Trains.
- **B to C:** 2 Buses.



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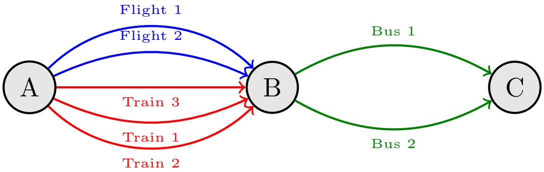

Solution:

1. **A to B (Addition):**
   $2\ (\text{F}) + 3\ (\text{T}) = 5$ ways.

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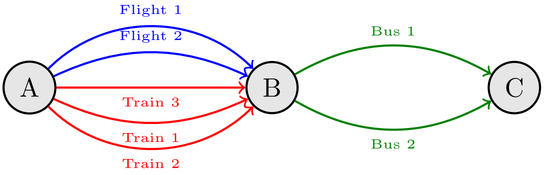

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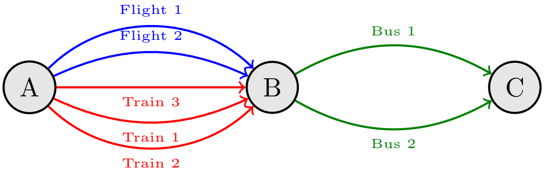

Solution:

1. **A to B (Addition):**
   $2\ (F) + 3\ (T) = 5$ ways.
2. **B to C:**
   2 ways.
3. **Total (Multiplication):**
   $5 \times 2 = 10$ routes.

Factorials

Before we discuss Permutations, we must define the **Factorial**.

Definition

The factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

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The Special Case of Zero: Why is $0! = 1$?

$$\text{Since } n! = n \times (n-1)! \implies 1! = 1 \times 0! \implies 1 = 1 \times 0! \implies 0! = 1$$

Examples:

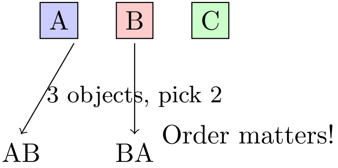
- $3! = 3 \times 2 \times 1 = 6$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Key Intuition

Permutation is about **arrangement**. The order in which you place objects changes the outcome (e.g., Password "123" $\neq$ "321").

Formula: The number of ways to arrange r objects chosen from n distinct objects:

$$P(n, r) = \frac{n!}{(n-r)!}$$



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$$P(12, 3) = 12 \times 11 \times 10 = 1,320 \text{ ways.}$$

4. Combinations (Order Doesn't Matter)

Key Intuition

Combination is about **selection**. "Fruit Salad" logic: Apple and Banana is the same as Banana and Apple.

Formula: The number of ways to choose r objects from n distinct objects:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note: Combinations result in fewer counts than permutations because duplicates (e.g., AB vs BA) are removed/divided out.

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Example 3: [Lottery] A lottery ticket requires you to choose 6 numbers from 1 to 49 (order of drawing doesn't matter). How many possible tickets exist?

$$C(49,6) = \frac{49!}{6!43!} = 13,983,816 \text{ tickets.}$$

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- **Ans:** $10 \times 15 = 150$ ways.

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- Treat (AB) as 1 unit. We arrange 4 units: $4! = 24$.
- A and B can swap places inside the unit: $2! = 2$.
- **Ans:** $24 \times 2 = 48$ ways.

Introduction to Probability Theory

Quantifying Uncertainty

1. Random Experiment & Sample Space

Random Experiment: A process where the outcome cannot be predicted with certainty.
Sample Space (S): The set of **all** possible outcomes.

Single Coin Toss:

$$S = \{H, T\}$$

$$|S| = 2$$

Rolling a Die:

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$|S| = 6$$

The elements of the Sample Space follows the *MECE Principle*: Mutually Exclusive (outcomes don't overlap) and Collectively Exhaustive (cover all possibilities).

2. Defining Probability

Probability of an Event A: For a finite sample space where all outcomes are equally likely (Count-based):

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}} = \frac{|A|}{|S|}$$

$$0 \leq P(A) \leq 1$$

Probability Examples (Count Based)

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- Favorable outcomes $A = \{HH, HT, TH\}$.
- $P(A) = 3/4 = 0.75$.

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Example 2: [Rolling Two Dice] What is the probability that the **sum of the dice is 7**?

- Total outcomes: $6 \times 6 = 36$.
- Favorable pairs (x, y): $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$.
- Count $= 6$.
- $P(Sum = 7) = 6/36 = 1/6 \approx 0.167$.

3. Kolmogorov's Axioms of Probability

The foundation of modern probability.

1. **Non-negativity:** For any event E, $P(E) \geq 0$.
2. **Normalization:** The probability of the sample space is 1. $P(S) = 1$.
3. **Additivity:** If A and B are mutually exclusive (cannot happen together), then:

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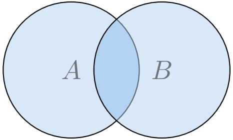
Useful Corollaries:

- **Complement Rule:** $P(A^c) = 1 - P(A)$.
- **Subset Rule:** If $A \subset B$, then $P(A) \leq P(B)$.
- (If A is part of B, A cannot be more probable than B).

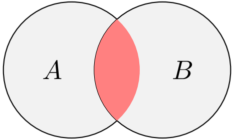
4. Events and Venn Diagrams

Union $(A \cup B)$: OR



Any shaded area

Intersection $(A \cap B)$: AND

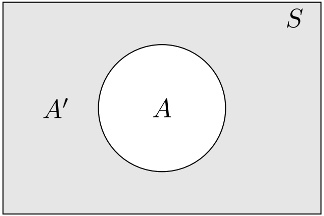


Only the overlap

Complementary Events

Complement (A' or A^c): NOT A

$$P(A') = 1 - P(A)$$



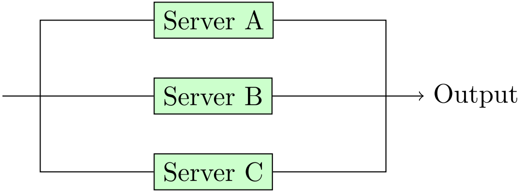
Physically: If probability of system failure is 0.01, reliability is 0.99.

5. Real World Example: Parallel Redundancy

Problem: Three servers (A, B, C) are connected in **Parallel**. System fails only if **ALL** servers fail.

- $P(\text{Fail}_A) = 0.1$
- $P(\text{Fail}_B) = 0.1$
- $P(\text{Fail}_C) = 0.1$

Find the probability of System Success.



Reliability Solution

Step 1: Find Probability of Total Failure Assuming failures are independent:

$$P(\text{System Fail}) = P(F_A) \times P(F_B) \times P(F_C)$$

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Step 2: Find Probability of Success (Complement Rule)

$$P(\text{Success}) = 1 - P(\text{System Fail})$$

$$P(\text{Success}) = 1 - 0.001 = 0.999$$

Conclusion: Redundancy improves reliability from 90% (single server) to 99.9%.

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- Use Complement. $P(\text{Miss}) = 1 - 0.4 = 0.6$.
- $P(\text{Miss All}) = 0.6 \times 0.6 \times 0.6 = 0.216$.
- **Ans:** $1 - 0.216 = 0.784$.

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Q3: [Difference]. Two dice are rolled. What is the probability the **difference between the numbers is exactly 3**?

- Pairs: (1,4), (2,5), (3,6), (4,1), (5,2), (6,3). Total 6 pairs.
- Total Outcomes: 36.
- **Ans:** $6/36 = 1/6$.

Thank You!

Next Steps: Conditional Probability and Bayes' Theorem.