Integration:

$$F(x) \xrightarrow{d/dx} f(x)$$

$$\int \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{differential}} = F(x) + c$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx = \alpha F(x) + C$$

$$\int f(\beta x)\,dx = \frac{1}{\beta} F(\beta x) + C$$

Few rules of integration

	$f(x)$	$\int f(x)\,dx$	example
1.	x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + c$	$\int x^2\,dx = \dfrac{x^3}{3} + c$; $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \dfrac{x^{1/2+1}}{1/2+1} = \dfrac{x^{3/2}}{3/2} = \dfrac{2}{3} x^{3/2}$
2.	$1/x$	$\ln(x) + c$	
3.	e^x	$e^x + c$	$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$
4.	$\sin(x)$	$-\cos(x) + C$	$\int e^{-5x}\,dx = -\frac{1}{5} e^{-5x} + C$
5.	$\cos(x)$	$\sin(x) + C$	

$$\int_2^5 x^2\,dx = \left.\frac{x^3}{3}\right]_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{1}{3}(125-8) = \frac{117}{3} = 39$$

$$\int_0^\infty e^{-x}\,dx = \left. -e^{-x}\right]_0^\infty = (-1)\left(e^{-\infty} - e^0\right)$$
$$= (-1)\,(0-1) = 1$$

e^{-x}

$e^{-\infty} = 0$

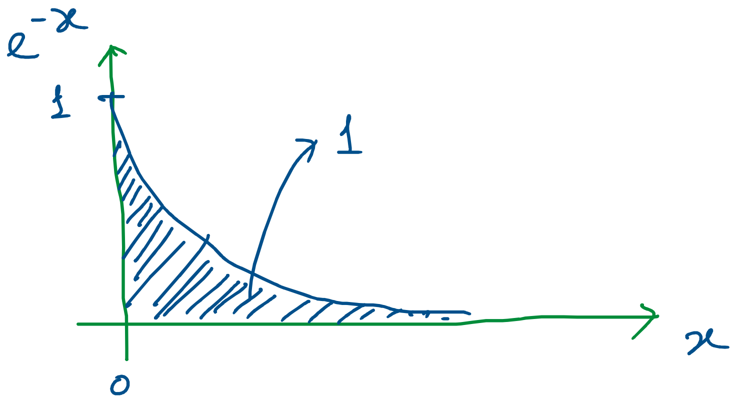
$2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024} < 0.001$

$2^{-200} \longrightarrow 0$

$2^{-(x)} \longrightarrow 0$

as $x \to \infty$

$e = 2.7\ldots$



$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}\,dx = 1$$

Function of multiple variables :

$f(x, y, z) \longrightarrow$ multivariate function.

$f(x, y) = x^2 y + y^2 x$, $g(x, y, z) = e^{-(x+y)} \cdot z^2 \cdot \sin(xz)$

$\Rightarrow g(1, 0, \pi/2) = e^{-1} \cdot \frac{\pi^2}{4} \cdot \sin(\pi/2) = \frac{\pi^2}{4e}$

$\Rightarrow f(1, 1) = 2$

$\Rightarrow f(2, 3) = 2^2 \cdot 3 + 3^2 \cdot 2$
$= 12 + 18 = 30$

Gradient of a function

Gradient is the generalization of first order derivative in multi-variable scenario.

Partial derivative of a function :

$f(x, y) = x^2 (\sin y)$.

Taking differentiation of the function wrt one variable keeping other variables constant.

$f(x, y) = x^2 \sin y$

If I want to take partial derivative of f wrt. x, then we need to consider y as const

I " " " " " " " y, " " " " x " "

$\frac{\partial f}{\partial x}$ → partial derivative of f wrt. x ; $\frac{\partial f}{\partial y}$ → partial derivative of f wrt. y

$$\frac{\partial f}{\partial x} = 2x \sin y \quad ; \quad \frac{\partial f}{\partial y} = x^2 \cos y$$

$$g(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2 e^{-(x_1 + x_2^2 + x_3^3)}$$

$$\frac{\partial g}{\partial x_1} = \frac{\partial}{\partial x_1}\left[(x_1+\alpha)^2 \cdot e^{-(x_1+\beta)}\right] = \left(\frac{\partial}{\partial x_1}(x_1+\alpha)^2\right) e^{-(x_1+\beta)} + (x_1+\alpha)^2 \cdot \frac{\partial}{\partial x_1} e^{-(x_1+\beta)}$$

$$= 2(x_1+\alpha) \cdot e^{-(x_1+\beta)} + (x_1+\alpha)^2 \cdot \left(-e^{-(x_1+\beta)}\right)$$

$$= (x_1+\alpha) e^{-(x_1+\beta)} \cdot \left[2 - (x_1+\alpha)\right]$$

$$= (x_1+x_2+x_3) \cdot e^{-(x_1+x_2^2+x_3^3)} \left\{2 - (x_1+x_2+x_3)\right\}$$

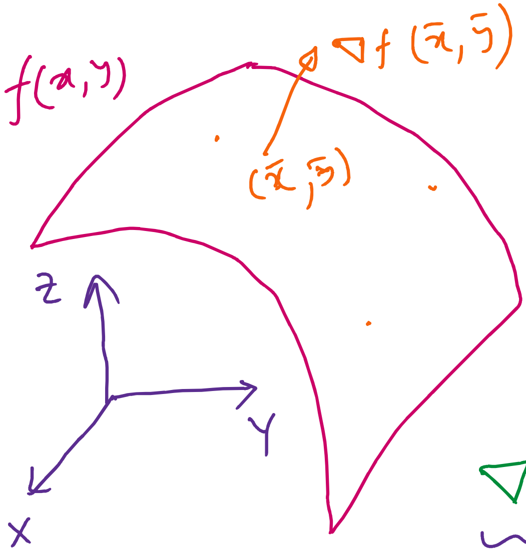
$$\frac{\partial g}{\partial x_2} = ?$$

$$\frac{\partial g}{\partial x_3} = ?$$

Gradient of a function of multiple variables

$f(x_1, x_2, x_3)$ $\qquad \left[\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dfrac{\partial f}{\partial x_3} \right]^T = \nabla f$

Example. $f(x, y) = x^2 \sin y$ $\qquad \therefore \nabla f = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix} = \begin{bmatrix} 2x \sin y \\ x^2 \cos y \end{bmatrix}$



Gradient is always perpendicular to the surface & it shows the rate of change of the function at that point.

$$\underbrace{\nabla f}_{\text{Gradient (grad) of } f} = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

I want to find out if $x_1 \to x_1 + \Delta x_1$, $x_2 \to x_2 + \Delta x_2$, $x_3 \to x_3 + \Delta x_3$ then what will be the change (Δf) of the function $f(x_1, x_2, x_3)$

$$\underline{\Delta f} = \underline{\left(\frac{\partial f}{\partial x_1}\right)} \underline{\Delta x_1} + \underline{\left(\frac{\partial f}{\partial x_2}\right)} \underline{\Delta x_2} + \underline{\left(\frac{\partial f}{\partial x_3}\right)} \underline{\Delta x_3} = (\nabla f)^T \cdot (\Delta \vec{x})$$

Software company: ✓ Staff salary /head, ✓ It cost /equip, ✓ licensing cost / license.

+10 new staff, −2 It equipment, +3 licenses.

ΔC = staff salary × 10 + It cost × (−2) + license cost × (3)

$$C = f(\text{salary}, \text{equip}, \text{license})$$

$$\Delta C = \left(\frac{\partial C}{\partial S}\right) \Delta S + \left(\frac{\partial C}{\partial e}\right) \Delta e + \left(\frac{\partial C}{\partial L}\right) \Delta L$$

So, in case of single variable function we used the first derivative to identify the points where the function will reach maxima / minima

We can extend this concept in case of multi variable function.

$\nabla f = 0$ } optimum points. $\qquad f(x_1, x_2, x_3)$

$$\nabla f = 0 \Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = 0 \Rightarrow \frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0 \ \& \ \frac{\partial f}{\partial x_3} = 0$$

Like, in single variable case we used second order derivative to compute the curvature of a function & to know whether an optimum point is max or minima, we can use similar concept in multivariable scenario.

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}_{3\times 3}$$

Hessian matrix

$(i, j)^{th}$ element

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$f(x,y,z) = 3\cdot\underbrace{(x-2)^2}_{\geq 0} + 4\underbrace{(y-3)^2}_{\geq 0} + 5\underbrace{(z-1)^2}_{\geq 0} + 12$$

$$\frac{\partial f}{\partial x} = 6(x-2) \quad ; \quad \frac{\partial f}{\partial y} = 8(y-3) \quad ; \quad \frac{\partial f}{\partial z} = 10(z-1)$$

$= 0$ $\quad\quad$ $= 0$ $\quad\quad$ $= 0$

$\Rightarrow x = 2$ $\quad\quad$ $y = 3$ $\quad\quad$ $z = 1$

$(2,3,1) \rightarrow$ at this point the function attains its minimum and the minimum value of the function is 12.