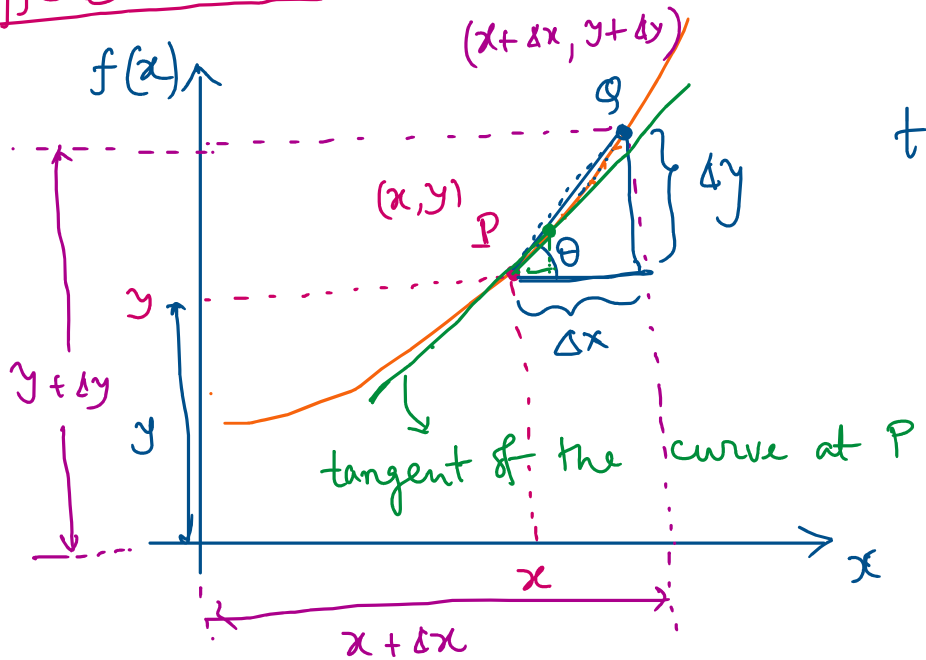
Differentiation

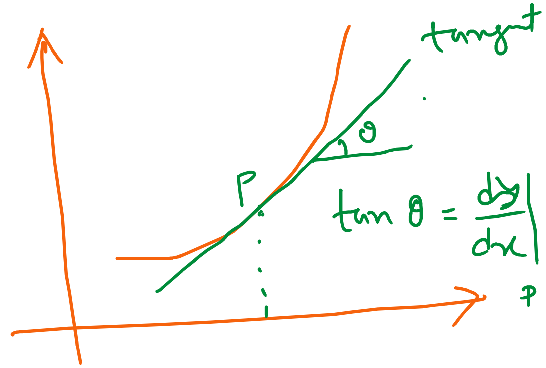


$\overline{PQ}$ chord, what is the slope?

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

So, if Δx is very very small (infinitesimally small) then the ratio $\frac{\Delta y}{\Delta x}$ denotes the slope of the tangent of the graph $y = f(x)$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$ } differentiation of y wrt x.



The geometrical significance of $\frac{dy}{dx}$ is that it is equal to the slope of the tangent of the graph $y = f(x)$ at any point.

Rules of differentiation

$$y = 3x^2 \Rightarrow \frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\delta x}$$ Δy } change in function; δx } change in independent variable.

$$y = f(x) = 3x^2$$

$$\therefore f(x+\Delta x) = 3(x+\Delta x)^2$$

$$\therefore \Delta y = f(x+\delta x) - f(x) = 3(x+\delta x)^2 - 3x^2 = 3\{(x+\delta x)^2 - x^2\}$$

$$[a^2 - b^2 = (a+b)(a-b)]$$

$$\therefore \Delta y = 3 \cdot (x+\delta x+x) \cdot (\cancel{x}+\Delta x - \cancel{x})$$

$$\therefore \delta y = 3(2x+\Delta x) \cdot \Delta x$$

$$\therefore \frac{\delta y}{\Delta x} = 3 \cdot (2x+\delta x)$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 3(2x+\Delta x)$$

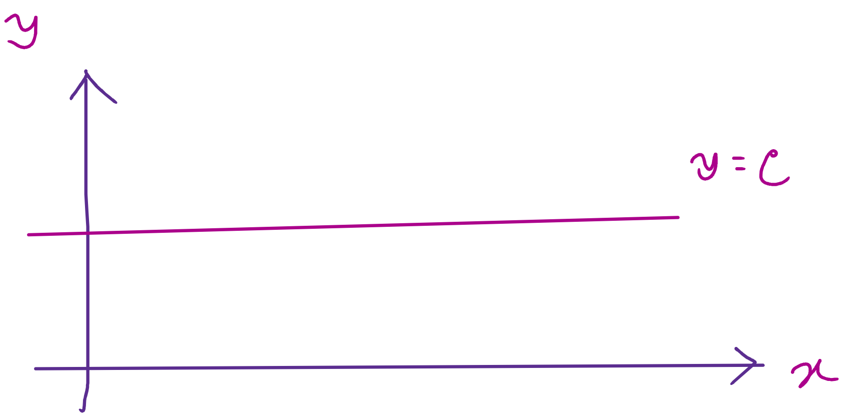
$$= 3 \cdot 2x = 6x$$

$$\therefore \boxed{\frac{d}{dx}(3x^2) = 6x}$$

Differentiation of some common functions

	$f(x)$	$f'(x)$
1)	x^n	nx^{n-1}
2)	c (constant)	0
3)	e^x	e^x
4)	$\log_e(x)$	$\dfrac{1}{x}$
5)	$\sin(x)$	$\cos(x)$
6)	$\cos(x)$	$-\sin(x)$
7)	$\tan(x)$	$\sec^2(x) = \dfrac{1}{\cos^2(x)}$

$\Rightarrow \quad x^2 \xrightarrow{\frac{d}{dx}} 2x \quad ; \quad x^{7/2} \xrightarrow{\frac{d}{dx}} \frac{7}{2} \cdot x^{5/2}$



Rules of differentiation

1. Linearity :

$$f(x) \xrightarrow{\frac{d}{dx}} f'(x) \qquad g(x) \xrightarrow{\frac{d}{dx}} g'(x)$$

$$\frac{d}{dx}\left[\alpha f(x) + \beta g(x)\right] = \alpha f'(x) + \beta g'(x)$$

$$\frac{d}{dx}\left[2x^2 + 3e^x\right] = 2 \cdot \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(e^x) = 4x + 3e^x$$

constant

$$\frac{d}{dx}\left[\log_e(\alpha x)\right] = \frac{d}{dx}\left[\log \alpha + \log(x)\right] \qquad \log(xy) = \log(x) + \log(y)$$

$$= 0 + \frac{d}{dx}\left(\log(x)\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left[\log_e(x^n)\right] = \frac{d}{dx}\left[n \log_e x\right] = n \frac{d}{dx}\left(\log_e x\right) = \frac{n}{x}$$

2. Multiplication Property

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot\frac{d}{dx}\left(g(x)\right) + g(x)\cdot\frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}\left(e^x \sin x\right) = e^x \frac{d}{dx}\left(\sin x\right) + \sin x \frac{d}{dx}\left(e^x\right) = e^x \cos x + e^x \sin x$$

$$= e^x\left(\sin x + \cos x\right)$$

$$\frac{d}{dx}\left(\sin x \cdot \cos x\right) = \sin(x)\cdot\frac{d}{dx}\left(\cos x\right) + \cos(x)\cdot\frac{d}{dx}\left(\sin(x)\right)$$

$$= -\sin^2(x) + \cos^2(x) = \cos^2(x) - \sin^2(x)$$

3. Division Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{\left[g(x)\right]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$\frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cdot\frac{d}{dx}(\sin(x)) - \sin(x)\cdot\frac{d}{dx}(\cos(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

<u>Composite functions</u>

$$f(x) = x^2 \quad \& \quad g(x) = \sin(x)$$

$$g\circ f(x) = g(f(x)) = g(x^2) = \sin(x^2)$$

$$f\circ g(x) = f(g(x)) = f(\sin(x)) = \sin^2(x)$$

$$\underbrace{f(x) = \alpha x + \beta, \quad g(x) = e^{-x}, \quad h(x)}_{} = \frac{1}{1+x}$$

$$h\circ g\circ f(x) = h\circ g(\alpha x+\beta) = h\left(e^{-(\alpha x+\beta)}\right) = \frac{1}{1+e^{-(\alpha x+\beta)}}$$

(sigmoid function)

4. Chain Rule of differentiation

$$\frac{d}{dx}\left(\sin(x^2)\right) = ?$$

$$\frac{d}{dx}\left[\frac{1}{1+e^{-(\alpha x+\beta)}}\right] = ?$$

$$\frac{d}{dx}\left(e^{\alpha x}\right) = ?$$

$$\frac{d}{dx}\left[f \circ g(x)\right] \Rightarrow \quad \text{let, } \quad g(x) = u$$

$$= \frac{d}{dx}\left[f(u)\right] = \frac{d f(u)}{du} \cdot \frac{du}{dx} = \frac{d f(u)}{du} \cdot \frac{d g(x)}{dx}$$

$e^{\alpha x} \Rightarrow f \circ g(x)$

where $f(x) = e^{x}$

& $g(x) = \alpha x = u$

$\therefore e^{\alpha x} = e^{u}$

$$\frac{du}{dx} = \alpha$$

$$\frac{d}{dx}\left(e^{\alpha x}\right) = \frac{d}{dx}\left(e^{u}\right) = \frac{d}{du}\left(e^{u}\right) \cdot \left(\frac{du}{dx}\right) = e^{u} \cdot \alpha$$

$$= \left(\alpha \cdot e^{\alpha x}\right)$$

$$\frac{d}{dx}\left(\sin \omega x\right) = ?$$

$$\frac{d}{dx}\left[\sin(\omega x)\right] = \frac{d}{dx}(\sin u) \qquad (\omega x = u) \qquad \therefore \frac{du}{dx} = \omega$$

$$= \frac{d}{du}(\sin u) \cdot \frac{du}{dx} = \omega \cos u = \omega \cos(\omega x)$$

$$f(x) \xrightarrow{d/dx} f'(x) \Rightarrow f(\alpha x) \xrightarrow{d/dx} \alpha f'(\alpha x)$$

$$\log(x) \xrightarrow{d/dx} \frac{1}{x} \Rightarrow \log(\alpha x) \xrightarrow{d/dx} \cancel{\alpha} \cdot \frac{1}{\cancel{\alpha} x} = \frac{1}{x}$$

$$\cos(x) \xrightarrow{d/dx} -\sin x \Rightarrow \cos(\pi x) \xrightarrow{d/dx} -\pi \sin(\pi x)$$

$$e^{x} \xrightarrow{d/dx} e^{x} \Rightarrow e^{-\alpha x} \xrightarrow{d/dx} (-\alpha) \cdot e^{-\alpha x}$$

$$\frac{d}{dx}\left[\frac{1}{1+e^{-(\alpha x+\beta)}}\right] = ?$$

$$\alpha x + \beta = u \qquad \therefore \frac{d}{dx}\left(\frac{1}{1+e^{-u}}\right)$$

$$\frac{du}{dx} = \alpha$$

$$\Rightarrow \frac{d}{du}\left(\frac{1}{1+e^{-u}}\right) \cdot \frac{du}{dx} = \alpha \frac{d}{du}\left(\frac{1}{1+e^{-u}}\right)$$

$$\frac{d}{dx}\left[\frac{1}{1+e^{-(\alpha x+\beta)}}\right] = \alpha \cdot \frac{d}{du}\left(\frac{1}{1+e^{-u}}\right) = \alpha \cdot \frac{d}{du}\left(\frac{1}{v}\right)$$

$$= \alpha \frac{d}{dv}\left(\frac{1}{v}\right) \cdot \frac{dv}{du}$$

Let, $1+e^{-u} = v$

$$\Rightarrow \frac{dv}{du} = -e^{-u}$$

$$= \alpha \cdot \left(-\frac{1}{v^2}\right) \cdot \left(-e^{-u}\right)$$

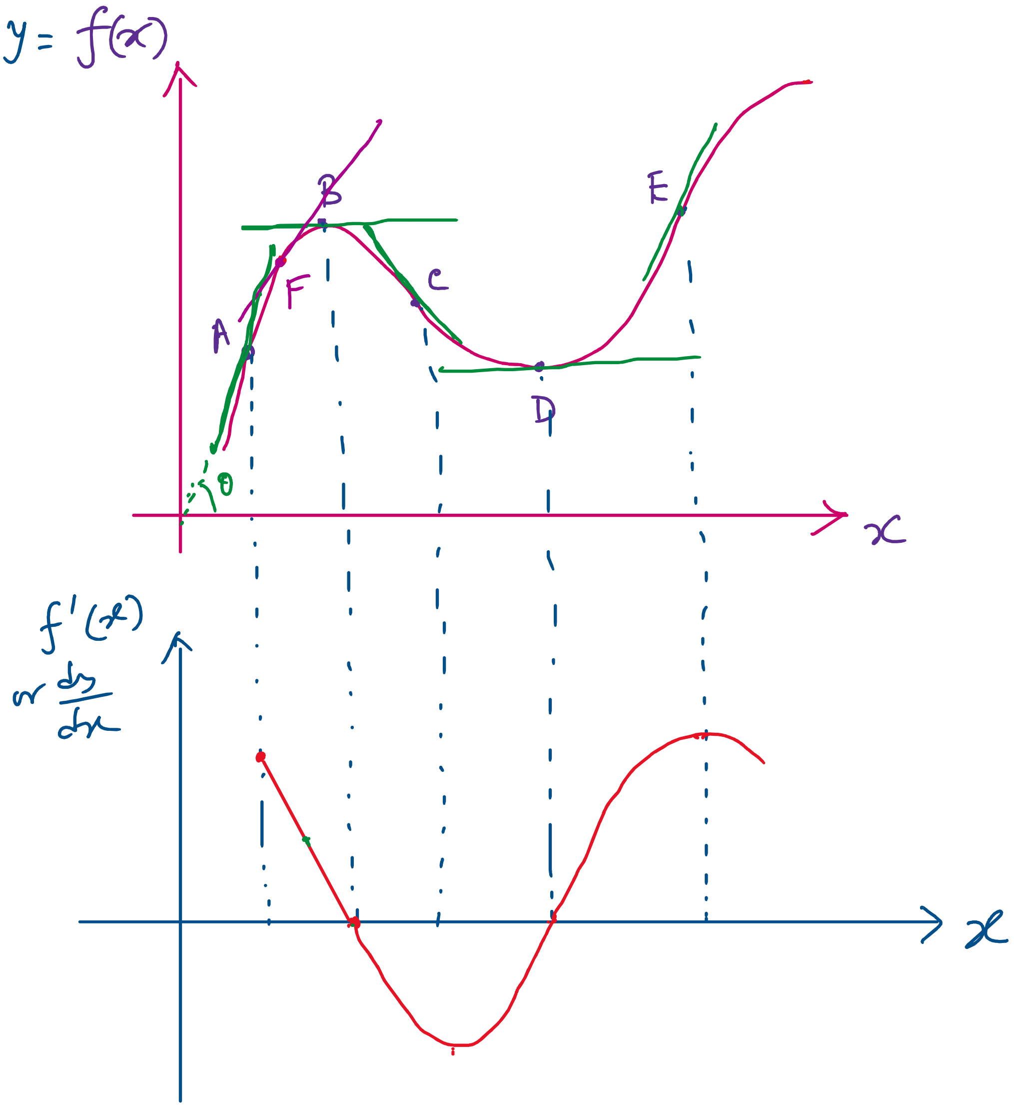
$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -1 \cdot x^{-2}$$

$$= -\frac{1}{x^2}$$

$$= \frac{\alpha e^{-u}}{v^2} = \frac{\alpha e^{-u}}{\left(1+e^{-u}\right)^2}$$

$$\therefore \frac{d}{dx}\left[\frac{1}{1+e^{-(\alpha x+\beta)}}\right] = \frac{\alpha \cdot e^{-(\alpha x+\beta)}}{\left[1+e^{-(\alpha x+\beta)}\right]^2}$$

$$\boxed{\frac{d[h(g(f(x)))]}{dx} = \frac{dh}{dg} \cdot \frac{dg}{df} \cdot \frac{df}{dx}} \longrightarrow \text{chain Rule of differentiation}$$



$\frac{dy}{dx}$ geometrically means the slope of the tangent.

$\left.\frac{dy}{dx}\right|_A > 0$ $\left.\frac{dy}{dx}\right|_C < 0$ $\left.\frac{dy}{dx}\right|_E > 0$

$\left.\frac{dy}{dx}\right|_B = \left.\frac{dy}{dx}\right|_D = 0$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

Successive Differentiation

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

$$y = \sin(x^2) \Rightarrow \frac{d^2y}{dx^2} = ?$$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin(x^2)\right) = \frac{d}{du}(\sin u) \cdot \frac{du}{dx} = 2x\cos(x^2)$$

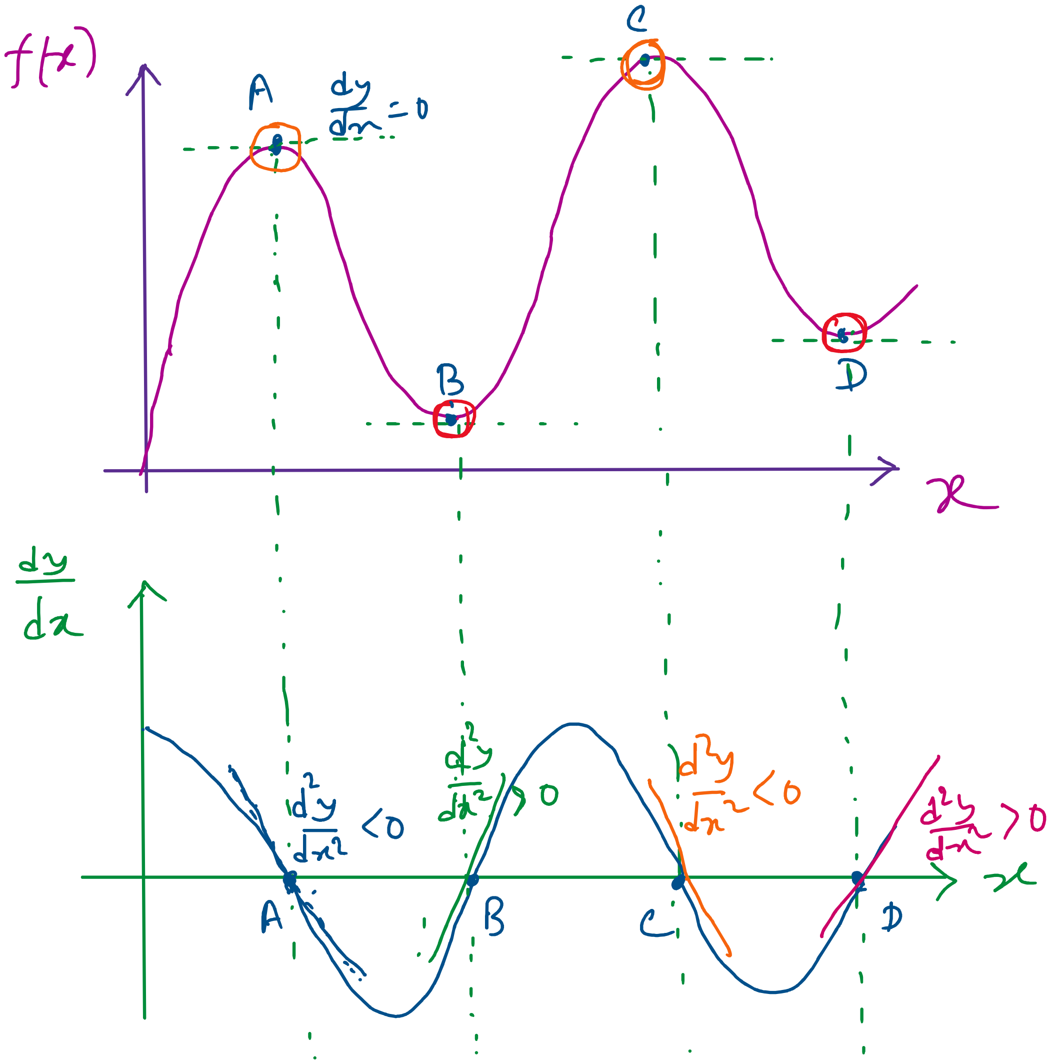
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\cos(x^2)\right) = 2\left[x \cdot \frac{d}{dx}\cos(x^2) + \cos(x^2)\frac{d}{dx}(x)\right]$$

$$= 2\left[x \cdot (2x) \cdot (-\sin(x^2)) + \cos(x^2)\right]$$

$$= 2\left[-2x^2\sin(x^2) + \cos(x^2)\right]$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$$

Maxima & Minima of a function



At maxima & minima the slope of the tangent of a function = 0

$\therefore \frac{dy}{dx} = 0$ at maxima & minima

B, D $\rightarrow$ local minima $\rightarrow \frac{d^2y}{dx^2} > 0$

A, C $\rightarrow$ local maxima $\rightarrow \frac{d^2y}{dx^2} < 0$

minima : $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} > 0$

maxima : $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} < 0$

$y = 4 - (x-2)^2 \Rightarrow$ find the minima / maxima of the function.

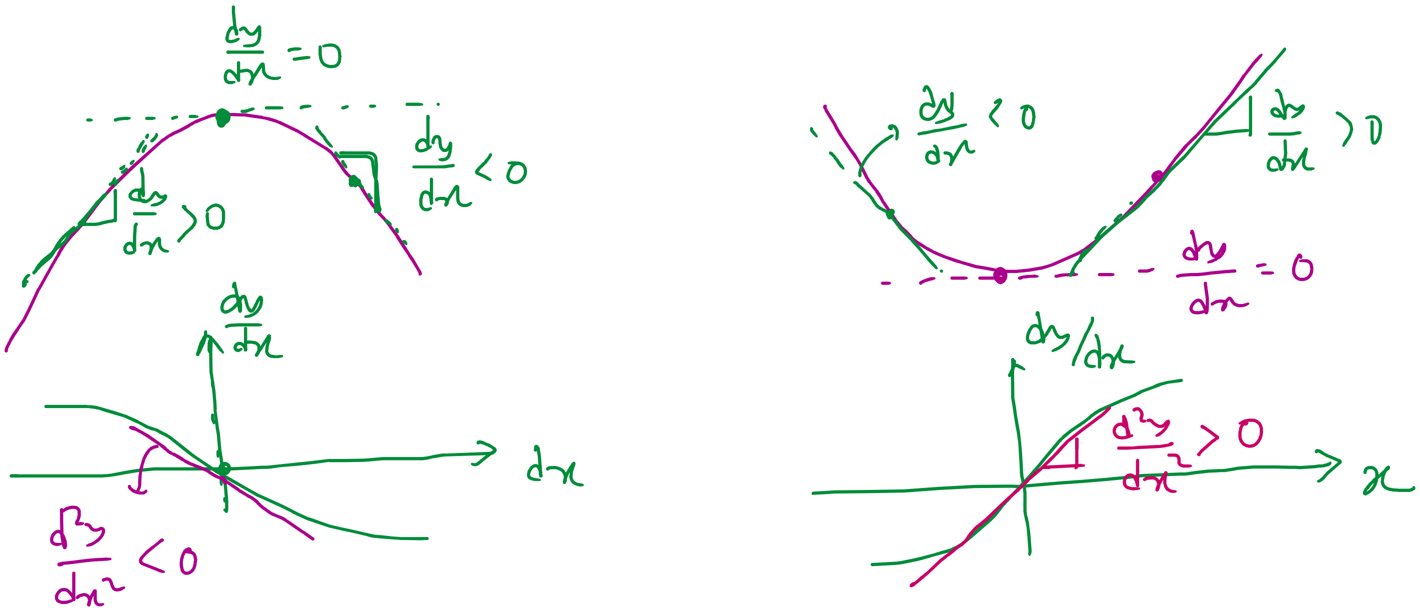
$$\frac{dy}{dx} = -\frac{d}{dx}(x-2)^2 = -2(x-2) = 0 \Rightarrow \boxed{x=2}$$

$$= -2x+4$$

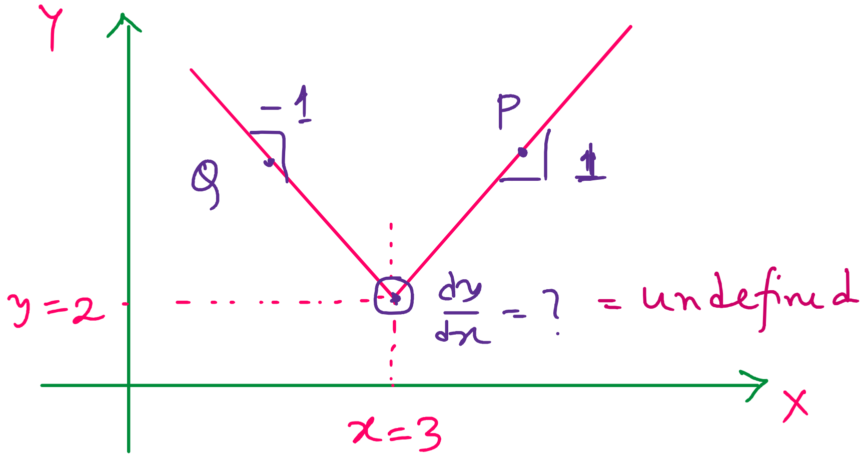
$$\frac{d^2y}{dx^2} = -2 < 0$$

at $x=2$ the function attains maxima

$\therefore$ maximum value of the function $= 4$



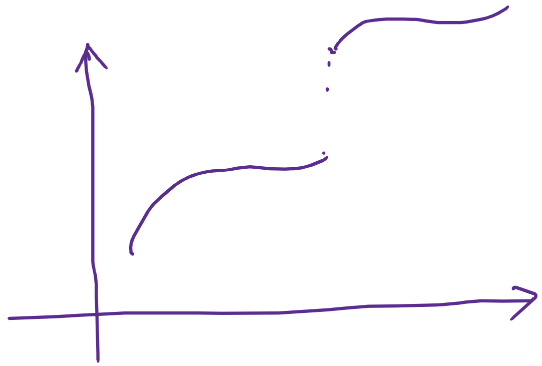
Does differentiation of a function always exist at a point?

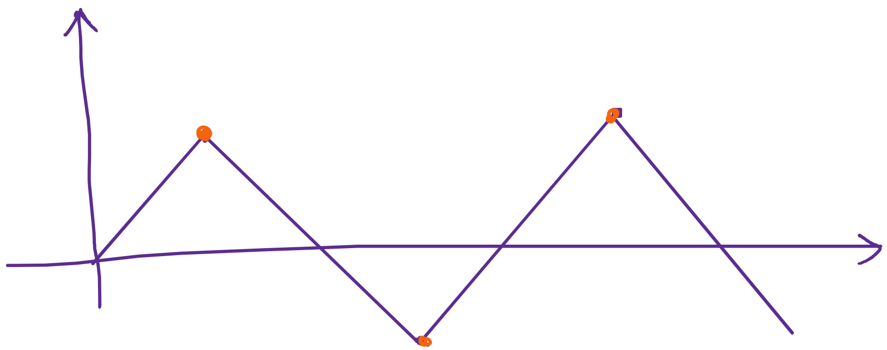


$$y = 2 + |x-3| \Rightarrow y = \begin{cases} x-1 & ; x \geq 3 \\ 5-x & ; x < 3 \end{cases}$$

$$\left.\frac{dy}{dx}\right|_P = ? \qquad \left.\frac{dy}{dx}\right|_Q = ?$$

$\therefore \frac{dy}{dx}$ does not exist at $x = 3$ for the curve $y = 2 + |x-3|$





$y = \lfloor x \rfloor \longrightarrow$ greatest integer function (floor function)

$x = 3.28$, $y = 3$

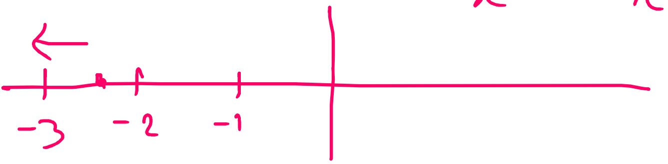
$x = 5.99$, $y = 5$

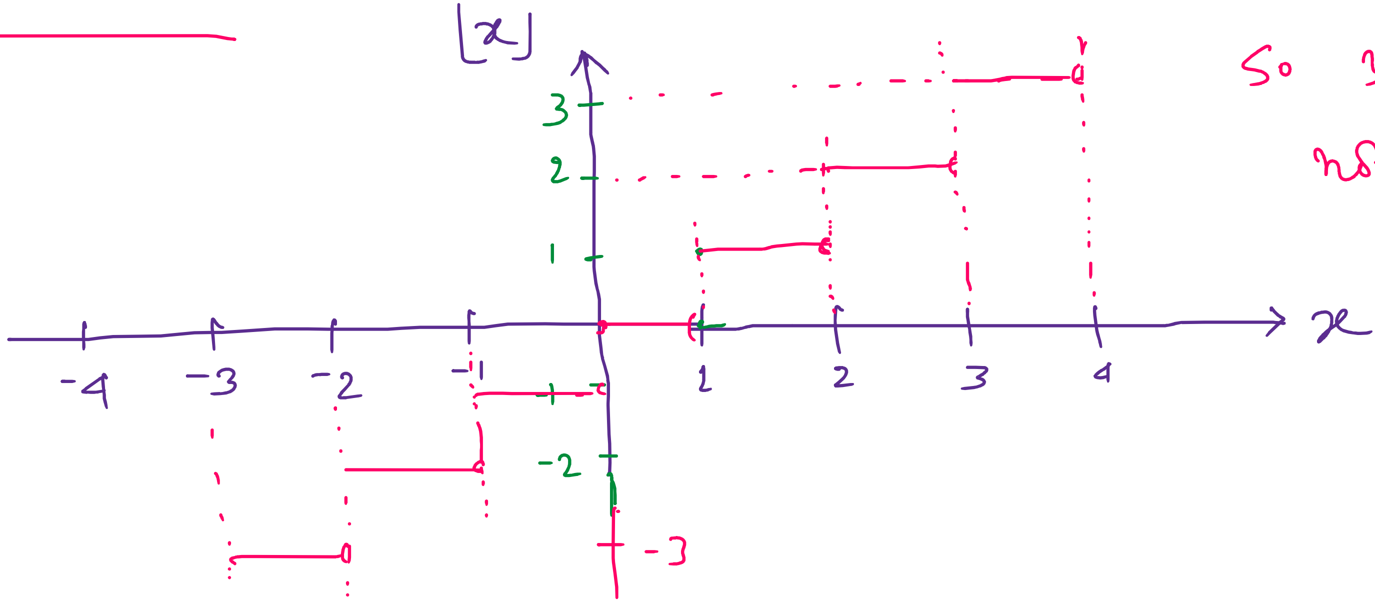
$x = -2.33$, $y = -3$

$x = -\pi$, $y = -4$

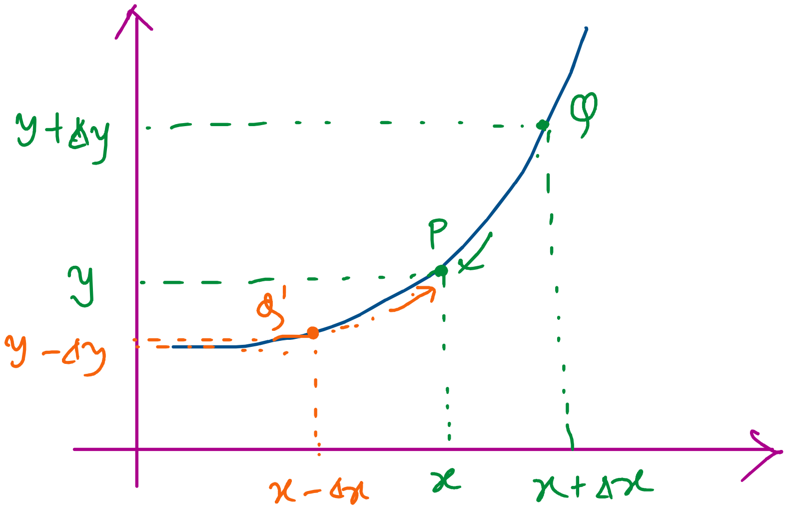
$y = \lfloor x \rfloor \rightarrow$ the integer value which is less than x but maximum among all the integers which are less than x.

$x = 1$, $\lfloor x \rfloor = 1$





So $y = \lfloor x \rfloor$ is not differentiable at $x = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$



When we approach from right (Q):

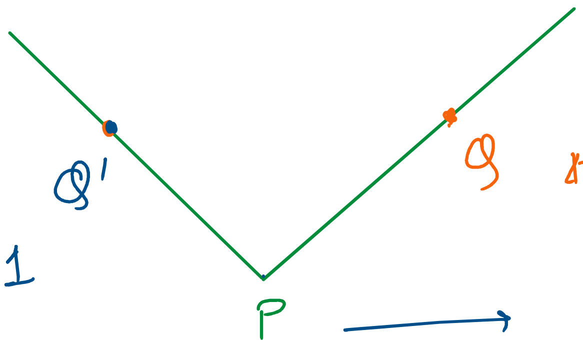
$\lim_{\Delta x \to 0+} \frac{\Delta y}{\Delta x}$ → right limiting value of diff.

When we approach from left (Q'):

$\lim_{\Delta x \to 0-} \frac{\Delta y}{\Delta x}$ → left limiting value of diff

now a function is differentiable if $\lim_{\Delta x \to 0+} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0-} \frac{\Delta y}{\Delta x}$

$\lim_{\Delta x \to 0-} \frac{\Delta y}{\Delta x} = -1$



$\lim_{\Delta x \to 0+} \frac{\Delta y}{\Delta x} = +1$

→ at P the differentiation does not exist.

Integration : There are two ways to approach integration

indefinite integral (anti - derivative)

definite integral (integration with limit)

Indefinite integral

$$\boxed{\frac{dy}{dx} = f(x)} \Rightarrow y = ?$$

$$\frac{dy}{dx} = 3x \Rightarrow y = \frac{3}{2}x^2 + c$$

family of curves

$$\therefore \quad y = \int f(x)\,dx + \underline{c}$$

constant of integration

$$y = \frac{3}{2}x^2 \Rightarrow \frac{dy}{dx} = 3x$$

$$y = \frac{3}{2}x^2 - 5 \Rightarrow \frac{dy}{dx} = 3x$$

$$y = \frac{3x^2}{2} + \pi/2 \Rightarrow \frac{dy}{dx} = 3x$$

Definite integral :

$$\int_a^b f(x)\,dx$$

$$= \lim_{\Delta x \to 0} \sum_{k=0}^{n-1} f(a + k\,\Delta x)\,\Delta x$$

$\therefore$ $\int_a^b f(x)\,dx$ represents the area under the curve from a to b

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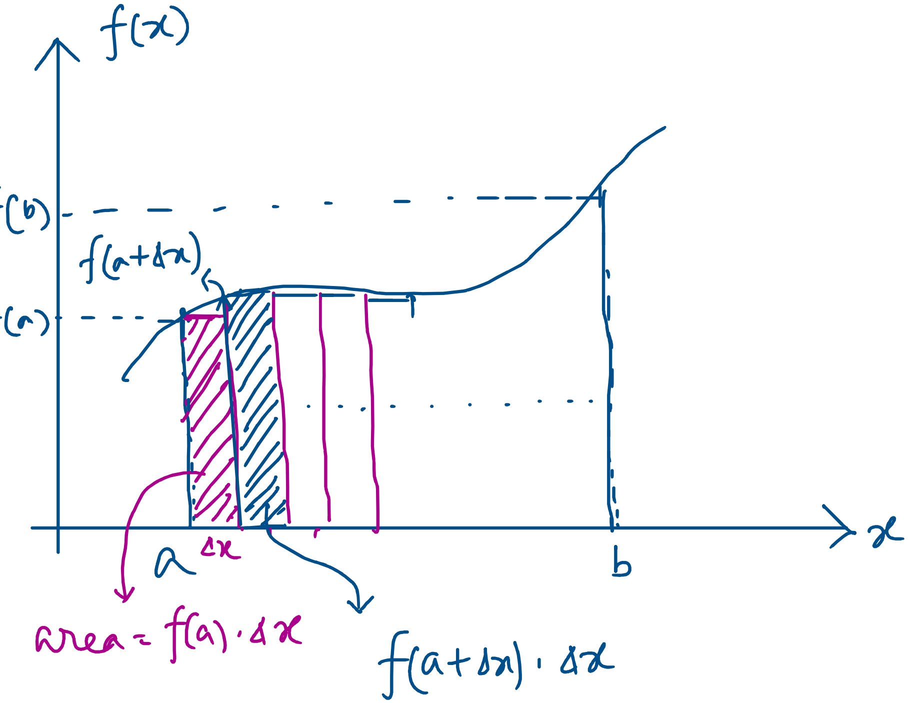
Geometric interpretation of definite integral.



$$f(a)\cdot\Delta x + f(a+\Delta x)\cdot\Delta x + f(a+2\Delta x)\cdot\Delta x + \cdots + f(a+\overline{n-1}\,\Delta x)\cdot\Delta x$$

$$= \left[\sum_{k=0}^{n-1} f(a+k\,\Delta x)\right]\Delta x$$

if $\Delta x \to 0$ then this sum represent area under the curve from a to b

Newton - Leibnitz Theorem / Fundamental theorem

Say, $F(x)$ is the anti-derivative of $f(x)$

i.e. $\int f(x)\,dx = F(x) + c$

then, $\int_a^b f(x)\,dx = F(b) - F(a)$

ex:

$$\int_2^5 (3x)\,dx = \frac{3}{2}x^2\Big]_2^5 = \frac{3}{2}(5^2) - \frac{3}{2}(2^2)$$

$$= \frac{3}{2}(25-4) = \frac{3\times 21}{2} = \frac{63}{2}$$

$$= 31.5$$

Improper integral :-

$$\int_a^b \underline{f(x)}\, dx$$

if any of the limit becomes $+\infty$ or $-\infty$

improper integral.

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = 1}_{\text{Gaussian integral}}$$

$$\int_0^{\infty} e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -(0 - 1) = 1$$